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On existence and asymptotic behavior of solutions of elliptic equations with nearly critical exponent and singular coefficients

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Abstract. In this paper we study the existence and asymptotic behavior of solutions of

$$-\Delta u = \mu \frac{u}{|x|^2} + |x|^\alpha u^{p(\alpha)-1-\varepsilon}, \qquad u > 0 \text{ in } B_R(0)$$

with Dirichlet boundary condition. Here, $-2 < \alpha < 0$, $p(\alpha) = \frac{2(N+\alpha)}{N-2}$, $0 < \varepsilon < p(\alpha) - 1$ and $p(\alpha) - 1 - \varepsilon$ is a nearly critical exponent. We combine variational arguments with the moving plane method to prove the existence of a positive radial solution. Moreover, the asymptotic behaviour of the solutions, as $\varepsilon \to 0$, is studied by using ODE techniques.

Keywords: asymptotic behavior, critical Sobolev exponent, Hardy exponents, singular coefficient.

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1 Introduction

In this paper, we consider the following elliptic problem:

$$\begin{cases} -\Delta u = \mu \dfrac{u}{|x|^2} + |x|^\alpha u^{p(\alpha)-1-\varepsilon}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a ball $B_R(0)$ in $\mathbb{R}^N (N \geq 3)$, $-2 < \alpha < 0$, $p(\alpha) = \frac{2(N+\alpha)}{N-2}$, $0 < \varepsilon < p(\alpha) - 1$, $0 \leq \mu < \overline{\mu} = \left(\frac{N-2}{2}\right)^2$.

The equation in problem (1.1) is the Euler–Lagrange equation of the energy functional $E : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$E(u) = \frac{1}{2}\left(\int_\Omega |\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) - \frac{1}{p(\alpha)-\varepsilon}\int_\Omega |x|^\alpha u^{p(\alpha)-\varepsilon}, \qquad \forall u \in H_0^1(\Omega).$$

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It is known that critical points of functional $E(u)$ correspond to solutions of (1.1).

We denote

$$\|u\| \triangleq \left(\int_\Omega |\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right)^{\frac{1}{2}}, \qquad \forall u \in H_0^1(\Omega).$$

Let us recall the Sobolev–Hardy inequality (see Lemma 2.1 in this paper), which using the fact $0 \le \mu < \bar{\mu}$ implies that $\|u\|$ is equivalent to the norm of $H_0^1(\Omega)$.

In the case $\mu = 0$ and $\alpha = 0$, a prototype of problem (1.1) is

$$\begin{cases} -\Delta u = u^{2^*-1-\varepsilon}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

When $\varepsilon = 0$, it is well known that the solution of problem (1.2) is bounded in the neighborhood of the origin. Gidas, Ni and Nirenberg [17] proved that all the solutions with reasonable behavior at infinity, namely

$$u = O(|x|^{2-N}), \tag{1.3}$$

are radially symmetric about some point. So, the form of the solutions may be assumed as

$$u(x) = \frac{[N(N-2)\lambda^2]^{\frac{N-2}{4}}}{(\lambda^2 + |x - x_0|^2)^{\frac{N}{2}-1}}$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^N$.

Later in [7, Corollay 8.2] and [9, Theorem 2.1], the growth assumption (1.3) was removed, which implies that, for positive C^2 solutions of problem (1.2), we have the same result.

