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Isolated periodic wave trains in a generalized Burgers–Huxley equation

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Abstract. We study the isolated periodic wave trains in a class of modified generalized Burgers–Huxley equation. The planar systems with a degenerate equilibrium arising after the traveling transformation are investigated. By finding certain positive definite Lyapunov functions in the neighborhood of the degenerate singular points and the Hopf bifurcation points, the number of possible limit cycles in the corresponding planar systems is determined. The existence of isolated periodic wave trains in the equation is established, which is universal for any positive integer n in this model. Within the process, one interesting example is obtained, namely a series of limit cycles bifurcating from a semi-hyperbolic singular point with one zero eigenvalue and one non-zero eigenvalue for its Jacobi matrix.

Keywords: generalized Burgers–Huxley equation, isolated periodic wave solution, positive definite Lyapunov function, degenerate singular point.

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1 Introduction

The Burgers–Huxley equation is a well-known nonlinear partial differential equation simulating nonlinear wave phenomena in physics, biology, economics and ecology. In the relation with the in-depth study of practical problems the following generalized Burgers–Huxley equation

$$u_t + \alpha u^n u_x - u_{xx} = \beta u(1 - u^n)(u^n - \gamma) \tag{1.1}$$

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was introduced in [25]. In this paper we consider the further generalization of (1.1) described by the equation

$$u_t + \Big(\alpha_0 + \sum_{i=1}^{n} \alpha_i u^i\Big) u_x - u_{xx} = \beta u(1-u^n)(u^n - \gamma) \tag{1.2}$$

where α_i, β, γ are real numbers and $n \in \mathbb{N}$. In equation (1.2) a more general version of the convective effect is introduced. Although this modification may go beyond the actual background, it is still related to some real models, for example, when $n = 1$ it corresponds to the single-species model with density-dependent migrations [24]. Of course, we also have to come up with some early models, for example, Burgers equations [3], Burgers–Huxley equation [30,32], Fitzhugh–Nagumo equation [8,11], Newwell–Whitehead equation [19].

Regarding to the exact solutions of equation (1.1) we note that some solitary wave solutions were obtained in [6,29]. Recently via the (G'/G)-expansion method the authors of [22] also obtained a series of exact solutions. In respect to the approximate analytical solutions and numerical solutions of equation (1.1) we can refer to the results in [4,10,21] and the references given there. As to the isolated periodic wave solutions, considering the bifurcations of codimension 1 and 2 in the traveling wave system the authors of [32] determined the existence of some bounded traveling waves for the Burgers–Huxley equation. Latter on computing the singular point quantities of the non-degenerate center-focus type equilibrium the authors of [26] proved the existence of the isolated periodic wave solution in the non-degenerate case for equation (1.1).

In this paper we continue the investigation of the isolated periodic wave solutions (which can also be called the isolated periodic wave trains, see [23]) of more general model (1.2). First, we apply the usual approach assuming that equation (1.2) has a travelling wave solution in the form

$$u(x,t) = v(\xi), \qquad \xi = x - ct \tag{1.3}$$

where c is the propagation speed of the wave. Substituting (1.3) into (1.2) we obtain

$$v''(\xi) = -cv'(\xi) + \Big(\alpha_0 + \sum_{i=1}^{n} \alpha_i v^i\Big) v'(\xi) - \beta v(1-v^n)(v^n - \gamma). \tag{1.4}$$

Then setting $y = v'(\xi)$ we reduce (1.4) to the planar dynamic system

$$\begin{cases} \frac{dv}{d\xi} = y \triangleq X(v,y), \\ \frac{dy}{d\xi} = -cy + (\alpha_0 + \sum_{i=1}^{n} \alpha_i v^i) y - \beta v(1-v^n)(v^n-\gamma) \triangleq Y(v,y). \end{cases} \tag{1.5}$$

Applying the bifurcation theory of the planar dynamical system to system (1.5) it is possible to investigate the existence of periodic wave trains which correspond to a family of periodic orbits in a neighborhood of a center, the solitary wave solutions of the peak type which correspond to the smooth homoclinic orbits, and the monotone kink solitary wave solutions which correspond to the heteroclinic orbits. As has been indicated in [14] these cases are usually considered for integrable systems. Integrability conditions of system (1.5) are similar to the ones of its special case which has been investigated in [26].

In this paper we focus on the existence of isolated periodic travelling trains which are caused by the presence of limit cycles. Our main idea is to keep track of the conditions of limit cycle bifurcations which can occur in the vicinity of the equilibriums, in particular near the degenerate singular points of the traveling wave system (1.5).

Bifurcations of isolated periodic wave trains for the reaction-diffusion equation have been extensively studied (see [12,13,23,27,28] and references therein). These bifurcations are caused mainly by Hopf bifurcation or Poincaré bifurcation around one non-degenerate equilibrium of the corresponding planar traveling wave system. However the bifurcations of isolated periodic wave trains due to limit cycles bifurcating from one degenerate equilibrium are not well investigated. In this work a particular attention is focused on the cases of a nilpotent focus, a nilpotent node and a semi-hyperbolic singular point whose Jacobi matrix has two eigenvalues: one is zero, another is non-zero. Our main approach is to determine the quasi-Lyapunov constants of system (1.5) by constructing Lyapunov functions not only for the cases of degenerate equilibriums, but also for the multiple Hopf bifurcations of non-degenerate equilibriums.

The paper is organized as follows. In Section 2 the equilibriums and degenerate cases of system (1.5) are determined. Section 3 is devoted to the study of the quasi-Lyapunov constants for the nilpotent critical point of the degenerate planar system (1.5). In Section 4 we apply the positive definite Lyapunov functions to determine the quasi-Lyapunov constants for multiple Hopf bifurcation of the degenerate system (1.5). In the last section using the above analysis we give a general result for the isolated periodic wave trains for any positive integer n in model (1.2).

