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The existence of ground state solutions for semi-linear degenerate Schrödinger equations with steep potential well

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Abstract. In this article, we study the following degenerated Schrödinger equations:

$$\begin{cases} -\Delta_\gamma u + \lambda V(x) u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in E_\lambda \,, \end{cases}$$

where $\lambda > 0$ is a parameter, Δ_γ is a degenerate elliptic operator, the potential $V(x)$ has a potential well with bottom and the nonlinearity $f(x,u)$ is either super-linear or sub-linear at infinity in u. The existence of ground state solution be obtained by using the variational methods.

Keywords: steep well potential, mountain pass theorem, strongly degenerate elliptic operator.

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1 Introduction

This article is concerned with a class of nonlinear Schrödinger equations:

$$\begin{cases} -\Delta_\gamma u + \lambda V(x) u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in E_\lambda \,, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, Δ_γ is a degenerate elliptic operator of the form

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}(\gamma_j^2 \partial_{x_j}), \qquad \partial_{x_j} = \frac{\partial}{\partial x_j}, \qquad \gamma = (\gamma_1(x), \gamma_2(x), \dots, \gamma_N(x)).$$

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Here, the functions $\gamma_j : \mathbb{R}^N \to \mathbb{R}$ are assumed to be continuous, different from zero and of class C^1 in $\mathbb{R}^N \setminus \Pi$, where

$$\Pi := \left\{ x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : \prod_{j=1}^{N} x_j = 0 \right\}.$$

Moreover, the function γ_j satisfy the following properties:

(i) There exists a semigroup of dilations $\{\delta_t\}_{t>0}$ such that

$$\delta_t : \mathbb{R}^N \to \mathbb{R}, \qquad \delta_t(x_1, \dots, x_N) = (t^{\varepsilon_1} x_1, \dots, t^{\varepsilon_N} x_N),$$

where $1 = \varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_N$, such that γ_j is δ_t-homogeneous of degree $\varepsilon_j - 1$, i.e.

$$\gamma_j(\delta_t(x)) = t^{\varepsilon_j - 1} \gamma_j(x), \qquad \forall x \in \mathbb{R}^N,\ \forall t > 0,\ j = 1, \dots, N.$$

The number

$$\tilde{N} := \sum_{j=1}^{N} \varepsilon_j$$

is called the homogeneous dimension of $\mathbb{R}^N$ with respect to $\{\delta_t\}_{t>0}$.

(ii) $\gamma_1 = 1,\ \gamma_j(x) = \gamma_j(x_1, x_2, \dots, x_{j-1}),\ j = 2, \dots, N.$

(iii) There exists a constant $\rho \ge 0$ such that

$$0 \le x_k \partial_{x_k} \gamma_j(x) \le \rho \gamma_j(x), \qquad \forall k \in \{1, 2, \dots, j-1\},\ \forall j = 2, \dots, N,$$

and for every $x \in \overline{\mathbb{R}}^N_+ = \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_j \ge 0,\ \forall j = 1, 2, \dots, N\}$.

(iv) Equalities $\gamma_j(x) = \gamma_j(x^*)(j = 1, 2, \dots, N)$ are satisfied for every $x \in \mathbb{R}^N$, where

$$x^* = (|x_1|, \dots, |x_N|), \quad \text{if } x = (x_1, x_2, \dots, x_N).$$

The Δ_γ-operator contains the following operator of Grušin-type

$$G_\alpha := \Delta_x + |x|^{2\alpha} \Delta_y, \qquad \alpha \ge 0,$$

where (x, y) denotes the point of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. This operator was studied by Grušin in [8] when α is an integer, and by Franchi and Lanconeli in [6,7], Loiudice in [11], Monti and Morbidelli in [13] when α is not an integer. The Δ_γ-operator also contains following semi-linear strongly degenerate operator

$$P_{\alpha,\beta} = \Delta_x + \Delta_y + |x|^{2\alpha} |y|^{2\beta} \Delta_z, \qquad (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

where α, β are nonnegative real numbers. The $P_{\alpha,\beta}$-operator was studied in [1]. For more information about the operator Δ_γ, please see [10].

In this paper, we study the existence of ground state solutions for the equation (1.1) under the assumptions that V is neither radially symmetric nor coercive. Precisely, we make the following assumptions.

$(V1)$ $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ satisfying $\inf_{x\in\mathbb{R}^N} V(x) > 0$.

$(V2)$ There exists $b > 0$ such that the set $\{x \in \mathbb{R}^N : V(x) < b\}$ is nonempty and has finite measure.

The conditions $(V1) \sim (V2)$ are special cases of steep potential well which were first introduced by Bartsch and Wang in [2]. In recent years, steep potential well are widely used in various equation, such as Schrödinger equations, Schrödinger–Poisson equations and Klein–Gordon–Maxwell system and so on (see [2–4,9,14,15]).

Nextly, wee will require that the nonlinear term satisfies either the assumptions:

$(f1)'$ $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there are constants $0 < a_1 < a_2 < a_3 \cdots < a_m < 1$ and functions $b_i(x) \in L^{\frac{2}{1-a_i}}(\mathbb{R}^N, (0, +\infty))$ such that

$$|f(x,z)| \le \sum_{i=1}^{m} (a_i + 1) b_i(x) |z|^{a_i}, \qquad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R};$$

$(f2)'$ There exist constants $\eta, \delta > 0, a_0 \in (1,2), \Omega \subset \mathbb{R}^N$ such that $\text{meas}(\Omega) \neq 0$ and

$$F(x,z) = \int_0^z f(x,t)dt \ge \eta |z|^{a_0}, \qquad \forall x \in \Omega \text{ and } \forall\ |z| \le \delta,$$

or the assumptions:

$(f1)$ $\lim_{|z|\to 0} \frac{f(x,z)}{|z|} = 0$ uniformly for $x \in \mathbb{R}^N$.

