The algebraic curves of planar polynomial differential systems with homogeneous nonlinearities

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Abstract. We consider planar polynomial systems of ordinary differential equations of the form $\dot{x} = x + P_n(x,y)$, $\dot{y} = y + Q_n(x,y)$, where $P_n(x,y)$, $Q_n(x,y)$ are homogeneous polynomials of degree n. We study the algebraic and non-algebraic invariant curves of these systems with emphasis on limit cycles.

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1 Introduction

The problem of studying the limit cycles is one of the central problems in the theory of ordinary differential equations. A significant subarea in this area is the study of the limit cycles of autonomous planar polynomial systems

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y). \tag{1.1}$$

Here $P(x,y)$, $Q(x,y)$ are real polynomials of the variables x, y; $t \in \mathbb{R}$ acts as an independent variable. The degree of the system is the maximum of the degrees of the polynomials $P(x,y)$, $Q(x,y)$.

A limit cycle of system (1.1) is a periodic solution whose trajectory is isolated among the trajectories of all periodic solutions. A limit cycle of system (1.1) is called algebraic of degree m if it is the real oval of an irreducible algebraic curve $H(x,y) = 0$ of degree m.

The problems of finding algebraic solutions to polynomial systems, in particular, of algebraic cycles, goes back to H. Poincaré and J.-G. Darboux, and are actively developing at present (see [5] and the literature cited therein).

In the present article, we study these problems in application to a differential system of the form

$$\dot{x} = x + P_n(x,y), \qquad \dot{y} = y + Q_n(x,y), \tag{1.2}$$

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which we call a Darboux-type system. Here $P_n(x,y)$, $Q_n(x,y)$ are homogeneous real polynomials of degree n of the variables x, y. Systems of such a kind appeared in Darboux's works on geometry.

We study necessary and sufficient conditions for the existence of a hyperbolic limit cycle for system (1.2), and this cycle turns out to be unique. The remaining trajectories (except for the singular point at the origin) have a limit cycle as the α- or ω-limit set and cannot be algebraic curves.

We prove that the degree of an algebraic limit cycle of system (1.2) is equal to 2 and obtain necessary and sufficient conditions for the existence of an algebraic limit cycle for (1.2).

The obtained results are illustrated by examples.

2   The main part

Consider a Darboux-type system of a more general kind than (1.2):

$$\dot{x} = sx + P_n(x,y), \qquad \dot{y} = sy + Q_n(x,y), \qquad s \neq 0, \ n > 1. \tag{2.1}$$

Consider the functions

$$f(\vartheta) = \cos\vartheta P_n(\cos\vartheta, \sin\vartheta) + \sin\vartheta Q_n(\cos\vartheta, \sin\vartheta),$$
$$g(\vartheta) = \cos\vartheta Q_n(\cos\vartheta, \sin\vartheta) - \sin\vartheta P_n(\cos\vartheta, \sin\vartheta).$$

Theorem 2.1.

(1) If n is even then system (2.1) has no periodic solutions.

(2) If n is odd and $g(\vartheta)$ has zeros on $[0, 2\pi]$ then system (2.1) has no periodic solutions.

(3) If there is a closed trajectory of system (2.1), then it contains inside itself the only singular point that coincides with the origin.

(4) System (2.1) has at most one limit cycle.

(5) For system (2.1) to have a unique limit cycle Γ, it is necessary and sufficient that the following conditions hold:

$$g(\vartheta) \neq 0, \qquad \vartheta \in [0, 2\pi]; \qquad sg(0) \int_0^{2\pi} \frac{f(\vartheta)}{g(\vartheta)} d\vartheta < 0. \tag{2.2}$$

(6) The cycle Γ is hyperbolic.

(7) If the cycle Γ is algebraic then its degree is equal to 2 and it is defined by an algebraic curve $H(x,y) = 0$, $H(x,y) = 1 + ax^2 + 2bxy + cy^2$.

Proof. Items (1)–(5) are proved in [1,7].

We will briefly give fragments of this proof in order to use them for proving items (6)–(7), which supplement the results of [1,7] concerning Darboux-type systems.

