Existence and Attractors of Solutions for Nonlinear Parabolic Systems

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Abstract: We prove existence and asymptotic behaviour results for weak solutions of a mixed problem (S). We also obtain the existence of the global attractor and the regularity for this attractor in $\left[H^2(\Omega)\right]^2$ and we derive estimates of its Haussdorf and fractal dimensions.

Keywords: Nonlinear parabolic systems; existence of solutions; global attractor; asymptotic behaviour; Haussdorf and fractal dimensions.
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0. Introduction

We consider the following nonlinear system

$$(S)\begin{cases} \frac{\partial b_1(u_1)}{\partial t} - \triangle u_1 + f_1(x,u_1,u_2) = 0 & \text{in } \Omega\times(0,T) \\ \frac{\partial b_2(u_2)}{\partial t} - \triangle u_2 + f_2(x,u_1,u_2) = 0 & \text{in } \Omega\times(0,T) \\ u_1 = u_2 = 0 & \text{in } \quad \partial\Omega\times(0,T) \\ (b_1(u_1(x,0), b_2(u_2(x,0)) = (b_1(\varphi_0(x)), b_2(\psi_0(x))) & \text{in } \Omega \end{cases}$$

where Ω is a bounded open subset in R^N , $N \geq 1$, with a smooth boundary $\partial\Omega$. (**S**) is an example of nonlinear parabolic systems modelling a reaction diffusion process for which many results on existence, uniqueness and regularity have been obtained in the case where $b_i(s) = s$ (see, for instance [6, 7, 18]).

The case of a single equation of the type (**S**) is studied in [1, 2, 3, 4, 5, 8, 9, 19]. The purpose of this paper is the natural extension to system (**S**) of the results by [8], which concerns the single equation $\frac{\partial\beta(u)}{\partial t} - \Delta u + f(x,t,u) = 0$.

Actually, our work generalizes the question of existence and regularity of the global attractor obtained therein.

In the first section of this paper, we give some assumptions and preliminaries and in section 2, we prove the existence of absorbing sets and the existence of the gobal attractor; while in section 3, we present the regularity of the attractor and show stabilization property. Finally, section 4 is devoted to estimates of the Haussdorf and fractal dimensions.

1. Preliminaries, Existence and Uniqueness

1.1 Notations and Assumptions

Let b_i, $(i = 1,2)$ be continuous functions with $b_i(0) = 0$. We define for $t \in R$

$\Psi_i(t) = \int_0^t b_i(\tau)d\tau$. Then the Legendre transform Ψ^* of Ψ is defined by $\Psi_i^*(\tau) = \sup\limits_{s\in R}\{\tau s - \Psi_i(s)\}$. Ω stands for a regular open bounded subset of

R^N and for any $T>0$, we set $Q_T=\Omega\times(0,T)$ and $S_T=\partial\Omega\times(0,T)$, where $\partial\Omega$ is the boundary of Ω . The norm in a space X will be denoted by : $\|.\|_r$ if $X=L^r(\Omega)$ for all r : $1\leq r\leq+\infty$, $\|.\|_X$ otherwise and $\langle .,.\rangle_{X,X'}$ will denote the duality product between X and its dual X'.

We start by introducing our assumptions and making precise the meaning of a solution of (**S**). Consider the system (S) under the following assumptions:

(H1) $(\varphi_0,\psi_0)\in L^2(\Omega)\times L^2(\Omega)$.

(H2) b_i is an increasing continuous function from R into R , $b_i(0)=0$, and there exists $c_{ij}>0$ such that : $|b_i(s)|\leq c_{i1}|s|+c_{i2}$, for all $s\in R$, $i=1,2$.

(H3) $f_i\in C^1(\overline{\Omega}\times R\times R)$.

(H4) $\forall x\in\Omega,\forall\xi\in R,\exists c_1>0,c_2>0$:

$$\begin{cases} sign(\xi)f_1(x,\xi,0)\geq -c_1\\ sign(\xi)f_2(x,0,\xi)\geq -c_2.\end{cases}$$

(H5) For any $N>0,\exists c_3>0,c_4>0,c_5>0$:

$$\begin{cases} sign(\xi)f_1(x,\xi,v)\geq c_3|\xi|^{p_1-1}-c_4\\ |f_1(x,\xi,v)|\leq c_5(|\xi|^{p_1-1}+1) \qquad p_1>2\\ |f(x,u,v)|\leq a_1(|u|),\text{ where } a:R^+\to R^+\ \ is\ \ increasing\\ \text{for any } v:|v|\leq N.\end{cases}$$

(H6) For any $M>0,\exists c_6>0,c_7>0,c_8>0$:

$$\begin{cases} sign(\xi)f_2(x,u,\xi)\geq c_6|\xi|^{p_2-1}-c_7\\ |f_2(x,u,\xi)|\leq c_8(|\xi|^{p_2-1}+1) \qquad p_2>2\\ |f_2(x,u,v)|\leq a_2(|v|),\text{ where } a_2:R^+\to R^+\ \ is\ \ increasing\\ \text{for any } u:|u|\leq M.\end{cases}$$

(H7) $0<\gamma_i\leq b_i'(s)$ for all $s\in R$.

Definition By a weak solution of (S), we mean an element

$u_i\in L^{p_i}(0,T;L^{p_i}(\Omega))\cap L^2(0,T;H_0^1(\Omega))\cap L^\infty(t_0,T;L^\infty(\Omega))$, for all $t_0>0$ such that

$\frac{\partial b_i(u_i)}{\partial t}\in L^{p_i^*}(0,T;L^{p_i^*}(\Omega))+L^2(0,T;H^{-1}(\Omega))$ and $\forall\phi_i\in L^2(0,T;H^{-1}(\Omega))$:

$\int_0^T\left\langle\frac{\partial b_i(u_i)}{\partial t},\phi_i\right\rangle_{V_i^*,V_i}dt+\int_0^T\int_\Omega\nabla u_i\nabla\phi_i dxdt+\int_0^T\int_\Omega f_i(x,u_1,u_2)dxdt=0,$

and if $(\phi_i)_t\in L^2(0,T;L^2(\Omega)),\phi_i(T)=0$

$\int_0^T\left\langle\frac{\partial b_i(u_i)}{\partial t},\phi_i\right\rangle_{V_i',V_i}dt=-\int_0^T\int_\Omega\left(b_i(u_i(t)-b_i(u_i(x,0))\right)(\phi_i)_t dxdt,$

where $V_i=L^{p_i}(\Omega)\cap H_0^1(\Omega)$, $V_i'=L^{p_i'}(\Omega)+H^{-1}(\Omega)$, $\frac{1}{p_i'}+\frac{1}{p_i}=1,\ \ i=1,2$.