When $\varepsilon > 0$, Atkinson and Peletier [2] used ODE arguments to obtain exact asymptotic estimates of the radially symmetric solution of problem (1.2) as $\varepsilon \to 0$. The following are their principal results

$$\lim_{\varepsilon \to 0} \varepsilon u^2(0, \varepsilon) = \frac{4}{N-2}\{N(N-2)\}^{\frac{N-2}{2}} \frac{\Gamma(N)}{[\Gamma(\frac{N}{2})]^2} \frac{1}{R^{N-2}}$$

and for $x \neq 0$

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{1}{2}} u(x, \varepsilon) = \frac{1}{2} N^{\frac{N-2}{4}} (N-2)^{\frac{N}{4}} R^{\frac{R-2}{2}} \frac{\Gamma(\frac{N}{2})}{[\Gamma(N)]^{\frac{1}{2}}} \left(\frac{1}{|x|^{N-2}} - \frac{1}{R^{N-2}}\right).$$

In the case $\mu = 0$ and $\alpha > 0$, problem (1.1) is known as the Hénon equation

$$\begin{cases} -\Delta u = |x|^\alpha u^{p-1}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $p \in (2, 2^*)$. Equation (1.4) was proposed by Hénon when he studied rotating stellar structures and readers can refer to Ni [24], Smets [26] and Cao–Peng [11]. Among these works, for equations with critical, supercritical and slightly subcritical growth, the existence and multiplicity of non-radial solutions, the symmetry and asymptotic behavior of ground states were studied by variational method (for $p \to \frac{2N}{N-2}$ or $\alpha \to \infty$).

In the case $0 \le \mu < \overline{\mu} = \left(\frac{N-2}{2}\right)^2$ and $\alpha = 0$, problem (1.1) can be written as

$$\begin{cases} -\Delta u = \mu \dfrac{u}{|x|^2} + u^{2^*-1-\varepsilon}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

By using Moser iteration and a generalized comparison principle, Cao and Peng [10] proved $u(x) \in H_0^1(\Omega)$ satisfying

$$\begin{cases} u(x)|x|^{\nu} \ge C_1, & \forall x \in \Omega' \subset\subset \Omega, \\ u(x)|x|^{\nu} \le C_2, & \forall x \in \Omega, \end{cases}$$

where C_1 and C_2 are two positive constants, $\nu = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}$. When $\Omega = B_R$, $u(x)$ is radially symmetric. Hence, they converted (1.5) to ODE and obtained the following:

$$\lim_{\varepsilon \to 0} \lim_{|x| \to 0} \varepsilon u_\varepsilon^2 |x|^{2\nu} = 4(2\sqrt{\overline{\mu} - \mu})^{N-1} N^{\frac{N-2}{2}} (N-2)^{-\frac{N+2}{2}} \frac{\Gamma(N)}{\left[\Gamma(\frac{N}{2})\right]^2} \frac{1}{R^{2\sqrt{\overline{\mu}-\mu}}}$$

and for $x \neq 0$

$$\begin{aligned} \lim_{\varepsilon \to 0} \varepsilon^{-\frac{1}{2}} u_\varepsilon(x) = &\frac{1}{2\sqrt{2}} (2\sqrt{\overline{\mu} - \mu})^{\frac{N-3}{2}} N^{\frac{N-2}{4}} (N-2)^{-\frac{N-6}{4}} R^{\sqrt{\overline{\mu}-\mu}} \frac{\Gamma(\frac{N}{2})}{[\Gamma(N)]^{\frac{1}{2}}} \\ &\times \left(\frac{1}{|x|^{\sqrt{\overline{\mu}} + \sqrt{\overline{\mu}-\mu}}} - \frac{1}{|x|^{\sqrt{\overline{\mu}} - \sqrt{\overline{\mu}-\mu}} |R|^{2\sqrt{\overline{\mu}-\mu}}} \right). \end{aligned}$$

Motivated by the previous works and remark 4.2 in [10], we first prove the existence and radial symmetry of positive solution of (1.1). Then we focus on the asymptotic behavior of the solutions of problem (1.1) as $\varepsilon \to 0$.

To state our main results, for convenience, we set $p = p(\alpha) - 1 - \varepsilon$, $\nu = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}$, $\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}$, $R > 0$. We denote by $u_\varepsilon(x)$ the solution of (1.1) and $\Gamma(x)$ is the Gamma function.