Henceforth, unless otherwise specified, we assume that n is odd and conditions (4) are fulfilled.

After passing to the polar coordinates $x=r\cos\vartheta, y=r\sin\vartheta$, system (2.1) turns into the system

$$\dot{r} = sr + r^n f(\vartheta), \qquad \dot{\vartheta} = r^{n-1} g(\vartheta), \tag{2.3}$$

which we replace by the linear equation

$$\frac{d\rho}{d\vartheta} = \frac{(n-1)f(\vartheta)}{g(\vartheta)}\rho + \frac{s(n-1)}{g(\vartheta)}, \ \rho = r^{n-1}, \tag{2.4}$$

where the functions $f(\vartheta)$, $g(\vartheta)$ are defined above.

The periodic solutions to this equation (with period $T = 2\pi$) generate periodic solutions to system (2.1).

Introduce the function

$$F(\vartheta) = \exp\left((n-1)\int_0^\vartheta \frac{f(\tau)}{g(\tau)}d\tau\right).$$

Denote the solution to equation (2.4) with $\rho(0) = \rho_0$ by $\rho(\vartheta; \rho_0)$:

$$\rho(\vartheta; \rho_0) = \left(\rho_0 + s(n-1)\int_0^\vartheta \frac{d\tau}{g(\tau)F(\tau)}\right)F(\vartheta). \tag{2.5}$$

For a periodic solution, we must take $\rho_0 = \rho_0^*$ from the condition $\rho(2\pi; \rho_0) = \rho_0$. In the case under consideration, there exists a unique such value:

$$\rho_0^* = s(n-1)\frac{F(2\pi)}{1 - F(2\pi)}\int_0^{2\pi} \frac{d\tau}{g(\tau)F(\tau)}. \tag{2.6}$$

For a solution $\rho = \rho(\vartheta, \rho_0^*)$ to define a periodic solution to (2.1), the condition $\rho(\vartheta, \rho_0^*) > 0$, $\vartheta \in [0, 2\pi]$ must be fulfilled. This condition holds by (2.2).

The orbits of system (2.1) have the parametric definition

$$x = \sqrt[n-1]{\rho(\vartheta; \rho_0)}\cos\vartheta, \qquad y = \sqrt[n-1]{\rho(\vartheta; \rho_0)}\sin\vartheta, \qquad \vartheta \in [0, 2\pi], \tag{2.7}$$

where $\rho(\vartheta; \rho_0)$ is from (7).

For a periodic orbit, we must take $\rho_0 = \rho_0^*$ in (2.6).

The cycle under consideration is hyperbolic. For showing this, calculate the derivative of the solution (7) with respect to ρ_0 for $\vartheta = 2\pi$ at the point $\rho_0 = \rho_0^*$.

$$\mu = \left.\frac{\partial\rho(\vartheta; \rho_0)}{\partial\rho_0}\right|_{\{\vartheta=2\pi, \rho_0=\rho_0^*\}} = F(2\pi).$$

By (2.2), we have that $\mu \neq 1$, i.e., the cycle Γ is hyperbolic.

Item (6) is proved.

For proving item (7), make use of another trick, proposed in [4].

The direction field of system (2.1) is symmetric with respect to the origin. In this case, the trajectories of this system and the formulas defining them must also possess the symmetry property. In particular, closed algebraic curves are defined by polynomials of the form $H(x, y) = h_0 + h_2(x, y) + h_4(x, y) + \ldots$, where $h_0 = \text{const} \neq 0$, $h_2(x, y)$, $h_4(x, y), \ldots$ are homogeneous polynomials of even degrees. Without loss of generality, we may assume that $h_0 = 1$.

If the limit cycle of system (2.1) is algebraic then consider an irreducible polynomial $H(x, y) = 1 + h_2(x, y) + \cdots + h_{2k}(x, y)$ such that $H(x, y) = 0$ contains the oval defined by (2.7)

with $\rho_0 = \rho_0^*$. $H(x,y) = 0$ is an invariant algebraic curve of (2.1), [4]. Then $H(r\cos\vartheta, r\sin\vartheta) = 0$ is an invariant curve of system (2.3). After inserting $R = r^2$ in $H(r\cos\vartheta, r\sin\vartheta)$, we obtain a polynomial $\tilde{H}(R,\vartheta)$ of the variable R whose coefficients are $h_i(\cos\vartheta, \sin\vartheta)$.

