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Alternative Results and Robustness for Fractional Evolution Equations with Periodic Boundary Conditions☆

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Abstract

In this paper, we study periodic boundary value problems for a class of linear fractional evolution equations involving the Caputo fractional derivative. Utilizing compactness of the constructed evolution operators and Fredholm alternative theorem, some interesting alternative results for the mild solutions are presented. Periodic motion controllers that are robust to parameter drift are also designed for given a periodic motion. An example is given to illustrate the results.

Keywords: Fractional evolution equations, Boundary value problems, Mild solutions, Alternative results, Robustness.
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1. Introduction

It is well known that periodic motion is a very important and special phenomena not only in the natural sciences but also in social sciences. There are a lot of periodic motions involved in climate, food supplement, insecticide population, sustainable development etc. The periodic solution theory of dynamic equations has been developed over the last serval decades. See, for example, [1, 2, 3, 4, 5, 6, 7].

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Recently, fractional differential equations have been proved to be effective tools in the modeling of many phenomena in various fields of physics, mechanics, chemistry, engineering, etc. There are many applications in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. For more details, one can see the monographs of Kilbas et al. [8], Lakshmikantham et al. [9], Miller and Ross [10], Podlubny [11] and Tarasov [12]. Recently, fractional differential systems and optimal controls in Banach spaces have been studied by many researchers such as Agarwal et al. [13, 14], Ahmad and Nieto [15, 16], Balachandran and Park [17], Bai [18], Benchohra et al. [19, 20], Chang and Nieto [21], Diagana et al. [22], El-Borai [23], Fečkan et al. [24], Henderson and Ouahab [25], Li et al. [26], Mophou and N'Guérékata [27], Tatar [28], Wang et al. [29, 30, 31, 32, 33, 34], Zhang [35, 36], and Zhou et al. [37, 38].

Periodic boundary problems for fractional differential equations serve as a class of important models to study the dynamics of processes that are subject to periodic changes in their initial state and final state. There are some papers discussing periodic (or anti-periodic) boundary problems for fractional differential equations in finite dimensional spaces [13, 16], however, there are few results on the theory on periodic boundary problems for fractional evolution equations in infinite dimensional spaces. Since the unbounded operator is involved in the fractional evolution equations, it is obvious that periodic boundary problems for fractional evolution equations are much more difficult than the same problems for fractional differential equations.

In this paper, we discuss periodic boundary valued problems (BVP for short) for fractional evolution equations such as

$$\begin{cases} {}^cD^q x(t) = Ax(t) + f(t), \ t \in J = [0, T], \ q \in (0, 1), \\ x(0) = x(T), \end{cases} \tag{1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, unbounded operator A is the generator of a strongly continuous semigroup $\{S(t), t \geq 0\}$ on a Banach space X, $f : J \to X$ will be specified later.

For that, we have to first discuss the homogeneous linear periodic BVP

$$\begin{cases} {}^cD^q x(t) = Ax(t), \ t \in J, \ q \in (0, 1), \\ x(0) = x(T), \end{cases} \tag{2}$$

and the associated Cauchy problem

$$\begin{cases} {}^cD^q x(t) = Ax(t), \ t \in J, \ q \in (0, 1), \\ x(0) = \bar{x}, \ \bar{x} \in X, \end{cases} \tag{3}$$

and introduce the suitable definition of mild solution for BVP (2). To study BVP (2), we construct the evolution operator $\{\mathscr{T}(\cdot)\}$ associated with the unbounded operator A, the order q, and a probability density function defined on $(0,\infty)$, which is very important in sequel. It can be deduced from the discussion on BVP (1) that the invertibility of $[I-\mathscr{T}(T)]$ is the key of the existence of mild solution of BVP (2). For the invertibility of $[I-\mathscr{T}(T)]$, compactness or other additional conditions on $\{S(t), t\geq 0\}$ generated by the unbounded operator A are needed.

We remark that the constructed evolution operators $\{\mathscr{T}(\cdot)\}$ can be used to reduce the existence of mild solution for BVP (2) to the existence of fixed points for an operator equation $\mathscr{T}(T)\bar{x}=\bar{x}$, where $\bar{x}$ can be acted as the initial value in Cauchy problem (3). Applying the well known Fredholm alternative theorem, we can present the alternative results on mild solution for BVP (1) and (2).