2 The equilibriums of system (1.5)

In this section we investigate the equilibriums of planar travelling system (1.5). Due to the practical background, the value of u in model (1.2) is nonnegative, thus we only investigate the dynamical behavior near the equilibrium points with $v \geq 0$ for system (1.5). We will focus on the limit cycle bifurcation in system (1.5) with one equilibrium as the degenerate singular point. It is easy to see that when $\gamma > 0$ system (1.5) has only three nonnegative equilibrium points: $(0,0), (1,0)$ and $(\sqrt[n]{\gamma}, 0)$, whereas when $\gamma \leq 0$, there exist only two nonnegative equilibrium points: $(0,0)$ and $(1,0)$.

For the Jacobian matrix at the origin we have

$$\begin{bmatrix} \frac{\partial X}{\partial v} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial v} & \frac{\partial Y}{\partial y} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 1 \\ \beta\gamma & \alpha_0 - c \end{bmatrix}. \tag{2.1}$$

Its two eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(\alpha_0 - c \pm \sqrt{(\alpha_0 - c)^2 + 4\beta\gamma}\right). \tag{2.2}$$

Thus, the origin of (1.5) is either a non-degenerate center or a weak focus if and only if $c = \alpha_0$ and $\beta\gamma < 0$. If $\beta\gamma = 0$ then the origin is a degenerate singular point, and the two eigenvalues are $\lambda_1 = 0$, $\lambda_2 = \alpha_0 - c$. When $\beta\gamma = 0$, according to the monographs [7, 31], we know that if $c \neq \alpha_0$ then the singular point is an elementary degenerate singular point, also called semi-hyperbolic, and when $c = \alpha_0$, the origin is a nilpotent critical point and the limit cycle bifurcation may happen at $(0,0)$.

For the Jacobian matrix at $(1,0)$,

$$\begin{bmatrix} \frac{\partial X}{\partial v} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial v} & \frac{\partial Y}{\partial y} \end{bmatrix}_{(1,0)} = \begin{bmatrix} 0 & 1 \\ n\beta(1-\gamma) & \sum\limits_{i=0}^{n} \alpha_i - c \end{bmatrix} \tag{2.3}$$

we obtain

$$\lambda_{1,2} = \frac{1}{2}\left[\sum_{i=0}^{n}\alpha_i - c \pm \sqrt{\left(\sum_{i=0}^{n}\alpha_i - c\right)^2 + 4n\beta(1-\gamma)}\right]. \tag{2.4}$$

It has a pair of conjugate pure imaginary eigenvalues and $(1,0)$ is either a non-degenerate center or a weak focus if and only if $c = \sum_{i=0}^{n}\alpha_i$ and $\beta(1-\gamma) < 0$. If $\beta(1-\gamma) = 0$ the singular point is degenerate since at least one of its two eigenvalues is zero. In this case, when $c \neq \sum_{i=0}^{n}\alpha_i$, the singular point is semi-hyperbolic, and when $c = \sum_{i=0}^{n}\alpha_i$, the equilibrium is a nilpotent critical point and limit cycle bifurcation may happen at $(1,0)$.

At the point $(\sqrt[n]{\gamma}, 0)$

$$\begin{bmatrix} \frac{\partial X}{\partial v} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial v} & \frac{\partial Y}{\partial y} \end{bmatrix}_{(\gamma^{\frac{1}{n}},0)} = \begin{bmatrix} 0 & 1 \\ n\beta\gamma(\gamma-1) & \sum\limits_{i=0}^{n}\alpha_i\gamma^{\frac{i}{n}} - c \end{bmatrix}. \tag{2.5}$$

Then the two eigenvalues are:

$$\lambda_{1,2} = \frac{1}{2}\left[\sum_{i=0}^{n}\alpha_i\gamma^{\frac{i}{n}} - c \pm \sqrt{\left(\sum_{i=0}^{n}\alpha_i\gamma^{\frac{i}{n}} - c\right)^2 + 4n\beta\gamma(\gamma-1)}\right]. \tag{2.6}$$

The Jacobian has a pair of conjugate pure imaginary eigenvalues and the point is non-degenerate, either a center or a focus if and only if $c = \sum_{i=0}^{n}\alpha_i\gamma^{\frac{i}{n}}$ and $\beta\gamma(\gamma-1) < 0$. Furthermore, if $\beta\gamma(\gamma-1) = 0$ the point is a degenerate singular point, and in this case, when $c \neq \sum_{i=0}^{n}\alpha_i\gamma^{\frac{i}{n}}$, the singular point is semi-hyperbolic, and when $c = \sum_{i=0}^{n}\alpha_i\gamma^{\frac{i}{n}}$, the equilibrium is a nilpotent critical point and the limit cycle bifurcation may occur at it.

Therefore we have the following conclusions.

Lemma 2.1. For system (1.5) there exists a degenerate nonnegative equilibrium point if and only if $\beta = 0$, or $\gamma = 0$ or $\gamma = 1$.

Lemma 2.2. For system (1.5), when $\gamma = 0$, only the origin is degenerate, and the equilibrium $(1,0)$ is non-degenerate. In this case, the origin is a nilpotent critical point if and only if $c = \alpha_0$, the equilibrium $(1,0)$ is a center or a focus if and only if $\beta < 0$.

Lemma 2.3. For system (1.5), when $\gamma = 1$, only the equilibrium $(v,y) = (1,0)$ is degenerate, and the origin is non-degenerate. In this case, the equilibrium $(1,0)$ is a nilpotent critical point if and only if $c = \sum_{i=0}^{n}\alpha_i$, the origin is a center or a focus if and only if $\beta < 0$.

Lemma 2.4. For system (1.5), when $\beta = 0$ the line $y = 0$ is a singular straight line on which each equilibrium point is non-isolated and degenerate, and there exists a first integral

$$H(v,y) = (c+\alpha_0)v + \frac{1}{2}\alpha_1 v^2 + \cdots + \frac{1}{n+1}\alpha_n v^{n+1} - y = h. \tag{2.7}$$

In this situation a limit cycle cannot exist.