Note that the polynomial $\tilde{H}(R,\vartheta)$ has only positive roots. Moreover, $R = (\rho(\vartheta;\rho_0^*))^{\frac{2}{n-1}}$ is its R-root. Each R-root of $\tilde{H}(R,\vartheta)$ generates a solution $\rho(\vartheta;\rho_0) = R^{\frac{n-1}{2}}$ to (2.4). If $\rho_0 \neq \rho_0^*$ then the solution $\rho(\vartheta;\rho_0)$ takes infinitely many different values at the points $\vartheta = 2\pi k$ for all integers k. On the other hand, $\tilde{H}(R, 2\pi k) = \tilde{H}(R, 0)$ for all integers k; therefore, these polynomials have the same roots. Hence, the polynomial $\tilde{H}(R, 0)$ has infinitely many roots $R = (\rho(2\pi k;\rho_0))^{\frac{2}{n-1}}$, which is impossible. We see that the R-root necessarily corresponds to a unique limit cycle system $R^{(n-1)/2} = \rho(\vartheta;\rho_0^*)$. Since the polynomial $\tilde{H}$ has only positive roots, we conclude that $\tilde{H}(R,\vartheta)$ has one and only one R-root and, thus, it takes the form $\tilde{H}(R,\vartheta) = 1 + h_2(\cos\vartheta, \sin\vartheta)R$. Then $H(r\cos\vartheta, r\sin\vartheta) = 1 + r^2 h_2(\cos\vartheta, \sin\vartheta)$, which implies that $H(x,y) = 1 + ax^2 + 2bxy + cy^2$.

Item (7) is proved.   □

Theorem 2.2. *System* (2.1) *has an algebraic limit cycle* $H(x,y) \equiv 1 + h_2(x,y) = 0$, $h_2(x,y) < 0$ *if and only if the conditions*

$$P_n\frac{\partial h_2}{\partial x} + Q_n\frac{\partial h_2}{\partial y} = 2s(-h_2)^{\frac{n+1}{2}}; \qquad xP_n - yQ_n \neq 0 \quad \text{for } (x,y) \neq (0,0)$$

are satisfied.

Proof. Recall that if the trajectory of a planar polynomial system of differential equations

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y) \tag{1.1}$$

is a part of an irreducible algebraic curve $H(x,y) = 0$ then there exists a polynomial $k(x,y)$ (cofactor) such that

$$\frac{\partial H(x,y)}{\partial x}P(x,y) + \frac{\partial H(x,y)}{\partial y}Q(x,y) = k(x,y)H(x,y). \tag{2.8}$$

Obviously, the degree of a cofactor is at most $n-1$ if n is the maximum of the degrees of the polynomials $P(x,y)$, $Q(x,y)$.

As follows from Theorem 2.1, for system (2.1), the closed algebraic curve (an algebraic limit cycle) is defined by the polynomial of the form $H(x,y) = 1 + h_2(x,y)$, where $h_2(x,y)$ is a homogeneous polynomial of degree 2.

Condition (2.8) takes the form

$$(sx + P_n)\frac{\partial}{\partial x}(1 + h_2) + (sy + Q_n)\frac{\partial}{\partial y}(1 + h_2) = k(1 + h_2).$$

Setting $n = 2m + 1$, we have $k = k_2 + k_4 + \cdots + k_{2m}$, and

$$(sx + P_{2m+1})\frac{\partial h_2}{\partial x} + (sy + Q_{2m+1})\frac{\partial h_2}{\partial y}$$
$$= k_2 + (h_2k_2 + k_4) + (h_2k_4 + k_6) + \cdots + (h_2k_{2m-2} + k_{2m}) + h_2k_{2m}.$$

We get

$$sx\frac{\partial h_2}{\partial x} + sy\frac{\partial h_2}{\partial y} = k_2,$$

$$h_2 k_2 + k_4 = 0,$$

$$h_2 k_4 + k_6 = 0,$$

$$\vdots$$

$$h_2 k_{2m-2} + k_{2m} = 0,$$

$$P_{2m+1}\frac{\partial h_2}{\partial x} + Q_{2m+1}\frac{\partial h_2}{\partial y} = h_2 k_{2m}.$$