$(f2)$ For some $2 < p < 2^*_\gamma$, $C_0 > 0$,

$$|f(x,z)| \le C_0(|z| + |z|^{p-1}), \qquad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R},$$

where $2^*_\gamma := \frac{2\tilde{N}}{\tilde{N}-2}$ is the critical Sobolev exponent;

$(f3)$ $F(x,z) := \int_0^z f(x,t)dt \ge 0$ for all $x \in \mathbb{R}^N$, and

$$\lim_{|z|\to+\infty} \frac{F(x,z)}{|z|^2} = +\infty, \qquad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R};$$

$(f4)$ There exist $a_1 > 0, L_0 > 0$ and $\tau > \frac{\tilde{N}}{2}$, such that

$$|f(x,z)|^\tau \le a_1 \mathcal{F}(x,z)|z|^\tau, \quad \text{for all } x \in \mathbb{R}^N \text{ and } |z| \ge L_0,$$

where

$$\mathcal{F}(x,z) := \frac{1}{2} f(x,z)z - F(x,z) \ge 0, \qquad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R};$$

$(f5)$ $\frac{f(x,z)}{|z|}$ is an increasing function of z on $\mathbb{R} \setminus \{0\}$ for every $x \in \mathbb{R}^N$.

Before stating our main results, we give several notations. For $\lambda > 0$, let

$$S^2_\gamma(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \gamma_j \partial_{x_j} u \in L^2(\mathbb{R}^N),\ j = 1, \dots, N \right\},$$
$$E_\lambda := \left\{ u \in S^2_\gamma(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda V(x) u^2 dx < +\infty \right\}.$$

Then, by assumption $(V1)$, E_λ is a Hilbert space with the inner product and norm respectively

$$(u,v)_\lambda = \int_{\mathbb{R}^N} (\nabla_\gamma u \nabla_\gamma v + \lambda V(x) uv) dx, \qquad \|u\|_\lambda = (u,u)_\lambda^{\frac{1}{2}},$$

where

$$\nabla_\gamma u = (\gamma_1 \partial_{x_1} u, \gamma_2 \partial_{x_2} u, \ldots, \gamma_N \partial_{x_N} u).$$

Obviously, the embedding $E_\lambda \hookrightarrow S^2_\gamma(\mathbb{R}^N)$ is continuous. It follows that $E_\lambda \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for each $s \in [2, 2^*_\gamma]$ (see [12]). Thus for each $2 \le s \le 2^*_\gamma$, there exists $d_s > 0$ such that

$$|u|_s \le d_s \|u\|_\lambda, \qquad \forall u \in E_\lambda, \tag{1.2}$$

where $L^s(\mathbb{R}^N)$ denote a Lebesgue space, the norm in $L^s(\mathbb{R}^N)$ is denoted by $|\cdot|_s$.

We point out that there are Rellich-type compact embeddings hold on bounded domains for subcritical exponents. By $S^2_\gamma(\Omega)$ we denote the set of all functions $u \in L^2(\Omega)$ such that $\gamma_j \partial_{x_j} u \in L^2(\Omega)$ for all $j = 1, \ldots, N$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$. The space $S^2_{\gamma,0}(\Omega)$ is defined as the closure of $C^1_0(\Omega)$ in the space $S^2_\gamma(\Omega)$. We define the norm on this space as

$$\int_\Omega (|\nabla_\gamma u|^2 + \lambda V(x) u^2) dx,$$

which is equivalent to $\int_\Omega |\nabla_\gamma u|^2 dx$, by $(V1)$. Then, we have that the embedding $S^2_{\gamma,0}(\Omega) \hookrightarrow L^s(\Omega)$ is compact for every $s \in [1, 2^*_\gamma)$ (see Proposition 3.2. in [10]).

We can now state the main result:

Theorem 1.1. *Assume $(V1)$ and $(f1)' \sim (f2)'$ are satisfied. Then $\forall \lambda > 0$, problem (1.1) admits at least a ground state solution in E_λ.*

Remark 1.2. To the best of our knowledge, it seems that Theorem 1.1 is the first result about the existence of ground state solutions for the semi-linear Δ_γ differential equation in $\mathbb{R}^N$. By the way, we would like to point out that in [12] the authors study existence of infinitely many solutions for semi-linear degenerate Schrödinger equations with the potential $V(x)$ satisfying the coercivity condition which implies $E_\lambda \hookrightarrow\hookrightarrow L^s(\mathbb{R}^N)$ for any $s \in [2, 2^*_\gamma)$.

Theorem 1.3. *Assume $(V1)$, $(V2)$ and $(f1) \sim (f5)$ are satisfied. Then there exists $\Lambda > 0$ such that problem (1.1) has at least a ground state solution in E_λ, for all $\lambda > \Lambda$.*

Remark 1.4. We point out that the Schrödinger equation with general steep potential well is considered in reference [3, 4], but they consider a special nonlinear term, where $f(x,z) = |z|^{p-2} z (2 < p < 2^*)$. At the same time, we also point out that although the Schrödinger equation with general steep potential well and the general nonlinear term are considered in reference [2, 9], the nonlinear term there satisfies the following Ambrosetti–Rabinowitz type condition:

(AR) There exist $\mu > 2$ and $L > 0$, such that

$$\mu F(x,z) \le z f(x,z), \qquad \forall x \in \mathbb{R}^N,\ \forall |z| \ge L.$$

The nonlinear term we consider here is not required to satisfy the Ambrosetti–Rabinowitz type condition, for example we allow nonlinearities of the type

$$f(x,z) = 2z \ln(1+z^2) + \frac{2z^3}{1+z^2}, \qquad \forall (x,z) \in \mathbb{R}^3 \times \mathbb{R}.$$

By a simple calculation, we have

$$F(x,z) = \int_0^z f(x,t)dt = z^2 \ln(1+z^2), \qquad \mathcal{F}(x,z) = \frac{2z^4}{1+z^2},$$

and

$$zf(x,z) - \mu F(x,z) = z^2 \left((2-\mu)\ln(1+z^2) + \frac{2z^2}{1+z^2} \right).$$

Now, it is easy to verify that the function f satisfies our assumptions and does not satisfy the Ambrosetti–Rabinowitz type condition.