3 Limit cycle bifurcations from the degenerate equilibriums

In this section we investigate the limit cycle bifurcations near the degenerate equilibriums of system (1.5). First we consider the real polynomial differential system

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = y + \sum_{k+j=2}^{\infty} a_{kj}x^k y^j = X(x,y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \lambda y + \sum_{k+j=2}^{\infty} b_{kj}x^k y^j = Y(x,y) \end{cases} \tag{3.1}$$

where $x, y, t, \lambda, a_{kj}, b_{kj} \in \mathbb{R}\,(k, j \in \mathbb{N})$. Obviously, when $\lambda = 0$ the origin $O(0,0)$ is a nilpotent critical point and when $\lambda \neq 0$ the origin is a semi-hyperbolic singular point, whose Jacobi matrix has two eigenvalues: one is zero, another is non-zero.

In the case $\lambda = 0$, according to the implicit function theorem there is the unique function $y = y(x)$ which satisfies $X(x, y(x)) \equiv 0,\ y(0) = 0$ and

$$\begin{aligned} Y(x, y(x)) &= A_k x^k + o(x^k), \qquad A_k \neq 0, \\ \left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right]_{y=y(x)} &= B_j x^j + o(x^j) \end{aligned} \tag{3.2}$$

where $k, j \in \mathbb{N}^+$. From [2,20,31], it is known that only when $k = 2m+1, m \in \mathbb{N}^+$, the origin of $(3.1)_{\lambda=0}$ may be a center or a focus or a node, and moreover we have

Proposition 3.1. Suppose that the origin of $(3.1)_{\lambda=0}$ is a nilpotent singular point with multiplicity $k = 2m+1, m \in \mathbb{N}^+$ and $B_j \neq 0$ in (3.2). Then it is a center or a focus if and only if one of the following two conditions is satisfied:

$$\begin{aligned} &\mathrm{C}_1: \quad m < j, \quad A_{2m+1} < 0; \\ &\mathrm{C}_2: \quad m = j, \quad A_{2m+1} < 0, \quad B_j^2 + 4(m+1)A_{2m+1} < 0. \end{aligned} \tag{3.3}$$

Proposition 3.2. Suppose that the origin of $(3.1)_{\lambda=0}$ is a nilpotent singular point with multiplicity $k = 2m+1, m \in \mathbb{N}^+$. Then it is a node if and only if $B_j \neq 0$ (with $j = 2N, N \in \mathbb{N}^+$ in (3.2)) and one of the following two conditions is satisfied:

$$\begin{aligned} &\mathrm{C}_3: \quad m > j, \quad A_{2m+1} < 0; \\ &\mathrm{C}_4: \quad m = j, \quad A_{2m+1} < 0, \quad B_j^2 + 4(m+1)A_{2m+1} \geq 0. \end{aligned} \tag{3.4}$$

For the case of Proposition 3.1 the limit cycle bifurcation can be determined by resolving the center-focus problem for the nilpotent critical point (see e.g. [1,5,15–17]). For the case of Proposition 3.2 the limit cycle bifurcation from a nilpotent node was studied in [18]. However, when $\lambda \neq 0$, the origin of (3.1) is a semi-hyperbolic singular point and it can be either a degenerate node or a degenerate saddle, or a saddle-node [2,20,31].

In our study we use the Lyapunov function method to investigate the existence of limit cycles bifurcating from the nilpotent critical points or semi-hyperbolic singular points. As a particular case of the Lyapunov stability theory we have the following statement.

Lemma 3.3. Suppose that the origin of system (3.1) is an isolated degenerate singular point, and there exists a positive definite Lyapunov function $V(x, y)$ such that

$$\left.\tfrac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.1)} = \tfrac{\mathrm{d}V}{\mathrm{d}x}\tfrac{\mathrm{d}x}{\mathrm{d}t} + \tfrac{\mathrm{d}V}{\mathrm{d}y}\tfrac{\mathrm{d}y}{\mathrm{d}t} = y^2[\beta_{2N}x^{2N} + o(x^{2N})], \qquad \beta_{2N} \neq 0, \tag{3.5}$$

where $(x, y) \in U(O, \delta)$ and δ is certain positive number. Then the origin is stable when $\beta_{2N} < 0$ and unstable when $\beta_{2N} > 0$.

It is easy to prove the following statement.

Theorem 3.4. Suppose that for system (3.1) there exists a positive definite Lyapunov function $V(x, y)$ such that

$$\begin{aligned} \left.\tfrac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.1)} &= \tfrac{\mathrm{d}V}{\mathrm{d}x}\tfrac{\mathrm{d}x}{\mathrm{d}t} + \tfrac{\mathrm{d}V}{\mathrm{d}y}\tfrac{\mathrm{d}y}{\mathrm{d}t} \\ &= y^2\Big[\sum_{i=0}^{N} \beta_{2i}(x^{2i} + O(x^{2i+1})) + o(x^{2N+1})\Big] = M(x, y), \qquad \beta_{2N} \neq 0, \end{aligned} \tag{3.6}$$

where $N \in \mathbb{N}^+$, $(x,y) \in U(O,\delta)$, δ is certain positive number, X^2+Y^2 is positive definite on $U(O,\delta)$ and β_{2i} are independent. Assume also that for a vector field χ^* from family (3.1) with fixed parameters $a_{ij}=a_{ij}^*$, $b_{ij}=b_{ij}^*$ the parameters β_{2i} in (3.6) satisfy $\beta_2=\beta_4=\cdots=\beta_{2N-2}=0, \beta_{2N}\neq 0$. Then it is possible to perturb vector field χ^* in such way, that the perturbed system has N small limit cycles in a neighborhood of the origin.

Proof. Let V be the Lyapunov function corresponding to vector field χ^*. Then for sufficiently small c the equation $V(x,y)=c$ defines a contour Γ located in U and surrounding the origin O. Since X^2+Y^2 is positive definite we can assume that inside Γ there no limit cycles and singular points different from O.

