Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **88**, 1–17; https://doi.org/10.14232/ejqtde.2021.1.88 www.math.u-szeged.hu/ejqtde/

Asymptotic behavior of solutions to difference equations in Banach spaces

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Received 1 October 2021, appeared 14 December 2021
Communicated by Stevo Stević

Abstract. We investigate the asymptotic properties of solutions to higher order nonlinear difference equations in Banach spaces. We introduce a new technique based on a vector version of discrete L'Hospital's rule, remainder operator, and the regional topology on the space of all sequences on a given Banach space. We establish sufficient conditions for the existence of solutions with prescribed asymptotic behavior. Moreover, we are dealing with the problem of approximation of solutions. Our technique allows us to control the degree of approximation of solutions.

Keywords: difference equation in Banach space, prescribed asymptotic behavior, degree of approximation, remainder operator, regional topology.

2020 Mathematics Subject Classification: 39A10.

1 Introduction

Let $\mathbb{N}$, $\mathbb{R}$ denote the set of positive integers and the set of real numbers respectively. In this paper we assume that $m \in \mathbb{N}$ is fixed and X is a real Banach space. We consider the equation

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n \tag{E}$$

$$n \in \mathbb{N}, \quad a_n \in \mathbb{R}, \quad b_n \in X, \quad f : \mathbb{N} \times X \to X, \quad \sigma : \mathbb{N} \to \mathbb{N}, \quad \lim \sigma(n) = \infty.$$

By a solution of (E) we mean a sequence $x : \mathbb{N} \to X$ satisfying (E) for all large n.

Nonlinear difference equations often appear in mathematical models used, for example, in technology, biology, physics, economics or medicine. Hence the study of behavior of solutions to difference equations is of great importance. Therefore, many papers are devoted to this topic, see for example [3, 4, 6, 12, 14, 15, 17–22]. In some papers the difference equations in Banach spaces are also investigated, see for example [1, 2, 5, 7–9, 16].

In this paper we deal with the problem of the existence of solutions to the equation (E), with prescribed asymptotic behavior and the problem of approximation of solutions to equation (E). More precisely, in Section 4 we establish conditions under which for a given sequence

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$y : \mathbb{N} \to X$ such that $\Delta^m x_n = b_n$ and a given number $s \in (-\infty, 0]$ there exists a solution x of (E) such that $x_n = y_n + \mathrm{o}(n^s)$ (then x is called a solution with prescribed asymptotic behavior, and y is called an approximative solution of (E)). Next, in Section 5, we establish conditions under which for a given solution x of (E) and a given number $s \in (-\infty, 0]$ there exists a sequence $y : \mathbb{N} \to X$ such that $\Delta^m y_n = b_n$ and $x_n = y_n + \mathrm{o}(n^s)$. By selecting the number s, we can control the degree of approximation of solution.

The paper is organized as follows. In Section 2, we introduce notation and terminology. In Section 3, we present our technical tools, i.e. vector version of discrete L'Hospital's rule, the regional topology on the space of all sequences on a given Banach space, and remainder operator which are needed to get the main results. The next two sections contain our main results. In Section 4 we establish sufficient conditions for the existence of solutions with prescribed asymptotic behavior. Section 5 is devoted to approximation of solutions.

2 Notation and terminology

Let $\mathbb{Z}$, denote the set of all integers. If $p, k \in \mathbb{Z}$, $p \le k$, then $\mathbb{N}(p)$, $\mathbb{N}(p,k)$ denote the sets defined by

$$\mathbb{N}(p) = \{p, p+1, \dots\}, \qquad \mathbb{N}(p,k) = \{p, p+1, \dots, k\}.$$

We use the symbol $|t|$ to denote the norm of a vector $t \in X$. The space of all sequences $x : \mathbb{N} \to \mathbb{R}$ we denote by $\mathbb{R}^{\mathbb{N}}$. Moreover, we use the symbol $X^{\mathbb{N}}$ to denote the space of all sequences $x : \mathbb{N} \to X$. If $a \in \mathbb{R}^{\mathbb{N}}$ and $x \in X^{\mathbb{N}}$, then ax denotes the sequence defined by pointwise multiplication

$$ax(n) = a_n x_n.$$

Moreover, $|x|$ denotes the sequence defined by $|x|(n) = |x_n|$ for every n. Let

$$\mathrm{Fin}(X) = \bigcup_{p=1}^{\infty} \{x \in X^{\mathbb{N}} : \ x_n = 0 \text{ for } n \ge p\},$$

$$\mathrm{o}_X(1) = \left\{ x \in X^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0 \right\}, \quad \mathrm{O}_X(1) = \left\{ x \in X^{\mathbb{N}} : x \text{ is bounded} \right\}$$

and for $a \in \mathbb{R}^{\mathbb{N}}$ let

$$\mathrm{o}_X(a) = \{ax : x \in \mathrm{o}_X(1)\} + \mathrm{Fin}(X),$$
$$\mathrm{O}_X(a) = \{ax : x \in \mathrm{O}_X(1)\} + \mathrm{Fin}(X).$$

For a sequence $a \in \mathbb{R}^{\mathbb{N}}$ and $x \in X^{\mathbb{N}}$ we write $x_n = \mathrm{o}(a_n)$ to denote the relation

$$x \in \mathrm{o}_X(a).$$

Analogously $x_n = \mathrm{O}(a_n)$ denotes the relation $x \in \mathrm{O}_X(a)$.