To obtain our main results, we have to overcome some difficulties in our proof. The main difficulty consists in the lack of compactness of the $E_\lambda \hookrightarrow L^s(\mathbb{R}^N)$ with $s \in [2, 2^*_\gamma]$. Since we assume that the potential is not radially symmetric, we cannot use the usual way to recover compactness, for example, restricting in the subspace of radial functions of E_λ. We also cannot borrow some ideas in [12] to recover compactness because the potential do not satisfied the coercivity condition. To recover the compactness, we establish the parameter dependent compactness conditions.

Now, we define the following energy functional

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla_\gamma u|^2 + \lambda V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx, \tag{1.3}$$

for any $u \in E_\lambda$. It is well known that J_λ is a C^1 functional with derivative given by

$$\langle J'_\lambda(u), v\rangle = \int_{\mathbb{R}^N} (\nabla_\gamma u \nabla_\gamma v + \lambda V(x)uv)dx - \int_{\mathbb{R}^N} f(x,u)v dx, \tag{1.4}$$

for any $u, v \in E_\lambda$. We have that u is a weak solution of equation (1.1) if only if it is a critical point of $J_\lambda(u)$ in E_λ.

2 The proof of main results for f sub-linear at infinity in u

Lemma 2.1 (see [17]). *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfy the (PS) condition. If J is bounded from below, then $c = \inf_E J$ is critical value of J.*

Lemma 2.2. *Assume that $(V1)$ and $(f1)'$ hold, then J_λ is bounded from below.*

Proof. It follows from $(f1)'$ that we can get

$$|F(x,z)| \le \sum_{i=1}^m b_i(x)|z|^{a_i+1}, \qquad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.1}$$

The above inequality combined with the Hölder inequality and (1.2) shows that

$$\begin{aligned} \int_{\mathbb{R}^N} |F(x,z)|dx &\le \int_{\mathbb{R}^N} \sum_{i=1}^m b_i(x)|z|^{a_i+1}dx \\ &\le \sum_{i=1}^m \left(\int_{\mathbb{R}^N} |b_i(x)|^{\frac{2}{1-a_i}}dx \right)^{\frac{1-a_i}{2}} \left(\int_{\mathbb{R}^N} |z|^2 dx \right)^{\frac{1+a_i}{2}} \\ &\le \sum_{i=1}^m d_2^{1+a_i} |b_i(x)|_{\frac{2}{1-a_i}} \|z\|_\lambda^{1+a_i}. \end{aligned} \tag{2.2}$$

Thus

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla_\gamma u|^2+\lambda V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \sum_{i=1}^m d_2^{1+a_i}|b_i(x)|_{\frac{2}{1-a_i}}\|u\|_\lambda^{1+a_i}. \end{aligned}$$

In view of $0<a_1<a_2<a_3<\cdots<a_m<1$ and $b_i(x)\in L^{\frac{2}{1-a_i}}(\mathbb{R}^N,(0,+\infty))$, it is clearly shows that J_λ is coercive, then J_λ is bounded from below. □

Lemma 2.3. *Assume that $(V1)$ and $(f1)'$ are satisfied, then J_λ satisfies the (PS) condition for each $\lambda>0$.*

Proof. We suppose that $\{u_n\}$ is a Palais–Smale sequence of J_λ, that is for some $c_\lambda\in\mathbb{R}$, $J_\lambda(u_n)\to c_\lambda$, $J'_\lambda(u_n)\to 0$, as $n\to\infty$. According to lemma 2.2, $\{u_n\}$ is bounded in E_λ. Therefore, up to a subsequence, there are $u\in E_\lambda$, we have

$$\begin{aligned} &u_n \rightharpoonup u, \quad \text{in } E_\lambda; \\ &u_n \to u, \quad \text{in } L^s_{loc}(\mathbb{R}^N),\ 2\leq s<2^*_\gamma. \end{aligned} \tag{2.3}$$

By $(f1)'$, for any fixed $\varepsilon>0$, we can choose $R_\varepsilon>0$ such that

$$\left(\int_{\mathbb{R}^N-B_{R_\varepsilon}}|b_i(x)|^{\frac{2}{1-a_i}}dx\right)^{\frac{1-a_i}{2}}<\varepsilon, \qquad i=1,2,\ldots,m. \tag{2.4}$$

It follows that (2.3), we obtain that

$$\lim_{n\to\infty}\int_{B_{R_\varepsilon}}|u_n-u|^2dx=0.$$

Hence, there exists $N_0\in\mathbb{N}$ such that we have

$$\int_{B_{R_\varepsilon}}|u_n-u|^2dx<\varepsilon^2, \qquad \forall n\geq N_0. \tag{2.5}$$

Combing this with the Hölder inequality and $(f1)'$, for any $n\geq N_0$ we have that

$$\begin{aligned} &\int_{B_{R_\varepsilon}}|f(x,u_n)-f(x,u)||u_n-u|dx \\ &\leq \left(\int_{B_{R_\varepsilon}}|f(x,u_n)-f(x,u)|^2dx\right)^{\frac{1}{2}}\left(\int_{B_{R_\varepsilon}}|u_n-u|^2dx\right)^{\frac{1}{2}} \\ &\leq \left(\int_{B_{R_\varepsilon}}|f(x,u_n)-f(x,u)|^2dx\right)^{\frac{1}{2}}\cdot\varepsilon \\ &\leq \left\{\int_{B_{R_\varepsilon}}2m\left[\sum_{i=1}^m(a_i+1)^2b_i^2(x)|u_n|^{2a_i}+\sum_{i=1}^m(a_i+1)^2b_i^2(x)|u|^{2a_i}\right]dx\right\}^{\frac{1}{2}}\cdot\varepsilon \\ &\leq \sqrt{2m}\left[\sum_{i=1}^m(a_i+1)^2|b_i(x)|^2_{\frac{2}{1-a_i}}(|u_n|_2^{2a_i}+|u|_2^{2a_i})\right]^{\frac{1}{2}}\cdot\varepsilon. \end{aligned} \tag{2.6}$$