1.2. Existence theorem.

Theorem 1 Let (H1) to (H6) be satisfied. Then there exists a solution (u_1,u_2) of problem (S) such that for $i=1,2$, we have

$u_i\in L^{p_i}(0,T;L^{p_i}(\Omega))\cap L^2(0,T;H_0^1(\Omega))\cap L^\infty(t_0,T;L^\infty(\Omega)),\forall t_0>0$

Proof: By theorem 3.2 in [8] ,we can choose $u_i^0\in L^{p_i}(Q_T)\cap L^2(0,T;H_0^1(\Omega))\cap L^\infty(\tau,T;L^\infty(\Omega))$, for any $\tau>0$ such that :

$$\begin{cases} \frac{\partial b_1(u_1^0)}{\partial t}-\triangle u_1^0+f_1(x,u_1^0,0)=0 & \text{in } Q_T\\ u_1^0=0 & \text{in } S_T\\ b_1(u_1^0)_{t=0}=b_1(\varphi_0) & \text{in } \Omega\end{cases}$$

$$\begin{cases} \frac{\partial b_2(u_2^0)}{\partial t} - \triangle u_2^0 + f_2(x,0,u_2^0) = 0 & \text{in } Q_T \\ u_2^0 = 0 & \text{in } S_T \\ b_1(u_2^0)_{t=0} = b_1(\psi_0) & \text{in } \Omega \end{cases}$$

and we construct two sequences of functions (u_1^n) and (u_2^n), such that :

$$\begin{cases} \frac{\partial b_1(u_1^n)}{\partial t} - \triangle u_1^n + f(x,u_1^n,u_2^{n-1}) = 0 & \text{in } Q_T & (1.1) \\ u_1^n = 0 & \text{in } S_T & (1.2) \\ b_1(u_1^n)_{t=0} = b_1(\varphi_0) & \text{in } \Omega & (1.3) \end{cases}$$

$$\begin{cases} \frac{\partial b_2(u_2^n)}{\partial t} - \triangle u_2^n + f_2(x,u_1^{n-1},u_2^n) = 0 & \text{in } Q_T & (1.4) \\ u_2^n = 0 & \text{in } S_T & (1.5) \\ b_2(u_2^n)_{t=0} = b_2(\psi_0) & \text{in } \Omega & (1.6) \end{cases}$$

We need lemma 1 and lemma 2 below to complete the proof of theorem 1. From now on we denote by c_i various positive constants independent of n.

Lemma 1

$$\forall \tau > 0, \exists c_\tau > 0 \quad such \quad that \quad \|u_i^n\|_{L^\infty(\tau,T;L^\infty(\Omega))} \leq c_\tau. \tag{1.7}$$

Proof : For $n = 0$, (1.7) is proved in [7]. So, suppose (1.7) for $(n-1)$. Multiplying (1.1) by $|b_1(u_1^n)|^k \, b_1(u_1^n)$ and using (H2),(H5) , we obtain :

$$\frac{1}{k+2}\int_\Omega |b_1(u_1^n)|^{k+2}\,dx + c_9\int_\Omega |b_1(u_1^n)|^{k+p_1}\,dx \leq c_{10}\int_\Omega |b_1(u_1^n)|^{k+1}\,dx.$$

Setting $y_{k,n}(t) = \|b_1(u_1^n)\|_{L^{k+2}(\Omega)}$ and using Holder's inequality on both sides, we have the existence of two constants $\lambda > 0$ and $\delta > 0$ such that

$$\frac{dy_{k,n}(t)}{dt} + \lambda y_{k,n}^{p_1-1}(t) \leq \delta,$$

which implies from lemma 5.1([22]) that $\forall t \geq \tau > 0$

$$y_{k,n}(t) \leq (\frac{\delta}{\lambda})^{\frac{1}{p_1-1}} + \frac{1}{[\lambda(p_1-2)t]^{\frac{1}{p_1-2}}} \quad .$$

As $k \to \infty$, we obtain

$$|u_1^n(t)| \leq c_\tau \qquad \forall t \geq \tau > 0\,.$$

The same holds also for u_2^n.■

Lemma 2 $\forall \tau > 0, \exists c_i = c_i\,(\tau,\varphi_0,\psi_0) > 0$:

$$\begin{aligned} \|u_i^n\|_{L^2(0,T;H_0^1(\Omega))} &\leq c_{11}, \\ \|u_i^n\|_{L^\infty(\tau,T;H_0^1(\Omega)\cap L^\infty(\Omega))} &\leq c_{12} \end{aligned}$$

and

$$\sum_{i=1}^{2}\left[\int_0^T\int_\Omega |\nabla u_i^n|^2\,dx + c_{13}\int_0^T\int_\Omega |u_i^n|^{p_i}\,dx\right] \leq c_{14}.$$

Proof of lemma 2 : Multiplying (1.1) by u_1^n and (1.4) by u_2^n, and adding , we get :

$$\frac{d}{dt}\sum_{i=1}^{2}\left[\int_\Omega \Psi_i^*(b_i(u_i^n))dx\right]+\sum_{i=1}^{2}\int_\Omega |\nabla u_i^n|^2\,dx+c_{15}\sum_{i=1}^{2}\int_\Omega |u_i^n|^{p_i}\,dx\le c_{16}. \quad (1.8)$$

But

$|\varphi_0|_{L^2(\Omega)}+|\psi_0|_{L^2(\Omega)}\le c\Rightarrow \int_\Omega \Psi_1^*(b_1(\varphi_0))dx+\int_\Omega \Psi_2^*(b_2(\psi_0))dx\le c$,

so we deduce that $\sum_{i=1}^{2}\int_0^T\int_\Omega |\nabla u_i^n|^2\,dx+c_{17}\sum_{i=1}^{2}\int_0^T\int_\Omega |u_i^n|^{p_i}\,dx\le c_{18}$.

