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Attractivity analysis on a neoclassical growth system incorporating patch structure and multiple pairs of time-varying delays

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Abstract. In this paper, we focus on the global dynamics of a neoclassical growth system incorporating patch structure and multiple pairs of time-varying delays. Firstly, we prove the global existence, positiveness and boundedness of solutions for the addressed system. Secondly, by employing some novel differential inequality analyses and the fluctuation lemma, both delay-independent and delay-dependent criteria are established to ensure that all solutions are convergent to the unique positive equilibrium point, which supplement and improve some existing results. Finally, some numerical examples are afforded to illustrate the effectiveness and feasibility of the theoretical findings.

Keywords: global attractivity, neoclassical growth system, patch structure, multiple pairs of time-varying delay.

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1 Introduction

Under the assumptions that labor and capital are fully allocated and the output market is adjusted immediately, Day proposed a discrete-time neoclassical growth model in literature [5], which has unimodal feedback production function. As we all know, there is an inevitable time lag between the acquisition of information and the implementation of decisions, but the model proposed by Day ignores the influence of delays and cannot fully explain the actual economic situation. To revise this drawback and better characterize the long-term behavior of economics, Matsumoto and Szidarovszky [25] introduced the delayed neoclassical growth equation

$$x'(t) = -\delta x(t) + Px^{\gamma}(t-\tau)e^{-\sigma x(t-\tau)}, \tag{1.1}$$

where $x(t)$ labels the capital per labor at time t, δ is the sum of labor growth rate and capital depreciation rate multiplied by average saving rate, τ designates the delay in the production

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function, γ denotes a proxy for measuring returns to scale of the production function, σ is regarded as a strength of a 'negative influence' produced by adding concentration of capital and is settled via a damaging degree of energy resources or natural environment. If $\gamma = 1$, the model (1.1) is the famous Nicholson's blowflies model, whose dynamic behavior has been extensively studied in recent years [1,3,13,15–20,22,23,27,31,32,37]. However, for the case of $\gamma \neq 1$, there are relatively few studies devoted to model (1.1) and its extended models [4,7,24,26,33,34].

Recently, regarding that the identical production function usually contains different delays, L. Berezansky and E. Braverman put forward a dynamic model of the form in [2],

$$x'(t) = \sum_{j=1}^{m} F_j(t, x(t-\tau_1(t)), \ldots, x(t-\tau_l(t))) - G(t, x(t)), \qquad t \geq t_0, \tag{1.2}$$

where l and m are positive integers, G describes the instantaneous mortality rate, and each $F_j (j \in I := \{1, 2, \cdots, m\})$ is the feedback control relying on the values of the stable variable with distinctive delays $\tau_1(t), \tau_2(t), \ldots, \tau_l(t)$. Manifestly, (1.2) contains the modified delayed differential neoclassical growth model

$$x'(t) = \beta(t)\left[-\delta x(t) + \sum_{j=1}^{m} P_j x^{\gamma}(t - g_j(t)) e^{-\sigma x(t-h_j(t))}\right], \qquad \gamma \in (0,\ 1), \tag{1.3}$$

which in the case $h_k \equiv g_k$ agrees with the traditional model [33].

In general, when each nonlinear function of the model contains only a small enough time delay, it will inherit some features of non time delay systems. For example, all the non-oscillatory solutions with respect to the unique positive equilibrium point are convergent. Moreover, as long as the time delay is small enough, the global attractivity for the positive equilibrium point has been shown in [2,30]. And the existence, oscillation, persistence, periodicity and stability of positive solutions have been widely explored for the single time-delay system (1.3) and similar models with $g_j(t) \equiv h_j(t)$ [4,7,24,26,33,34]. However, when the same nonlinear function of the model incorporates two or more time delays, chaotic oscillation of the system will occur, which will increase the difficulty in the study of the dynamics of such systems. Therefore, this issue has attracted the attention of many scholars. More recently, Huang et al. [21] studied the attractivity for the scalar equation (1.3). Meanwhile, since the financial environment of some capitals is fragmented, and the natural separation of the space area is separate, the above scalar neoclassical growth model can be naturally generalized to the patch structure system [8,36], the scalar equation (1.3) can be normally extended to the following system incorporating patch structure and multiple pairs of time-varying delays:

$$x_i'(t) = \beta(t)\left[-\bar{\delta}_i x_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij} x_j(t) + \sum_{j=1}^{m} P_{ij} x_i^{\gamma}(t - g_{ij}(t)) e^{-\sigma_{ij} x_i(t-h_{ij}(t))}\right], \quad \gamma \in (0,\ 1), \tag{1.4}$$

where $i \in Q := \{1, 2, \ldots, n\}$, x_i stands for the amount of the capital per labor in the patch i, a_{ij} designates the dispersal coefficient of the capital from patch j to patch i, m accounts for the number of population reproductive types, $P_{ij} x_i^{\gamma}(t - g_{ij}(t)) e^{-\sigma_{ij} x_i(t-h_{ij}(t))}$ describes the time-dependent reproduction function which is related to the incubation delay $h_{ij}(t)$ and the maturation delay $g_{ij}(t)$, and $x_i^{\gamma} e^{-\sigma_{ij} x_i}$ acquires the maximum reproduce rate at $x_i(t) = \frac{\gamma}{\sigma_{ij}}$. For more detailed biological significance, one can directly refer to [8,21,36] and their references quoted therein.

Hereafter, by changing the variables

$$\bar{\delta}_i = \delta_i - a_{ii} \quad \text{with } a_{ii} < 0,$$

(1.4) can be rewritten as

$$x_i'(t) = \beta(t)\left[-\bar{\delta}_i x_i(t) + \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{m} P_{ij}x_i^{\gamma}(t - g_{ij}(t))e^{-\sigma_{ij}x_i(t-h_{ij}(t))}\right], \quad \gamma \in (0,1),\ i \in Q. \tag{1.5}$$

It should be pointed out that, the dynamic characteristics of neoclassical growth model incorporating patch structure and multiple pairs of time-varying delays have not been fully studied. To the best of our knowledge, we have only found that the author of [36] established the attractivity results of the system (1.5) when $g_{ij}(t) \equiv h_{ij}(t)\ (i \in Q,\ j \in I)$. However, there is no research on the dynamic behavior of the model (1.5) with $g_{ij}(t) \neq h_{ij}(t)\ (i \in Q,\ j \in I)$.