Assume for determinacy that $\beta_{2N}<0$. Then Γ bounds a positive invariant set Ω and all trajectories in Ω tend to the origin. Note also that $M(x,y)/y^2<\delta<0$ on Γ. Therefore under sufficiently small perturbations the vector field of the perturbed system is still directed inside Γ.

Therefore, since β_{2i} are independent we can perturb the vector field χ^* keeping $\beta_2=\cdots=\beta_{2N-4}=0$ and choose $\beta_{2N-2}>0$ and sufficiently small such that the system still has the positively invariant set Ω. Since as the result of the perturbation the origin is now unstable singular point, the perturbed system has in Ω a limit cycle surrounding the origin. Repeating the procedure we obtain N small limit cycles around the origin. $\square$

Definition 3.1. For system (3.1) with condition (3.6) satisfied if $\beta_0=\beta_2=\cdots=\beta_{2N-2}=0$ and $\beta_{2N}\neq 0$, then the quantity β_{2N} is called the N-th quasi-Lyapunov constant at the origin.

(I) When $\gamma=0$ system (1.5) has the form

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}\xi}=y=X(v,y) \\ \frac{\mathrm{d}y}{\mathrm{d}\xi}=-cy+(\sum_{i=0}^{n}\alpha_i v^i)y-\beta v^{n+1}(1-v^n)=Y(v,y). \end{cases} \tag{3.7}$$

From Lemma 2.2, only when $c=\alpha_0$, that is, $\lambda=\alpha_0-c=0$, the origin of system (3.7) is a nilpotent critical point. From (3.2) we have

$$\begin{aligned} Y(v,y(v))=Y(v,0)&=-\beta v^{n+1}+\beta v^{2n+1}, \\ \left[\frac{\partial X}{\partial v}+\frac{\partial Y}{\partial y}\right]_{y=0}&=\alpha_1 v+\alpha_2 v^2+\cdots+\alpha_n v^n. \end{aligned} \tag{3.8}$$

Moreover, according to Propositions 3.1 and 3.2, only when $n=2m$, $m\in\mathbb{N}^+$, that is, $A_{2m+1}=-\beta<0$, the origin is a center, or a focus or a node. When $c\neq\alpha_0$, that is $\lambda=\alpha_0-c\neq 0$, the origin of system (3.7) is a node.

When $\beta>0$ for system (3.7) with $n=2m$ there exists a positive definite Lyapunov function

$$V(v,y)=\frac{1}{2}\left[y^2+\beta\left(\frac{1}{m+1}-\frac{v^{2m}}{2m+1}\right)v^{2m+2}\right] \tag{3.9}$$

such that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\xi}\right|_{(3.7)}=\frac{\mathrm{d}V}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}\xi}+\frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}\xi}=y^2[(\alpha_0-c)+\alpha_1 v+\alpha_2 v^2+\cdots+\alpha_{2m}v^{2m}] \tag{3.10}$$

where

$$|v|<\left(\frac{2m+1}{m+1}\right)^{\frac{1}{2m}}=:\delta_m.$$

Observing that δ_m is a strictly monotonically decreasing with respect to $m \in \mathbb{N}^+$, we have

$$\lim_{m\to\infty} \delta_m = 1 < \delta_m \leq \sqrt{\frac{3}{2}} = \delta_1.$$

Thus $V(v, y)$ is a positive definite Lyapunov function in the neighborhood $U(O, \delta_m)$. Since α_i are independent, we can choose the perturbations in such way that

$$0 < |\alpha_0 - c| \ll |\alpha_2| \ll \cdots \ll |\alpha_{2m}|, \qquad \alpha_{2i}\alpha_{2i+2} < 0, \qquad n = 1, 2, \ldots, m-1. \tag{3.11}$$

Then from Theorem 3.4 we obtain the following conclusion.

Theorem 3.5. For system (3.7) with $\gamma = 0$, if $\beta > 0$, $n = 2m$, $\alpha_{2m} \neq 0, m \in \mathbb{N}^+$ then for a suitable choice of α_i there exist m limit cycles in a neighborhood of origin of system (3.7).

Denote the second function in the product on the right side of (3.10) by $g(v)$,

$$g(v) = (\alpha_0 - c) + \alpha_1 v + \alpha_2 v^2 + \cdots + \alpha_{2m} v^{2m}. \tag{3.12}$$

The following statement shows that m small amplitude limit cycles can appear in the system under small perturbations.

Corollary 3.6. In Theorem 3.5, write $\alpha_n = \alpha_{2m} = K$ and let

$$\begin{aligned} \alpha_{2m-2} &= \frac{K}{(2m-2)!} f^{(2m-2)}(0), \ldots, \alpha_{2j} = \frac{K}{(2j)!} f^{(2j)}(0), \ldots, \\ \alpha_0 &= c - Kf(0), \qquad \alpha_{2j+1} = 0, \qquad j = 0, 1, \ldots, m-1 \end{aligned} \tag{3.13}$$

where

$$f(v) = \prod_{j=1}^{m} (v^2 - r_j^2\varepsilon^2) = (v^2 - r_1^2\varepsilon^2)(v^2 - r_2^2\varepsilon^2)\cdots(v^2 - r_j^2\varepsilon^2) \tag{3.14}$$

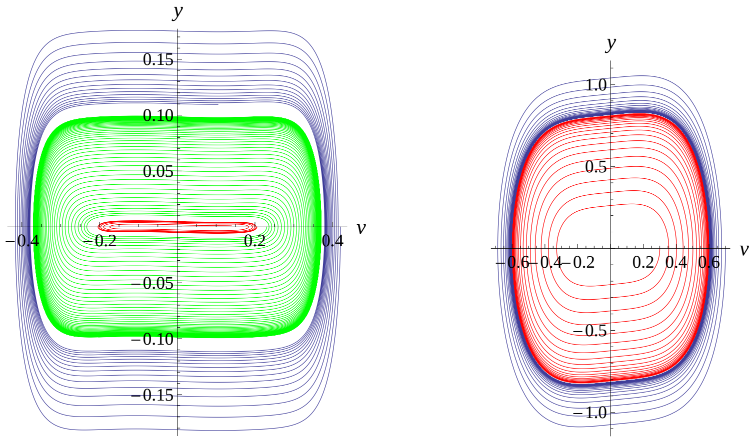
where $0 < r_1 < r_2 < \cdots < r_m$. When $0 < \varepsilon << |K|$, there are m limit cycles in a small enough neighborhood of the origin for system (3.7).