Whence lemma 2.

From lemma 2 and Lemma 1, there is a subsequence u_i^n $(i=1,2)$ with the following properties:

$u_i^n\to u_i$ weakly in $L^2(0,T;H_0^1(\Omega))\cap L^{p_i}(0,T;L^{p_i}(\Omega))$

$b_i(u_i^n)\to\chi_i$weakly in $L^2(0,T;L^2(\Omega))$

$b_i(u_i^n)\to\chi_i$strongly in $L^2(\tau,T;H^{-1}(\Omega))$ (by the compactness result of Aubin (see [22])). By lemma 7([9]), we have $\chi_i=b_i(u_i)$. Moreover,

$f_1(.,u_1^n,u_2^{n-1})$ converges to $f_1(.,u_1,u_2)$ in $L^r(\tau,T;L^r(\Omega)),\forall r\ge 1,\forall\tau\ge 1$

and $f_2(.,u_1^{n-1},u_2^n)$ converges to $f_2(.,u_1,u_2)$ in $L^r(\tau,T;L^r(\Omega)),\forall r\ge 1$;

taking the limit as n goes to ∞ , we deduce that (u_1,u_2) is a weak solution of (S).

1.3. Uniqueness

Theorem 2.

Assume that f_1 and f_2 verify :

$$(\mathrm{H8})\begin{cases}\forall M>0,\forall N>0,\exists c_M>0,c_N>0:\\ \forall u,\overline{u},v,\overline{v}:|u|+|\overline{u}|\le M \text{ and } |v|+|\overline{v}|\le N, \text{ we have}\\ |f_1(x,u,v)-f_1(x,\overline{u},\overline{v})|^2+|f_2(x,u,v)-f_2(x,\overline{u},\overline{v})|^2\le\\ c_M(b_1(u)-b_1(\overline{u}))(u-\overline{u})+c_N(b_2(v)-b_2(\overline{v}))(v-\overline{v}).\end{cases}$$

Then (S) has a unique solutions (u,v) in Q_T.

Proof : Let (u,v) and ($\overline{u},\overline{v}$) be solutions of (S); then we have :

$$\frac{\partial(b_1(u)-b_1(\overline{u}))}{\partial t}-\triangle(u-\overline{u})=f_1(x,\overline{u},\overline{v})-f_1(x,u,v) \quad (1.9)$$

and

$$\frac{\partial(b_2(v)-b_2(\overline{v}))}{\partial t}-\triangle(v-\overline{v})=f_2(x,\overline{u},\overline{v})-f_2(x,u,v). \quad (1.10)$$

Multiplying (1.9) by $w_1=(-\triangle)^{-1}(b_1(u)-b_1(\overline{u}))$ and (1.10) by $w_2=(-\triangle)^{-1}(b_2(v)-b_2(\overline{v}))$ and adding, we get

$$\frac{1}{2}\frac{d}{dt}\left[\|b_1(u)-b_1(\overline{u})\|_{H^{-1}(\Omega)}^2+\|b_2(v)-b_2(\overline{v})\|_{H^{-1}(\Omega)}^2\right]+$$
$$\big(b_1(u)-b_1(\overline{u}),u-\overline{u}\big)_{L^2(\Omega)}+\big(b_2(v)-b_2(\overline{v}),v-\overline{v}\big)_{L^2(\Omega)}\le$$
$$c\,\|f_1(x,u,v)-f_1(x,\overline{u},\overline{v})\|_{H^{-1}(\Omega)}\,\|b_1(u)-b_1(\overline{u})\|_{H^{-1}(\Omega)}+$$

$$c\,\|f_2(x,u,v)-f_2(x,\overline{u},\overline{v})\|_{H^{-1}(\Omega)}\,\|b_2(v)-b_2(\overline{v})\|_{H^{-1}(\Omega)}\,. \qquad (1.11)$$

From hypothesis (H8) we obtain

$$\|f_1(x,u,v)-f_1(x,\overline{u},\overline{v})\|^2_{H^{-1}(\Omega)}+\|f_2(x,u,v)-f_2(x,\overline{u},\overline{v})\|^2_{H^{-1}(\Omega)}$$

$$\leq c\left[\|f_1(x,u,v)-f_1(x,\overline{u},\overline{v})\|^2_{L^2(\Omega)}+\|f_2(x,u,v)-f_2(x,\overline{u},\overline{v})\|^2_{L^2(\Omega)}\right]$$

$$\leq cc_M(b_1(u)-b_1(\overline{u}),u-\overline{u})_{L^2(\Omega)}+cc_N(b_2(v)-b_2(\overline{v}),v-\overline{v})_{L^2(\Omega)}\ , \qquad (1.12)$$

where $M=\|u\|_{L^\infty(0,T;L^\infty(\Omega))}+\|\overline{u}\|_{L^\infty(0,T;L^\infty(\Omega))}$ and $N=\|v\|_{L^\infty(0,T;L^\infty(\Omega))}+\|\overline{v}\|_{L^\infty(0,T;L^\infty(\Omega))}$.

Therefore, using Schwartz inequality in (1.11), the fact that ($b_i, i=1,2$) is increassing and (1.12), we deduce that

$$\frac{d}{dt}\left[\|b_1(u)-b_1(\overline{u})\|^2_{H^{-1}(\Omega)}+\|b_2(v)-b_2(\overline{v})\|^2_{H^{-1}(\Omega)}\right]\leq$$
$$c\frac{d}{dt}\left[\|b_1(u)-b_1(\overline{u})\|^2_{H^{-1}(\Omega)}+\|b_2(v)-b_2(\overline{v})\|^2_{H^{-1}(\Omega)}\right].$$

Thus, we deduce that $b_1(u)=b_1(\overline{u})$ and $b_2(v)=b_2(\overline{v})$, hence $u=\overline{u}$ and $v=\overline{v}$.

Remark 1. Theorem 1 establishes the existence of dynamical system $\{S(t)\}_{t\geq 0}$ which maps $L^2(\Omega)\times L^2(\Omega)$ into $L^2(\Omega)\times L^2(\Omega)$ such that $S(t)(\varphi_0,\psi_0)=(u_1(t),u_2(t))$.