According to the above discussions, our goal is to establish the global attractivity conditions of the unique positive equilibrium point for the system (1.5) under $g_{ij}(t) \neq h_{ij}(t)(i \in Q,\ j \in I)$. Briefly speaking, the contributions of this article can be summarized as below. 1) The boundedness and persistence on the solutions of system (1.5) are established by exploiting some novel differential inequality analyses; 2) Under certain assumptions, with the aid of the fluctuation lemma, some sufficient criteria ensuring the global attractivity of system (1.5) are obtained for the first time, which improve and generalize all recent works reported in [21,36]; 3) Numerical simulations involving comparison discussions are afforded to reveal the obtained theoretical results.

The remaining of this work is arranged as follows. In Section 2, some necessary lemmas and assumptions are listed. In Section 3, the global attractivity of the unique positive equilibrium point for the addressed system is demonstrated. To evidence our theoretical results, some numerical experiments are carried out in Section 4. Conclusions are given in Section 5.

2 Preliminary results

Throughout this manuscript, $\mathbb{N}^+$ labels the set of all positive integers and $\mathbb{R}^n$ ($\mathbb{R}^1 = \mathbb{R}$) designates the n-dimensional real vectors set. For a bounded real function u, let $u^+ = \sup_{\vartheta\in\mathbb{R}} u(\vartheta)$, $u^- = \inf_{\vartheta\in\mathbb{R}} u(\vartheta)$.

With the biological applications in mind, we assume that $\delta_i > 0,\ P_{ij} > 0,\ \sigma_{ij} > 0,\ \beta^- > 0$ and

$$r_i = \max\left\{\max_{1\le j\le m}\sup_{t\in\mathbb{R}} g_{ij}(t),\ \max_{1\le j\le m}\sup_{t\in\mathbb{R}} h_{ij}(t)\right\}, \qquad r = \max_{1\le i\le n}\{r_i\}.$$

Likewise, $g_{ij},\ h_{ij},\ \beta : \mathbb{R} \longrightarrow (0, +\infty)\ (i \in Q,\ j \in I)$ are bounded and continuous functions, $A = (a_{ij})_{n\times n}$ is an irreducible and cooperative matrix with $a_{ij} \ge 0\ (i \neq j)$, and

$$\sum_{j=1, j\neq i}^{n} a_{ij} = -a_{ii}, \quad \text{for all } i \in Q. \tag{2.1}$$

In addition, suppose that there exists a positive constant N^* such that

$$-\delta_i(N^*)^{1-\gamma} + \sum_{j=1}^{m} P_{ij}e^{-\sigma_{ij}N^*} = 0, \quad \text{for all } i \in Q, \tag{2.2}$$

which implies that $(N^*, N^*, \ldots, N^*)$ is a positive equilibrium point of system (1.5).

Denote $C = \prod_{i=1}^{n} C([-r_i, 0], \mathbb{R})$ be a Banach space involving the supremum norm $\|\cdot\|$, and $C_+ = \prod_{i=1}^{n} C([-r_i, 0], [0, +\infty))$. Also, we set $x_t(t_0, \varphi)(x(t; t_0, \varphi))$ for an admissible solution of (1.5) obeying the initial conditions:

$$x_{t_0} = \varphi, \qquad \varphi \in C_+ \quad \text{and} \quad \varphi_i(0) > 0, \qquad i \in Q, \tag{2.3}$$

and $[t_0, \eta(\varphi))$ be the maximal right-interval of existence.

Now, we present two lemmas to reveal the positiveness and boundedness of (1.5).

Lemma 2.1. *$x(t) = x(t; t_0, \varphi)$ has positiveness and boundedness on $[t_0, +\infty)$.*

Proof. By Theorem 5.2.1 in [28], we have that $x_t(t_0, \varphi) \in C_+$ for all $t \in [t_0, \eta(\varphi))$. This, together with (1.5) and (2.3), follows that

$$\begin{aligned} x_i(t) &= \varphi_i(0) e^{-\int_{t_0}^{t} (\delta_i - a_{ii})\beta(s) ds} + e^{-\int_{t_0}^{t} (\delta_i - a_{ii})\beta(s) ds} \int_{t_0}^{t} \beta(s) \\ &\quad \times \left[\sum_{j=1, j \neq i}^{n} a_{ij} x_j(s) + \sum_{j=1}^{m} P_{ij} x_i^{\gamma}(s - g_{ij}(s)) e^{-\sigma_{ij} x_i(s - h_{ij}(s))} \right] e^{\int_{t_0}^{s} (\delta_i - a_{ii})\beta(v) dv} ds \\ &> 0 \quad \text{for all } t \in [t_0, \eta(\varphi)) \text{ and } i \in Q. \end{aligned} \tag{2.4}$$

For $t > t_0$, let $i_0 \in Q$ and $T_{i_0} \in [t_0 - r_{i_0}, t]$ such that

$$x_{i_0}(T_{i_0}) = \max_{t_0 - r_{i_0} \le s \le t} x_{i_0}(s) = \max_{i \in Q} \left\{ \max_{t_0 - r_i \le s \le t} x_i(s) \right\}.$$

When $T_{i_0} \in [t_0 - r_{i_0}, t_0]$, it is easily seen that

$$\|x_s(t_0, \varphi)\| \le x_{i_0}(T_{i_0}) = \|\varphi\| \quad \text{for all } s \in [t_0, t]. \tag{2.5}$$

If $T_{i_0} \in (t_0, t]$, (1.5), (2.1) and (2.4) lead to

$$\begin{aligned} 0 &\le x_{i_0}'(T_{i_0}) \\ &= \beta(T_{i_0}) \left[-\delta_{i_0} x_{i_0}(T_{i_0}) + \sum_{j=1}^{n} a_{i_0 j} x_j(T_{i_0}) + \sum_{j=1}^{m} P_{i_0 j} x_{i_0}^{\gamma}(T_{i_0} - g_{i_0 j}(T_{i_0})) e^{-\sigma_{i_0 j} x_{i_0}(T_{i_0} - h_{i_0 j}(T_{i_0}))} \right] \\ &\le \beta(T_{i_0}) \left[-\delta_{i_0} x_{i_0}(T_{i_0}) + \sum_{j=1}^{n} a_{i_0 j} x_{i_0}(T_{i_0}) + \sum_{j=1}^{m} P_{i_0 j} x_{i_0}^{\gamma}(T_{i_0}) e^{-\sigma_{i_0 j} x_{i_0}(T_{i_0} - h_{i_0 j}(T_{i_0}))} \right] \\ &\le \beta(T_{i_0}) x_{i_0}^{\gamma}(T_{i_0}) \left[-\delta_{i_0} x_{i_0}^{1-\gamma}(T_{i_0}) + \sum_{j=1}^{m} P_{i_0 j} \right], \end{aligned}$$

which yields

$$\|x_s(t_0, \varphi)\| \le x_{i_0}(T_{i_0}) \le \max_{i \in Q} \left(\frac{\sum_{j=1}^{m} P_{ij}}{\delta_i} \right)^{\frac{1}{1-\gamma}} \quad \text{for all } \ s \in (t_0, t]. \tag{2.6}$$