Again by $(f1)'$, the Hölder inequality and (2.4), we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^N - B_{R_\varepsilon}} |f(x,u_n) - f(x,u)||u_n - u| dx \\
&\quad \le \int_{\mathbb{R}^N - B_{R_\varepsilon}} \sum_{i=1}^{m} (a_i+1) b_i(x) (|u_n|^{a_i+1} + |u|^{a_i}|u_n| + |u_n|^{a_i}|u| + |u|^{a_i+1}) dx \\
&\quad \le \sum_{i=1}^{m} (a_i+1) \left(\int_{\mathbb{R}^N - B_{R_\varepsilon}} |b_i|^{\frac{2}{1-a_i}} dx \right)^{\frac{1-a_i}{2}} \left(|u_n|_2^{a_i+1} + |u|_2^{a_i}|u_n|_2 + |u_n|_2^{a_i}|u|_2 + |u|_2^{a_i+1} \right) \\
&\quad \le \varepsilon \sum_{i=1}^{m} (a_i+1) \left(|u_n|_2^{a_i+1} + |u|_2^{a_i}|u_n|_2 + |u_n|_2^{a_i}|u|_2 + |u|_2^{a_i+1} \right).
\end{aligned} \tag{2.7}$$

Since ε is arbitrary, by (2.6) and (2.7), we known that

$$\int_{\mathbb{R}^N} |f(x,u_n) - f(x,u)||u_n - u| dx \to 0, \quad \text{as } n \to \infty. \tag{2.8}$$

Thus, from (1.4) and (2.3), it holds

$$\|u_n - u\|_\lambda^2 = \langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u \rangle + \int_{\mathbb{R}^N} |f(x,u_n) - f(x,u)||u_n - u| dx \to 0, \quad \text{as } n \to \infty.$$

So, $u_n \to u$ in E_λ. □

Proof of Theorem 1.1. By Lemmas 2.1, 2.2 and 2.3, we known that $c_\lambda = \inf_{E_\lambda} J_\lambda(u)$ is critical value of J_λ. Next, we will prove $c_\lambda \neq 0$. Let $u \in E_\lambda$ and $\|u\|_\lambda = 1$, by $(f2)'$, we can get

$$\begin{aligned}
J_\lambda(tu) &= \frac{t^2}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x,tu) dx \\
&\le \frac{t^2}{2} - \eta |t|^{a_0} \int_\Omega |u|^{a_0} dx.
\end{aligned}$$

Since $1 < a_0 < 2$, as $t > 0$ small enough, $J_\lambda(tu) < 0$. Hence $c_\lambda = \inf_{E_\lambda} J_\lambda(u) < 0$, equation (1.1) possesses at least a nontrivial ground state solution u_λ for every $\lambda > 0$. Then the proof of Theorem 1.1 is completed. □

3 The proof of main results for f super-linear at infinity in u

To complete the proof of our theorem, we need the following definition of Cerami condition and critical point theorem(see [16]).

If any sequence $\{u_n\} \subset H$ such that $J(u_n) \to c$ and $J'(u_n)(1 + \|u_n\|) \to 0$, then this sequence is called a $(C)_c$ sequence. If any $(C)_c$ sequence $\{u_n\} \subset H$ of J has a convergent subsequence, then this C^1 functional J satisfies $(C)_c$ condition.

Theorem 3.1 (Mountain Pass Theorem). *Let H be a real Banach space and $J \in C^1(H, \mathbb{R})$. Assume that there exist $v_0 \in H, v_1 \in H$, and a bounded open neighborhood Ω of v_0 such that $v_1 \notin \Omega$ and*

$$\inf_{u \in \partial\Omega} J(u) > \max\{J(v_0), J(v_1)\}.$$

Let

$$\Gamma := \{\gamma \in C([0,1]), H) : \gamma(0) = v_0, \gamma(1) = v_1\}$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)).$$

If J satisfies the $(C)_c$ condition, then c is a critical value of J and $c > \max\{J(v_0), J(v_1)\}$.

We choose $H = E_\lambda, J = J_\lambda, v_0 = 0$ and define $\Omega = B(0,\rho)$ is a ball with radius ρ and origin at $0 \in H$, where radius ρ is given in following lemma.

Lemma 3.2. *Assume $(V1)$ and $(f1),(f2)$ are satisfied, then for each $\lambda > 0$, there exist $\rho > 0$ such that*

$$\inf_{\|u\|_\lambda=\rho} J_\lambda(u) > 0.$$

Proof. According to $(f1)$, for any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$, such that

$$|f(x,z)| \le \varepsilon|z|, \qquad \forall x \in \mathbb{R}^N \text{ and } |z| \le \delta. \tag{3.1}$$

By $(f2)$ we can obtain that

$$|f(x,z)| \le C_0(|z| + |z|^{p-1}) \le |z|^{p-1}\left(C_0\frac{1}{\delta^{p-2}} + 1\right) := C_\varepsilon|z|^{p-1}, \qquad \forall x \in \mathbb{R}^N,\ |z| \ge \delta. \tag{3.2}$$

Combining this with (3.1), (3.2) and $F(x,z) = \int_0^1 f(x,tz)zdt$, we get

$$|f(x,z)| \le C_\varepsilon|z|^{p-1} + \varepsilon|z|, \qquad \forall(x,z) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.3}$$

and

$$|F(x,z)| \le \frac{C_\varepsilon}{p}|z|^p + \frac{\varepsilon}{2}|z|^2, \qquad \forall(x,z) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.4}$$