Theorem 1.1. *Suppose that $-2 < \alpha < 0$, $0 \le \mu < \bar{\mu}$, $0 < \varepsilon < p(\alpha) - 1$. Then problem (1.1) has a radially symmetric solution in $H_0^1(\Omega)$.*

For the proof of this Theorem 1.1, we first obtain a solution by the Mountain Pass Lemma. Then, by moving plane method for elliptic equations with variable coefficients in [14], we can prove that the posotive solution is radially symmetric. For problem (1.2), the solution satisfies Gidas–Ni–Nirenberg Theorem in [17] and hence all solutions of (1.2) are radial symmetric. However, here we cannot use Gidas–Ni–Nirenberg theorem directly since problem (1.1) includes the hardy term $\mu \frac{u}{|x|^2}$ and singular coefficient $|x|^\alpha$. Luckily, through a transformation of the original solution $u_\varepsilon(x)$, the new equation satisfied by the new solution $v(x)$ satisfies the conditions of a Corollary in [14] and we obtain the result. To be more precise, set

$$v(x) = |x|^{-\sqrt{\overline{\mu}} + \sqrt{\overline{\mu}-\mu}} u_\varepsilon(x),$$

using Moser iteration and a generalized comparison principle introduced by Merle and Peletier [22], we prove that $v \in L^\infty(\Omega)$ and is bounded from below and above. Thus we obtain that the precise singularity of $u_\varepsilon(x)$ at the origin is like $|x|^{-\sqrt{\overline{\mu}} + \sqrt{\overline{\mu}-\mu}}$. Then applying

Lemma 2.5 in Section 2 to this new equation, we deduce that $v(x)$ is radially symmetric and satisfies the following ODE:

$$\begin{cases} v'' + \frac{N-1-2\nu}{r} v' + \frac{1}{r^{(p(\alpha)-2-\varepsilon)\nu-\alpha}} v^{p(\alpha)-1-\varepsilon} = 0, & \text{for } 0 < r < R, \\ v(r) > 0, & \text{for } 0 < r < R, \\ v(R) = 0. \end{cases} \tag{1.6}$$

Because (1.6) is still singular at the origin, we can use the well-known shooting argument introduced by Atkinson and Peletier [2] to convert (1.6) to the following ODE:

$$\begin{cases} y''(t) = -t^{-k(\alpha,\varepsilon)y^{p(\alpha)-1-\varepsilon}}, \\ y(t) > 0, \text{ for } T < t < \infty, \\ y(T) = 0, \end{cases} \tag{1.7}$$

where $k(\alpha,\varepsilon) = \frac{2m+\alpha}{m-1} - \frac{(p(\alpha)-2-\varepsilon)\nu}{m-1}, m = 1 + 2\sqrt{\bar{\mu}-\mu} = N - 2\nu - 1, T = (\frac{m-1}{R})^{m-1}, p(\alpha) - 1 = 2k(\alpha,\varepsilon) - 3 - \frac{2\nu\varepsilon}{m-1}$.

Till now, study on behaviors and precise properties of the original solution $u_\varepsilon(x)$ can be reduced to deal with (1.7). Based on this, we have

Theorem 1.2. *Let $u_\varepsilon(x) \in H_0^1(\Omega)$ be a solution of problem (1.1). Then*

$$\lim_{\varepsilon\to 0} \lim_{|x|\to 0} \varepsilon u_\varepsilon^2 |x|^{2\nu} = 2(\alpha+2)(2\sqrt{\bar{\mu}-\mu})^{\frac{2N+\alpha-2}{\alpha+2}} (N+\alpha)^{\frac{N-2}{\alpha+2}} (N-2)^{-\frac{2\alpha+N+2}{\alpha+2}} \frac{\Gamma(\frac{2(N+2)}{\alpha+2})}{\left[\Gamma(\frac{N+\alpha}{\alpha+2})\right]^2} \frac{1}{R^{2\sqrt{\bar{\mu}-\mu}}}.$$