Using Euler's formula for the homogeneous polynomial $h_2(x, y)$, we have

$$2sh_2 = k_2.$$

Further, we put

$$k_4 = -h_2 k_2 = -2sh_2^2, \quad k_6 = -h_2 k_4 = 2sh_2^3, \quad \ldots, \quad k_{2m} = -h_2 k_{2m-2} = (-1)^{m+1}2sh_2^m.$$

Based on these equalities, we have

$$P_{2m+1}\frac{\partial h_2}{\partial x} + Q_{2m+1}\frac{\partial h_2}{\partial y} = 2s(-1)^{m+1}h^{2m+1}.$$

The last equality proves the invariance of the curve $H(x, y) = 0$.
There are no singular points on the curve.
The theorem is proved. □

In the case of a cubic Darboux system ($n = 3$), we can give an exhaustive solution the problem under consideration.

In [2], Theorem 3.2 was proved, which classifies planar homogeneous cubic vector fields. In our case, this theorem has the following consequence:

Proposition 2.3. *The system*

$$\dot{x} = x + P_3(x, y), \qquad \dot{y} = y + Q_3(x, y) \tag{2.9}$$

has a limit cycle only if there exists a linear transformation $\sigma \in GL(2; \mathbb{R})$ and a time scaling taking system (2.9) *into the system of the form*

$$\dot{x} = sx + p_1 x^3 + (p_2 - \alpha)x^2 y + p_3 xy^2 - \alpha y^3 \equiv sx + \tilde{P}_3(x, y),$$
$$\dot{y} = sy + \alpha x^3 + p_1 x^2 y + (p_2 + \alpha)xy^2 + p_3 y^3 \equiv sy + \tilde{Q}_3(x, y), \tag{2.10}$$
$$p_1, \ p_2, \ p_3, \ s \in \mathbb{R}, \quad \alpha = \pm 1.$$

For system (2.10),

$$f(\vartheta) = \frac{1}{2}(p_1 + p_3 + (p_1 - p_3)\cos 2\vartheta + p_2 \sin 2\vartheta), \qquad g(\vartheta) \equiv \alpha,$$

$$sg(0)\int_0^{2\pi}\frac{f(\vartheta)}{g(\vartheta)}d\vartheta = \pi s(p_1 + p_3).$$

By Theorem 2.1, we infer that, for the existence of a unique hyperbolic limit cycle for system (2.10), it is necessary and sufficient that

$$s(p_1 + p_3) < 0. \tag{2.11}$$

Consider the question of the existence of an algebraic limit cycle for system (2.10). Suppose that system (2.10) has a quadratic limit cycle $H = 1 + h_2 \equiv 1 + ax^2 + 2bxy + cy^2$. Condition (2.8) takes the form

$$\frac{\partial H}{\partial x}(sx + \tilde{P}_3) + \frac{\partial H}{\partial y}(sy + \tilde{Q}_3) = k_2 H.$$

From Theorem 2.2 we obtain

$$\tilde{P}_3 \frac{\partial h_2}{\partial x} + \tilde{Q}_3 \frac{\partial h_2}{\partial y} = 2sh_2^2. \tag{2.12}$$

Equating the coefficients at the same degrees of the variables x, y on the left- and right-hand sides of (2.12), after easy transformations, we obtain the system of equalities

$$\begin{aligned}
-a^2 s + ap_1 + \alpha b &= 0, \\
-4abs + ap_2 - \alpha a + 2bp_1 + \alpha c &= 0, \\
-2acs + ap_3 - 4b^2 s + 2bp_2 + cp_1 &= 0, \\
-\alpha a - 4bcs + 2bp_3 + cp_2 + \alpha c &= 0, \\
-\alpha b - c^2 s + cp_3 &= 0.
\end{aligned} \tag{2.13}$$

Here and below, we have used the *Mathematica* system for implementing symbols and numerical calculations.