From (2.5) and (2.6), we obtain that $x(t)$ has boundedness on $[t_0, \eta(\varphi))$, and

$$\|x_t(t_0, \varphi)\| \le x_{i_0}(T_{i_0}) \le \max_{i \in Q} \left(\frac{\sum_{j=1}^{m} P_{ij}}{\delta_i} \right)^{\frac{1}{1-\gamma}} + \|\varphi\| =: X^{\varphi} \quad \text{for all } \ t \in [t_0, \eta(\varphi)). \tag{2.7}$$

This, together with Theorem 2.3.1 in [9], follows $\eta(\varphi) = +\infty$, and finishes the evidence of Lemma 2.1. □

Lemma 2.2. $\liminf_{t\to+\infty} x_i(t) > 0$ *for all* $i \in Q$.

Proof. To obtain a contradiction, we suppose that $l = \min_{i\in Q} \liminf_{t\to+\infty} x_i(t) = 0$. Let

$$m(t) = \max\left\{\xi : \xi \le t \;\middle|\; \text{there is } \hat{i} \in Q \text{ satisfying } x_{\hat{i}}(\xi) = \min_{i\in Q}\left\{\min_{t_0\le s\le t} x_i(s)\right\}\right\}.$$

Then, $\lim_{t\to+\infty} m(t) = +\infty$. Likewise, for a strictly monotone increasing infinite sequence $\{t_p\}_{p\ge1}$, there are $\hat{i} \in Q$ and a subsequence $\{t_{p_k}\}_{k\ge1} \subseteq \{t_p\}_{p\ge1}$ agreeing with

$$x_{\hat{i}}(m(t_{p_k})) = \min_{t_0\le s\le t_{p_k}} x_{\hat{i}}(s) = \min_{i\in Q}\left\{\min_{t_0\le s\le t_{p_k}} x_i(s)\right\} \quad \text{and} \quad \lim_{k\to+\infty} x_{\hat{i}}(m(t_{p_k})) = 0. \tag{2.8}$$

Owing to (1.5), (2.1), (2.7) and (2.8), we derive

$$\begin{aligned}
0 &\ge x'_{\hat{i}}(m(t_{p_k})) \\
&\ge \beta(m(t_{p_k}))\Bigg[-\delta_{\hat{i}} x_{\hat{i}}(m(t_{p_k})) + x_{\hat{i}}(m(t_{p_k}))\sum_{j=1}^{n} a_{\hat{i}j} \\
&\quad + \sum_{j=1}^{m} P_{\hat{i}j} x_{\hat{i}}^{\gamma}(m(t_{p_k}) - g_{\hat{i}j}(m(t_{p_k})))e^{-\sigma_{\hat{i}j} x_{\hat{i}}(m(t_{p_k}) - h_{\hat{i}j}(m(t_{p_k})))}\Bigg] \\
&\ge \beta(m(t_{p_k}))\left[-\delta_{\hat{i}} x_{\hat{i}}(m(t_{p_k})) + \sum_{j=1}^{m} P_{\hat{i}j} x_{\hat{i}}^{\gamma}(m(t_{p_k}))e^{-\sigma_{\hat{i}j} X^{\varphi}}\right] \quad \text{for all } m(t_{p_k}) > t_0,
\end{aligned}$$

and

$$\delta_{\hat{i}} \ge \sum_{j=1}^{m} P_{\hat{i}j} \frac{1}{x_{\hat{i}}^{1-\gamma}(m(t_{p_k}))} e^{-\sigma_{\hat{i}j} X^{\varphi}}, \quad \text{for all } m(t_{p_k}) > t_0. \tag{2.9}$$

By taking limits, (2.8) and (2.9) give us $\delta_{\hat{i}} \ge +\infty$, which yields a contradiction and finishes the proof.

Lemma 2.3. *Lemma 2.2 indicates that* $(0, 0, \dots, 0)$ *is unstable.*

3 Global attractivity analysis

First, we present a delay-independent criterion to assure the attractivity for nonoscillatory solutions of system (1.5).

Proposition 3.1. *If*

$$\min_{i\in Q} \liminf_{t\to+\infty} x_i(t) \ge N^* \quad (\text{or } \max_{i\in Q} \limsup_{t\to+\infty} x_i(t) \le N^*),$$

then $\limsup_{t\to+\infty} x_i(t) = N^*$ *(or* $\liminf_{t\to+\infty} x_i(t) = N^*$*) for all* $i \in Q$.

Proof. We just need to deal with the case that

$$\min_{i\in Q} \liminf_{t\to+\infty} x_i(t) \ge N^*,$$

since the situation is entirely analogous for the case that $\max_{i\in Q} \limsup_{t\to+\infty} x_i(t) \le N^*$.

Set $y_i(t) = x_i(t) - N^*(i \in Q)$, it is evident that

$$\limsup_{t\to+\infty} y_i(t) \geq 0 \quad \text{for all } i \in Q. \tag{3.1}$$

Let $i^* \in Q$ be such an index as $\limsup_{t\to+\infty} y_{i^*}(t) = \max_{i\in Q} \limsup_{t\to+\infty} y_i(t)$. We state that

$$\limsup_{t\to+\infty} y_{i^*}(t) = 0.$$

Otherwise, $\limsup_{t\to+\infty} y_{i^*}(t) > 0$. Owing to the fluctuation lemma [29, Lemma A.1.], it is an easy matter to find a sequence $\{t_k\}_{k\geq 1}$ obeying

$$\lim_{k\to+\infty} t_k = +\infty, \quad \lim_{k\to+\infty} y_{i^*}(t_k) = \limsup_{t\to+\infty} y_{i^*}(t), \quad \lim_{k\to+\infty} y'_{i^*}(t_k) = 0. \tag{3.2}$$

Due to (1.5) and (2.1), we gain

$$y'_{i^*}(t_k) = \beta(t_k)\left[-\delta_{i^*} x_{i^*}(t_k) + \sum_{j=1}^{n} a_{i^*j} y_j(t_k) + \sum_{j=1}^{m} P_{i^*j} x^{\gamma}_{i^*}(t_k - g_{i^*j}(t_k)) e^{-\sigma_{i^*j} x_{i^*}(t_k - h_{i^*j}(t_k))}\right]. \tag{3.3}$$