2. Global attractor

Proposition 1 Assume that (H1)-(H8) hold; then the solution (u_1,u_2) of system (**S**) satisfies :

$$|u_1(t)|_{L^\infty(\Omega)}+|u_2(t)|_{L^\infty(\Omega)}\leq c(t_0) \qquad \forall t\geq t_0 \qquad (2.0)$$

and

$$\sum_{i=1}^{2}\int_\Omega |\nabla u_i|^2\,dx\leq c \qquad \forall t\geq t_0+r \qquad (2.1)$$

Proof : Reasoning as the proof of lemma 1, we also have(2.0).

Multiplying the first equation of (**S**) by u_1 and the second by u_2, by (H2) and (2.5), we get :

$$\frac{d}{dt}\sum_{i=1}^{2}\int_{\Omega}\Psi_i^*(b_i(u_i))dx+\sum_{i=1}^{2}\frac{1}{2}\int_{\Omega}\left|\nabla u_i\right|^2dx=-\sum_{i=1}^{2}\int_{\Omega}f_i(x,u)u_idx\leq c. \quad (2.2)$$

For fixed $r>0$ and $\tau>0$, integrate (2.2) on $]t,t+r[$

$$\forall t\geq\tau>0\quad\sum_{i=1}^{2}\int_{t}^{t+r}\int_{\Omega}\left|\nabla u_i\right|^2dxds\leq c(\tau). \quad (2.3)$$

Multiplying the first equation of (S) by $(u_1)_t$ and the second by $(u_2)_t$, we get

$$\frac{1}{2}\frac{d}{dt}(\sum_{i=1}^{2}\int_{\Omega}\left|\nabla u_i\right|^2dx)+\sum_{i=1}^{2}\int_{\Omega}(b_i'(u_i)(\frac{\partial u_i}{\partial t})^2dx= \quad (2.4)$$

$$\sum_{i=1}^{2}\int_{\Omega}f_i(x,u)(u_i)_t\leq\quad\frac{1}{2}\sum_{i=1}^{2}\int_{\Omega}\frac{f_i^2(x,u)}{b_i'(u_i)}dx+\frac{1}{2}\sum_{i=1}^{2}\int_{\Omega}(b_i'(u_i)(\frac{\partial u_i}{\partial t})^2dx.$$

By (H7) and the properties of functions f_i , we obtain:

$$\frac{d}{dt}\left[\sum_{i=1}^{2}\int_{\Omega}\left|\nabla u_i\right|^2dx\right]\leq c(\tau). \quad (2.5)$$

From the uniform Gronwall's lemma see [22], we get (2.1).■

Remark 2. By proposition1 we deduce that there exist absorbing sets in $L^{\sigma_1}(\Omega)\times L^{\sigma_1}(\Omega)$ for any $\sigma_i:1\leq\sigma_i\leq+\infty$ and absorbing sets in $H_0^1(\Omega)\times H_0^1(\Omega)$; then assumptions (1.1) ,(1.4) and (1.12) in theorem 1.1 [22, *p*.23] are satisfied with $U=\left[L^2(\Omega)\right]^2$, so we have the following :

Theorem 2. Assume that (H1) - (H7) are satisfied. Then the semi-group $S(t)$ associated with the boundary value problem (**S**) possesses a maximal attractor A, which is bounded in $\left[H_0^1(\Omega)\cap L^\infty(\Omega)\right]\times\left[H_0^1(\Omega)\cap L^\infty(\Omega)\right]$, compact and connected in $\left[L^2(\Omega)\right]^2$. Its domain of attraction is the whole space $\left[L^2(\Omega)\right]^2$.

3. A regularity property of the attractor

In this section we shall show supplementary regularity estimates on the solution of problem (**S**) and by use of them, we shall obtain more regularity on the attractor obtained in section 3. We shall assume that

$$(H9)\left\{N\leq 3 \text{ and } b_i \text{ is of class } \mathcal{C}^3.\right.$$

Hereafter, we shall assume that there exist positive constants $\delta_i > 0$ and a function Φ from R^{N+2} to R such that :

$$(H10)\left\{\begin{array}{c} f_i(x,u) = f_i(u) - h_i(x) = \delta_i \frac{\partial \Phi}{\partial u_i} \\ f_i \text{ satisfying (H3) to (H6) and } h_i \in L^\infty(\Omega). \end{array}\right.$$

We shall denote : $r(t) = \sum_{i=1}^{2} \int_\Omega b_i'(u_i) \left(u_i'\right)^2 dx.$

Theorem 3 Let f_i and b_i satisfies hypothesis (H1) to (H10). Then the solution (u_1, u_2) of problem (**S**) satisfies the following regularity estimates:

$$\frac{\partial\, b_i(u_i)}{\partial t} \in L^2(t_0, +\infty; L^2(\Omega)\ ), \tag{3.0}$$

$$\frac{\partial\, \nabla u_i}{\partial t} \in L^2(t_0, +\infty; L^2(\Omega)\,), \tag{3.1}$$

and

$$u_i \in H^2\,(\Omega)\,. \tag{3.2}$$

To prove this theorem, we need the following lemma:

Lemma 3 Under the assumptions of theorem 3, there exist constants $C = C(\varphi_0, \psi_0)$, such that for any $T > 0$:

$$\|u_i\|_{L^\infty(0,T,H_0^1(\Omega))} \leq C < \infty \tag{3.3}$$

and

$$\left\|\frac{\partial u_i}{\partial t}\right\|_{L^2(Q_T)} \leq C < \infty. \tag{3.4}$$

Proof of lemma 3 : Multiplying the equation $\frac{\partial b_i(u_i)}{\partial t} - div\,[\nabla u_i] + \delta_i \frac{\partial \Phi}{\partial u_i} = 0$ by $\frac{1}{\delta_i}\,(u_i)_t$ and adding the two equations, we obtain :

$$\sum_{i=1}^{2} \frac{1}{\delta_i} \int_{Q_T} b_i'(u_i)(\frac{\partial u_i}{\partial t})^2 dxdt + \sum_{i=1}^{2} \frac{1}{2\delta_i} \int_\Omega |\nabla u_i(.,T)|^2\, dx =$$

$$\int_\Omega [-\Phi(., u_1(T), u_2(T)) + \Phi(., \varphi_0, \psi_0]\, dx = \frac{1}{2\delta_1} \int_\Omega |\nabla \varphi_0|^2\, dx + \frac{1}{2\delta_2} \int_\Omega |\nabla \psi_0|^2\, dx. \tag{3.5}$$