To evaluate the probability of BVP (1), we study the following periodic BVP with parameter perturbations

$$\begin{cases} {}^cD^qx(t)=Ax(t)+f(t)+p(t,x(t),\xi),\ t\in J,\ q\in(0,1), \\ x(0)=x(T), \end{cases} \tag{4}$$

where p is a given function and $\xi\in\Lambda=(-\tilde{\xi},\tilde{\xi})$ is a small parameter perturbation that may be caused by some adaptive control algorithms or parameter drift.

For given a periodic motion of BVP (1), periodic motion controllers that are robust to parameter drift are designed by virtue of BVP with parameter perturbations (4).

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some concepts and preparation results. In Section 3, the alternative results on mild solution for BVP (2) are given. In Section 4, the alternative results on mild solution for BVP (1) are obtained either the invertibility of $[I-\mathscr{T}(T)]$ exists or the invertibility of $[I-\mathscr{T}(T)]$ does not exist. An example is given at last to demonstrate the application of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let X and Y be two Banach spaces and let $L_b(X,Y)$ denote the space of bounded linear operators from X to Y. Suppose $T>0$, let $J=[0,T]$. Let us put $M=\sup_{t\in J}\|S(t)\|_{L_b(X,X)}$, which is a finite number. We denote $C(J,X)$ to be the Banach space of all continuous functions from J into X with the norm $\|x\|_C=\sup\{\|x(t)\|:t\in$

$J\}$. For measurable functions $l : J \to R$, define the norm $\|l\|_{L^p(J,R)} = \Big(\int_J |l(t)|^p dt\Big)^{\frac{1}{p}}$, $1 \leq p < \infty$. We denote $L^p(J,R)$ the Banach space of all Lebesgue measurable functions l with $\|l\|_{L^p(J,R)} < \infty$.

Let us recall the following known definitions.

Definition 2.1. *([8]) The fractional integral of order γ with the lower limit zero for a function f is defined as*

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \ t > 0, \ \gamma > 0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. *([8]) The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0,\infty) \to R$ can be written as*

$${}^L D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \ t > 0, \ n-1 < \gamma < n.$$

Definition 2.3. *([8]) The Caputo derivative of order γ for a function $f : [0,\infty) \to R$ can be written as*

$${}^c D^\gamma f(t) = {}^L D^\gamma \Big(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \Big), \ t > 0, \ n-1 < \gamma < n.$$

Remark 2.4. *(i) If $f(t) \in C^n[0,\infty)$, then*

$${}^c D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I^{n-\gamma} f^{(n)}(t), \ t > 0, \ n-1 < \gamma < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Theorem 2.5. *([39]) Let X be a reflexive Banach space, $\{S(t), t \geq 0\}$ be a strongly continuous semigroup on X with A as its infinitesimal generator. Then the adjoint semigroup $\{S^*(t), t \geq 0\}$ is a strongly continuous semigroup on X^* and its generator is just A^*.*

3. Homogeneous Periodic BVP

In this section, we consider the homogeneous linear periodic BVP (2).

We make the following assumption.

[HA]: $A : D(A) \to X$ is the infinitesimal generator of a strongly continuous semigroup $\{S(t), t \geq 0\}$.

We introduce the following definition of a mild solution for our problem.

Definition 3.1. *By the mild solution of the periodic BVP (2), we mean the function $x \in C(J, X)$ satisfying*

$$x(t) = \mathscr{T}(t)x(0), \ t \in (0, T], \ \textit{and } x(0) = x(T),$$

where

$$\mathscr{T}(t) = \int_0^\infty \xi_q(\theta) S(t^q\theta) d\theta, \ \xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\varpi_q(\theta^{-\frac{1}{q}}) \geq 0,$$

$$\varpi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0, \infty),$$

ξ_q is a probability density function defined on $(0, \infty)$, that is

$$\xi_q(\theta) \geq 0, \quad \theta \in (0, \infty) \quad \textit{and} \quad \int_0^\infty \xi_q(\theta) d\theta = 1.$$

The following results will be used throughout this paper.

Lemma 3.2. *(Lemma 2.9, [29]) The operator $\mathscr{T}$ has the following properties:*

(i) For any fixed $t \geq 0$, $\mathscr{T}(t)$ is a linear and bounded operator, i.e., for any $x \in X$,

$$\|\mathscr{T}(t)x\| \leq M\|x\|.$$

(ii) $\{\mathscr{T}(t), t \geq 0\}$ is also strongly continuous.