Then, from (3.4) and (1.2), we have that

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla_\gamma u|^2 + \lambda V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx \\
&\ge \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N}\frac{\varepsilon}{2}|u|^2dx - \int_{\mathbb{R}^N}\frac{C_\varepsilon}{p}|u|^p dx \\
&\ge \frac{1}{2}\|u\|_\lambda^2 - \frac{\varepsilon}{2}d_2^2\|u\|_\lambda^2 - \frac{C_\varepsilon}{p}d_p^p\|u\|_\lambda^p \\
&\ge \frac{1}{4}\|u\|_\lambda^2 - \frac{C_\varepsilon}{p}d_p^p\|u\|_\lambda^p,
\end{aligned}$$

where $2 < p < 2_\gamma^*$ and $0 < \varepsilon < \frac{1}{2d_2^2}$. Choosing $\rho = \|u\|_\lambda$ small enough concludes the proof. $\square$

Lemma 3.3. *Under assumption $(V1)$ and $(f3)$, there exist $v_1 \in E_\lambda$, such that $\|v_1\|_\lambda > \rho$ and $J_\lambda(v_1) < 0$.*

Proof. Let $u \in E_\lambda$ satisfied $u \neq 0$, then $\text{meas}(\{x \in \mathbb{R}^N : u(x) \neq 0\}) > 0$. If there exists $M_0 > 0$ such that $J_\lambda(tu) > -M_0$, then by $(f3)$ and the Fatou lemma, we have that

$$\begin{aligned}
0 = \lim_{t\to+\infty}\frac{-M_0}{t^2} &\le \limsup_{t\to+\infty}\frac{J_\lambda(tu)}{t^2} \\
&= \limsup_{t\to+\infty}\left(\frac{\frac{t^2}{2}\|u\|_\lambda^2}{t^2} - \int_{\mathbb{R}^N}\frac{F(x,tu)}{t^2}dx\right) \\
&\le \frac{1}{2}\|u\|_\lambda^2 - \liminf_{t\to+\infty}\int_{u(x)\neq 0}\frac{F(x,tu)}{(tu)^2}u^2dx \\
&= -\infty.
\end{aligned}$$

Obviously, this is a contradiction. So $J_\lambda(tu) \to -\infty$, as $t \to +\infty$. Let $v_1 = tu$, for large enough t, we have $\|v_1\|_\lambda > \rho$ and $J_\lambda(v_1) < 0$. The proof is complete. $\square$

It is clear that

$$\inf_{u\in\partial\Omega} J_\lambda(u) = \inf_{\|u\|_\lambda=\rho} J_\lambda(u) > 0 = \max\{J_\lambda(0), J_\lambda(v_1)\} = \max\{J_\lambda(v_0), J_\lambda(v_1)\}.$$

That is, the geometric conditions of mountain pass theorem are satisfied. Thus, the mountain pass value

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)).$$

exists.

Lemma 3.4. *Let $(V1), (V2)$ and $(f1) \sim (f4)$ be satisfied. For any $M > c_\lambda$, the $(C)_{c_\lambda}$ sequence of J_λ is bounded in E_λ for enough large λ.*

Proof. Let $\{u_n\} \subset E_\lambda$ be a $(C)_{c_\lambda}$ sequence of J_λ, that is

$$J_\lambda(u_n) \to c_\lambda, \qquad J'_\lambda(u_n)(1 + \|u_n\|_\lambda) \to 0, \quad \text{as } n \to \infty. \tag{3.5}$$

Arguing by contradiction, up to subsequence, we assume that $\|u_n\|_\lambda \to \infty$ as $n \to \infty$. Let $w_n = \frac{u_n}{\|u_n\|_\lambda}$, then $\|w_n\|_\lambda = 1$, $\{w_n\}$ is bounded. Going if necessary to a subsequence, there exists a $w \in E_\lambda$ such that we have

$$\begin{aligned} w_n &\to w, \qquad \text{in } L^s_{loc}(\mathbb{R}^N), \text{ for } 2 \le s < 2^*_\gamma; \\ w_n(x) &\to w(x), \quad \text{a.e. } x \in \mathbb{R}^N. \end{aligned} \tag{3.6}$$

Firstly, we consider the case $w = 0$. By (1.4) and (3.5), we obtain that

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)u_n}{\|u_n\|^2_\lambda} dx = 1 - \frac{\langle J'_\lambda(u_n), u_n\rangle}{\|u_n\|^2_\lambda} \to 1, \quad \text{as } n \to \infty. \tag{3.7}$$

From $(f1)$, there exist $\delta > 0$, such that

$$\left|\frac{f(x,z)z}{z^2}\right| = \left|\frac{f(x,z)}{z}\right| \le 1, \qquad \forall x \in \mathbb{R}^N,\ 0 < |z| < \delta. \tag{3.8}$$

By $(f2)$, there exist $C > 0$ satisfy

$$\left|\frac{f(x,z)z}{z^2}\right| \le \left|\frac{C_0(|z|^2 + |z|^p)}{z^2}\right| \le C, \qquad \forall x \in \mathbb{R}^N,\ \delta \le |z| \le L_0. \tag{3.9}$$

Hence, from (3.8) and (3.9), we have that

$$|f(x,z)z| \le (C+1)z^2, \qquad \forall x \in \mathbb{R}^N,\ 0 < |z| \le L_0. \tag{3.10}$$

By $(V2)$, (3.6) and $\|w_n\|_\lambda = 1$, we get that

$$\begin{aligned} \int_{\mathbb{R}^N} w_n^2 dx &= \int_{V(x)\ge b} w_n^2 dx + \int_{V(x)<b} w_n^2 dx \\ &\le \frac{1}{\lambda b}\int_{V(x)\ge b} \lambda V(x) w_n^2 dx + \int_{V(x)<b} w_n^2 dx \\ &\le \frac{1}{\lambda b}\int_{\mathbb{R}^N} \lambda V(x) w_n^2 dx + \int_{V(x)<b} w_n^2 dx \\ &\le \frac{1}{\lambda b} + \int_{V(x)<b} w_n^2 dx \to 0, \quad \text{as } n \to \infty,\ \lambda \to +\infty. \end{aligned} \tag{3.11}$$