Because $\beta(t)$, $x_{i^*}(t - g_{i^*j}(t))$ and $x_{i^*}(t - h_{i^*j}(t))$ are bounded on $[t_0, +\infty)$, we can select a subsequence of $\{t_k\}$ (for convenience of exposition, we still label by $\{t_k\}$) satisfying that $\lim_{k\to+\infty} \beta(t_k)$, $\lim_{k\to+\infty} y_l(t_k)$, $\lim_{k\to+\infty} x_{i^*}(t_k - g_{i^*j}(t_k))$ and $\lim_{k\to+\infty} x_{i^*}(t_k - h_{i^*j}(t_k))$ exist for all $l \in Q\backslash\{i^*\}$ and $j \in I$. Moreover, $0 < \beta^- \leq \lim_{k\to+\infty} \beta(t_k)$, and

$$N^* \leq \lim_{k\to+\infty} x_{i^*}(t_k - h_{i^*j}(t_k)), \qquad \lim_{k\to+\infty} x_{i^*}(t_k - g_{i^*j}(t_k)) \leq N^* + \lim_{k\to+\infty} y_{i^*}(t_k). \tag{3.4}$$

With the help of (3.4), we regard two cases as follow.

Case 1. If $\lim_{k\to+\infty} x_{i^*}(t_k - h_{i^*j}(t_k)) = N^*$ for all $j \in I$, by taking limits, (2.1), (2.2), (3.2), and (3.3) reveal that

$$\begin{aligned}
0 &= \lim_{k\to+\infty} y'_{i^*}(t_k) \\
&\leq \lim_{k\to+\infty} \beta(t_k)\left[-\delta_{i^*}\left(\limsup_{t\to+\infty} y_{i^*}(t) + N^*\right) + \limsup_{t\to+\infty} y_{i^*}(t) \sum_{j=1}^{n} a_{i^*j}\right. \\
&\quad \left. + \sum_{j=1}^{m} P_{i^*j}\left(\limsup_{t\to+\infty} y_{i^*}(t) + N^*\right)^{\gamma} e^{-\sigma_{i^*j} N^*}\right] \\
&\leq \lim_{k\to+\infty} \beta(t_k)\left(\limsup_{t\to+\infty} y_{i^*}(t) + N^*\right)^{\gamma}\left[-\delta_{i^*}\left(\limsup_{t\to+\infty} y_{i^*}(t) + N^*\right)^{1-\gamma} + \sum_{j=1}^{m} P_{i^*j} e^{-\sigma_{i^*j} N^*}\right] \\
&< \lim_{k\to+\infty} \beta(t_k)\left(\limsup_{t\to+\infty} y_{i^*}(t) + N^*\right)^{\gamma}\left[-\delta_{i^*}(N^*)^{1-\gamma} + \sum_{j=1}^{m} P_{i^*j} e^{-\sigma_{i^*j} N^*}\right] \\
&= 0,
\end{aligned}$$

which leads to a contradiction, and suggests that $\limsup_{t\to+\infty} y_{i^*}(t) = 0$.

Case 2. If for some $j \in I$, $N^* < \lim_{k\to+\infty} x_{i^*}(t_k - h_{i^*j}(t_k))$, it follows from (2.1), (2.2), (3.2) and (3.3) that

$$\begin{aligned}
0 &= \lim_{k\to+\infty} y'_{i^*}(t_k)\\
&< \lim_{k\to+\infty} \beta(t_k)\left[-\delta_{i^*}\lim_{k\to+\infty} x_{i^*}(t_k) + \sum_{j=1}^{n} a_{i^*j}\lim_{k\to+\infty} y_j(t_k)\right.\\
&\quad \left. + \sum_{j=1}^{m} P_{i^*j}\left(\lim_{k\to+\infty} x^{\gamma}_{i^*}(t_k - g_{i^*j}(t_k))\right) e^{-\sigma_{i^*j}N^*}\right]\\
&< \lim_{k\to+\infty} \beta(t_k)\left(\limsup_{k\to+\infty} y_{i^*}(t) + N^*\right)^{\gamma}\left[-\delta_{i^*}(N^*)^{1-\gamma} + \sum_{j=1}^{m} P_{i^*j}e^{-\sigma_{i^*j}N^*}\right]\\
&= 0,
\end{aligned}$$

which is also a contradiction and proves the above statement. This finishes the proof of Proposition 3.1. □

Corollary 3.2. *If for any $i \in Q$, $x_i(t)$ is eventually nonoscillatory about N^*, i.e., there is T^* obeying that*

$$x_i(t) \geq N^* (\text{or } x_i(t) \leq N^*) \quad \text{for all } t \geq T^* \text{ and } i \in Q.$$

Then $\lim_{t\to+\infty} x_i(t) = N^*$ *for all* $i \in Q$.

Remark 3.3. Corollary 3.2 shows that a delay-independent criterion has been established to guarantee that all non-oscillatory solutions of the system (1.5) are convergent to its unique positive equilibrium point.

Remark 3.4. It is obvious that all conclusions in Theorem 3.1, Theorem 3.2 of [21] and the results of Theorem 3.1 in [36] are special ones of Proposition 3.1.

Theorem 3.5. *Let* $\sigma = \max_{i\in Q}\max_{j\in I}\sigma_{ij}$, *suppose that, for all* $i \in Q$,

$$\frac{\delta_i \sigma N^*(e^{(\delta_i - a_{ii})\beta^+ r} - 1)}{\delta_i - a_{ii}} \leq 1, \tag{3.5}$$

and

$$0 < \sigma N^* \delta_i \frac{1 - e^{-r(\delta_i - a_{ii})\beta^+}}{\delta_i[1 - e(1 - e^{-r(\delta_i - a_{ii})\beta^+})] - a_{ii}e^{-r(\delta_i - a_{ii})\beta^+}} \leq 1, \tag{3.6}$$

hold. Then $\lim_{t\to+\infty} x_i(t) = N^*$ *for all* $i \in Q$.