We use the symbol Δ to denote the difference operator defined by

$$\Delta : X^{\mathbb{N}} \to X^{\mathbb{N}}, \qquad (\Delta x)(n) = x_{n+1} - x_n.$$

As usual we use Δx_n to denote the value $(\Delta x)(n)$. For $k \in \mathbb{N}$ we denote by Δ^k the k-th iteration of the operator Δ. Moreover, Δ^0 denotes the identity operator. For $k \in \mathbb{N}(0)$ we define

$$\mathrm{Pol}_X(k-1) = \mathrm{Ker}(\Delta^k) = \left\{ x \in X^{\mathbb{N}} : \Delta^k x = 0 \right\}.$$

Then $\mathrm{Pol}_X(k-1)$ is the space of all polynomial sequences of degree less than k. Note that

$$\mathrm{Pol}_X(-1) = \mathrm{Ker}(\Delta^0) = 0$$

is the zero space. It is easy to see that $\varphi \in \mathrm{Pol}_X(k-1)$ if and only if there exist vectors $x_0, x_1, \ldots, x_{k-1} \in X$ such that

$$\varphi(n) = x_{k-1}n^{k-1} + x_{k-2}n^{k-2} + \cdots + x_1 n + x_0$$

for any $n \in \mathbb{N}$. For $b \in X^{\mathbb{N}}$ we use the symbol $\Delta^{-k}b$ to denote the set

$$\Delta^{-k}b = \left\{ x \in X^{\mathbb{N}} : \Delta^k x = b \right\}.$$

Remark 2.1. If y is an arbitrary element of $\Delta^{-k}b$, then

$$\Delta^{-k}b = y + \mathrm{Pol}_X(k-1).$$

Let H be a metric space. For a subset A of H and $\varepsilon > 0$, we define an ε-ball about A by

$$\mathrm{B}(A,\varepsilon) = \bigcup_{a\in A} \mathrm{B}(a,\varepsilon).$$

where $\mathrm{B}(a,\varepsilon)$ denotes an open ball of radius ε centered at a. We say that a subset U of H is a uniform neighborhood of A if there exists a positive ε such that

$$\mathrm{B}(A,\varepsilon) \subset U.$$

A subset A of H is called an ε-net for a subset Z of H if $Z \subset \mathrm{B}(A,\varepsilon)$. A subset Z of H is said to be totally bounded if for any $\varepsilon > 0$ there exist a finite ε-net for Z.

3 Preliminaries

In this section, we introduce the technical tools that form the basis of our technique for studying the asymptotic properties of solutions to difference equations.

3.1 Discrete L'Hospital's rule

Lemma 3.1. *Assume a, b, r are positive real numbers, $c \in X$, $a_1, a_2, \ldots, a_n$ are real numbers with the same nonzero sign. Then*

$$a\mathrm{B}(c,r) = \mathrm{B}(ac, ar), \quad a\mathrm{B}(c,r) + b\mathrm{B}(c,r) = (a+b)\mathrm{B}(c,r), \tag{3.1}$$

and

$$a_1\mathrm{B}(c,r) + a_2\mathrm{B}(c,r) + \cdots + a_n\mathrm{B}(c,r) = (a_1 + \cdots + a_n)\mathrm{B}(c,r). \tag{3.2}$$

Proof. The assertion (3.1) is an easy exercise, (3.2) is a consequence of (3.1). □

Lemma 3.2. *Assume $x \in X^{\mathbb{N}}$, $p \in \mathbb{N}$, $r, L \in \mathbb{R}$,*

$$c \in X, \quad r > 0, \quad L \geq |c| + r,$$

(y_n) is a sequence of real numbers, strictly monotonic for $n \geq p$. Moreover,

$$y_n \neq 0 \quad \textit{and} \quad \frac{\Delta x_n}{\Delta y_n} \in \mathrm{B}(c, r) \tag{3.3}$$

for $n \geq p$. Then

$$\left| \frac{x_n}{y_n} - c \right| < r + L \left| \frac{y_k}{y_n} \right| + \left| \frac{x_k}{y_n} \right| \tag{3.4}$$

for $n, k \geq p$.

Proof. Assume the sequence (y_n) is increasing for $n \geq p$. Choose $n, k \geq p$. For $i \geq p$ we have $\Delta x_i \in (\Delta y_i)\mathrm{B}(c, r)$. Hence, using Lemma 3.1, we obtain

$$\begin{aligned} x_n - x_k = \Delta x_k + \cdots + \Delta x_{n-1} \ &\in \ (\Delta y_k)\mathrm{B}(c, r) + \cdots + (\Delta y_{n-1})\mathrm{B}(c, r) \\ &= (\Delta y_k + \cdots + \Delta y_{n-1})\mathrm{B}(c, r) = (y_n - y_k)\mathrm{B}(c, r). \end{aligned}$$

for $n \geq k$. Similarly, for $k \geq n$, we have $x_k - x_n \in (y_k - y_n)\mathrm{B}(c, r)$. Hence

$$x_n - x_k \in (y_n - y_k)\mathrm{B}(c, r) \quad \text{and} \quad \frac{x_n}{y_n} - \frac{x_k}{y_n} \in \left(1 - \frac{y_k}{y_n}\right)\mathrm{B}(c, r).$$

Therefore, there exists a vector $b \in B(c, r)$ such that

$$\frac{x_n}{y_n} - \frac{x_k}{y_n} = \left(1 - \frac{y_k}{y_n}\right) b = b - \left(\frac{y_k}{y_n}\right) b.$$

Hence

$$\frac{x_n}{y_n} - c = b - c - \left(\frac{y_k}{y_n}\right) b + \frac{x_k}{y_n}.$$

Since $|b - c| < r$ and $|b| \leq |c| + r \leq L$, we have

$$\left| \frac{x_n}{y_n} - c \right| < r + L \left| \frac{y_k}{y_n} \right| + \left| \frac{x_k}{y_n} \right|$$

The case when (y_n) is decreasing for $n \geq p$ is analogous. □

Theorem 3.3 (Discrete L'Hospital's rule)**.** *Assume $(x_n) \in X^{\mathbb{N}}$, (y_n) is a sequence of real numbers which is strictly monotonic for large n. Moreover, we assume that the sequence $(\Delta x_n / \Delta y_n)$ is convergent and one of the following conditions is satisfied:*