Proof. Substituting α_j $(j = 0, 1, \ldots, 2m)$ into (3.12) we have

$$g(v) = Kf(0) + \frac{K}{2!} f^{(2)}(0)\, v^2 + \frac{K}{4!} f^{(4)}(0)\, v^4 + \cdots + \frac{K}{(2j)!} f^{(2j)}(0)\, v^{2j} + \cdots + Kv^{2m}.$$

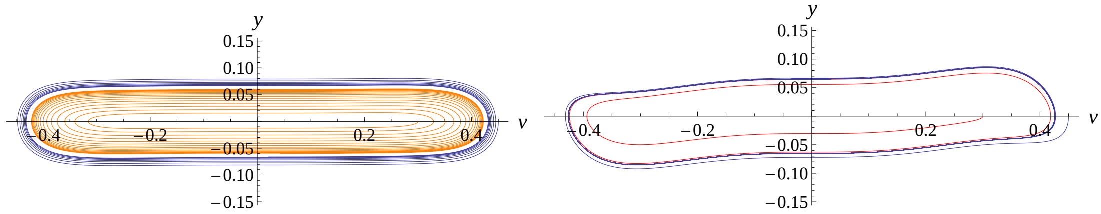
Note that each $\frac{1}{(2j)!} f^{(2j)}(0)$ in the above expression corresponds to the coefficient of the term v^{2j} of $f(v)$ in (3.14) and $\frac{1}{(2m)!} f^{(2m)}(0) = 1$. That is, $g(v) = Kf(v)$. Because $f(v)$ has just m simple positive roots, $v = r_j\varepsilon$, $j = 1, \cdots, m$, the coefficients of $f(v)$ have alternating signs, namely $\alpha_{2i}\alpha_{2i+2} < 0$. Since the other conditions of Theorem 3.5 are also satisfied, the proof is completed. □

Remark 3.1. When $\alpha_0 - c = 0$, the origin is a nilpotent critical point and there is one less perturbation coefficient, so in such situation there exist just $m - 1$ limit cycles in a neighborhood of origin of system (3.7).



(a) (b)

Figure 3.1: (a) The phase portrait showing two limit cycles with the inner one unstable and the outer one stable, where three trajectories have initial points $(0.205, 0), (0.23, 0), (0.42, 0)$ in Case (i). (b) The phase portrait showing one stable limit cycle, where two trajectories have initial points $(0.3, 0)$ and $(0.7, 0)$ in Case (ii).



(a) (b)

Figure 3.2: (a) The phase portrait showing one stable limit cycle, where two trajectories have initial points $(0.3, 0)$ and $(0.5, 0)$ in Case (iii). (b) The phase portrait showing one stable limit cycle, where two trajectories have initial points $(0.3, 0)$ and $(0.45, 0)$ in Case (iv).

Applying Corollary 3.6, Propositions 3.1 and 3.2 we obtain several examples of system (3.7) with $\alpha_{2i-1} = 0$, $i = 1, 2, \ldots$, as follows:

(i) when $n = 4$, setting $c - \alpha_0 = -0.01$, $\alpha_2 = 1$, $\alpha_4 = K = -10$ and $\beta = 10$, as a semi-hyperbolic singular point, the origin is a stable node and there exist 2 limit cycles, where one is stable, another is unstable, see Fig. 3.1 (a);

(ii) when $n = 2$, setting $c - \alpha_0 = 0.1$, $\alpha_2 = K = -1$ and $\beta = 10$, as a semi-hyperbolic singular point, the origin is an unstable node and there exists one stable limit cycle, see Fig. 3.1 (b);

(iii) when $n = 4$, setting $\alpha_0 - c = 0$, $\alpha_2 = 0.4$, $\alpha_4 = K = -4$ and $\beta = 10$, the origin is a unstable nilpotent focus and there exists one stable limit cycle, see Fig. 3.2 (a);

(iv) when $n = 4$, setting $\alpha_0 - c = 0$, $\alpha_2 = 5$, $\alpha_4 = K = -50$ and $\beta = 2$, the origin is a unstable nilpotent node and there exists one stable limit cycle, see Fig. 3.2 (b).

(II) We consider the degenerate equilibrium $(1, 0)$ of system (1.5) in the case $\gamma = 1$. Applying the translation $v \mapsto v + 1$ and keeping the notation v for the translated system we obtain from (1.5) the system

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\xi} = y = X(v,y), \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} = -cy + (\sum_{i=0}^{n} \alpha_i (v+1)^i) y + \beta(v+1)[(v+1)^n - 1]^2 = Y(v,y). \end{cases} \tag{3.15}$$

We see that the solution to $X(v, y(v)) = 0, y(0) = 0$ is $y(v) = 0$, thus from (3.2), we have

$$\begin{aligned} Y(v, y(v)) = Y(v, 0) &= \beta(n^2 v^2 + \cdots + v^{2n+1}), \\ \left[\frac{\partial X}{\partial v} + \frac{\partial Y}{\partial y}\right]_{y=0} &= \sum_{i=0}^{n} \alpha_i (v+1)^i - c, \end{aligned} \tag{3.16}$$

and from Lemma 2.2, we obtain that only when $c = \sum_{i=0}^{n} \alpha_i$ the origin of (3.15) is a nilpotent critical point. Since $n > 1$ by Propositions 3.1 and 3.2, when $\beta \neq 0$ and $A_2 = n^2\beta \neq 0$ in (3.2), the origin cannot be a center or a focus, or a node. When $\beta = 0$, from Lemma 2.4 there is no limit cycle bifurcating from the degenerate equilibrium $(1, 0)$ of system (1.5).