(iii) For every $t > 0$, $\mathscr{T}(t)$ is also a compact operator if $S(t)$ is compact.

Theorem 3.3. *Assume that [HA] holds and $\{S(t), t \geq 0\}$ is a compact semigroup in X. Then either the homogeneous linear periodic BVP (2) has a unique trivial mild solution or it has finitely many linearly independent nontrivial mild solutions in $C(J, X)$.*

Proof. The periodic BVP (2) has a mild solution x if and only if $\mathscr{T}(T)$ has a fixed point. In fact, if the periodic BVP (2) has a mild solution x, then we have $x(T) = \mathscr{T}(T)x(0) = x(0)$. Thus, $\bar{x} := x(0)$ is a fixed point of $\mathscr{T}(T)$. On the other hand, if $\bar{x} = x(0)$ is a fixed point of $\mathscr{T}(T)$, the mild solution of the Cauchy problem

$$\begin{cases} {}^cD^q x(t) = Ax(t), \ t \in J, \ q \in (0, 1), \\ x(0) = \bar{x}, \end{cases} \tag{5}$$

is given by $x(t) = \mathscr{T}(t)\bar{x}$, which implies that $x(T) = \mathscr{T}(T)\bar{x} = x(0)$. This yields that x is just the mild solution of the periodic BVP (2).

By (iii) of Lemma 3.2, $\mathscr{T}(T)$ is a compact operator. By the Fredholm alternative theorem, either (i) $\mathscr{T}(T)x(0) = x(0)$ only has trivial mild solution and $[I - \mathscr{T}(T)]^{-1}$ exists, or (ii) $\mathscr{T}(T)x(0) = x(0)$ has nontrivial mild solutions which form a finite dimensional subspace of X. This implies that every mild solution of the periodic BVP (2) can be written as $x(t) = \sum_{i=1}^{m} \alpha_i \mathscr{T}(t)x(0)^i$ where m is finite and $\alpha_1, \alpha_2, \cdots, \alpha_m$ are constants. $\square$

Remark 3.4. *(i) If $\|S(t)\| < 1$ for $t \in (0, T]$, then $\mathscr{T}(nT) \to 0$ as $n \to \infty$ and the operator $I - \mathscr{T}(T)$ is invertible and $[I - \mathscr{T}(T)]^{-1} \in L_b(X)$. (ii) If $\|\mathscr{T}(T)\| < 1$, then the operator $I - \mathscr{T}(T)$ is invertible and $[I - \mathscr{T}(T)]^{-1} \in L_b(X)$.*

4. Nonhomogeneous Periodic BVP

In this section, we consider the nonhomogeneous linear periodic BVP (1). For that, we need the following assumption.

[HF]: Input $f : J \to X$ is measurable for $t \in J$ and there exists a constant $q_1 \in (0, q)$ and real-valued function $h(\cdot) \in L^{\frac{1}{q_1}}(J, R)$ such that $\|f(t)\| \leq h(t)$, for each $t \in J$.

We use the following definition of a mild solution for our problem.

Definition 4.1. *By the mild solution of the periodic BVP (1), we mean the function $x \in C(J, X)$ satisfying*

$$x(t) = \mathscr{T}(t)x(0) + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)f(s)ds, \ t \in (0, T], \ \text{and } x(0) = x(T),$$

where

$$\mathscr{S}(t) = q\int_0^{\infty} \theta\xi_q(\theta)S(t^q\theta)d\theta,$$

while $\mathscr{T}(t)$, ξ_q and $\varpi_q(\theta)$ are as in Definition 3.1.

We collect the properties of operator $\mathscr{S}$ as follows.

Lemma 4.2. *(Lemma 2.9, [29]) The operator $\mathscr{S}$ has the following properties:*

(i) For any fixed $t \geq 0$, $\mathscr{S}(t)$ is a linear and bounded operator, i.e., for any $x \in X$,

$$\|\mathscr{S}(t)x\| \leq \frac{M}{\Gamma(q)}\|x\|.$$

(ii) $\{\mathscr{S}(t), t \geq 0\}$ is also strongly continuous.

(iii) For every $t > 0$, $\mathscr{S}(t)$ is a compact operator if $S(t)$ is compact.

Now we are ready to state and prove the main result in this paper.