System (2.13) can be regarded as an inhomogeneous system of linear equations $AX = B$ with respect to the variables p_1, p_2, p_3 with the parameters a, b, c, s, α

$$A = \begin{pmatrix} a & 0 & 0 \\ 2b & a & 0 \\ c & 2b & a \\ 0 & c & 2b \\ 0 & 0 & c \end{pmatrix}, \qquad B = \begin{pmatrix} a^2 s - \alpha b \\ \alpha(a - c) + 4abs \\ 2acs + 4b^2 s \\ \alpha(a - c) + 4bcs \\ \alpha b + c^2 s \end{pmatrix}, \qquad X = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

The system is solvable if and only if the rank of the matrix A is equal to the rank of the extended matrix $(A|B)$.

Obviously, $a \neq 0$ in (2.13). Therefore, rank $A = 3$. Then all minors of order 4 in $(A|B)$ must be zero. These minors are

$$\begin{aligned}
(A|B)_1 &= \alpha(a^2 c^2 - 8ab^2 c - 2ac^3 + 8b^4 + 4b^2 c^2 + c^4), \\
(A|B)_2 &= \alpha(-3a^2 bc + 4ab^3 + 2abc^2 + bc^3), \\
(A|B)_3 &= \alpha(-a^3 c + 2a^2 b^2 + 2a^2 c^2 - ac^3 + 2b^2 c^2), \\
(A|B)_4 &= \alpha(a^3 b + 2a^2 bc - 3abc^2 + 4b^3 c), \\
(A|B)_5 &= \alpha(a^4 - 2a^3 c + 4a^2 b^2 + a^2 c^2 - 8ab^2 c + 8b^4),
\end{aligned}$$

where $(A|B)_i$ stands for the minor obtained from $(A|B)$ by deleting the ith row. Note that the obtained expressions do not depend on s and contain α as a factor.

Since $\alpha = \pm 1$, it suffices to consider the system of homogeneous equations

$$a^2c^2 - 8ab^2c - 2ac^3 + 8b^4 + 4b^2c^2 + c^4 = 0,$$
$$-3a^2bc + 4ab^3 + 2abc^2 + bc^3 = 0,$$
$$-a^3c + 2a^2b^2 + 2a^2c^2 - ac^3 + 2b^2c^2 = 0, \qquad (2.14)$$
$$a^3b + 2a^2bc - 3abc^2 + 4b^3c = 0,$$
$$a^4 - 2a^3c + 4a^2b^2 + a^2c^2 - 8ab^2c + 8b^4 = 0.$$

The second and fourth equations have the form

$$b(-3a^2c + 4ab^2 + 2ac^2 + c^3) = 0, \qquad b(a^3 + 2a^2c - 3ac^2 + 4b^2c) = 0. \qquad (2.15)$$

Case 1. $b = 0$.

In this case, system (2.14) is reduced to the system

$$c^2(a - c)^2 = ac(a - c)^2 = a^2(a - c)^2 = 0,$$

which implies that we have a nonzero solution $a = c \neq 0$, $b = 0$ to system (2.14).

Case 2.

If $b \neq 0$ then $-3a^2c + 4ab^2 + 2ac^2 + c^3 = a^3 + 2a^2c - 3ac^2 + 4b^2c = 0$, which implies that $(a - c)(a + c)^3 = 0$. If $a = c$ then the fourth equation of (2.14) gives $c = 0$, and hence the fifth equation yields $b = 0$, which contradicts the assumption $b \neq 0$. If $a = -c$ then the only real solution to (2.15) is $a = b = c = 0$, which again leads to a contradiction.

For the above-found values of the parameters a, b, c, system (2.13) is reduced to the form

$$ap_1 = a^2s, \quad ap_2 = 0, \quad ap_1 + ap_3 = 2a^2s, \quad ap_2 = 0, \quad ap_3 = a^2s, \quad a \neq 0,$$

and has a nonzero solution $p_1 = p_3 \neq 0$, $p_2 = 0$, $a = c = p_1/s$, $b = 0$.

Then $H(x, y) = 1 + \frac{p_1}{s}(x^2 + y^2)$, and the algebraic curve $H(x, y) = 0$ defines a real oval (circle) under the condition $p_1/s < 0$ (cf. (2.11)).