Proof. Let

$$z_i(t) = \sigma(x_i(t) - N^*), \qquad i \in Q,$$

we have from (1.5) that

$$\begin{aligned}
&z'_i(t) + \sigma\delta_i\beta(t)N^* + \delta_i\beta(t)z_i(t)\\
&\quad = \beta(t)\sum_{j=1}^{n} a_{ij}z_j(t) + \sigma\beta(t)\sum_{j=1}^{m} P_{ij}\left[\frac{z_i(t - g_{ij}(t))}{\sigma} + N^*\right]^{\gamma} e^{-\frac{\sigma_{ij}z_i(t-h_{ij}(t))}{\sigma} - \sigma_{ij}N^*},
\end{aligned} \tag{3.7}$$

and

$$\left(z_i(t)e^{\int_{t_0}^{t}(\delta_i-a_{ii})\beta(v)dv}\right)' = \left[\sum_{j=1,j\neq i}^{n} a_{ij}\beta(t)z_j(t) + \sigma\beta(t)\sum_{j=1}^{m} P_{ij}\left(\frac{z_i(t-g_{ij}(t))}{\sigma}+N^*\right)^{\gamma} \times e^{-\frac{\sigma_{ij}z_i(t-h_{ij}(t))}{\sigma}-\sigma_{ij}N^*} - \sigma\beta(t)\delta_i N^*\right] e^{\int_{t_0}^{t}(\delta_i-a_{ii})\beta(v)dv}, \quad t\geq t_0,\ i\in Q. \tag{3.8}$$

To finish the verification, we shall reveal that

$$\min_{i\in Q}\liminf_{t\to+\infty} z_i(t) = \max_{i\in Q}\limsup_{t\to+\infty} z_i(t) = 0.$$

In view of Corollary 3.2, we only need to treat the case that for each $T^* > t_0$, there are $t^*, t^{**} \in (T^*, +\infty)$ such that

$$\min_{i\in Q} z_i(t^*) < 0 \quad \text{and} \quad \max_{i\in Q} z_i(t^{**}) > 0. \tag{3.9}$$

Set

$$\mu = \limsup_{t\to+\infty} z_{i_1}(t) = \max_{i\in Q}\limsup_{t\to+\infty} z_i(t), \qquad \lambda = \liminf_{t\to+\infty} z_{i_2}(t) = \min_{i\in Q}\liminf_{t\to+\infty} z_i(t). \tag{3.10}$$

Owing to (3.9), we gain

$$\lambda \leq 0 \leq \mu.$$

Now, it suffices to evidence that $\lambda = \mu = 0$. Contrarily, either $\mu > 0$ or $\lambda < 0$ is valid.

We only deal with the case that $\mu > 0$ occurs. ($\lambda < 0$ can be treated similarly.)

If $\lambda = 0$, i.e., $\lambda = \min_{i\in Q}\liminf_{t\to+\infty} z_i(t) = 0$. By Proposition 3.1, one can see that $\mu = \limsup_{t\to+\infty} z_{i_1}(t) = 0$.

When $\mu > 0$ and $\lambda < 0$, on account of the fluctuation lemma [29, Lemma A.1.], one can take two strictly monotone increasing infinite sequences $\{l_q\}_{q\geq 1}$, $\{s_q\}_{q\geq 1}$ satisfying that

$$z_{i_1}(l_q) > 0,\ l_q \to +\infty, \quad z_{i_1}(l_q) \to \mu, \quad z'_{i_1}(l_q) \to 0 \quad \text{as } q\to+\infty, \tag{3.11}$$

and

$$z_{i_2}(s_q) < 0,\ s_q \to +\infty, \quad z_{i_2}(s_q) \to \lambda, \quad z'_{i_2}(s_q) \to 0 \quad \text{as } q\to+\infty. \tag{3.12}$$

Note that a bounded sequence has a convergent subsequence, we can presume that for all $j \in I$,

$$\lim_{q\to+\infty}\beta(l_q) = \beta^*, \quad \lim_{q\to+\infty} z_{i_1}(l_q - g_{i_1 j}(l_q)) = z_{i_1}^j, \quad \lim_{q\to+\infty} z_i(l_q) = z_i^l \quad (i\in Q\setminus\{i_1\}), \tag{3.13}$$

and

$$\lim_{q\to+\infty}\beta(s_q) = \beta^{**}, \quad \lim_{q\to+\infty} z_{i_2}(s_q - g_{i_2 j}(s_q)) = z_{i_2}^j, \quad \lim_{q\to+\infty} z_i(s_q) = z_i^s \quad (i\in Q\setminus\{i_2\}). \tag{3.14}$$

To obtain a contradiction, we divide our proof into three steps.

First, we assert that there exists $H_1 > 0$ obeying that, for any $q \geq H_1$, there is $L_q \in [l_q - r_{i_1}, l_q)$ agreeing with

$$z_{i_1}(L_q) = 0, \quad \text{and} \quad z_{i_1}(t) > 0, \quad \text{for all } t\in(L_q, l_q). \tag{3.15}$$

If not, there exists a subsequence of $\{l_q\}$ (do not relabel) such that

$$z_{i_1}(t) > 0, \quad \text{for all } t \in [l_q - r_{i_1}, l_q),\ q = 1, 2, \ldots \tag{3.16}$$

Subsequently,

$$0 \le \lim_{q\to+\infty} z_{i_1}(l_q - g_{i_1 j}(l_q)) \le \mu \quad \text{for all } j \in I, \tag{3.17}$$

and

$$\begin{aligned} z'_{i_1}(l_q) &= \beta(l_q)\sum_{j=1}^{n} a_{i_1 j} z_j(l_q) + \sigma\beta(l_q)\sum_{j=1}^{m} P_{i_1 j}\left[\frac{z_{i_1}(l_q - g_{i_1 j}(l_q))}{\sigma} + N^*\right]^{\gamma} e^{-\frac{\sigma_{i_1 j} z_{i_1}(l_q - h_{i_1 j}(l_q))}{\sigma} - \sigma_{i_1 j} N^*} \\ &\quad - \sigma\delta_{i_1}\beta(l_q)N^* - \delta_{i_1}\beta(l_q) z_{i_1}(l_q) \\ &< \beta(l_q)\sum_{j=1}^{n} a_{i_1 j} z_j(l_q) + \sigma\beta(l_q)\sum_{j=1}^{m} P_{i_1 j}\left[\frac{z_{i_1}(l_q - g_{i_1 j}(l_q))}{\sigma} + N^*\right]^{\gamma} e^{-\sigma_{i_1 j} N^*} \\ &\quad - \sigma\delta_{i_1}\beta(l_q)N^* - \delta_{i_1}\beta(l_q) z_{i_1}(l_q). \end{aligned} \tag{3.18}$$

By taking limit, (3.11), (3.13), (3.17) and (3.18) lead to

$$\begin{aligned} 0 &\le a_{i_1 i_1}\beta^* \lim_{q\to+\infty} z_{i_1}(l_q) + \beta^* \sum_{j=1, j\ne i_1}^{n} a_{i_1 j} \lim_{q\to+\infty} z_j(l_q) \\ &\quad + \sigma\beta^* \sum_{j=1}^{m} P_{i_1 j}\left[\frac{\lim_{q\to+\infty} z_{i_1}(l_q - g_{i_1 j}(l_q))}{\sigma} + N^*\right]^{\gamma} e^{-\sigma_{i_1 j}N^*} - \sigma\delta_{i_1}\beta^* N^* - \delta_{i_1}\beta^* \lim_{q\to+\infty} z_{i_1}(l_q) \\ &\le \sigma\beta^* \sum_{j=1}^{m} P_{i_1 j}\left[\frac{\lim_{q\to+\infty} z_{i_1}(l_q - g_{i_1 j}(l_q))}{\sigma} + N^*\right]^{\gamma} e^{-\sigma_{i_1 j}N^*} - \sigma\beta^*\delta_{i_1}\left(N^* + \frac{\mu}{\sigma}\right) \\ &\le \sigma\beta^*\left(N^* + \frac{\mu}{\sigma}\right)^{\gamma}\left[\sum_{j=1}^{m} P_{i_1 j} e^{-\sigma_{i_1 j}N^*} - \delta_{i_1}\left(N^* + \frac{\mu}{\sigma}\right)^{1-\gamma}\right] \\ &< 0, \end{aligned}$$

which is a contradiction and validates the above assertion.