Theorem 1.3. *Let $u_\varepsilon(x) \in H_0^1(\Omega)$ be a solution of problem (1.1). Then, for every $x \neq 0$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{1}{2}} u_\varepsilon(x) = \frac{1}{2}(\alpha+2)^{-\frac{1}{2}} (2\sqrt{\bar{\mu}-\mu})^{\frac{2N-\alpha-6}{2\alpha+4}} (N+\alpha)^{\frac{N-2}{2\alpha+4}} (N-2)^{\frac{2\alpha-N+6}{2\alpha+4}} R^{\sqrt{\bar{\mu}-\mu}} \frac{\Gamma(\frac{N+\alpha}{\alpha+2})}{\left[\Gamma\left(\frac{2(N+\alpha)}{\alpha+2}\right)\right]^{\frac{1}{2}}}$$
$$\times \left(\frac{1}{|x|^{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}} - \frac{1}{|x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}} |R|^{2\sqrt{\bar{\mu}-\mu}}} \right).$$

Notations:

- $C, C_i, i = 0, 1, 2, \ldots$ denote positive constants, which may vary from line to line;
- $\|\cdot\|$ and $\|\cdot\|_{L^q}$ denote the usual norms of the spaces $H_0^1(\Omega)$ and $L^q(\Omega)$, respectively, $\Omega \in \mathbb{R}^N$;
- Some of the notations that will appear in the following paragraphs:

$$m = 1 + 2\sqrt{\bar{\mu}-\mu} = N - 2\nu - 1, \qquad T = \left(\frac{m-1}{R}\right)^{m-1},$$
$$k = k(\alpha,\varepsilon) = \frac{2m+\alpha}{m-1} - \frac{(p(\alpha)-2-\varepsilon)\nu}{m-1}, \quad k_1(\alpha,\varepsilon) = (k-1)^{\frac{1}{k-2}},$$
$$k_2(\alpha,\varepsilon) = \frac{k-1}{k-2}, \qquad T_{\alpha,\varepsilon} = \frac{\gamma^{\frac{p(\alpha)-2-\varepsilon}{k-2}}}{k_1(\alpha,\varepsilon)} = \frac{\gamma^{2-\frac{m-1-2\nu}{(m-1)(k-2)}\varepsilon}}{k_1(\alpha,\varepsilon)},$$
$$\tau(\alpha,\varepsilon) = \left(\frac{t}{T_{\alpha,\varepsilon}}\right)^{k-2}, \qquad \varphi(\alpha,\varepsilon) = \frac{m-1+2\nu}{(m-1)(k-2)}\varepsilon,$$
$$C_{\alpha,\beta,\varepsilon} = \frac{\beta}{(1+\beta^{k-2})^{\frac{1}{k-2}}}, \qquad d_{\alpha,\beta,\varepsilon} = \frac{(1-C_{\alpha,\beta,\varepsilon})(1+2\nu/(m-1))}{C_{\alpha,\beta,\varepsilon}^{2+(1+2\nu/(m-1))\varepsilon}}.$$

2 Preliminary results and existence of solution

In this section,we shall provide some preliminaries which will be used in the sequel and prove the existence of solution to problem (1.1).

Lemma 2.1 (see [16, Lemma 3.1 and 3.2]). *Suppose $-2 < \alpha < 0$, $2 \leq q \leq p(\alpha)$, and $0 \leq \mu < \overline{\mu}$. Then*

(i) (Hardy inequality)

$$\int_\Omega \frac{u^2}{|x|^2} \leq \frac{1}{\mu}\int_\Omega |\nabla u|^2, \qquad \forall u \in H_0^1(\Omega);$$

(ii) (Sobolev–Hardy inequality)

there exists a constant $C > 0$ such that

$$\left(\int_\Omega |x|^\alpha |u|^q\right)^{\frac{1}{q}} \leq C\|u\|, \qquad \forall u \in H_0^1(\Omega);$$

(iii) the map $u \mapsto |x|^{\frac{\alpha}{q}} u$ from $H_0^1(\Omega)$ into $L^q(\Omega)$ is compact for $q < p(\alpha)$.