(*a*) $\lim_{n\to\infty} x_n = 0$ *and* $\lim_{n\to\infty} y_n = 0$,

(*b*) *the sequence (y_n) is unbounded.*

Then the sequence (x_n / y_n) is convergent and

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\Delta x_n}{\Delta y_n}.$$

Proof. Let $\varepsilon > 0$. There exists an index p such that

$$\left| \frac{\Delta x_n}{\Delta y_n} - \frac{\Delta x_k}{\Delta y_k} \right| < \varepsilon$$

for $n, k \geq p$. Let $c = \Delta x_p / \Delta y_p$. Then $\Delta x_n / \Delta y_n \in \mathrm{B}(c, \varepsilon)$ for $n \geq p$. If condition (a) is satisfied and $n \geq p$, then taking sufficiently large k and using Lemma 3.2 we obtain $|x_n / y_n - c| < 2\varepsilon$. Similarly, if condition (b) is satisfied, then using Lemma 3.2 we obtain an index $q \geq p$ such that $|x_n / y_n - c| < 2\varepsilon$ for $n \geq q$. Then

$$\left| \frac{x_n}{y_n} - \frac{\Delta x_n}{\Delta y_n} \right| \leq \left| \frac{x_n}{y_n} - c \right| + \left| c - \frac{\Delta x_n}{\Delta y_n} \right| < 2\varepsilon + \varepsilon. \qquad \square$$

Lemma 3.4. *If $x \in X^{\mathbb{N}}$, $m \in \mathbb{N}$, $s \in (-1, \infty)$, and $\Delta^m x_n = \mathrm{o}(n^s)$, then*

$$x_n = \mathrm{o}(n^{s+m}).$$

Proof. Induction on m. Let $m = 1$. Using L'Hospital's rule we obtain

$$\begin{aligned}
\lim_{t \to \infty} \frac{(t+1)^{s+1} - t^{s+1}}{t^s} &= \lim_{t \to \infty} \frac{(t+1)^{s+1} - t^{s+1}}{t^{-1} t^{s+1}} = \lim_{t \to \infty} \frac{(1 + t^{-1})^{s+1} - 1}{t^{-1}} \\
&= \lim_{t \to \infty} \frac{(s+1)(1 + t^{-1})^s (-t^{-2})}{-t^{-2}} = s + 1.
\end{aligned}$$

Hence

$$\lim_{n \to \infty} \frac{\Delta n^{s+1}}{n^s} = s + 1.$$

So by assumption $\Delta x = \mathrm{o}(n^s)$ we obtain

$$\lim \frac{\Delta x_n}{\Delta n^{s+1}} = \lim \frac{\Delta x_n}{n^s} \frac{n^s}{\Delta n^{s+1}} = \lim \frac{\Delta x_n}{n^s} \lim \frac{n^s}{\Delta n^{s+1}} = \frac{0}{s+1} = 0.$$

Since $s > -1$, the sequence (n^{s+1}) is increasing to infinity. By Theorem 3.3, we obtain $x_n = \mathrm{o}(n^{s+1})$. Hence the assertion is true for $m = 1$. Assume it is true for certain $m \geq 1$ and let $\Delta^{m+1} x_n = \mathrm{o}(n^s)$. Then $\Delta^m \Delta x_n = \mathrm{o}(n^s)$ and by inductive hypothesis we get $\Delta x_n = \mathrm{o}(n^{s+m})$. Hence by the first part of the proof we obtain $x_n = \mathrm{o}(n^{s+m+1})$. $\square$

3.2 Regional topology

Let Y be a real vector space. We say that a function $\|\cdot\| : Y \to [0, \infty]$ is regional norm if the condition $\|x\| = 0$ is equivalent to $x = 0$ and for any $x, y \in Y$ and $\alpha \in \mathbb{R}$ we have

$$\|\alpha x\| = |\alpha| \|x\|, \quad \|x + y\| \leq \|x\| + \|y\|.$$

Hence, the notion of regional norm generalizes the notion of usual norm. If a regional norm on Y is given, then we say that Y is a regional normed space. If there exists a vector $x \in Y$ such that $\|x\| = \infty$, then we say that Y is extraordinary.

Assume Y is a regional normed space. We say that a subset Z of Y is ordinary if $\|x - y\| < \infty$ for any $x, y \in Z$. We regard every ordinary subset Z of Y as a metric space with metric defined by

$$d(x, y) = \|x - y\|.$$

Let $U \subset Y$. We say that U is regionally open if $U \cap Z$ is open in Z for any ordinary subset Z of Y. The family of all regionally open subsets is a topology on Y which we call the regional topology. We regard any subset of Y as a topological space with topology induced by the regional topology. The subset

$$Y_0 = \{y \in Y : \|y\| < \infty\},$$

is a linear subspace of Y and regional norm induces an usual norm on Y_0. We say that Y is a regional Banach space if Y_0 is a Banach space.

An important special case of a regional Banach space we obtain as follows. Let D be an arbitrary nonempty set and let $\mathrm{F}(D, X)$ denote the space of all functions $f : D \to X$. Then the formula

$$\|f\| = \sup\{|f(p)| : p \in D\}$$

defines a regional norm on $\mathrm{F}(D, X)$. This space is extraordinary if D is infinite. In particular, we obtain the regional topology on the space

$$X^{\mathbb{N}} = \mathrm{F}(\mathbb{N}, X).$$

The regional topology in $\mathrm{F}(D, X)$ is, simply, the topology of uniform convergence. In extraordinary case this topology is not linear but almost linear. For more details and for the proof of the following theorem we refer to [13].

Theorem 3.5 (Generalized Schauder theorem)**.** *Assume Q is a closed and convex subset of a regional Banach space Y, a map $A : Q \to Q$ is continuous and the set $A(Q)$ is ordinary and totally bounded. Then there exists a point $x \in Q$ such that $A(x) = x$.*

We say that a family $T \subset X^{\mathbb{N}}$ is pointwise totally bounded if for any n the set $T(n) = \{t_n : t \in T\}$ is totally bounded. We say that T is stable at infinity if for any $\varepsilon > 0$ there exists an index p such that $|x_n - y_n| < \varepsilon$ for any $n > p$ and any $x, y \in T$.