In view of (3.10) and (3.11), we obtain that

$$\begin{aligned}\int_{|u_n|\le L_0}\frac{|f(x,u_n)u_n|}{\|u_n\|_\lambda^2}dx &\le (C+1)\int_{|u_n|\le L_0}\frac{u_n^2}{\|u_n\|_\lambda^2}dx\\ &=(C+1)\int_{|u_n|\le L_0}w_n^2dx\\ &\le (C+1)\int_{\mathbb{R}^N}w_n^2dx\to 0,\quad \text{as } n\to\infty,\ \lambda\to+\infty.\end{aligned}\tag{3.12}$$

Combing the Hölder inequality, (1.2), $\|w_n\|_\lambda=1$ and (3.11), for any $s\in(2,2_\gamma^*)$ we have that

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|w_n|^sdx\right)^{\frac{1}{s}} &= \left(\int_{\mathbb{R}^N}|w_n|^{\theta s}|w_n|^{(1-\theta)s}dx\right)^{\frac{1}{s}}\\ &\le \left(\int_{\mathbb{R}^N}|w_n|^{\theta s\cdot\frac{2}{\theta s}}dx\right)^{\frac{\theta s}{2}\cdot\frac{1}{s}}\left(\int_{\mathbb{R}^N}|w_n|^{(1-\theta)s\cdot\frac{2_\gamma^*}{(1-\theta)s}}dx\right)^{\frac{(1-\theta)s}{2_\gamma^*}\cdot\frac{1}{s}}\\ &=\left(\int_{\mathbb{R}^N}|w_n|^2dx\right)^{\frac{\theta}{2}}\left(\int_{\mathbb{R}^N}|w_n|^{2_\gamma^*}dx\right)^{\frac{1-\theta}{2_\gamma^*}}\\ &\le d_{2_\gamma^*}^{1-\theta}\left(\int_{\mathbb{R}^N}|w_n|^2dx\right)^{\frac{\theta}{2}}\to 0,\quad \text{as } n\to\infty,\ \lambda\to+\infty,\end{aligned}\tag{3.13}$$

where $\theta=\frac{2(2_\gamma^*-s)}{s(2_\gamma^*-2)}$. By (3.5) and $(f4)$, we get that for n large enough

$$M>J_\lambda(u_n)-\frac{1}{2}\langle J'_\lambda(u_n),u_n\rangle=\int_{\mathbb{R}^N}\mathcal{F}(x,u_n)dx\ge 0.\tag{3.14}$$

From $\tau>\frac{\tilde N}{2}$, we easily obtain $\frac{2\tau}{\tau-1}\in(2,2_\gamma^*)$. So, by the Hölder inequality, $(f4)$, (3.14) and (3.13) with $s=\frac{2\tau}{\tau-1}$, we get that

$$\begin{aligned}\int_{|u_n|\ge L_0}\frac{|f(x,u_n)u_n|}{\|u_n\|_\lambda^2}dx &= \int_{|u_n|\ge L_0}\left|\frac{f(x,u_n)}{u_n}\right|w_n^2dx\\ &\le \left(\int_{|u_n|\ge L_0}\left|\frac{f(x,u_n)}{u_n}\right|^\tau dx\right)^{\frac{1}{\tau}}\left(\int_{|u_n|\ge L_0}|w_n|^{2\cdot\frac{\tau}{\tau-1}}dx\right)^{\frac{\tau-1}{\tau}}\\ &\le \left(\int_{|u_n|\ge L_0}a_1\mathcal{F}(x,u_n)dx\right)^{\frac{1}{\tau}}\left(\int_{\mathbb{R}^N}|w_n|^{\frac{2\tau}{\tau-1}}dx\right)^{\frac{\tau-1}{\tau}}\\ &\le a_1^{\frac{1}{\tau}}\left(\int_{\mathbb{R}^N}\mathcal{F}(x,u_n)dx\right)^{\frac{1}{\tau}}\left(\left(\int_{\mathbb{R}^N}|w_n|^{\frac{2\tau}{\tau-1}}dx\right)^{\frac{\tau-1}{2\tau}}\right)^2\\ &\le (a_1M)^{\frac{1}{\tau}}\left(\left(\int_{\mathbb{R}^N}|w_n|^{\frac{2\tau}{\tau-1}}dx\right)^{\frac{\tau-1}{2\tau}}\right)^2\to 0,\quad \text{as } n\to+\infty,\ \lambda\to+\infty.\end{aligned}$$

Thus, combining with (3.12), we obtain that

$$\int_{\mathbb{R}^N}\frac{f(x,u_n)u_n}{\|u_n\|_\lambda^2}dx=\int_{|u_n|\le L_0}\frac{f(x,u_n)u_n}{\|u_n\|_\lambda^2}dx+\int_{|u_n|\ge L_0}\frac{f(x,u_n)u_n}{\|u_n\|_\lambda^2}dx\to 0,\quad \text{as } n\to\infty,\ \lambda\to+\infty,$$

which is a contradiction with (3.7).