Theorem 4.3. *Let [HA], [HF] hold and $\{S(t), t \geq 0\}$ be a compact semigroup in X. If the periodic BVP (2) has no non-trivial mild solutions, then the periodic BVP (1) has a unique mild solution given by*

$$x_T(t) = \mathscr{T}(t)[I - \mathscr{T}(T)]^{-1}z + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)f(s)ds \tag{6}$$

where

$$z = \int_0^T (T-s)^{q-1}\mathscr{S}(T-s)f(s)ds. \tag{7}$$

Further, we have the estimate

$$\|x_T(t)\| \leq \frac{M\,(ML_1+1)}{\Gamma(q)} \frac{T^{q-q_1}H}{(\frac{q-q_1}{1-q_1})^{1-q_1}}, \tag{8}$$

where $L_1 = \|[I - \mathscr{T}(T)]^{-1}\|$ and $H = \|h\|_{L^{\frac{1}{q_1}}(J,R)}$.

Proof. By the Fredholm alternative theorem, $[I - \mathscr{T}(T)]^{-1}$ exists and is bounded. Since the periodic BVP (2) has no non-trivial mild solutions, the operator equation $[I - \mathscr{T}(T)]x(0) = z$ has an unique solution $x(0) = [I - \mathscr{T}(T)]^{-1}z := \bar{x}$. Now we consider the Cauchy problem

$$\begin{cases} {}^cD^q x(t) = Ax(t) + f(t), \ t \in J, \ q \in (0,1), \\ x(0) = [I - \mathscr{T}(T)]^{-1}z := \bar{x}. \end{cases} \tag{9}$$

One can verify that the mild solution $x_T(\cdot)$ of (9) corresponding to initial value $\bar{x}$ is just the mild solution of the periodic BVP (1). Moreover, the uniqueness of the mild solution is due to the Fredholm alternative theorem again.

For the estimation of $x_T(\cdot)$, by Lemma 3.2 and Lemma 4.2,

$$\begin{aligned}
\|x_T(t)\| &\leq \left\|\mathscr{T}(t)[I - \mathscr{T}(T)]^{-1}z\right\| + \int_0^t (t-s)^{q-1}\|\mathscr{S}(t-s)f(s)\|ds \\
&\leq M\left\|[I - \mathscr{T}(T)]^{-1}\right\|\|z\| + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds \\
&\leq M\left\|[I - \mathscr{T}(T)]^{-1}\right\|\int_0^T (T-s)^{q-1}\|\mathscr{S}(T-s)f(s)\|ds \\
&\quad + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds \\
&\leq \frac{M^2\|[I - \mathscr{T}(T)]^{-1}\|}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds \\
&\quad + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{M^2\left\|[I-\mathscr{T}(T)]^{-1}\right\|}{\Gamma(q)}\left(\int_0^T (T-s)^{\frac{q-1}{1-q_1}}ds\right)^{1-q_1}\left(\int_0^T h(s)^{\frac{1}{q_1}}ds\right)^{q_1}\\
&\quad+\frac{M}{\Gamma(q)}\left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}}ds\right)^{1-q_1}\left(\int_0^t h(s)^{\frac{1}{q_1}}ds\right)^{q_1}\\
&\leq \frac{M\left(ML_1+1\right)}{\Gamma(q)}\frac{T^{q-q_1}H}{(\frac{q-q_1}{1-q_1})^{1-q_1}},
\end{aligned}
$$

where $L_1=\|[I-\mathscr{T}(T)]^{-1}\|$ and $H=\|h\|_{L^{\frac{1}{q_1}}(J,R)}$. This completes the proof. □

Remark 4.4. *By Remark 3.4, we can replace the assumption of $\{S(t),t\geq 0\}$ being compact by $\|S(t)\|<1$ for $t\in(0,T]$ or $\|\mathscr{T}(T)\|<1$ directly. It is obvious that all the results in Theorem 4.3 also hold.*

Let X be a Hilbert space. In case $[I-\mathscr{T}(T)]^{-1}$ does not exist, we need to consider the adjoint problem of BVP (1):

$$
\begin{cases}
{}^cD^q y(t)=-A^*y(t),\ t\in J,\\
y(T)=y(0)\in X^*,
\end{cases}
\tag{10}
$$

where A^* is the adjoint operator of A respectively. Due to reflexivity of X and Lemma 2.5, A^* is the infinitesimal generator of a strongly semigroup $\{S^*(t),t\geq 0\}$ in X^*.