Lemma 3.6. *If a family $T \subset X^{\mathbb{N}}$ is pointwise totally bounded and stable at infinity, then T is totally bounded with respect to regional norm.*

Proof. Let $t \in T$ and $\varepsilon > 0$. Choose an index p such that

$$|x_n - y_n| < \varepsilon$$

for any $x, y \in T$ and any $n > p$. For any $i = 1, \dots, p$ choose a finite ε-net G_i for the set

$$T(i) = \{x_i : x \in T\}.$$

Let

$$G = \left\{ z \in X^{\mathbb{N}} : z_n \in G_n \text{ for } n \le p \text{ and } z_n = t_n \text{ for } n > p \right\}.$$

Fix an $x \in T$. For any $i \in \mathbb{N}(1, p)$ choose $g_i \in G_i$ such that $|x_i - g_i| < \varepsilon$. Let $h \in X^{\mathbb{N}}$ be defined by

$$h_n = g_n \text{ for } n \le p, \quad h_n = t_n \text{ for } n > p.$$

Then $h \in G$ and $|x - h| < \varepsilon$. Hence G is a finite ε-net for T. □

3.3 Remainder operator

In this section we define the iterated remainder operator. This operator will be used in the proofs of our main results. In Lemmas 3.7 and 3.8 we establish some basic properties this operator. Next in Lemma 3.10 we show that if $x \in X^{\mathbb{N}}$ and $\Delta^m x$ is asymptotically zero, then x is asymptotically polynomial. In Lemmas 3.11 and 3.12 we present some useful consequences of Lemma 3.10.

Now we define the spaces $\mathrm{S}_X(m)$ of m-times summable sequences and the remainder operator. Let

$$\mathrm{S}_X(0) = \mathrm{o}_X(1), \qquad \mathrm{S}_X(1) = \left\{ x \in X^{\mathbb{N}} : \text{ the series } \sum_{n=1}^{\infty} x_n \text{ is convergent} \right\}.$$

For $x \in \mathrm{S}_X(1)$, we define the sequence $r(x)$ by the formula

$$r(x)(n) = \sum_{j=n}^{\infty} x_j.$$

Then $r(x) \in \mathrm{S}_X(0)$ and we obtain the remainder operator

$$r : \mathrm{S}_X(1) \to \mathrm{S}_X(0).$$

For $m \in \mathbb{N}$, by induction, we define the linear space $\mathrm{S}_X(m+1)$ and the linear operator

$$r^{m+1} : \mathrm{S}_X(m+1) \to \mathrm{S}_X(0)$$

by

$$\mathrm{S}_X(m+1) = \{x \in \mathrm{S}_X(m) : \ r^m(x) \in \mathrm{S}_X(1)\}, \qquad r^{m+1}(x) = r(r^m(x)).$$

Note that

$$r^m(x)(n) = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} x_{i_m}$$

for any $x \in \mathrm{S}_X(m)$ and any $n \in \mathbb{N}$.

In the proof of the next lemma we use the fact that in Banach space absolute convergence implies convergence of a series.

Lemma 3.7. *Assume $x \in X^{\mathbb{N}}$, $m \in \mathbb{N}$, $p \in \mathbb{N}$, and $s \in (-\infty, 0]$. Then*

- (a) *if $|x| \in \mathrm{S}_{\mathbb{R}}(m)$, then $x \in \mathrm{S}_X(m)$ and $|r^m(x)| \le r^m(|x|)$,*
- (b) *$|x| \in \mathrm{S}_{\mathbb{R}}(m)$ if and only if $\sum_{n=1}^{\infty} n^{m-1}|x_n| < \infty$,*
- (c) *if $|x| \in \mathrm{S}_{\mathbb{R}}(m)$, then $r^m(|x|)(p) \le \sum_{n=p}^{\infty} n^{m-1}|x_n|$,*
- (d) *if $x \in \mathrm{S}_X(m)$, then $\Delta^m(r^m(x)) = (-1)^m x$,*
- (e) *if $x \in \mathrm{o}_X(1)$, then $\Delta^m x \in \mathrm{S}_X(m)$ and $r^m(\Delta^m(x)) = (-1)^m x$,*
- (f) *if $\sum_{n=1}^{\infty} n^{m-1-s}|x_n| < \infty$, then $x \in \mathrm{S}_X(m)$ and $r^m(x)(n) = \mathrm{o}(n^s)$.*

Proof. Using our notation, the assertion (a) may be proved by repeating the proof of [10, Lemma 1]. Analogously, repeating the proof of [10, Lemma 2] we obtain (b). Similarly, we can obtain (c), (d), and (e) from [10, Lemma 2], [10, Lemma 5] and [10, Lemma 6] respectively. The assertion (f) we can obtain from [12, Lemma 4.2]. □

Lemma 3.8. *If $x \in X^{\mathbb{N}}$ and $|x| \in \mathrm{S}_{\mathbb{R}}(m)$, then*

$$r^m(x)(n) = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} x_{i_m} = \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} x_{n+k}$$
$$= \sum_{k=0}^{\infty} \frac{(k+1)(k+2)\cdots(k+m-1)}{(m-1)!} x_{n+k} = \sum_{j=n}^{\infty} \frac{(j-n+1)\cdots(j-n+m-1)}{(m-1)!} x_j.$$

Proof. See [11, Lemma 4]. □

Lemma 3.9. *If $a, b \in \mathrm{S}_{\mathbb{R}}(m)$ and $a \leq b$, then $r^m(a) \leq r^m(b)$.*

Proof. See [12, Lemma 4.1 (h)]. □

Lemma 3.10. *Assume $a \in \mathbb{R}^{\mathbb{N}}$, $x \in X^{\mathbb{N}}$, $m \in \mathbb{N}$, $s \in (-\infty, m-1]$,*

$$\sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty, \quad \textit{and} \quad \Delta^m x_n = \mathrm{O}(a_n).$$