Thus, we have proved

Theorem 2.4. *System* (2.10) *admits a hyperbolic algebraic cycle if and only if*

$$p_1 = p_3, \qquad p_2 = 0, \qquad p_1s < 0. \qquad (2.16)$$

Moreover, the cycle is defined by the algebraic curve

$$H \equiv 1 + \frac{p_1}{s}(x^2 + y^2) = 0.$$

System (2.10) for which conditions (2.16) are fulfilled has the form

$$\dot{x} = sx + px^3 - \alpha x^2y + pxy^2 - \alpha y^3, \qquad \dot{y} = sy + \alpha x^3 + px^2y + \alpha xy^2 + py^3. \qquad (2.17)$$

Put $\delta = \sqrt{-s/p}$.

A straightforward check shows that

$$x(t) = \delta \cos \delta^2 t, \qquad y(t) = \alpha\delta \sin \delta^2 t$$

is a periodic solution to system (2.17). This solution is a suitable parametrization for the circle $H = 0$ mentioned in the theorem. The period of the obtained cycle is equal to $T = 2\pi/\delta^2$.

The cycle is stable if $s > 0$ and unstable if $s < 0$.

Using Therem 2.2 and Theorem 2.4, in [8], we proved

Theorem 2.5. *System* (2.10) *has an algebraic limit cycle if and only if the coefficients* p_{ij}, q_{ij}, $i, j = 0, 1, 2, 3$, $i + j = 3$ *are representable as*

$$p_{30} = -s\frac{(c^2 + d^2)\,(\alpha(ac + bd) + p(ad - bc))}{(bc - ad)^3},$$

$$p_{21} = s\frac{a^2(\alpha(3c^2 + d^2) + 2cdp) + 2ab(2\alpha cd + p(d^2 - c^2)) + b^2(\alpha(c^2 + 3d^2) - 2cdp)}{(bc - ad)^3},$$

$$p_{12} = -s\frac{(a^2 + b^2)(3\alpha ac + adp + 3\alpha bd - bcp)}{(bc - ad)^3},$$

$$p_{03} = s\frac{\alpha(a^2 + b^2)^2}{(bc - ad)^3}, \qquad q_{30} = -s\frac{\alpha\,(c^2 + d^2)^2}{(bc - ad)^3},$$

$$q_{21} = s\frac{(c^2 + d^2)(3\alpha(ac + bd) - p(ad - bc))}{(bc - ad)^3},$$

$$q_{12} = -s\frac{a^2(\alpha(3c^2 + d^2) - 2cdp) + 2ab(2\alpha cd + p(c^2 - d^2)) + b^2(\alpha(c^2 + 3d^2) + 2cdp)}{(bc - ad)^3},$$

$$q_{03} = s\frac{(a^2 + b^2)(\alpha(ac + bd) - p(ad - bc))}{(bc - ad)^3},$$

where $a, b, c, d, p, s \in \mathbb{R}$, $ad - bc \neq 0$, $ps < 0$, $\alpha = \pm 1$.

Moreover, the cycle is defined by the algebraic curve

$$H \equiv 1 + \frac{ps}{(ad - bc)^2}((ay - cx)^2 + (by - dx)^2).$$

Consider several examples illustrating the obtained results.

Example 2.6. ([1]).

$$\dot{x} = -x + x^3 - x^2y + xy^2 - y^3, \qquad \dot{y} = -y + x^3 + x^2y + xy^2 + y^3. \tag{2.18}$$

The system can be written down in the form (2.10) if we take $s = -1$, $p_1 = p_3 = 1$, $p_2 = 0$, $\alpha = 1$. Hence, by Theorem 2.4, system (2.18) has the hyperbolic algebraic limit cycle $1 - x^2 - y^2 = 0$. The cycle is unstable since it contains a stable singular point, the origin. This cycle was presented in [1].