Before we prove the main theorems in this section, we need the following results.

Lemma 4.5. *(Lemma 2.11, [32]) In the Hilbert space X, we have the following results.*

(i) For any fixed $t\geq 0$, $\mathscr{T}^*(t)$ and $\mathscr{S}^*(t)$ are linear and bounded operators.

(ii) $\{\mathscr{T}^*(t),t\geq 0\}$ and $\{\mathscr{S}^*(t),t\geq 0\}$ are strongly continuous.

(iii) For every $t>0$, $\mathscr{T}^*(t)$ and $\mathscr{S}^*(t)$ are compact operators, where

$$
\mathscr{T}^*(t)=\int_0^\infty \xi_q(\theta)T^*(t^q\theta)d\theta,\quad \mathscr{S}^*(t)=q\int_0^\infty \theta\xi_q(\theta)T^*(t^q\theta)d\theta.
$$

Now, we are ready to prove our main results in this section.

Theorem 4.6. *Let [HA], [HF] hold and $\{S(t),t\geq 0\}$ be a compact semigroup in a Hilbert space X. Suppose $[I-\mathscr{T}(T)]^{-1}$ does not exist. Then the adjoint equation (10) has m linearly independent mild solutions $y^1,y^2,\cdots,y^m$.*

Proof. Since the operators $\mathscr{T}(T)$ and $\mathscr{T}^*(T)$ are compact, $\mathscr{T}(T)y(0)=y(0)$ has m nontrivial mild solutions $y^i(0)$ $(i=1,2,\cdots,m)$, which form a finite dimensional subspace of X. It comes from dim ker$[I-\mathscr{T}(T)]=$ dim ker$[I-\mathscr{T}^*(T)]=m<+\infty$

that $[I - \mathscr{T}^*(T)]y(0) = 0$ has m nontrivial mild solutions $y^i(0)$ $(i = 1, 2, \cdots, m)$, that is, $\mathscr{T}^*(T)y^i(0) = y^i(0)$ $(i = 1, 2, \cdots, m)$. Consider the Cauchy problem

$$\begin{cases} {}^cD^q y(t) = -A^* y(t), \ t \in J, \\ y(T) = y^i(0). \end{cases} \tag{11}$$

It is easy to see that the mild solutions of Cauchy problem (11) are just the mild solutions of the periodic BVP (10). This implies that every mild solution of the periodic BVP (10) can be written as $y(t) = \sum_{i=1}^m \alpha_i \mathscr{T}(t) y(0)^i$ where m is finite and $\alpha_1, \alpha_2, \cdots, \alpha_m$ are constants. This completes the proof. □

Theorem 4.7. *Let [HA], [HF] hold and $\{S(t), t \geq 0\}$ be a compact semigroup in a Hilbert space X. Suppose $[I - \mathscr{T}(T)]^{-1}$ does not exist. Then the periodic BVP (1) has a mild solution if and only if*

$$\langle z, y^i(0) \rangle_{X^*, X} = 0, \ i = 1, 2, \cdots, m$$

which is equivalent to

$$\int_0^T \langle (T-\theta)^{q-1} \mathscr{S}(T-\theta) f(\theta), y^i(0) \rangle_{X, X^*} d\theta = 0.$$

Otherwise, the periodic BVP (1) has no mild solution.

Further, every mild solution of the periodic BVP (1) can be given by

$$x(t) = x_T(t) + \sum_{i=1}^m \alpha_i x^i(t)$$

where $t \in J$, x_T is a mild solution of the periodic BVP (1), $x^1, x^2, \cdots, x^m$ are m linearly independent mild solutions of the periodic BVP (2), and $\alpha_1, \cdots, \alpha_m$ are constants.

Proof. From compactness of $\mathscr{T}(T)$, $\mathscr{T}^*(T)$ is compact and dim ker$[I - \mathscr{T}(T)] =$ dim ker$[I - \mathscr{T}(T)] = m < +\infty$. The operator equation $[I - \mathscr{T}(T)]y(0) = 0$ have m nontrivial linearly independent solutions $\{y^i(0)\}_{i=1}^m$. Let y^i be the solution of the periodic BVP (10) corresponding to $y^i(0)$ $(i = 1, 2, \cdots, m)$ which is just a mild solution of the Cauchy problem (11). It is well known that the operator equation

$$[I - \mathscr{T}(T)]\bar{x} = z$$

has a solution if and only if

$$\langle z, y^i(0) \rangle_{X^*, X} = 0, i = 1, 2, \cdots, m,$$

which is equivalent to

$$\begin{aligned} 0 &= \langle z, y^i(0) \rangle_{X, X^*} \\ &= \int_0^T \langle (T-\theta)^{q-1} \mathscr{S}(T-\theta) f(\theta), y^i(0) \rangle_{X, X^*} d\theta. \end{aligned}$$