Φ being continuous and (u_1, u_2) bounded, we then obtain:

$$\sum_{i=1}^{2} \frac{\gamma_i}{\delta_i} \int_{Q_T} (\frac{\partial u_i}{\partial t})^2 dxdt + \sum_{i=1}^{2} \frac{1}{2\delta_i} \int_\Omega |\nabla u_i(.,T)|^2\, dx \leq C(\varphi_0, \psi_0)\ , \tag{3.6}$$

whence (3.3) and (3.4).■

Proof of theorem 3 : Differentiating equation $\frac{\partial b_i(u_i)}{\partial t} - div\,[\nabla u_i] + f_i(u_1, u_2) = h_i$ we obtain

$$b_i'(u_i)u_i'' + b_i''(u_i)(u_i')^2 - div\,(\nabla u_i))' + \sum_{j=1}^{2} \frac{\partial f_i(u)}{\partial u_j} u_j' = 0\ . \tag{3.7}$$

Now multiplying (3.7) by u_i', and integrating over Ω gives

$$\frac{1}{2}r'(t) + \frac{1}{2}\sum_{i=1}^{2}\int_\Omega b_i''(u_i)(u_i')^3 dx + \sum_{i=1}^{2}\|u_i{}'\|_{H_0^1(\Omega)} + \int_\Omega \left(\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial f_i(u)}{\partial u_j}u_j'\right) u_i' dx = 0\,. \tag{3.8}$$

The L^∞ estimate and hypothesis (H9) imply successively :

$$\int_\Omega \left(\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial f_i(u)}{\partial u_j}u_j'\right) u_i' dx \le M\sum_{i=1}^{2}\int_\Omega (u_i')^2\, dx, \tag{3.9}$$

$$\gamma\sum_{i=1}^{2}\int_\Omega (u_i')^2\, d \le r(t) \le M\sum_{i=1}^{2}\|u_i{}'\|^2_{H_0^1(\Omega)}\,, \tag{3.10}$$

and

$$-\frac{1}{2}\sum_{i=1}^{2}\int_\Omega b''(u_i)(u_i')^3 dx \le \sum_{i=1}^{2}\frac{M_i}{2}\,|u_i'|^3_{L^3(\Omega)}\,. \tag{3.11}$$

Since for $N \le 3, H_0^1(\Omega)$ is continuously imbedded in $L^6(\Omega)$, we then obtain by Young's inequality that :

$$|u_i{}'|^3_{L^3(\Omega)} \le c\,|u_i'|^{\frac{9}{4}}_{L^2(\Omega)}\,|u_i'|^3_{L^3(\Omega)} \le c\,|u_i'|^{\frac{18}{5}}_{L^2(\Omega)} + \frac{1}{2}\,\|u_i{}'\|^2_{H_0^1(\Omega)}\,. \tag{3.12}$$

By (3.9),(3.10),(3.11) and (3.12), (3.7) becomes :

$$r'(t) + \frac{1}{2}\sum_{i=1}^{2}\|u_i{}'\|^2_{H_0^1(\Omega)} + cr(t) \le cr(t)^{\frac{9}{5}} + cr(t) \le cr(t)^2 + c. \tag{3.13}$$

On the other hand, using (2.4) we obtain :

$$\sum_{i=1}^{2}\int_\tau^{\tau+r}\int_\Omega b_i'(u_i)\,(u_i')^2\, dxdt \le c_\tau, \text{ for any } \tau \ge t_0\ . \tag{3.14}$$

Estimates (3.13) and (3.14) and the use of the uniform Gronwall' lemma thus gives

$$r(t) \leq c(t_0), \quad \text{for any} \qquad \forall t \geq t_0. \tag{3.15}$$

Now, by (3.15) and hypothesis (H1) , we get :

$$\sum_{i=1}^{2} \int_{\Omega} \left(\frac{\partial b_i(u_i)}{\partial t} \right)^2 dx \leq Mr(t) \leq c(t_0) \quad for \;\; any \;\; t \geq t_0 \;\; .$$

Then (3.0) is satisfied. Now, as we have :
$-\triangle u_i = -f_i(x,u) - b_i(u_i)_t \in L^{\infty}(t_0, +\infty; L^2(\Omega))$,
then by (S) $u_i(.,t)$ is in bounded subset of $H^2(\Omega)$. Hence estimate (3.2) follows.■

For a solution (u_1, u_2) of (**S**), we define the $\omega -$ limit set by :

$$\omega(\varphi_0, \psi_0) = \left\{ \begin{array}{c} w = (w_1, w_2) \in \left(H_0^1(\Omega) \times L^{\infty}(\Omega)\right) \cap \left(H_0^1(\Omega) \times L^{\infty}(\Omega)\right) \\ \exists t_n \to +\infty \quad u_1(.,t_n) \to w_1 \text{ in } L^2(\Omega), \;\; u_2(.,t_n) \to w_2 \text{ in } L^2(\Omega) \end{array} \right\}.$$

Corollary 1. Under the assumptions (H1) to (H10), we have $\omega(\varphi_0, \psi_0) \neq \emptyset$ and any $(w_1, w_2) \in \omega(\varphi_0, \psi_0)$ is a bounded weak solution of the stationary problem

$$\left\{ \begin{array}{l} -\Delta u_i + f_i(x, u_1, u_2) = 0 \quad \text{in } \Omega \\ \qquad u_i = 0 \qquad \text{on} \qquad \partial\Omega \end{array} \right.$$

Proof : From (3.3) we obtain $\omega(\varphi_0, \psi_0) \neq \emptyset$. Setting $w_i = \underset{n\to\infty}{Lim}\; u_i(.,t_n)$ and $w = (w_1, w_2) \in \omega(\varphi_0, \psi_0)$, we get that $w = (w_1, w_2)$ is a solution of the Dirichlet problem for elliptic system. The proof is analogous to El Ouardi and de Thélin [12] and is omitted here.