Example 2.7.

$$\dot{x} = x - 2x^2y - 4xy^2 - 2y^3, \qquad \dot{y} = y + 2x^3 + 2xy^2 - 4y^3. \tag{2.19}$$

We have

$$f(\vartheta) = -4\sin^2\vartheta, \qquad g(\vartheta) \equiv 2, \qquad F(\vartheta) = e^{\sin 2\vartheta - 2\vartheta}.$$

By Theorem 2.1, we conclude that the system has a hyperbolic stable limit cycle. By (2.5), (2.6), the cycle is written as

$$r = \sqrt{\rho}, \qquad \rho = \left(\rho_0^* + \int_0^\vartheta e^{2\tau - \sin 2\tau}d\tau\right)e^{\sin 2\vartheta - 2\vartheta},$$

$$\rho_0^* = \frac{1}{e^{4\pi} - 1}\int_0^{2\pi} e^{\sin 2\tau - 2\tau}d\tau.$$

An attempt to find a quadratic limit cycle in the form $1 + ax^2 + 2bxy + cy^2 = 0$ leads to the system of equations

$$a^2 - 2b = 0, \quad a + 2ab - c = 0, \quad 2a + 2b^2 + ac = 0, \quad a + 4b - c + 2bc = 0, \quad 2b + 4c + c^2 = 0.$$

Express b, c in terms of a from the first two equations and insert them in the remaining three equations. After easy calculations, it is not hard to see that the system has only the zero solution $a = b = c = 0$. Hence, the limit cycle of system (2.19) is non-algebraic.

Example 2.8.

$$\dot{x} = x + (x - y)(x^2 + 2y^2)^2, \qquad \dot{y} = y - (x + y)(x^2 + 2y^2)^2.$$

We have

$$f(\vartheta) = g(\vartheta) = -\frac{1}{4}(\cos 2\vartheta - 3)^2, \qquad F(\vartheta) = e^{4\vartheta}.$$

By Theorem 2.1, we conclude that system has a hyperbolic stable limit cycle. The cycle is written as

$$r = \sqrt{\rho}, \qquad \rho = \left(\rho_0^* - 8 \int_0^\vartheta \frac{d\tau}{e^{4\tau}(\cos 2\tau - 3)^2}\right) e^{4\vartheta},$$

$$\rho_0^* = \frac{8e^{8\pi}}{e^{8\pi} - 1} \int_0^{2\pi} \frac{d\tau}{e^{4\tau}(\cos 2\tau - 3)^2}.$$

Just as in the previous example, it can be shown that the cycle is non-algebraic due to Theorem 2.2.

Using Example 2.6 and Theorem 2.2, we can construct a Darboux system of any (odd) degree with an algebraic limit cycle.

Example 2.9.

$$\dot{x} = -x + (x^3 - x^2y + xy^2 - y^3)(x^2 + y^2)^{2k}, \qquad \dot{y} = -y + (x^3 + x^2y + xy^2 + y^3)(x^2 + y^2)^{2k}.$$

The system has the algebraic limit cycle $H(x, y) \equiv 1 - x^2 - y^2 = 0$.

As we already observed, in the presence of a limit cycle, all the remaining trajectories of system (1.2) are non-algebraic curves. In a neighborhood of the cycle, these trajectories are spirals with infinitely many helices and intersect a straight line transversal to the cycle infinitely many times. In this case, the corresponding algebraic equation has infinitely many roots, which is impossible.

Example 2.6 shows that the coexistence of algebraic and non-algebraic curves is possible for cubic Darboux systems. Example 2.7 demonstrates that there exist cubic Darboux systems having no algebraic curve. With account taken of Theorem 2.2, this property is possessed by all Darboux systems (1.2) with $n \neq 3$ for which $g(\vartheta) \neq 0$, $\vartheta \in [0, 2\pi]$ (see Example 2.8).

It was proved in [1] that system (1.2) is Darboux integrable and a formula for its first integral was given.

System (1.2) has no polynomial integral (and, generally, no integral defined on the whole phase plane) since its singular point at the origin is a node.

Under certain conditions, system (1.2) has a rational integral. In this case, all its trajectories are algebraic curves.

Recall that if system (1.1) has $N = n(n + 1) + 2$ algebraic invariants then it admits a rational first integral (see [3]).