Then

$$x \in \mathrm{Pol}_X(m-1) + \mathrm{o}_X(n^s).$$

Proof. Let $s \leq 0$. The condition $\Delta^m x_n = \mathrm{O}(a_n)$ implies

$$\sum_{n=1}^{\infty} n^{m-1-s}|\Delta^m x_n| < \infty.$$

Let $u = \Delta^m(x)$. By Lemma 3.7 (f), $u \in \mathrm{S}_X(m)$ and $r^m(u)(n) = \mathrm{o}(n^s)$. Let $w = (-1)^m r^m(u)$. Then $w_n = \mathrm{o}(n^s)$ and, by Lemma 3.7 (d), $\Delta^m(w) = u = \Delta^m(x)$. Hence

$$x - w \in \mathrm{Ker}(\Delta^m) = \mathrm{Pol}_X(m-1)$$

and we obtain

$$x = x - w + w \in \mathrm{Pol}_X(m-1) + \mathrm{o}_X(n^s).$$

Let $s \in (0, m-1]$. Choose $k \in \mathbb{N}(1, m-1)$ such that $k - 1 < s \leq k$. Then

$$\sum_{n=1}^{\infty} n^{(m-k)-1-(s-k)}|u_n| < \infty$$

and, by Lemma 3.7 (f), $u \in \mathrm{S}(m-k)$ and $r^{m-k}(u)(n) = \mathrm{o}(n^{s-k})$. Let $w = (-1)^{m-k} r^{m-k}(u)$. Then $w_n = \mathrm{o}(n^{s-k})$ and, by Lemma 3.7 (d), $\Delta^{m-k} w = u$. Choose $z \in X^{\mathbb{N}}$ such that $\Delta^k z_n = w_n = \mathrm{o}(n^{s-k})$. Since $s - k > -1$, so by Lemma 3.4 we have $z_n = \mathrm{o}(n^s)$. Moreover

$$\Delta^m z = \Delta^{m-k}\Delta^k z = \Delta^{m-k} w = u = \Delta^m x$$

and

$$x = x - z + z \in \mathrm{Pol}_X(m-1) + \mathrm{o}_X(n^s).$$ □

Lemma 3.11. *Assume $a \in \mathbb{R}^{\mathbb{N}}$, $b, x \in X^{\mathbb{N}}$, $m \in \mathbb{N}$, $s \in (-\infty, m-1]$,*

$$\sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty, \quad \textit{and} \quad \Delta^m x \in \mathrm{O}_X(a) + b.$$

Then

$$x \in \Delta^{-m} b + \mathrm{o}_X(n^s).$$

Proof. Choose $u \in \Delta^{-m} b$. Then

$$\Delta^m (x - u) = \Delta^m x - \Delta^m u = \Delta^m x - b \in \mathrm{O}_X(a).$$

Hence, by the previous lemma,

$$x - u \in \mathrm{Pol}_X(m-1) + \mathrm{o}_X(n^s).$$

On the other hand,

$$u + \mathrm{Pol}_X(m-1) = \Delta^{-m} b.$$

Hence

$$x \in u + \mathrm{Pol}_X(m-1) + \mathrm{o}_X(n^s) = \Delta^{-m} b + \mathrm{o}_X(n^s).$$

□

Lemma 3.12. *Assume $b \in X^{\mathbb{N}}$, $m \in \mathbb{N}$, $s \in (-\infty, m-1]$, and*

$$\sum_{n=1}^{\infty} n^{m-1-s} |b_n| < \infty.$$

Then

$$\Delta^{-m} b + \mathrm{o}_X(n^s) = \mathrm{Pol}_X(m-1) + \mathrm{o}_X(n^s).$$

Proof. Let $x \in \Delta^{-m} b$ and $z \in \mathrm{o}_X(n^s)$. By Lemma 3.10,

$$x \in \mathrm{Pol}_X(m-1) + \mathrm{o}_X(n^s).$$

Hence $x + z \in \mathrm{Pol}_X(m-1) + \mathrm{o}_X(n^s)$ and we have

$$\Delta^{-m} b + \mathrm{o}_X(n^s) \subset \mathrm{Pol}_X(m-1) + \mathrm{o}_X(n^s).$$

By Lemma 3.7 (f), $b \in \mathrm{S}_X(m)$ and $r^m(b)(n) = \mathrm{o}(n^s)$. Let

$$u = (-1)^m r^m(b) \quad \text{and} \quad \varphi \in \mathrm{Pol}_X(m-1).$$

Then $u = \mathrm{o}(n^s)$ and using Lemma 3.7 (d), we have

$$\Delta^m(\varphi + u) = \Delta^m u = b.$$

Hence

$$\varphi + u \in \Delta^{-m} b \quad \text{and} \quad \varphi \in \Delta^{-m} b + \mathrm{o}(n^s).$$

Therefore

$$\mathrm{Pol}_X(m-1) + \mathrm{o}_X(n^s) \subset \Delta^{-m} b + \mathrm{o}_X(n^s).$$

□

4 Solutions with prescribed asymptotic behavior

We say that a map $f : Y \to Z$ from a metric space Y to a metric space Z is a Heine map if it is completely continuous and is uniformly continuous on any bounded subset of Y. We define a metric d on $\mathbb{N} \times X$ by

$$d((k,s),(n,t)) = \max(|n-k|, |t-s|).$$

Note that if the dimension of the space X is finite then any continuous map $f : \mathbb{N} \times X \to X$ is a Heine map.

Theorem 4.1. *Assume f is a Heine map, $s \in (-\infty, 0]$,*

$$\sum_{n=1}^{\infty} n^{m-s-1}|a_n| < \infty, \tag{4.1}$$

$w \in \mathbb{R}^{\mathbb{N}}$ is positive and bounded, $g : [0,\infty) \to [0,\infty)$ is locally bounded,

$$|f(n,t)| \leq g(w_n|t|) \tag{4.2}$$

for $(n,t) \in \mathbb{N} \times X$, $y \in X^{\mathbb{N}}$, $\Delta^m y = b$ and

$$w_n y_{\sigma(n)} = \mathrm{O}(1). \tag{4.3}$$

Then there exists a solution x of (E) such that $x = y + \mathrm{o}(n^s)$.