Corollary 2. Under the assumptions (H1) to (H10), we have

$\mathcal{A} \subset \left(\mathcal{W}^{2,6}(\Omega)\right)^2$ if $N = 3$

and

$\mathcal{A} \subset \left(\mathcal{W}^{2,r}(\Omega)\right)^2$ for all $r < \infty$ if $N \leq 2$.

Proof: Taking the inner product of (4.7) with u_i'', we get
$\frac{d}{dt}(\sum\limits_{i=1}^{2} \|u_i'\|^2_{H_0^1(\Omega)}) \leq c(\sum\limits_{i=1}^{2} |u_i'|^4_{H_0^1(\Omega)} + \sum\limits_{i=1}^{2} |u_i'|^2_{L^2(\Omega)})$.

By uniform Gronwall's lemma, we get $\sum\limits_{i=1}^{2} \|u_i'\|^2_{H_0^1(\Omega)} \leq c$, $\forall t \geq T$.

Then $\sum\limits_{i=1}^{2} \|u_i'\|^2_{L^{\alpha_i}(\Omega)} \leq c$, $\forall t \geq T$ for all $t \geq \tau$ and $\alpha_i = 6$ if $N = 3$ or $1 \leq \alpha_i < \infty$ if $N \leq 2$.

4. Dimension of the attractor $\mathcal{A}$

4.1 Linearized problem

Let $(\varphi_0, \psi_0) \in \mathcal{A}$; then by theorem3, $u(t) = (u_1(t), u_2(t))$ belongs to a bounded subset of $\left[H^2(\Omega)\right]^2$. This fact allows us to linearize the system (S) along $u(t)$. Formally, the candidate for the linearized problem is

$$(\mathrm{S}_L) \qquad \begin{cases} U = (U_1, U_2) \in \left[L^2(0,T; H_0^1(\Omega)\right]^2 \\ \frac{\partial}{\partial t}\left(b_i'(u_i U_i) - \triangle U_i + \sum_{j=1}^{2} \frac{\partial f_i}{\partial u_j} U_j = 0 \\ U(0) = (U_1(0), U_2(0)) = U_0 \end{cases}$$

The existence and uniqueness of solution can be deduced from (4.0) below

$$U_i \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)). \tag{4.0}$$

To deduce (4.0), Multiply the equation in (S_L) by $b_i'(u_i)U_i$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{2} |b_i'(u_i)U_i|^2_{L^2(\Omega)}\right) + \sum_{i=1}^{2} (\nabla u_i, \nabla(b_i'(u_i)U_i))_{L^2(\Omega)}$$

$$= \sum_{i=1}^{2}\Big(\sum_{j=1}^{2} \frac{\partial f_i}{\partial u_j} U_j, b_i'(u_i)U_i\Big)_{L^2(\Omega)} \tag{4.1}$$

By the hypothesis on b_i, we have

$$\nabla(b_i'(u_i)U_i) = b_i'(u_i)\nabla U_i + b_i''(u_i)\nabla u_i.U_i, \tag{4.2}$$

$$(\nabla U_i, b_i''(u_i)\nabla u_i.U_i)_{L^2(\Omega)} \le c\,|\nabla u_i|_{L^4(\Omega)}\,|U_i|_{L^4(\Omega)}\,|\nabla U_i|_{L^2(\Omega)}, \tag{4.3}$$

and

$$\sum_{i=1}^{2}\Big(\sum_{j=1}^{2} \frac{\partial f_i}{\partial u_j} U_j, b_i'(u_i)U_i\Big)_{L^2(\Omega)} \le M \sum_{i=1}^{2}\sum_{j=1}^{2} |U_j U_i|_{L^1(\Omega)} \le c \sum_{i=1}^{2} |U_i|^2_{L^2(\Omega)}. \tag{4.4}$$

From (4.2) to (4.4), (4.1) becomes

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{2} |b_i'(u_i)U_i|^2_{L^2(\Omega)}\right) + \gamma \sum_{i=1}^{2} |U_i|^2_{H_0^1(\Omega)} \le \quad c \sum_{i=1}^{2} |U_i|^2_{L^2(\Omega)} \le c \sum_{i=1}^{2} |b_i'(u_i)U_i|^2_{L^2(\Omega)}.$$

By standard application of Gronwall's inequality, we get (4.0).

4.2 Differentiability of the Semigroup

We assume that $f_i \in \mathcal{C}^2(R \times R)$ $(\forall i = 1, 2)$. Let $u_0 = (\varphi_0, \psi_0), v_0 = (\overline{\varphi}_0, \overline{\psi}_0)$, $S(t)$ be the solution of (S) and $S'(t, u_0)$ the solution of (S_L). The results of [6] imply the following proposition :

Proposition 3.
Assume (H1) to (H10), then for any $(u_0, v_0) \in \left[L^\infty(\Omega) \times H^2(\Omega)\right]^2$, we have $|S(t)v_0 - S(t)u_0 - S'(t,u_0)(v_0-u_0)|_{(L^2(\Omega))^2} \leq c(t)o(|v_0-u_0|_{(L^2(\Omega))^2})$

We need the lemma 3 for the proof of proposition 3:

Let (u_1, u_2) and (v_1, v_2) be two solutions of (S) in $\mathcal{A}$ with $(u_1(0), u_2(0)) = (\varphi_0, \psi_0)$ and $(v_1(0), v_2(0)) = (\varphi_1, \psi_1)$. Setting $w_1 = u_1 - v_1$ and $w_2 = u_2 - v_2$, we have

Lemme 3 Assume (H1) to (H10). For all $T > 0$, there exists $c(T) > 0$ such that for all $t \in [0, T]$,

$$\sum_{i=1}^{2} |w_i(t)|^2_{H_0^1(\Omega)} \leq c(T) \sum_{i=1}^{2} |w_i(0)|^2_{H_0^1(\Omega)}, \tag{4.7}$$

$$t \sum_{i=1}^{2} |w_i(t)|^2_{H_0^1(\Omega)} \leq c(T) \sum_{i=1}^{2} |w_i(0)|^2_{L^2(\Omega)}, \tag{4.8}$$

and

$$\sum_{i=1}^{2} |w_i'(t)|^2_{L^2(0,T;L^2(\Omega))} \leq c(T) \sum_{i=1}^{2} |w_i(0)|^2_{H_0^1(\Omega)} \tag{4.9}$$