Remark 3.2. When $\beta \neq 0$ and $c \neq \sum_{i=0}^{n} \alpha_i$ we verified that there does not exist a limit cycle bifurcating from the origin of (3.15) for some concrete values of n, for which the origin is a degenerate point of saddle-node type. However, for general values of n the problem is open.

4 Hopf bifurcation for the non-degenerate equilibriums

In this section we apply the Lyapunov function method to investigate the Hopf bifurcations in system (1.5), that is, the bifurcations of small amplitude limit cycles from the non-degenerate equilibrium $(1, 0)$ or the origin $(0, 0)$ under the condition $\gamma = 0$ or $\gamma = 1$.

Consider first the real polynomial differential system

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda x - y + \sum_{k+j=2}^{\infty} a_{kj} x^k y^j = X(x,y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} = x + \lambda y + \sum_{k+j=2}^{\infty} b_{kj} x^k y^j = Y(x,y) \end{cases} \tag{4.1}$$

where $x, y, t, \lambda, a_{kj}, b_{kj} \in \mathbb{R}\, (k, j \in \mathbb{N})$.

With a similar reasoning as in the proof of Theorem 3.4 we obtain the following theorem.

Theorem 4.1. Suppose that for system (4.1) there exists a positive definite function $V(x, y)$ such that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.1)} = \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = y^2\left[\sum_{i=0}^{N} \beta_{2i}(x^{2i} + O(x^{2i+1})) + o(x^{2N+1})\right] \tag{4.2}$$

where $N \in \mathbb{N}^+$ and $(x, y) \in U(O, \delta)$, δ is certain positive number and β_{2i} are independent. Assume also that for a vector field χ^* from family (4.1) with fixed parameters $a_{ij} = a_{ij}^*$, $b_{ij} = b_{ij}^*$ the parameters β_{2i} in (4.2) satisfy $\beta_0 = \beta_2 = \beta_4 = \cdots = \beta_{2N-2} = 0, \beta_{2N} \neq 0$. Then it is possible to perturb vector field χ^* in such way, that the perturbed system has N small limit cycles in a neighborhood of the origin.

Definition 4.1. For system (4.1) under condition (4.2), if $\beta_0 = \beta_2 = \cdots = \beta_{2N-2} = 0$ and $\beta_{2N} \neq 0$, then the quantity $\beta_{2N} = V_N$ is called the N-th quasi-Lyapunov constant at the origin $(N = 1, 2, \ldots)$.

(I) For the case $\gamma = 1$ when $\beta < 0$ system (1.5) has a Hopf bifurcation at the singular point $(0,0)$. Indeed, in this case system (1.5) has the form

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}\xi} = y = X(v,y), \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} = -cy + (\sum_{i=0}^{n} \alpha_i v^i)y + \beta v(1-v^n)^2 = Y(v,y). \end{cases} \tag{4.3}$$

Thus, when $n \in \mathbb{N}^+$, there exists a positive definite Lyapunov function

$$V(v,y) = \frac{1}{2}\left[y^2 - \beta v^2\left(1 - \frac{4v^n}{2+n} + \frac{v^{2n}}{1+n}\right)\right], \qquad |v| < \left[\frac{2+n}{4}\right]^{\frac{1}{n}} \tag{4.4}$$

such that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\xi}\right|_{(4.3)} = \frac{\mathrm{d}V}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}\xi} + \frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}\xi} = y^2\left(\sum_{i=0}^{n}\alpha_i v^i - c\right). \tag{4.5}$$

Indeed, for each $n \in \mathbb{N}^+$ we choose $\delta_n = [\frac{2+n}{4}]^{\frac{1}{n}}$ yielding $\frac{3}{4} \le \delta_n \le (\frac{3}{2})^{2/7}$. Then it is easy to verify that $V(v,y)$ is a positive definite Lyapunov function in the neighborhood $U(O,\delta_n)$.

Using Theorem 4.1 we have the following conclusion.

Theorem 4.2. For system (4.3) with $\gamma = 1$, if $\beta < 0$ and $\alpha_{2m} \neq 0$ for either $n = 2m$ or $n = 2m+1$, $m \in \mathbb{N}^+$, we can choose perturbations such that

$$0 < |\alpha_0 - c| \ll |\alpha_2| \ll |\alpha_4| \ll \cdots \ll |\alpha_{2m}|, \qquad \alpha_{2i-2}\alpha_{2i} < 0, \qquad i = 1,2,\dots,m, \tag{4.6}$$

and there exist m limit cycles in a neighborhood of origin of system (4.3).

(II) For the case of $\gamma = 0$, when $\beta < 0$ system (1.5) has the Hopf bifurcation at the singular point $(1,0)$. After the translation $v \mapsto v+1$ keeping the notation v for the new variable we obtain the system

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}\xi} = y = X(v,y), \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} = -cy + (\sum_{i=0}^{n} \alpha_i (v+1)^i)y - \beta(v+1)^{n+1}[1-(v+1)^n] = Y(v,y). \end{cases} \tag{4.7}$$

We show that in the neighborhood $U(O,2)$ of the origin there exists a positive definite Lyapunov function

$$V(v,y) = \frac{1}{2}y^2 - \frac{\beta}{2}\left[\frac{n}{(2+n)(1+n)} - \frac{2(v+1)^{n+2}}{2+n} + \frac{(v+1)^{2n+2}}{1+n})\right] \tag{4.8}$$

such that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\xi}\right|_{(4.3)} = \frac{\mathrm{d}V}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}\xi} + \frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}\xi} = y^2\left(\sum_{i=0}^{n}\alpha_i (v+1)^i - c\right). \tag{4.9}$$

Clearly, $V(0,0) = 0$. Thus, we need to prove that when $(v,y) \in U(O,\delta_n)$ and $(v,y) \neq (0,0)$ it holds that $V(v,y) > 0$.