Proof. For $x \in X^{\mathbb{N}}$ let $\bar{x} \in X^{\mathbb{N}}$ be defined by

$$\bar{x}_n = f(n, x_{\sigma(n)}).$$

Choose a positive constant c. Let

$$T = \left\{ x \in X^{\mathbb{N}} : |x - y| \leq c \right\}.$$

By boundedness of w and (4.3), there exists a constant K such that if $x \in T$ and $n \in \mathbb{N}$, then

$$\begin{aligned} |w_n x_{\sigma(n)}| &= |w_n x_{\sigma(n)} - w_n y_{\sigma(n)} + w_n y_{\sigma(n)}| \\ &\leq |w_n||x_{\sigma(n)} - y_{\sigma(n)}| + |w_n y_{\sigma(n)}| \leq K. \end{aligned}$$

Since g is locally bounded, there exists $M > 0$ such that $g([0,K]) \subset [0,M]$. Therefore, we have

$$g(|w_n x_{\sigma(n)}|) \leq M \qquad \text{and} \qquad |\bar{x}_n| \leq g(|x_{\sigma(n)} w_n|) \leq M \tag{4.4}$$

for $x \in T$ and $n \in \mathbb{N}$. Since $r^m(|a|)(n) = \mathrm{o}(1)$, there exists an index $p \geq 1$ such that

$$M r^m(|a|)(n) \leq c \quad \text{for} \quad n \geq p. \tag{4.5}$$

Let $\mu, \rho \in \mathbb{R}^{\mathbb{N}}$,

$$\mu_n = \begin{cases} 0 & \text{for } n < p, \\ 1 & \text{for } n \geq p, \end{cases} \qquad \rho = \mu M r^m(|a|). \tag{4.6}$$

Now, we define a subset S of $X^{\mathbb{N}}$ and a map $A : S \to X^{\mathbb{N}}$ by

$$S = \left\{ x \in X^{\mathbb{N}} : \ |x - y| \leq \rho \right\}, \qquad A(x) = y + (-1)^m \mu r^m(a\bar{x}).$$

Then $S \subset T$. Obviously, S is convex, closed and ordinary subset of $X^{\mathbb{N}}$. If $x \in S$, then, using Lemma 3.7 (a), Lemma 3.9, (4.4) and (4.6) we get

$$|Ax - y| = |\mu r^m(a\bar{x})| \leq \mu r^m(|a\bar{x}|) \leq \rho.$$

Hence $A(S) \subset S$. Choose $\varepsilon > 0$. There exists $q \geq p$ and $\alpha > 0$ such that

$$2M \sum_{n=q}^{\infty} n^{m-1}|a_n| < \varepsilon \qquad \text{and} \qquad \alpha q^{m-1} \sum_{n=1}^{q} |a_n| < \varepsilon. \tag{4.7}$$

Let

$$L = \max\{|y_{\sigma(n)} - y_n| : n \in \mathbb{N}(1, q)\},$$

and

$$W = \{(n,t) \in \mathbb{N} \times X : n \in \mathbb{N}(1,q), |t - y_n| \le L + c\}.$$

The function f is uniformly continuous on W. Hence, there exists a $\delta > 0$ such that

$$\text{if} \quad (n,s),(n,t) \in W \quad \text{and} \quad |s-t| < \delta, \quad \text{then} \quad |f(n,s) - f(n,t)| < \alpha. \tag{4.8}$$

Assume $x, z \in S$, $|x - z| < \delta$. Let $u = \bar{x} - \bar{z}$. Then

$$|Ax - Az| = |\mu r^m(au)|.$$

Using Lemma 3.7 we get

$$\begin{aligned} d(Ax, Az) &= \sup_{n \in \mathbb{N}} |Ax_n - Az_n| = \sup_{n \in \mathbb{N}} |r^m(au)(n)| \\ &\le \sup_{n \in \mathbb{N}} r^m(|au|)(n) \le \sum_{n=1}^{\infty} n^{m-1} |a_n u_n|. \end{aligned}$$

Hence

$$d(Ax, Az) \le \sum_{n=1}^{q} n^{m-1} |a_n u_n| + \sum_{n=q}^{\infty} n^{m-1} |a_n u_n|. \tag{4.9}$$

By (4.4), $|u| \le 2M$. If $n \in \mathbb{N}(1, q)$, then

$$|x_{\sigma(n)} - y_n| \le |x_{\sigma(n)} - y_{\sigma(n)}| + |y_{\sigma(n)} - y_n| \le \rho(n) + L \le L + c.$$

Hence $(n, x_{\sigma(n)}) \in W$. Analogously $(n, z_{\sigma(n)}) \in W$. Therefore, by (4.8), $|u_n| \le \alpha$ for $n \le q$. By (4.7) and (4.9) we get

$$d(Ax, Az) \le \alpha q^{m-1} \sum_{n=1}^{q} |a_n| + 2M \sum_{n=q}^{\infty} n^{m-1} |a_n| < \varepsilon + \varepsilon.$$

Thus the map A is continuous. Now, we will show that the family $A(S)$ is pointwise totally bounded. Fix an $n \in \mathbb{N}$. Then

$$A(S)(n) = \{y_n + (-1)^m \mu_n r^m(a\bar{x})(n) : x \in S\}$$

and, by Lemma 3.8,

$$r^m(a\bar{x})(n) = \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} a_{n+k} f(n+k, x_{\sigma(n+k)}).$$

Let

$$Q_n = \{r^m(a\bar{x})(n) : x \in S\}.$$

For $k \in \mathbb{N}(0)$ let

$$\lambda_k = \binom{m+k-1}{m-1},$$

$$V_k = \{(n+k, x_{\sigma(n+k)}) : x \in S\} = \{n+k\} \times S(\sigma(n+k))$$

and

$$U_k = \{\lambda_k a_{n+k} \bar{x}_{n+k} : x \in S\}$$

Then V_k is a bounded subset of $\mathbb{N} \times X$ and, since f is completely continuous, the set $f(V_k)$ is totally bounded. Hence

$$U_k = \{\lambda_k a_{n+k} f(n+k, x_{\sigma(n+k)}) : x \in S\} = \lambda_k a_{n+k} f(V_k)$$

is also totally bounded. Let $\varepsilon > 0$. By (4.1) and Lemma 3.7 (b), $|a| \in \mathrm{S}_{\mathbb{R}}(m)$. By Lemma 3.8 there exists an index n_1 such that

$$M \sum_{k=n_1}^{\infty} \lambda_k |a_{n+k}| < \varepsilon.$$

Let

$$D = \left\{ \sum_{k=n_1}^{\infty} \lambda_k a_{n+k} \bar{x}_{n+k} : x \in S \right\} \quad \text{and} \quad U = U_0 + U_1 + \cdots + U_{n_1}.$$