This completes the proof. □

5. Parameter Perturbation Methods for Robustness

Define

$$C_T(J, X) = \{x \in C(J, X) : x(0) = x(T)\}$$

with $\|x\|_{C_T} = \sup\{\|x(t)\| : t \in J\}$ for $x \in C_T(J, X)$. It can be seen that endowed with the norm $\|\cdot\|_{C_T}$, $C_T(J, X)$ is a Banach space.

Denote

$$\begin{aligned} \mathcal{S}_\rho &= \{x \in C_T(J, X) : \|x\|_{C_T} < \rho\}, \\ \mathcal{B}(x, \rho_1) &= \{x \in C_T(J; X) : \|x - x_T\|_{C_T} \le \rho_1\}, \end{aligned}$$

where

$$\begin{aligned} \rho &= \frac{M\,(ML_1 + 1)}{\Gamma(q)} \left[\frac{T^{q-q_1} H}{(\frac{q-q_1}{1-q_1})^{1-q_1}} + \frac{T^q}{q} \sup_{|\xi| \le \tilde{\xi}} \chi(\xi) \right], \\ \rho_1 &= \frac{M\,(ML_1 + 1)\,T^q}{\Gamma(1+q)} \sup_{|\xi| \le \tilde{\xi}} \chi(\xi), \end{aligned}$$

and χ is a nonnegative function.

We introduce assumption [HP]:

[HP1]: $p : J \times \mathcal{S}_\rho \times \Lambda \to X$ is measurable in t.

[HP2]: There exists a nonnegative function ϖ such that $\lim_{\xi \to 0} \varpi(\xi) = \varpi(0) = 0$ and for any $t \in J$, x, $y \in \mathcal{S}_\rho$, and $\xi \in \Lambda$, we have

$$\|p(t, x, \xi) - p(t, y, \xi)\| \le \varpi(\xi)\|x - y\|.$$

[HP3]: There exists a nonnegative function χ such that $\lim_{\xi \to 0} \chi(\xi) = \chi(0) = 0$ and for any $t \in J$, $x \in \mathcal{S}_\rho$, and $\xi \in \Lambda$, we have

$$\|p(t, x, \xi)\| \le \chi(\xi).$$

Now we introduce the mild solution of the periodic BVP (4).

Definition 5.1. *By the mild solution of the periodic BVP (4), we mean the function $x \in C(J, X)$ satisfying*

$$x(t) = \mathscr{T}(t)x(0) + \int_0^t (t-s)^{q-1} \mathscr{S}(t-s)[f(s) + p(s, x(s), \xi)]ds, \ t \in (0, T], \quad \text{and } x(0) = x(T).$$

The following result shows that given a periodic motion we can design periodic motion controllers that are robust whith respect to parameter drift.

Theorem 5.2. *Let [HA], [HF] and [HP] hold, $\{S(t), t \geq 0\}$ be a compact semigroup in X, and the periodic BVP (2) have no trivial mild solution. Then there is a $\xi_0 \in (0, \tilde{\xi})$ such that for $|\xi| \leq \xi_0$, the periodic BVP (4) has a unique mild solution x_T^{ξ} satisfying*

$$\left\| x_T^{\xi} - x_T \right\|_{C_T} \leq \rho_1 \quad \text{and} \quad \lim_{\xi \to 0} x_T^{\xi}(t) = x_T(t)$$

uniformly on $t \in J$ where x_T is the mild solution of the periodic BVP (1).

Proof. Let

$$x_0 = [I - \mathscr{T}(T)]^{-1} \Big[z + \int_0^T (T-s)^{q-1} \mathscr{S}(T-s) p(s, x(s), \xi) ds \Big] \in X$$

be fixed. Define the map $\mathcal{O}$ on $\mathcal{B}(x_T, \rho_1)$ by

$$(\mathcal{O}x)(t) \quad = \quad \mathscr{T}(t) x_0 + \int_0^t (t-s)^{q-1} \mathscr{S}(t-s)[f(s) + p(s, x(s), \xi)] ds.$$

It is not difficult to show that $\mathcal{O}x \in C_T(J, X)$.