Letting $r = v+1$ we have

$$\frac{n}{(2+n)(1+n)} - \frac{2(v+1)^{n+2}}{2+n} + \frac{(v+1)^{2n+2}}{1+n} = \frac{n}{(2+n)(1+n)} - \frac{2r^{n+2}}{2+n} + \frac{r^{2n+2}}{1+n} \triangleq L_r$$

so we will verify that when $r \neq \pm 1$, the inequality $L_r > 0$ holds. From

$$\frac{\mathrm{d}L_r}{\mathrm{d}r} = 2r^{n+1}(r^n - 1) = 0, \tag{4.10}$$

we find the stationary point. Namely, (i) if n is an odd number, only $r = 0$ and $r = 1$ are the stationary points of L_r; (ii) if n is an even number, then only $r = 0$ and $r = \pm 1$ are the stationary points of L_r.

Performing the monotonicity analysis of the function L_r we see that it has the minimum at $r = 1$, i.e. $L_r(1) = 0$, but there is no extreme at $r = 1$ for the case (i). The function has a minimum at $r = \pm 1$, i.e. $L_r(1) = 0$ and a local maximum at $r = 1$, i.e., $L_r(1) = \frac{n}{(2+n)(1+n)}$ for the case (ii). In summary, for arbitrary $n \in \mathbb{N}$ and $\beta < 0$, when $r \neq \pm 1$, i.e., $v \neq 0, -2$, we have

$$L_r > 0 \quad \text{or} \quad V(v, y) > 0. \tag{4.11}$$

We now can conclude that when the conditions of Theorem 4.1 are satisfied the following statement holds.

Theorem 4.3. For system (4.7) with $\gamma = 0$, if $\beta < 0$, $n = 2m$ or $2m + 1$, $\beta_{2m} \neq 0, m \in \mathbb{N}^+$ we can choose perturbations such that

$$\begin{aligned} &0 < |\beta_0| \ll |\beta_2| \ll |\beta_4| \ll \cdots \ll |\beta_{2m}|, \qquad \beta_{2i-2}\beta_{2i} < 0, \\ &\beta_1 \leq |\beta_0|, |\beta_3| \leq |\beta_2|, \ldots, |\beta_{2i+1}| \leq |\beta_{2i}|, \qquad i = 1, 2, \ldots, m \end{aligned} \tag{4.12}$$

where

$$\beta_0 = \sum_{j=0}^{n} \alpha_j - c,\ \beta_1 = \sum_{j=1}^{n} j\alpha_j,\ \beta_2 = \sum_{j=2}^{n} j(j-1)\alpha_j,\ \ldots, \beta_k = \sum_{j=k}^{n} \begin{pmatrix} k \\ j \end{pmatrix} \alpha_j, \ldots,\ \beta_n = \alpha_n$$

and there exist m limit cycles in a neighborhood of the origin of system (4.7).

5 Illustration of the quasi-Lyapunov constant and isolated periodic wave trains

The readers may be confused by the quasi-Lyapunov constant given in the Definitions 3.1 and 4.1, what is the difference between it and the Lyapunov constant? Here we try to illustrate this. In fact, within the above process certain positive definite Lyapunov functions $V(v, y)$ with finite terms are constructed to investigate limit cycle bifurcations, but these are different from first integral or integrating factor with formal series form to be determined for the Lyapunov constants or focus values, see e.g. [9, 17, 31]. For the later case, in general, the formal series of first integral with infinite terms is also positive definite in the neighborhood of the origin, which terms are derived successively, in a sense, it is a relatively complete sequence of positive definite Lyapunov function. However, for the the former, it is not a complete sequence necessarily. The highest order Lyapunov constant or focus value can be determined under the later case, just at most limit cycles is revealed, while only the quasi-Lyapunov constant are determined under the former case, and the highest order Lyapunov constant can not be determined necessarily.

Obviously, from Theorem 4.3, for the case $n = 1$ in system (4.7) we cannot obtain a limit cycle, and for the case $n = 2$ we can obtain only one limit cycle bifurcating from the origin. However, we have the following complete results by utilizing method of first integral formal series, namely determining the complete sequence of positive definite Lyapunov function.

Proposition 5.1. Under the degenerate condition of $\beta < 0$ and $\gamma = 0$, there exist at least one limit cycle for the case $n = 1$ and two limit cycles for the case $n = 2$, respectively, bifurcating from the origin of system (4.7) as a Hopf bifurcation point.

Proof. (i) When $n = 1$, system (4.7) has the form

$$\dot{v} = y, \qquad \dot{y} = \beta v + (\alpha_0 + \alpha_1 - c)y + h_1(v, y) \tag{5.1}$$

where $h_1(v, y) = \alpha_1 vy + 2\beta v^2 + \beta v^3$. If $c = \alpha_0 + \alpha_1$ holds, the origin is a center or a focus. For system (5.1) there is a formal series

$$H(v, y) = y^2 - \frac{\beta x^4}{2} - \frac{4\beta x^3}{3} - \beta x^2 + \frac{\alpha_1^2 y^4}{2\beta^2} - \frac{2\alpha_1 y^3}{3\beta} + \text{h.o.t} \tag{5.2}$$

such that

$$\frac{\mathrm{d}H}{\mathrm{d}\xi}\bigg|_{(4.3)} = \frac{\mathrm{d}H}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}\xi} + \frac{\mathrm{d}H}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}\xi} = -4\alpha_1 v^2 y^2 + \text{h.o.t.} \tag{5.3}$$

Thus, the first Lyapunov constant is $L_1 = -4\alpha_1$. When $c - \alpha_0 - \alpha_1 \neq 0$ and sufficiently small the origin becomes a weak focus and there exists one limit cycle bifurcated in a neighborhood of it.