Then

$$Q_n = \left\{ \sum_{k=0}^{\infty} \lambda_k a_{n+k} \bar{x}_{n+k} : x \in S \right\} \subset U + D.$$

By (4.4), $|\bar{x}_{n+k}| \leq M$ for any k. Hence $|z| < \varepsilon$ for any $z \in D$. Moreover, U is totally bounded and there exists a finite ε-net H for U. If $u \in U$, then there exists $h \in H$ such that $|u - h| \leq \varepsilon$. Moreover, if $z \in D$, then

$$|u + z - h| \leq |u - h| + |z| \leq 2\varepsilon.$$

Hence H is a finite 2ε-net for $U + D$ and for $Q_n \subset U + D$. Therefore Q_n is totally bounded. Thus

$$A(S)(n) = y_n + (-1)^m \mu_n Q_n$$

is also totally bounded. Obviously the family $A(S)$ is stable at infinity. Hence, by Lemma 3.6, $A(S)$ is totally bounded. Therefore, by Theorem 3.5, there exists a sequence $x \in S$ such that $A(x) = x$. Then

$$x_n = y_n + (-1)^m r^m (a\bar{x})(n)$$

for $n \geq p$. This means that there exists a sequence $u \in X^{\mathbb{N}}$ such that $u_n = 0$ for $n \geq p$ and

$$x = y + (-1)^m r^m (a\bar{x}) + u. \tag{4.10}$$

Hence, by Lemma 3.7 (d),

$$\Delta^m x = \Delta^m y + a\bar{x} + \Delta^m u = a\bar{x} + b + \Delta^m u.$$

It is easy to see that $\Delta^m u_n = 0$ for $n \geq p$ and we obtain

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n$$

for $n \geq p$. Moreover, using (4.10) and Lemma 3.7 (f), we get $x_n = y_n + \mathrm{o}(n^s)$. □

Theorem 4.2. *Assume $s \in (-\infty, 0]$, $y \in X^{\mathbb{N}}$, $\Delta^m y = b$,*

$$\sum_{n=1}^{\infty} n^{m-s-1} |a_n| < \infty,$$

$U \subset X$ is a uniform neighborhood of the set $y(\mathbb{N})$, and the map $f|\mathbb{N} \times U$ is Heine and bounded. Then there exists a solution x of (E) such that $x_n = y_n + \mathrm{o}(n^s)$.

Proof. For $x \in X^{\mathbb{N}}$ let $\bar{x} \in X^{\mathbb{N}}$ be defined by

$$\bar{x}_n = f(n, x_{\sigma(n)}).$$

Choose a positive constant c such that $\mathrm{B}(y(\mathbb{N}), c) \subset U$. Let

$$T = \{x \in X^{\mathbb{N}} : |x - y| \le c\} \quad \text{and} \quad M = \sup\{|f(n,t)| : (n,t) \in \mathbb{N} \times U\}.$$

If $x \in T$ and $n \in \mathbb{N}$, then $x_n \in \mathrm{B}(y(\mathbb{N}), c) \subset U$. Hence

$$|\bar{x}_n| \le M$$

for any $x \in T$ and $n \in \mathbb{N}$. There exists an index $p \ge 1$ such that

$$Mr^m(|a|)(n) \le c \quad \text{for} \quad n \ge p.$$

The rest of the proof is analogous to the second part of the proof of Theorem 4.1. □

Corollary 4.3. *Assume the map f is Heine, $s \in (-\infty, 0]$, and*

$$\sum_{n=1}^{\infty} n^{m-s-1}|a_n| < \infty.$$

Moreover, for any bounded subset Z of X, f is bounded on $\mathbb{N} \times Z$. Then for any bounded solution y of the equation $\Delta^m y = b$ there exists a solution x of (E) such that $x_n = y_n + \mathrm{o}(n^s)$.

Proof. The assertion is an easy consequence of Theorem 4.2. □

5 Approximations of solutions

Theorem 5.1. *Assume x is a solution of (E), $s \in (-\infty, m-1]$, $p \in \mathbb{N}$, $U \subset X$,*

$$\sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty, \qquad g : [0,\infty) \to [0,\infty), \qquad w \in \mathbb{R}^{\mathbb{N}},$$

and one of the following conditions is satisfied:

(1) *the sequence $\bar{x}_n = f(n, x_{\sigma(n)})$ is bounded,*

(2) *f is bounded on $\mathbb{N}(p) \times U$ and $x_{\sigma(n)} \in U$ for large n,*

(3) *f is bounded on $\mathbb{N}(p) \times U$ and $x_n \in U$ for large n,*

(4) *f is bounded,*

(5) *g is locally bounded, $x_{\sigma(n)} = \mathrm{O}(w_n^{-1})$ and $|f(n,t)| \le g(|w_n t|)$ on $\mathbb{N} \times X$.*

Then $x \in \Delta^{-m}b + \mathrm{o}_X(n^s)$. If, moreover,

$$\sum_{n=1}^{\infty} n^{m-1-s}|b_n| < \infty,$$

then $x \in \mathrm{Pol}_X(m-1) + \mathrm{o}(n^s)$.