Similarly, from (3.12) and (3.14), one can find $H_1^* > 0$ such that for any $q \ge H_1^*$, there is $S_q \in [s_q - r_{i_2}, s_q)$ such that

$$z_{i_2}(S_q) = 0, \quad \text{and} \quad z_{i_2}(t) < 0, \quad \text{for all } t \in (S_q, s_q). \tag{3.19}$$

Secondly, we show

$$e^{-\mu} - 1 \le \lambda \le 0 \le \mu \le e^{-\lambda} - 1. \tag{3.20}$$

For any $0 < \varepsilon < \sigma(N^* + \frac{\lambda}{\sigma}) = \sigma \liminf_{t\to+\infty} x_{i_2}(t)$, (3.10) suggests that one can select a positive integer $q^* > H_1 + H_1^*$ satisfying

$$\lambda - \varepsilon < z_i(t) < \mu + \varepsilon \quad \text{for all } t > \min\{l_{q^*}, s_{q^*}\} - 2r \ \text{ and } \ i \in Q. \tag{3.21}$$

With the aid of (2.1), (2.2), (3.8), (3.19), (3.21) and (3.23), we obtain

$$
\begin{aligned}
& z_{i_2}(s_q)e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv} \\
&= -\sigma\delta_{i_2}N^*\frac{e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}-e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}}{\delta_{i_2}-a_{i_2i_2}} \\
&\quad+\sum_{j=1,j\neq i_2}^{n}a_{i_2j}\int_{S_q}^{s_q}z_j(t)\beta(t)e^{\int_{t_0}^{t}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}dt+\sigma\sum_{j=1}^{m}P_{i_2j}\int_{S_q}^{s_q}\left[N^*+\frac{z_{i_2}(t-g_{i_2j}(t))}{\sigma}\right]^{\gamma} \\
&\quad\times e^{-\sigma_{i_2j}N^*-\frac{\sigma_{i_2j}}{\sigma}z_{i_2}(t-h_{i_2j}(t))}\beta(t)e^{\int_{t_0}^{t}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}dt \\
&> -\sigma\delta_{i_2}N^*\frac{e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}-e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}}{\delta_{i_2}-a_{i_2i_2}} \\
&\quad+(\lambda-\varepsilon)\frac{e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}-e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}}{\delta_{i_2}-a_{i_2i_2}.}\sum_{j=1,j\neq i_2}^{n}a_{i_2j} \\
&\quad+\sigma\sum_{j=1}^{m}P_{i_2j}\int_{S_q}^{s_q}(N^*)^{\gamma}\left[\frac{N^*+\frac{\lambda-\varepsilon}{\sigma}}{N^*}\right]^{\gamma}e^{-\sigma_{i_2j}N^*-\frac{\sigma_{i_2j}}{\sigma}(\mu+\varepsilon)}\beta(t)e^{\int_{t_0}^{t}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}dt \\
&> -\sigma\delta_{i_2}N^*\frac{e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}-e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}}{\delta_{i_2}-a_{i_2i_2}} \\
&\quad+(\lambda-\varepsilon)\frac{e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}-e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}}{\delta_{i_2}-a_{i_2i_2}}\sum_{j=1,j\neq i_2}^{n}a_{i_2j} \\
&\quad+\sigma\sum_{j=1}^{m}P_{i_2j}\int_{S_q}^{s_q}(N^*)^{\gamma-1}\left(N^*+\frac{\lambda-\varepsilon}{\sigma}\right)e^{-\sigma_{i_2j}N^*-\frac{\sigma_{i_2j}}{\sigma}(\mu+\varepsilon)}\beta(t)e^{\int_{t_0}^{t}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}dt \\
&\geq \sigma\delta_{i_2}N^*\frac{e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}-e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}}{\delta_{i_2}-a_{i_2i_2}}[e^{-(\mu+\varepsilon)}-1] \\
&\quad+(\lambda-\varepsilon)\left(e^{\int_{t_0}^{s_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}-e^{\int_{t_0}^{S_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}\right), \qquad q>q^*
\end{aligned}
$$

and

$$
\begin{aligned}
& z_{i_2}(s_q)+(\lambda-\varepsilon)\left(e^{-(\delta_{i_2}-a_{i_2i_2})\beta^+r}-1\right) \\
&\quad\geq z_{i_2}(s_q)+(\lambda-\varepsilon)\left(e^{-\int_{S_q}^{s_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}-1\right) \\
&\quad> \sigma N^*\left(1-e^{-\int_{S_q}^{s_q}(\delta_{i_2}-a_{i_2i_2})\beta(v)dv}\right)\frac{\delta_{i_2}}{\delta_{i_2}-a_{i_2i_2}}\left[e^{-(\mu+\varepsilon)}-1\right] \\
&\quad\geq \sigma N^*\left(1-e^{-(\delta_{i_2}-a_{i_2i_2})\beta^+r}\right)\frac{\delta_{i_2}}{\delta_{i_2}-a_{i_2i_2}}\left[e^{-(\mu+\varepsilon)}-1\right], \qquad q>q^*.
\end{aligned} \tag{3.22}
$$

Letting $q\to\infty$ and $\varepsilon\to 0$, (3.5) and (3.22) give us

$$
\lambda\geq\sigma N^*\left(e^{(\delta_{i_2}-a_{i_2i_2})\beta^+r}-1\right)\frac{\delta_{i_2}}{\delta_{i_2}-a_{i_2i_2}}(e^{-\mu}-1)\geq(e^{-\mu}-1)\geq -1. \tag{3.23}
$$