Secondly, we consider the case $w \neq 0$. Evidently, $\mathrm{meas}(\{x \in \mathbb{R}^N : w(x) \neq 0\}) > 0$ and $|u_n(x)| \to \infty$ as $n \to \infty$, for a.e. $x \in \{x \in \mathbb{R}^N : w(x) \neq 0\}$. Thus, from $(f3)$ and Fatou's lemma, we can get

$$\begin{aligned} \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(x,u_n)dx}{\|u_n\|_\lambda^2} &\geq \liminf_{n\to\infty} \int_{w(x)\neq 0} \frac{F(x,u_n)}{u_n^2} w_n^2 dx \\ &\geq \int_{w(x)\neq 0} \liminf_{n\to\infty} \frac{F(x,u_n)}{u_n^2} w_n^2 dx \\ &= +\infty. \end{aligned} \tag{3.15}$$

By (3.5), we have

$$\begin{aligned} \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(x,u_n)dx}{\|u_n\|_\lambda^2} &\leq \limsup_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(x,u_n)dx}{\|u_n\|_\lambda^2} \\ &= \limsup_{n\to\infty} \left(\frac{1}{2} - \frac{J_\lambda(u_n)}{\|u_n\|_\lambda^2} \right) \\ &= \frac{1}{2}, \end{aligned}$$

which is contradiction with (3.15).

So $\{u_n\}$ is bounded. □

Lemma 3.5. *Assume $(V1), (V2)$ and $(f1) \sim (f4)$ be satisfied, then for any $M > c_\lambda$, there exist $\Lambda = \Lambda(M) > 0$ such that J_λ satisfies $(C)_{c_\lambda}$ condition for all $\lambda > \Lambda$.*

Proof. Let $\{u_n\} \subset E_\lambda$ satisfies (3.5). By Lemma 3.4, we known that $\{u_n\}$ is bounded in E_λ. Thus, up to a subsequence, we have that

$$u_n \rightharpoonup u, \quad \text{in } E_\lambda; \tag{3.16}$$

$$u_n \to u, \quad \text{in } L^s_{loc}(\mathbb{R}^N), \text{ for } 2 \leq s < 2^*_\gamma; \tag{3.17}$$

$$u_n(x) \to u(x), \quad \text{a.e. } x \in \mathbb{R}^N. \tag{3.18}$$

Let $v_n := u_n - u$, then $v_n \rightharpoonup 0$ in E_λ by (3.16), which implies that

$$\|u_n\|_\lambda^2 = (v_n + u, v_n + u)_\lambda = \|v_n\|_\lambda^2 + \|u\|_\lambda^2 + o(1). \tag{3.19}$$

Next, by using the similar proof method of Proposition A.1 in the literature [5], we can get that

$$\int_{\mathbb{R}^N} F(x,u_n)dx = \int_{\mathbb{R}^N} F(x,v_n)dx + \int_{\mathbb{R}^N} F(x,u)dx + o(1), \tag{3.20}$$

and

$$\int_{\mathbb{R}^N} f(x,u_n)\varphi dx = \int_{\mathbb{R}^N} f(x,v_n)\varphi dx + \int_{\mathbb{R}^N} f(x,u)\varphi dx + o(1), \tag{3.21}$$

for any $\varphi \in E_\lambda$. By (3.19) and (3.20), we can obtain that

$$J_\lambda(u_n) = J_\lambda(v_n) + J_\lambda(u) + o(1). \tag{3.22}$$

Combing with (3.21) and $u_n = v_n + u$, for any $\varphi \in E_\lambda$ we have that

$$\langle J'_\lambda(u_n), \varphi \rangle = \langle J'_\lambda(v_n), \varphi \rangle + \langle J'_\lambda(u), \varphi \rangle + o(1). \tag{3.23}$$

From (3.3), (3.18) and the dominated convergence theorem, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we obtain that

$$\int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))\varphi dx = \int_{\Omega_\varphi} (f(x,u_n) - f(x,u))\varphi dx \to 0, \quad \text{as } n \to \infty, \tag{3.24}$$

here Ω_φ is the support set of φ. For each $\varphi \in C_0^\infty(\mathbb{R}^N)$, by (3.16) we have

$$(u_n - u, \varphi)_\lambda \to 0, \quad \text{as } n \to \infty. \tag{3.25}$$

By (3.25), (3.24), (3.5) and the dense of $C_0^\infty(\mathbb{R}^N)$ in E_λ, it shows that

$$\lim_{n\to\infty} \langle J'_\lambda(u_n), \varphi\rangle = \langle J'_\lambda(u), \varphi\rangle = 0, \qquad \forall \varphi \in E_\lambda. \tag{3.26}$$

Hence, $J'_\lambda(u) = 0$ and from $(f4)$ we can obtain that

$$J_\lambda(u) = J_\lambda(u) - \frac{1}{2}\langle J'_\lambda(u), u\rangle = \int_{\mathbb{R}^N} \mathcal{F}(x,u)dx \geq 0.$$

So, by (3.22), (3.23), (3.26) and the boundedness of $\{v_n\}$, we get that

$$\begin{aligned}\int_{\mathbb{R}^N} \mathcal{F}(x,v_n)dx &= J_\lambda(v_n) - \frac{1}{2}\langle J'_\lambda(v_n), v_n\rangle \\ &= J_\lambda(u_n) - J_\lambda(u) - \frac{1}{2}\langle J'_\lambda(u_n) - J'_\lambda(u), v_n\rangle + o(1) \\ &\leq J_\lambda(u_n) + o(1).\end{aligned}$$

Thus, for enough large n, we have that

$$\int_{\mathbb{R}^N} \mathcal{F}(x,v_n)dx < M. \tag{3.27}$$

Now, we will show that $v_n \to 0$ in E_λ. By $(V2)$ and (3.17) that

$$\int_{\mathbb{R}^N} v_n^2 dx = \int_{V(x)\geq b} v_n^2 dx + \int_{V(x)<b} v_n^2 dx \leq \frac{1}{\lambda b}\|v_n\|_\lambda^2 + o(1). \tag{3.28}$$

Thus, combing with the Hölder inequality and (1.2), for any $s \in (2, 2_\gamma^*)$ we have