Proof. Obviously (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Assume (5). Then the sequence

$$z_n = g(|w_n x_{\sigma(n)}|)$$

is bounded and $|f(n, x_{\sigma(n)})| \leq g(|w_n x_{\sigma(n)}|) = z_n$. Hence (5) $\Rightarrow$ (1). If the sequence $\bar{x}$ is bounded, then by the equality

$$\Delta^m x_n = a_n \bar{x}_n + b_n$$

for large n we obtain $\Delta^m x = \mathrm{O}(a) + b$. Hence the assertion follows from Lemma 3.11. □

Corollary 5.2. *Assume f is bounded on $\mathbb{N} \times Z$ for any bounded subset Z of X, $s \leq 0$,*

$$\sum_{n=1}^{\infty} n^{m-s-1}|a_n| < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} n^{m-s-1}|b_n| < \infty.$$

Then any bounded solution x of (E) is convergent. More precisely, there exists a vector $c \in X$ such that $x = c + \mathrm{o}(n^s)$.

Proof. Let x be a bounded solution of (E) and let $Z = x(\mathbb{N})$. Then f is bounded on $\mathbb{N} \times Z$, and, by Theorem 5.1, $x \in \mathrm{Pol}_X(m-1) + \mathrm{o}(n^s)$. Using the boundedness of x and assumption $s \leq 0$ we see that there exists a vector $c \in X$ such that $x = c + \mathrm{o}(n^s)$. □

Corollary 5.3. *Assume that for any bounded subset Z of X, f is bounded on $\mathbb{N} \times Z$, $s \leq 0$, $q \in \mathbb{N}$, y is a q-periodic solution of the equation $\Delta^m y = b$ and*

$$\sum_{n=1}^{\infty} n^{m-s-1}|a_n| < \infty.$$

Then any bounded solution x of (E) is asymptotically q-periodic. More precisely, there exists a vector $c \in X$ such that $x = c + y + \mathrm{o}(n^s)$.

Proof. If x is a bounded solution of (E), then, by Theorem 5.1,

$$x \in \Delta^{-m} b + \mathrm{o}(n^s) = y + \mathrm{Pol}_X(m-1) + \mathrm{o}_X(n^s).$$

Using boundedness of x and y and assumption $s \leq 0$ we see that there exists a vector $c \in X$ such that $x = c + y + \mathrm{o}(n^s)$. □

Lemma 5.4. *Assume a, u are nonnegative sequences, $p \in \mathbb{N}$, $\lambda, \mu > 0$, and $b \geq 0$. Let $g : [0, \infty) \to [0, \infty)$ be nondecreasing, $g(b) > 0$,*

$$\sum_{k=0}^{\infty} a_k < \infty, \quad \int_b^{\infty} \frac{dt}{g(t)} = \infty, \quad \text{and} \quad u_n \leq b + \lambda \sum_{k=p}^{n-1} a_k g(\mu u_k)$$

for $n \geq p$. Then the sequence u is bounded.

Proof. See [12, Lemma 7.2]. □

Lemma 5.5. *If $x \in X^{\mathbb{N}}$, $m \in \mathbb{N}$ and $p \in \mathbb{N}(m)$ then there exists a positive constant L such that*

$$|x_n| \leq n^{(m-1)} \left(L + \sum_{i=p}^{n-1} |\Delta^m x_i| \right) \quad \text{for} \quad n \geq p.$$

Proof. The proof is analogous to the proof of [12, Lemma 7.3]. □

Theorem 5.6. *Assume $\sigma(n) \leq n$ for large n, $s \in (-\infty, m-1]$,*

$$g : [0,\infty) \to [0,\infty), \qquad w \in \mathbb{R}^{\mathbb{N}}, \qquad w = \mathrm{O}(n^{1-m}),$$

$|f(n,t)| \leq g(|w_n t|)$ on $\mathbb{N} \times X$, g is nondecreasing, $g(t) > 0$ for $t > 1$,

$$\sum_{n=0}^{\infty} n^{m-1-s}|a_n| < \infty, \qquad \sum_{n=0}^{\infty} n^{m-1-s}|b_n| < \infty, \qquad \int_1^{\infty} \frac{dt}{g(t)} = \infty$$

and x is a solution of (E). Then $x \in \mathrm{Pol}_X(m-1) + \mathrm{o}_X(n^s)$.

Proof. Choose $M > 0$ such that $|w_n|n^{m-1} \leq M$. Then $|w_n|n^{(m-1)} \leq M$. By assumption

$$\begin{aligned} |\Delta^m x_n| &= |a_n f(n, x_{\sigma(n)}) + b_n| \leq |a_n||f(n, x_{\sigma(n)})| + |b_n| \\ &\leq |a_n||g(|w_n x_{\sigma(n)}|)| + |b_n|. \end{aligned}$$

By Lemma 5.5, there exists a positive constant L such that

$$|x_{\sigma(n)}| \leq \sigma(n)^{(m-1)} \left(L + \sum_{i=p}^{\sigma(n)-1} |\Delta^m x_i| \right) \leq n^{(m-1)} \left(L + \sum_{i=p}^{n-1} |\Delta^m x_i| \right).$$

Hence

$$|w_n x_{\sigma(n)}| \leq ML + M \sum_{j=1}^{n-1} |\Delta^m x_j|.$$

Then

$$\begin{aligned} |w_n x_{\sigma(n)}| &\leq ML + M \sum_{j=1}^{n-1} |a_j| g(|w_j x_{\sigma(j)}|) + M \sum_{j=1}^{n-1} |b_j| \\ &\leq K + M \sum_{j=1}^{n-1} |a_j| g(|w_j x_{\sigma(j)}|), \end{aligned}$$

where

$$K = ML + M \sum_{j=1}^{n-1} |b_j|.$$

Obviously $\int_K^{\infty} g(t)^{-1} dt = \infty$. By Lemma 5.4, the sequence $(w_n x_{\sigma(n)})$ is bounded. Choose $Q > 0$ such that $|w_n x_{\sigma(n)}| \leq Q$ for every n. Choose $P \geq 1$ such that $g(Q) \leq P$. Then $g(|w_n x_{\sigma(n)}|) \leq P$ for every n. Hence

$$|\Delta^m x_n| \leq |a_n| g(|w_n x_{\sigma(n)}|) + |b_n| \leq P|a_n| + |b_n| \leq P(|a_n| + |b_n|).$$

Therefore $\Delta^m x_n = \mathrm{O}(|a_n| + |b_n|)$. Now the conclusion follows from Lemma 3.10. □

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