Lemma 2.2 (see [5, Theorem 2.2]). *Let J be a C^1 function on a Banach space X. Suppose there exists a neighborhood U of 0 in X and a constant ρ such that $J(u) \geq \rho$ for every u in the boundary of U,*

$$J(0) < \rho \quad \text{and} \quad J(v) < \rho \quad \text{for some} \quad v \notin U.$$

Set

$$c = \inf_{g\in\Gamma} \max_{\omega \in g} J(\omega) \geq \rho,$$

where $\Gamma = \{g \in C([0,1], X) : g(0) = 0, g(1) = v, J(g(1)) < \rho\}$.

Conclusion: there is a sequence $\{u_n\}$ in X such that $J(u_n) \to c$ and $J'(u_n) \to 0$ in X^*.

Lemma 2.3 (The Caffarelli–Kohn–Nirenberg inequalities, see [8] and [12]). *For all $u \in C_0^\infty(\mathbb{R}^N)$,*

$$\left(\int_{\mathbb{R}^N} |x|^{-bq}|u^q|\right)^{\frac{p}{q}} \leq C_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|Du|^p dx,$$

where (i) for $n > p$,

$$-\infty < a < \tfrac{n-p}{p}, \quad 0 \leq b-a \leq 1, \quad \text{and} \quad q = \tfrac{np}{n-p+p(b-a)}$$

and (ii) for $n \leq p$,

$$-\infty < a < \tfrac{n-p}{p}, \quad \tfrac{p-n}{p} \leq b-a \leq 1, \quad \text{and} \quad q = \tfrac{np}{n-p+p(b-a)}.$$

Lemma 2.4 (see [25, page 4]). *Suppose V is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subset V$ be a weakly closed subset of V. Suppose $E : M \to \mathbb{R} \cup \{+\infty\}$ is coercive and (sequentially) weakly lower semi-continuous on M with respect to V, that is, suppose the following conditions are fulfilled:*

(1) (coercive) $E(u) \to \infty$ as $\|u\| \to \infty$, $u \in M$.

(2) (W.S.L.S.C) For any $u \in M$, any sequence $\{u_m\}$ in M such that $u_m \rightharpoonup u$ weakly in V there holds:

$$E(u) \leq \liminf_{m\to\infty} E(u_m).$$

Then E is bounded from below on M and attains its infimum in M.

Lemma 2.5 (see [14, Corollary 1.6]). *Let u be a bounded $C^2(B_R\backslash\{0\})\cap C^1(\overline{B_R}\backslash\{0\})$ solution of*

$$\begin{cases}\partial_i(|x|^b\partial_i u)+K|x|^a u^q=0, & x\in B_R\backslash\{0\},\\ u>0, & x\in B_R\backslash\{0\},\\ u=0, & x\in\partial B_R\backslash\{0\},\end{cases}$$

where K is a positive constant.Then u is radially symmetric in B_R provided $q\geq 1$, $b(\frac{1}{2}b+N-2)\leq 0$ and $\frac{1}{2}b\geq\frac{a}{q}$.

Proof. When $b<0$, we have $|x|^b$ is singular at origin.

It's clear that

$$\frac{S(x)}{|x|^b}-\frac{S(x^\lambda)}{|x^\lambda|^b}=\frac{1}{2}b\left(\frac{1}{2}b+N-2\right)(|x|^{b-2}-|x^\lambda|^{b-2})\geq 0$$

and

$$K\left(\frac{|x|^b}{|x^\lambda|^b}\right)^{\frac{1}{2}}|x^\lambda|^a u^q-K|x|^a\left[\left(\frac{|x|^b}{|x^\lambda|^b}\right)^{\frac{1}{2}}u\right]^q=K|x|^{\frac{1}{2}b}|x^\lambda|^{a-\frac{1}{2}b}\left[1-\left(\frac{|x^\lambda|}{|x|}\right)^{\frac{1}{2}bq-a}\right]u^q\geq 0,$$

where $S(x)=\frac{1}{2}(\Delta|x|^b-\frac{1}{2|x|^b}|\nabla|x|^b|^2)$.