(ii) When $n = 2$, system (4.7) becomes

$$\dot{v} = y, \ \ \dot{y} = 2\beta v + (\alpha_0 + \alpha_1 + \alpha_2 - c)y + h_2(v, y) \tag{5.4}$$

where $h_2(v, y) = \beta v^2(7 + 9v + 5v^2 + v^3) + vy(\alpha_1 + 2\alpha_2 + \alpha_2 v)$. Similarly as above there exists a series

$$\begin{aligned} H(v, y) = {} & y^2 - 2\beta v^2 - \tfrac{14\beta}{3} v^3 - \tfrac{(\alpha_1 + 2\alpha_2)}{3\beta} y^3 + \tfrac{(\alpha_1 + 2\alpha_2)^2}{8\beta^2} y^4 - \tfrac{9\beta}{2} v^4 \\ & + \tfrac{2}{15}(7\alpha_1^2 + 26\alpha_2\alpha_1 + 24\alpha_2^2 - 15\beta)v^5 + \tfrac{3(\alpha_1 + 2\alpha_2)}{2\beta} v^2 y^3 - \tfrac{(7\alpha_1^2 + 26\alpha_2\alpha_1 + 24\alpha_2^2)}{6\beta} v^3 y^2 \\ & - \tfrac{1}{20\beta^3}(\alpha_1 + 2\alpha_2)\left(\alpha_1^2 + 4\alpha_2\alpha_1 + 4\alpha_2^2 + 6\beta\right)y^5 + \tfrac{1}{18}(49\alpha_1^2 + 182\alpha_2\alpha_1 + 168\alpha_2^2 - 6\beta)v^6 \\ & + \tfrac{1}{12\beta^2}(\alpha_1 + 2\alpha_2)\left(7\alpha_1^2 + 26\alpha_2\alpha_1 + 24\alpha_2^2 + 42\beta\right)v^3 y^3 - \tfrac{9(\alpha_1 + 2\alpha_2)^2}{8\beta^2} v^2 y^4 \\ & + \tfrac{1}{48\beta^4}(\alpha_1 + 2\alpha_2)^2\left(\alpha_1^2 + 4\alpha_2\alpha_1 + 4\alpha_2^2 + 15\beta\right)y^6 + \text{h.o.t} \end{aligned}$$

such that

$$\frac{\mathrm{d}H}{\mathrm{d}\xi}\bigg|_{(4.3)} = L_1 v^2 y^2 + L_2 v^4 y^4 + \text{h.o.t.} \tag{5.5}$$

where $L_1 = -(7\alpha_1 + 12\alpha_2)$ and $L_2 = -\frac{1}{6\beta}(\alpha_1 + 2\alpha_2)(7\alpha_1^2 + 26\alpha_2\alpha_1 + 24\alpha_2^2 + 285\beta)$ are the first and second Lyapunov constants, respectively. Similarly as above we conclude that there exists two limit cycles bifurcated from the origin. □

Next, we consider the isolated periodic wave trains of the degenerate generalized Burgers–Huxley equation (1.2). As it is known a small amplitude limit cycle corresponds to an isolate bounded periodic solution of system (3.7). Thus from Theorems 3.5, 4.2 and 4.3 we have the following conclusion.

Theorem 5.2. In the Burgers–Huxley equation (1.2), for any $n = 2m$ or $2m + 1$, $m \in \mathbb{N}^+$, at least m isolated periodic wave trains can bifurcate from $u(x, t) = 0$ under the degenerate condition of $\beta > 0, \gamma = 0$ or $\beta < 0, \gamma = 1$; at least m isolated periodic wave trains can bifurcate from $u(x, t) = 1$ under the degenerate condition of $\beta < 0, \gamma = 0$.

Though the exact explicit expressions of the above isolated periodic wave trains cannot be given, some approximation methods can be used. Here we apply numerical computations to get the solutions for the four examples given in Remark 3.1 and shown in Fig. 5.1.

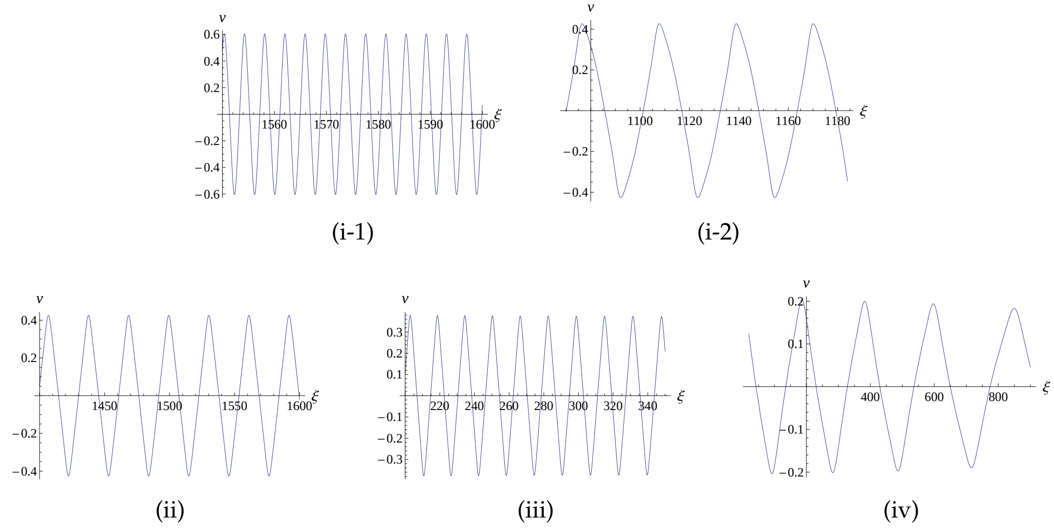


Figure 5.1: The periodic waves in cases (i-1,i-2), (ii), (iii), (iv) corresponding to the isolated periodic wave trains in Cases (i), (ii), (iii), (iv), respectively.

6 Conclusion

We studied the isolated periodic wave trains for a class of modified generalized Burgers–Huxley equation by focusing on the limit cycle bifurcations in a neighborhood of the degenerate equilibrium points. For any positive integer n the number of small amplitude limit cycles bifurcating from the nilpotent point or semi-hyperbolic singular point of system (1.5) is estimated. Finally, the existences of corresponding multiple isolated periodic wave trains in the original model (1.2) is established.

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