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |v_n|^s dx\right)^{\frac{1}{s}} &= \left(\int_{\mathbb{R}^N} |v_n|^{\theta s}|v_n|^{(1-\theta)s} dx\right)^{\frac{1}{s}} \\ &\leq \left(\int_{\mathbb{R}^N} |v_n|^{\theta s\cdot\frac{2}{\theta s}} dx\right)^{\frac{\theta s}{2}\cdot\frac{1}{s}} \left(\int_{\mathbb{R}^N} |v_n|^{(1-\theta)s\cdot\frac{2_\gamma^*}{(1-\theta)s}} dx\right)^{\frac{(1-\theta)s}{2_\gamma^*}\cdot\frac{1}{s}} \\ &= \left(\int_{\mathbb{R}^N} |v_n|^2 dx\right)^{\frac{\theta}{2}} \left(\int_{\mathbb{R}^N} |v_n|^{2_\gamma^*} dx\right)^{\frac{1-\theta}{2_\gamma^*}} \\ &\leq d_{2_\gamma^*}^{1-\theta}(\lambda b)^{-\frac{\theta}{2}}\|v_n\|_\lambda + o(1),\end{aligned} \tag{3.29}$$

where $\theta = \frac{2(2_\gamma^*-s)}{s(2_\gamma^*-2)}$. According to (3.28) and (3.10), we obtain that

$$\begin{aligned}\int_{v_n\leq L_0} f(x,v_n)v_n dx &\leq (C+1)\int_{v_n\leq L_0} v_n^2 dx \\ &\leq (C+1)\int_{\mathbb{R}^N} v_n^2 dx \\ &\leq \frac{C+1}{\lambda b}\|v_n\|_\lambda^2 + o(1).\end{aligned} \tag{3.30}$$

By $\tau > \frac{\tilde{N}}{2}$, it is easy obtained that $\frac{2\tau}{\tau-1} \in (2, 2^*_\gamma)$. Thus, from the Hölder inequality, (3.27), (3.29) with $s = \frac{2\tau}{\tau-1}$ and the boundedness of $\{v_n\}$, we can see that

$$\begin{aligned}
\int_{v_n \geq L_0} f(x, v_n) v_n dx &\leq \int_{|u_n| \geq L_0} \left| \frac{f(x, v_n)}{v_n} \right| v_n^2 dx \\
&\leq \left(\int_{v_n \geq L_0} \left| \frac{f(x, v_n)}{v_n} \right|^\tau dx \right)^{\frac{1}{\tau}} \left(\int_{v_n \geq L_0} |v_n|^{2 \cdot \frac{\tau}{\tau-1}} dx \right)^{\frac{\tau-1}{\tau}} \\
&\leq \left(\int_{v_n \geq L_0} a_1 \mathcal{F}(x, v_n) dx \right)^{\frac{1}{\tau}} \left(\int_{\mathbb{R}^N} |v_n|^{\frac{2\tau}{\tau-1}} dx \right)^{\frac{\tau-1}{\tau}} \\
&\leq a_1^{\frac{1}{\tau}} \left(\int_{\mathbb{R}^N} \mathcal{F}(x, v_n) dx \right)^{\frac{1}{\tau}} \left(\left(\int_{\mathbb{R}^N} |v_n|^{\frac{2\tau}{\tau-1}} dx \right)^{\frac{\tau-1}{2\tau}} \right)^2 \\
&\leq (a_1 M)^{\frac{1}{\tau}} d_{2^*_\gamma}^{2(1-\theta)} (\lambda b)^{-\theta} \|v_n\|_\lambda^2 + o(1).
\end{aligned} \tag{3.31}$$

Therefore, by (3.30) and (3.31), we have

$$\begin{aligned}
o(1) &= \langle J'_\lambda(v_n), v_n \rangle \\
&= \|v_n\|_\lambda^2 - \int_{\mathbb{R}^N} f(x, v_n) v_n dx \\
&= \|v_n\|_\lambda^2 - \int_{v_n \leq L_0} f(x, v_n) v_n dx - \int_{v_n \geq L_0} f(x, v_n) v_n dx \\
&\geq \left[1 - \frac{C+1}{\lambda b} - (a_1 M)^{\frac{1}{\tau}} d_{2^*_\gamma}^{2(1-\theta)} (\lambda b)^{-\theta} \right] \|v_n\|_\lambda^2 + o(1).
\end{aligned}$$

So, there exist $\Lambda = \Lambda(M) > 0$ such that $v_n \to 0$ in E_λ as $n \to \infty$ for any $\lambda > \Lambda$. The proof is complete. □

Proof of Theorem 1.3. By Lemma 3.2, 3.3, 3.4 and 3.5, all condition of Theorem 3.1 are satisfied. Thus equation (1.1) possesses at least a nontrivial solution $u_\lambda \in E_\lambda$ and $J_\lambda(u_\lambda) = c_\lambda$ is a critical value, as $\lambda > \Lambda$. Set $S = \{u \in E_\lambda - \{0\} : J'_\lambda(u) = 0\}$. Evidently, by $u_\lambda \in S$ we have that

$$\inf_{u \in S} J_\lambda(u) \leq J_\lambda(u_\lambda) = c_\lambda.$$

For any $u \in S$, let $\gamma_u(t) = tt_0 u, t \in [0, 1]$, then $\gamma \in \Gamma$ for enough large t_0 by Lemma 3.3. Thus, according to the definition of c_λ for any $u \in S$ we have

$$c_\lambda \leq \max_{t \in [0,1]} J_\lambda(\gamma_u(t)) = \max_{t \in [0,1]} J_\lambda(tt_0 u) = \max_{t \in [0, t_0]} J_\lambda(tu) = \max_{t \geq 0} J_\lambda(tu).$$

It is easy obtained that $J_\lambda(u) = \max_{t \geq 0} J_\lambda(tu)$ by $(f5)$ for any $u \in S$. So, from the arbitrariness of u, we obtain

$$\inf_{u \in S} J_\lambda(u) \geq c_\lambda.$$

Thus,

$$c_\lambda = \inf_{u \in S} J_\lambda(u),$$

and we can conclude that u_λ is the ground state solution, then the proof of Theorem 1.3 is completed. □

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