By assumption [HP], we can choose a $\xi_0 \in (0, \tilde{\xi})$ such that

$$\frac{M\,(ML_1 + 1)\,T^q}{\Gamma(1+q)} \sup_{|\xi| \leq \xi_0} \chi(\xi) \leq \rho_1$$

and

$$\eta = \frac{M\,(ML_1 + 1)\,T^q}{\Gamma(1+q)} \sup_{|\xi| \leq \xi_0} \varpi(\xi) < 1.$$

For $\xi \in (-\xi_0, \xi_0)$ and x, $y \in \mathcal{B}(x_T, \rho_1)$, one can verify that

$$\|\mathcal{O}x - x_T\|_{C_T} \quad \leq \quad \frac{M\,(ML_1 + 1)\,T^q}{\Gamma(1+q)} \sup_{|\xi| \leq \xi_0} \chi(\xi) \leq \rho_1, \tag{12}$$

and

$$\|\mathcal{O}x - \mathcal{O}y\|_{C_T} \leq \eta \|x - y\|_{C_T}.$$

This implies that $\mathcal{O}$ is a contraction mapping on $\mathcal{B}(x_T, \rho_1)$. Then, $\mathcal{O}$ has a unique fixed point $x_T^{\xi} \in \mathcal{B}(x_T, \rho_1)$ given by

$$x_T^{\xi}(t) \quad = \quad \mathscr{T}(t) x_0 + \int_0^t (t-s)^{q-1} \mathscr{S}(t-s)[f(s) + p(s, x_T^{\xi}(s), \xi)] ds, \tag{13}$$

which is just the unique mild solution of the periodic BVP (4).

From the expression (12) and (13), one can get $\left\| x_T^{\xi} - x_T \right\|_{C_T} \leq \rho_1$. It is easy to see that $\lim_{\xi \to 0} x_T^{\xi}(t) = x_T(t)$ uniformly on $t \in J$. $\square$

6. Example

In this section, an example is given to illustrate our theory.

Consider the problem

$$\begin{cases} {}^cD^q x(t,y) = \frac{\partial^2}{\partial y^2}x(t,y) + |\sin(t,y)| + \xi|\cos x(t,y)|, \ y \in \Omega, \ 0 < t \leq 2\pi, \\ x(0,y) = x(2\pi,y), \ y \in \Omega, \\ x(t,y) = 0, \ y \in \partial\Omega, \ t > 0, \end{cases} \tag{14}$$

where $q \in (0,1)$, $\Omega = (0,\pi)$ and $\xi \in (-1,1)$.

Let $X = Y = L^2(0,\pi)$ and $A : D(A) \to X$ be defined by $Ax = x_{yy}$, $x \in D(A)$ where $D(A) = \{x \in X : x, x_y$ are absolutely continuous, $x(0) = x(\pi) = 0$ and $x_{yy} \in X\}$. Then $Ax = \sum_{n=1}^{\infty} n^2(x, x_n)x_n$, $x \in X$, where $x_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns)$, $n = 1, 2, 3, \cdots$ is the orthogonal set of eigenfunctions of A. It can be easily shown that A is the infinitesimal generator of a compact analytic semigroup $\{S(t), t \geq 0\}$ in X and is given by $S(t)x = \sum_{n=1}^{\infty} e^{-n^2t}(x, x_n)x_n$. So there exists a constant $M \geq 1$ such that $\|S(t)\| \leq M$. As a matter of fact it holds $\|S(t)\| \leq e^{-t}$ for any $t \geq 0$, so $M = 1$.

Let us set $J = [0, 2\pi]$, $x(\cdot)(y) = x(\cdot, y)$, $f(\cdot)(y) = |\sin(\cdot, y)|$ and $p(\cdot, x(\cdot, y), \xi) = \xi|\cos x(\cdot, y)|$.

Thus, problem (14) can be rewritten as

$$\begin{cases} {}^cD^q x(t) = Ax(t) + f(t) + p(t, x(t), \xi), \ t \in J, \ q \in (0,1), \\ x(0) = x(2\pi), \end{cases} \tag{15}$$

It satisfies all the assumptions given in Remark 3.4 (i) and Theorem 5.2, so our results can be applied to problem (14).

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