In view of (2.1), (2.2), (3.8), (3.15) and (3.21), we acquire

$$
\begin{aligned}
&z_{i_1}(l_q)e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}\\
&= -\sigma\delta_{i_1}N^*\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}}{\delta_{i_1}-a_{i_1i_1}}\\
&\quad+\sum_{j=1,j\neq i_1}^{n}a_{i_1j}\int_{L_q}^{l_q}z_j(t)\beta(t)e^{\int_{t_0}^{t}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}dt\\
&\quad+\sigma\sum_{j=1}^{m}P_{i_1j}\int_{L_q}^{l_q}\left[N^*+\frac{z_{i_1}(t-g_{i_1j}(t))}{\sigma}\right]^{\gamma}\\
&\quad\times e^{-\sigma_{i_1j}N^*-\frac{\sigma_{i_1j}}{\sigma}z_{i_1}(t-h_{i_1j}(t))}\beta(t)e^{\int_{t_0}^{t}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}dt\\
&< -\sigma\delta_{i_1}N^*\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}}{\delta_{i_1}-a_{i_1i_1}}\\
&\quad+(\mu+\varepsilon)\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}}{\delta_{i_1}-a_{i_1i_1}}\sum_{j=1,j\neq i_1}^{n}a_{i_1j}\\
&\quad+\sigma\sum_{j=1}^{m}P_{i_1j}\int_{L_q}^{l_q}\left[N^*+\frac{\mu+\varepsilon}{\sigma}\right]^{\gamma}e^{-\sigma_{i_1j}N^*-\frac{\sigma_{i_1j}}{\sigma}(\lambda-\varepsilon)}\beta(t)e^{\int_{t_0}^{t}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}dt\\
&= -\sigma\delta_{i_1}N^*\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}}{\delta_{i_1}-a_{i_1i_1}}\\
&\quad+(\mu+\varepsilon)\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}}{\delta_{i_1}-a_{i_1i_1}}\sum_{j=1,j\neq i_1}^{n}a_{i_1j}\\
&\quad+\sigma\sum_{j=1}^{m}P_{i_1j}\int_{L_q}^{l_q}(N^*)^{\gamma}\left[\frac{N^*+\frac{\mu+\varepsilon}{\sigma}}{N^*}\right]^{\gamma}e^{-\sigma_{i_1j}N^*-\frac{\sigma_{i_1j}}{\sigma}(\lambda-\varepsilon)}\beta(t)e^{\int_{t_0}^{t}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}dt\\
&< -\sigma\delta_{i_1}N^*\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}}{\delta_{i_1}-a_{i_1i_1}}\\
&\quad+(\mu+\varepsilon)\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}}{\delta_{i_1}-a_{i_1i_1}}\sum_{j=1,j\neq i_1}^{n}a_{i_1j}\\
&\quad+\sigma\sum_{j=1}^{m}P_{i_1j}\int_{L_q}^{l_q}(N^*)^{\gamma-1}\left(N^*+\frac{\mu+\varepsilon}{\sigma}\right)e^{-\sigma_{i_1j}N^*-\frac{\sigma_{i_1j}}{\sigma}(\lambda-\varepsilon)}\beta(t)e^{\int_{t_0}^{t}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}dt\\
&\le -\sigma\delta_{i_1}N^*\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}}{\delta_{i_1}-a_{i_1i_1}}\\
&\quad+(\mu+\varepsilon)\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}}{\delta_{i_1}-a_{i_1i_1}}\sum_{j=1,j\neq i_1}^{n}a_{i_1j}\\
&\quad+\frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}-e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}}{\delta_{i_1}-a_{i_1i_1}}\\
&\quad\times\left[\sigma\sum_{j=1}^{m}P_{i_1j}(N^*)^{\gamma}e^{-\sigma_{i_1j}N^*}e^{-(\lambda-\varepsilon)}+(\mu+\varepsilon)e^{1+\varepsilon}\sum_{j=1}^{m}P_{i_1j}(N^*)^{\gamma-1}e^{-\sigma_{i_1j}N^*}\right]
\end{aligned}
$$

$$= \sigma\delta_{i_1} N^* \frac{e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv} - e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}}{\delta_{i_1}-a_{i_1i_1}} \left[e^{-(\lambda-\varepsilon)}-1\right]$$
$$+ (\mu+\varepsilon)\frac{e^{1+\varepsilon}\delta_{i_1}-a_{i_1i_1}}{\delta_{i_1}-a_{i_1i_1}}\left(e^{\int_{t_0}^{l_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv} - e^{\int_{t_0}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}\right), \qquad q > q^*,$$

and

$$\begin{aligned} z_{i_1}(l_q) &< \sigma N^*\delta_{i_1}\frac{1-e^{\int_{l_q}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}}{\delta_{i_1}-a_{i_1i_1}}\left[e^{-(\lambda-\varepsilon)}-1\right] \\ &\quad + (\mu+\varepsilon)\frac{e^{1+\varepsilon}\delta_{i_1}-a_{i_1i_1}}{\delta_{i_1}-a_{i_1i_1}}\left(1-e^{\int_{l_q}^{L_q}(\delta_{i_1}-a_{i_1i_1})\beta(v)dv}\right) \\ &\le \sigma N^*\delta_{i_1}\frac{1-e^{-r(\delta_{i_1}-a_{i_1i_1})\beta^+}}{\delta_{i_1}-a_{i_1i_1}}\left[e^{-(\lambda-\varepsilon)}-1\right] \\ &\quad + (\mu+\varepsilon)\frac{e^{1+\varepsilon}\delta_{i_1}-a_{i_1i_1}}{\delta_{i_1}-a_{i_1i_1}}\left(1-e^{-r(\delta_{i_1}-a_{i_1i_1})\beta^+}\right),\ q>q^*. \end{aligned} \tag{3.24}$$

Letting $q \to \infty$ and $\varepsilon \to 0$, (3.6) and (3.24) entail that

$$\mu \le \sigma N^*\delta_{i_1}\frac{1-e^{-r(\delta_{i_1}-a_{i_1i_1})\beta^+}}{\delta_{i_1}[1-e(1-e^{-r(\delta_{i_1}-a_{i_1i_1})\beta^+})]-a_{i_1i_1}e^{-r(\delta_{i_1}-a_{i_1i_1})\beta^+}}(e^{-\lambda}-1) \le (e^{-\lambda}-1), \tag{3.25}$$

which, together with (3.23), involves that (3.20) holds.

Finally, from the proof in Theorem 4.1 of [30], (3.20) implies that $\lambda = \mu = 0$, which yields a clear contradiction of the fact that $\mu > 0$. This finishes the proof. □

Remark 3.6. Apparently, $\lim_{r\to 0^+} e^{(\delta_i - a_{ii})\beta^+ r} = 1$, then the conditions (3.5) and (3.6) naturally hold, which means that sufficiently small pairs of timing-varying delays have little influence on the global attractivity of the positive equilibrium point for system (1.5). On the other hand, $\lim_{r\to+\infty} e^{(\delta_i-a_{ii})\beta^+ r} = +\infty$, then the assumptions (3.5) and (3.6) do not hold, which indicates that large enough pairs of time-varying delays will lead to chaotic oscillation of the system (1.5). We will verify this through some numerical simulations in the next section.