Example 2.10.

$$\dot{x} = x - x^3, \qquad \dot{y} = y - y^3. \tag{2.20}$$

The singular points of system (2.20) are:
$O(0,0), O_1(1,1), O_2(-1,1), O_3(-1,-1), O_4(1,-1)$ are star nodes;
$O_5(1,0), O_6(0,1), O_7(-1,0), O_8(0,-1)$ are hyperbolic saddles.

System (2.20) has 8 invariant straight lines $x=0$, $x=\pm 1$, $y=0$, $y=\pm 1$, $y=\pm x$, using which one can construct the rational Darboux integral

$$V(x,y) = \frac{y^2(1-x^2)}{x^2(1-y^2)}.$$

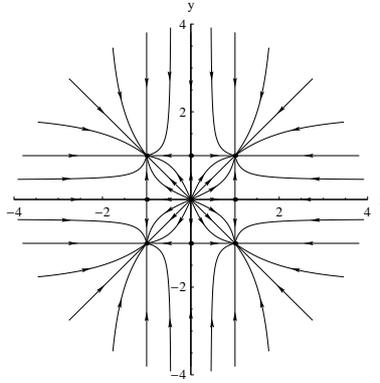The phase portrait of system (2.20) is given in Figure 2.1.

Figure 2.1: The phase portrait of system (2.20)

In some cases, a rational integral for system (1.2) can be found with the use of a smaller number of invariants than N.

Suppose that the homogeneous polynomials $P_n(x,y), Q_n(x,y)$ in system (1.2) satisfy the Cauchy–Riemann conditions: $P_{nx} = Q_{ny}$, $P_{ny} = -Q_{nx}$. Introducing the complex variable $z = x + iy$, we can write down this system in the form

$$\dot{z} = z + \mathcal{P}_n(z), \tag{2.21}$$

where $\mathcal{P}_n(z)$ is a complex polynomial of degree n : $\mathcal{P}_n(z) = P_n((z + \bar{z})/2, (z - \bar{z})/2i) + iQ_n((z + \bar{z})/2, (z - \bar{z})/2i)$.

Theorem 2.11 ([6])**.** *Suppose that all the singular points of system (2.21) are star nodes and all the eigenvalues $\lambda_{1,2}^k = \omega_k$ are rationally commensurable: $\omega_k \in \omega\mathbb{Q}$, $\omega \in \mathbb{R}$, $k = 1, 2, \ldots, n$. Then system (2.21) admits a rational first integral.*

In this case, the integral can be constructed with the use of n complex invariants of the form $f_k = z - z_k$, where z_k are the roots of the equation $z + \mathcal{P}_n(z) = 0$ (see [6]).

Example 2.12. Consider the system

$$\dot{x} = x - x^5 + 10x^3y^2 - 5x^4y, \qquad \dot{y} = y - 5x^4y - 10x^2y^3 - y^5, \tag{2.22}$$

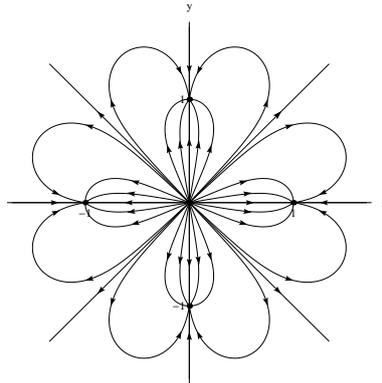which corresponds to the complex system $\dot{z} = z - z^5$.

Figure 2.2: The phase portrait of the system (2.22)

The singular points of system (2.22) are found from the relation $z - z^5 = 0$ and have the form $O(0,0)$, $O_1(1,0)$, $O_2(0,1)$, $O_3(-1,0)$, $O_4(0,-1)$. The origin is an unstable star node; the remaining four singular points are stable star nodes. The system has 5 invariants $f_1=z$, $f_{2,3}=z\pm1$, $f_{4,5}=z\pm i$, using which we can construct the rational first integral

$$V(x,y) = \frac{xy(x^2 - y^2)}{(x^2 + y^2)^4 - (x^2 + y^2)^2 + 8x^2y^2}.$$

Figure 2.2 contains a phase portrait of system (2.22).

Acknowledgments

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