Proof : We have

$$\begin{cases} \frac{\partial b_i(u_i)}{\partial t} - \triangle u_i + f_i(x,u) = 0 \\ u(0) = (u_1(0), u_2(0)) = (\varphi_0, \psi_0) \end{cases} \qquad \begin{cases} \frac{\partial b_i(v_i)}{\partial t} - \triangle v_i + f_i(x,v) = 0 \\ v(0) = (v_1(0), v_2(0)) = (\varphi_1, \psi_1)\ . \end{cases}$$

Thus, the difference $w_i = u_i - v_i$ satisfies

$$b_i'(u_i)w_i' - \triangle w_i = \left[b_i'(v_i) - b_i'(u_i)\right] v_i' + f_i(v) - f_i(u). \tag{4.10}$$

Setting

$F_{11} = \int_0^1 \frac{\partial f_1}{\partial u_1}(x, u_1 + \theta(u_2 - u_1), u_2) d\theta$, $F_{21} = \int_0^1 \frac{\partial f_1}{\partial u_2}(x, u_1, u_2 + \theta(u_2 - u_1)) d\theta$,
$F_{12} = \int_0^1 \frac{\partial f_2}{\partial u_1}(x, u_1 + \theta(u_2 - u_1), u_2) d\theta$ and $F_{22} = \int_0^1 \frac{\partial f_2}{\partial u_2}(x, u_1, u_2 + \theta(u_2 - u_1)) d\theta$,
(4.10) becomes

$$b_i'(u_i)w_i' - \triangle w_i = \left[b_i'(v_i) - b_i'(u_i)\right] v_i' + \sum_{i=1}^{2} F_{ij} w_j. \tag{4.11}$$

We multiply (4.11) by $\frac{w_i}{b_i'(u_i)}$

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{2}|w_i|^2_{L^2(\Omega)}+\sum_{i=1}^{2}(\nabla w_i,\nabla(\frac{w_i}{b_i'(u_i)}))_{L^2(\Omega)}=$$

$$\sum_{i=1}^{2}(\frac{b_i'(v_i)-b_i'(u_i)}{b_i'(u_i)}v_i',w_i)_{L^2(\Omega)}+\sum_{i=1}^{2}\sum_{j=1}^{2}(F_{ij}w_j,\frac{w_i}{b_i'(u_i)})_{L^2(\Omega)} \qquad (4.12)$$

And parallel to lemma 16 in [8], it is easy to see that (4.7), (4.8) and (4.9) hold.

Proof of proposition 3 : It is similar to the proof for the lemma 15 in [8, p.125] and is omited.

4.3 Dimension Estimates

Consider the linearized problem

$$(S_L)\quad\begin{cases} U_i' = F_i'(u_i(\tau))U_i & \text{in } \Omega\times R^+\\ U_i=0 & \text{on } \partial\Omega\times R\\ U_i(0)=\xi_i \end{cases}$$

where $F_i'(u_i(\tau))U_i=\frac{1}{b_i'(u_i(\tau))}\triangle U_i-\frac{b_i''(u_i(\tau))}{b_i'(u_i(\tau))}u_i'U_i-\frac{1}{b_i'(u_i(\tau))}\sum_{j=1}^{2}\frac{\partial f_i}{\partial u_j}U_j$.

This problem can be rewritten as

$$\text{(L)}\quad\begin{cases} U'=F'(u(\tau))U\\ U=0\\ U(0)=\xi \end{cases}$$

where $U=\begin{pmatrix}U_1\\U_2\end{pmatrix}$, $\qquad F'(u(\tau))=\begin{pmatrix}F_1'(u_1(\tau)) & 0\\ 0 & F_1'(u_1(\tau))\end{pmatrix}$.

Let $\overline{U}_1,.......,\overline{U}_m$ be m solutions of (L) corresponding to the initial data $\xi_,,........,\xi_m$ and $\mathcal{Q}_m(\tau)$ be the orthogonal projector in $H=L^2(\Omega)\times L^2(\Omega)$ such that $\mathcal{Q}_mH\subset V=H_0^1(\Omega)\times H_0^1(\Omega)$. If $\left\{W^k=(w_1^k,w_2^k\right\}_{k=1}^{m}$ is an orthonormal basis of $\mathcal{Q}_m(\tau)H$; then

$Tr(F'(u(\tau))\circ\mathcal{Q}_m(\tau)=\sum_{k=1}^{m}(F'(u(\tau))W^k,W^k)_H=\sum_{i=1}^{2}\sum_{k=1}^{m}(F_i'(u_i(\tau))W_i^k,W_i^k)_{L^2(\Omega)}$

and

$(F_i'(u_i(\tau))W_i^k,W_i^k)_{L^2(\Omega)}=(\triangle w_i^k,\frac{w_i^k}{b_i'(u_i(\tau))})_{L^2(\Omega)}-(\frac{b_i''(u_i(\tau))}{b_i'(u_i(\tau))}u_i'w_i^k,w_i^k)_{L^2(\Omega)}-$ $(\frac{1}{b_i'(u_i(\tau))}\sum_{j=1}^{2}\frac{\partial f_i}{\partial u_j}w_j^k,w_i^k)_{L^2(\Omega)}$.

Since $u=(u_1,u_2)\in(L^\infty(0,+\infty,L^\infty(\Omega)))^2$, we have $b_i(u_i),b_i'(u_i)$,

$b_i^{''}(u_i) \in (L^\infty(0,+\infty, L^\infty(\Omega)))^2$, so

$$\sum_{i=1}^{2}(F_i'(u_i(\tau))W_i^k, W_i^k)_{L^2(\Omega)} \leq \sum_{i=1}^{2}(\triangle w_i^k, \frac{w_i^k}{b_i'(u_i(\tau))})_{L^2(\Omega)} + c\sum_{i=1}^{2}\int_\Omega |u_i'| \left|w_i^k\right|^2 dx +$$

$$\int_\Omega \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial f_i(x,u)}{\partial u_j} w_j^k w_i^k)dx. \tag{4.13}$$