4 Numerical example

Example 4.1. Regard the following patch structure neoclassical growth model incorporating multiple pairs of time-varying delays:

$$\begin{cases} x_1'(t) = (3+\sin^2(t))\Big[(-\frac{1}{20}x_1(t)+\frac{1}{20}x_2(t)) - \frac{1}{20}x_1(t) \\ \qquad + \frac{11}{100}e^{\frac{8}{5}}x_1^{\frac{1}{3}}(t-g_{11}(t))e^{-\frac{1}{5}x_1(t-h_{11}(t))} \\ \qquad + \frac{9}{100}e^{\frac{16}{5}}x_1^{\frac{1}{3}}(t-g_{12}(t))e^{-\frac{2}{5}x_1(t-h_{12}(t))}\Big], \\ x_2'(t) = (3+\sin^2(t))\Big[(-\frac{1}{80}x_2(t)+\frac{1}{80}x_1(t)) - \frac{1}{80}x_2(t) \\ \qquad + \frac{3}{200}e^{\frac{8}{3}}x_2^{\frac{1}{3}}(t-g_{21}(t))e^{-\frac{1}{3}x_2(t-h_{21}(t))} \\ \qquad + \frac{7}{200}e^{2}x_2^{\frac{1}{3}}(t-g_{22}(t))e^{-\frac{1}{4}x_2(t-h_{22}(t))}\Big], \end{cases} \tag{4.1}$$

which possesses a unique positive equilibrium point $(N^*, N^*) = (8, 8)$.

Now, one can easily check that

$$g_{ij}(t) = \frac{1}{20}|\cos(i+j)t|, \; h_{ij}(t) = \frac{1}{40}|\sin(i+j)t|, \qquad i, \; j = 1, 2. \tag{4.2}$$

satisfy (3.5) and (3.6). By Theorem 3.5, we obtain that the positive equilibrium point $(8, 8)$ is a global attractor of (4.1) incorporating delays (4.2). The numeric simulations in Figure 4.1 support this theoretical assertions.

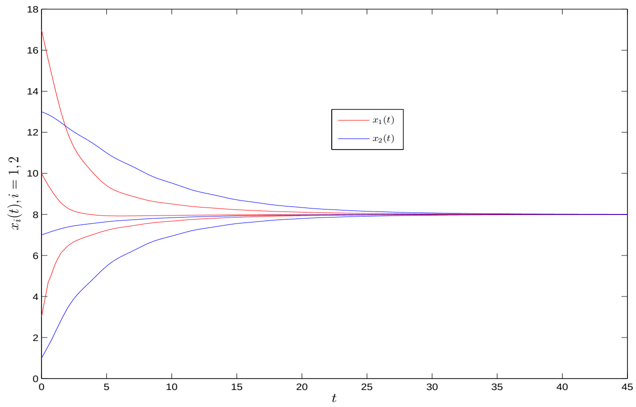

Figure 4.1: Numerical solutions of example (4.1) obeying (4.2) and the initial values: $(3, 1), (10, 7), (17, 13)$.

Moreover, if we choose

$$g_{ij}(t) = 40j, \qquad h_{ij}(t) = 60j, \qquad i, j = 1, 2, \tag{4.3}$$

it is an elementary computation to show that (3.5) and (3.6) do not hold for system (4.1) with delays (4.3). It can be seen from Figure 4.2 that $(8, 8)$ maybe not the global attractor of (4.1) with delays (4.3). This confirms the conclusions reached in Remark 3.6.

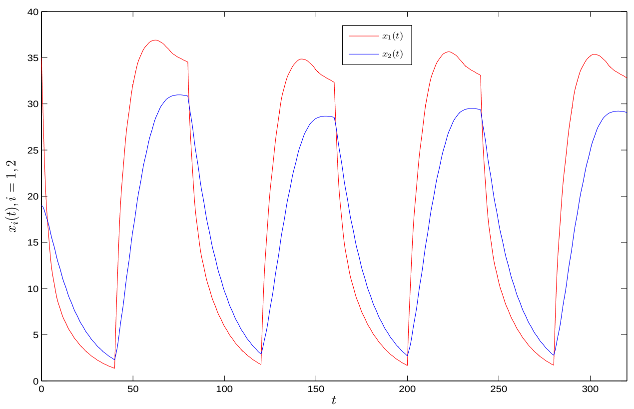

Figure 4.2: Numerical solutions of example (4.1) satisfying (4.3) and the initial value $(35, 19)$.

Remark 4.2. From the above simulations, we can make the following observations. First, small delays will make the positive equilibrium point be attractive. Second, big delays maybe yield complex dynamic behavior. In addition, the latest literature [8,21,36] and [6,10–12,14,35] have not touched the global attractivity of the positive equilibrium point for the patch structure neoclassical growth system with multiple pairs of time-varying delays. It can be found that all the conclusions in the above mentioned literature and the references cited therein cannot be used to reveal the global attractivity of (4.1). It should be pointed out that, in equations (20) and (21) on page 3861 of [36],

$$\lim_{q\to+\infty} y'_{i_1}(l_q) \geq 0 \quad \text{and} \quad \lim_{q\to+\infty} y'_{i_2}(s_q) \leq 0$$

maybe not hold. For a counterexample, consider $y_{i_1}(t) = 1 + \frac{1}{1+t^2}$ and $y_{i_2}(t) = -1 - \frac{1}{1+t^2}$. In the proof of Theorem 3.5, we have successfully corrected these errors by adopting new proof strategies and ideas. This implies that our results generalize and improve all the ones in the above-mentioned references.

5 Conclusions

By introducing two time-varying delays in the same time-dependent reproduction function, this paper proposed a neoclassical growth system incorporating patch structure and multiple pairs of time-varying delays. Via some novel differential inequality analyses and the fluctuation lemma, the persistence on the positive solutions, as well as the global attractivity on the positive equilibrium point have firstly been established for the addressed model. The obtained results reveal that, by controlling labor growth rate, capital depreciation rate and the related parameters in the reproduction function, the attractivity of the positive equilibrium point can be guaranteed if the time-varying delays are sufficiently small in the development process. The adopted strategies could be taken into consideration in the area of dynamics problems on other patch structure population systems incorporating two or more distinctive delays in the same time-dependent reproduction function.

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