Now, we have

$$\sum_{i=1}^{2}(\triangle w_i^k, \frac{w_i^k}{b_i'(u_i(\tau))})_{L^2(\Omega)} \leq -c\sum_{i=1}^{2}\left\|w_i^k\right\|^2_{H_0^1(\Omega)} + cJ$$

where

$$J = \sum_{i=1}^{2}\int_\Omega \left|\nabla w_i^k\right| \left|w_i^k\right| |\nabla u_i|\, dx \leq c(\sum_{i=1}^{2}\left\|w_i^k\right\|^2_{H_0^1(\Omega)})^{\frac{1}{2}}(\int_\Omega \sum_{i=1}^{2}\left|w_i^k\right|^2 dx)^{\frac{1}{2}}$$

$$\leq \frac{c}{2}\sum_{i=1}^{2}\left\|w_i^k\right\|^2_{H_0^1(\Omega)} + c(\int_\Omega \sum_{i=1}^{2}\left|w_i^k\right|^2 dx) \tag{4.14}$$

and

$$\int_\Omega \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial f_i(u)}{\partial u_j} w_j^k w_i^k)dx \leq c(\int_\Omega \sum_{i=1}^{2}\left|w_i^k\right|^2 dx). \tag{4.15}$$

According to (4.14) and (4.15), relation (4.13) becomes

$$\sum_{i=1}^{2}(F_i'(u_i(\tau))W_i^k, W_i^k)_{L^2(\Omega)} \leq -c\sum_{i=1}^{2}\left\|w_i^k\right\|^2_{H_0^1(\Omega)} + c\sum_{i=1}^{2}\left\|w_i^k\right\|^2_{H_0^1(\Omega)} +$$

$$c(\int_\Omega \sum_{i=1}^{2}\left|w_i^k\right|^2 dx) + c\sum_{i=1}^{2}\int_\Omega |u_i'| \left|w_i^k\right|^2 dx \tag{4.16}$$

Which leads to

$$Tr(F'(u(\tau)) \circ \mathcal{Q}_m(\tau) \leq -c_{105}\sum_{i=1}^{2}\sum_{k=1}^{m}\left\|w_i^k\right\|^2_{H_0^1(\Omega)} + c\sum_{i=1}^{2}\sum_{k=1}^{m}\left\|w_i^k\right\|^2_{H_0^1(\Omega)} +$$

$$c(\int_\Omega \sum_{i=1}^{2}\sum_{k=1}^{m}\left|w_i^k\right|^2 dx) + c\sum_{i=1}^{2}\sum_{k=1}^{m}\int_\Omega |u_i'| \left|w_i^k\right|^2 dx. \tag{4.17}$$

We set $\rho(x) = \sum\limits_{k=1}^{m} \left|w^k(x)\right|^2 = \sum\limits_{i=1}^{2}\sum\limits_{k=1}^{m} \left|w_i^k(x)\right|^2$ and $\gamma(t) = \max(|u_1'(t)|, |u_2'(t)|,$ $\theta(t) = \int_\Omega \gamma(t)^{\frac{5}{2}} dx$. So, we get

$$\sum_{i=1}^{2}\sum_{k=1}^{m} \int_\Omega |u_i'| \left|w_i^k\right|^2 dx \leq \int_\Omega \gamma(t)\rho(x) dx \leq c \left[\int_\Omega \gamma(t)^{\frac{5}{2}} dx\right]^{\frac{2}{5}} \left[\int_\Omega \rho^{\frac{5}{3}} dx\right]^{\frac{3}{5}}. \tag{4.18}$$

Therefore by theorem 4.1 in [22] , there exists $c_1' > 0$ such that :

$$\int_\Omega \rho^{\frac{5}{3}} dx \leq c_1' \sum_{k=1}^{m} \left\|w^k(x)\right\|^2_{H_0^1(\Omega)}. \tag{4.19}$$

So, we get

$$\sum_{i=1}^{2}\sum_{k=1}^{m} \int_\Omega |u_i'| \left|w_i^k\right|^2 dx \leq c \int_\Omega \gamma(t)^{\frac{5}{2}} dx + \frac{c}{5} \sum_{k=1}^{m} \left\|w^k(x)\right\|^2_{H_0^1(\Omega)}.$$

From (4.18) to (4.19), (4.16) becomes

$$Tr(F'(u(\tau)) \circ \mathcal{Q}_m(\tau) \leq -c \sum_{k=1}^{m} \left\|w^k\right\|^2_{H_0^1(\Omega)} + c \int_\Omega \rho dx + c \int_\Omega \gamma(t)^{\frac{5}{2}} dx,$$

and as in [22] , we obtain :

$$Tr(F'(u(\tau)) \circ \mathcal{Q}_m(\tau) \leq -cm^{1+\frac{2}{N}} + c'm + \theta(t). \tag{4.20}$$

Setting $\quad q_m(t) = \sup\limits_{u_0 \in \mathcal{A}} \quad \sup\limits_{\substack{\xi_i \in H, \ |\xi_i| \leq 1 \\ i=1,\ldots,m}} \left\{ \frac{1}{t} \int_0^t Tr(F'(S(\tau)u_0)) \circ \mathcal{Q}_m(\tau) d\tau \right\}$

and

$$q_m = \underset{t \to +\infty}{Lim} \sup q_m(t).$$

Then, by lemma 15 in [8, $p.119$] , we have $\int_0^\eta \theta d\tau \leq c(\eta)$ and $u_i' \in L^\infty(\eta, +\infty, L^\infty(\Omega))$. Thus, $q_m(t) \leq \frac{c(\eta)}{t} - cm^{1+\frac{2}{N}} + c'm + c'(\eta)$ and $q_m \leq -cm^{1+\frac{2}{N}} + c'm + c'(\eta)$

and for all integers $j > 0$, *we get* $\mu_1 + \mu_2 \ldots\ldots\ldots + \mu_j \leq q_j \leq -cj^{1+\frac{2}{N}} + c'j + c'(\eta)$.

Hence

$$\mu_1 + \mu_2 \ldots\ldots\ldots + \mu_m < 0 \quad \text{for any} \quad m < c''. \tag{4.21}$$

This shows that the fractal dimension of the attractor $\mathcal{A}$ is finite and arguing as for theorem V3.3 in [22], we conclude to the following :

Theorem 4. Assume (H1) to (H10) and let m be an integer satisfying (4.21). Then

(i) $\dim_{Fractale}(\mathcal{A}) \leq \ 2m$

(ii) $\dim_{H}(\mathcal{A}) \leq m.$

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