From [14], we have

$$h_\lambda(x)=\left(\frac{|x^\lambda|^b}{|x|^b}\right)^{\frac{1}{2}}u(x^\lambda)-u(x),$$

where $x^\lambda=(2\lambda-x_1,x_2,\ldots,x_N)$.

By Lemma 4.2 in [14], we can obtain u has a positive lower bound near the origin. Hence, we can get the estimate of $h_\lambda(x)$ near the origin. Furthermore, if $x^\lambda=0$, we have $h_\lambda(x)=\infty$.

Now, we consider the case of $u\in C^2(B_1\backslash\{0\})\cap C^1(\overline{B_1}\backslash\{0\})$ in Proposition 1.3 of [14]. Analogically, we can also obtain $u(x_1,x_2,\ldots,x_N)\leq u(-x_1,x_2,\ldots,x_N)$ for $x_1\in(-1,0)$ and $x_1\in(0,1)$. Hence, u is symmetric in x_1. By Lemma 1.1 in [14], the above analysis and scaling transformation, $u(x)$ is radially symmetric in B_R. □

Next, we shall prove the existence of solution to the problem (1.1). To start with, we prove the existence of nonnegative solution to the following Dirichlet problem:

$$\begin{cases}-\Delta u=\mu\frac{u}{|x|^2}+|x|^\alpha|u|^{p(\alpha)-2-\varepsilon}u, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}\tag{2.1}$$

where Ω is a ball in $\mathbb{R}^N(N\geq 3)$ centered at the origin, $-2<\alpha<0$, $p(\alpha)=\frac{2(N+\alpha)}{N-2}$, $0<\varepsilon<p(\alpha)-1$,$0\leq\mu<\overline{\mu}=(\frac{N-2}{2})^2$.

The energy functional corresponding to problem (2.1) is

$$J(u)=\frac{1}{2}\|u\|^2-\frac{1}{p(\alpha)-\varepsilon}\int_\Omega|x|^\alpha|u|^{p(\alpha)-\varepsilon},\qquad u\in H_0^1(\Omega).$$

Lemma 2.6. *The function J satisfies $(PS)_c$ condition for every $c\in\mathbb{R}$.*

Proof. Take $c \in \mathbb{R}$ and assume that $\{u_n\}$ is a Palais–Smale sequence at level c, namely such that

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \quad (\text{in } H^{-1}(\Omega)).$$

This implies that there is a constant $M > 0$ such that

$$|J(u_n)| \leq M. \tag{2.2}$$

From $J'(u_n) \to 0$, we obtain

$$o(1)\|u_n\| = \langle J'(u_n), u_n \rangle = \|u_n\|^2 - \int_\Omega |x|^\alpha |u_n|^{p(\alpha)-\varepsilon}. \tag{2.3}$$

Calculating (2.2) $- \frac{1}{p(\alpha)-\varepsilon}$(2.3), we have

$$\begin{aligned} M + o(1)\|u_n\| &\geq \frac{1}{2}\|u_n\|^2 - \frac{1}{p(\alpha)-\varepsilon}\int_\Omega |x|^\alpha |u_n|^{p(\alpha)-\varepsilon} \\ &\quad - \frac{1}{p(\alpha)-\varepsilon}\|u_n\|^2 + \frac{1}{p(\alpha)-\varepsilon}\int_\Omega |x|^\alpha |u_n|^{p(\alpha)-\varepsilon} \\ &= \left(\frac{1}{2} - \frac{1}{p(\alpha)-\varepsilon}\right)\|u_n\|^2, \end{aligned}$$

which implies the boundedness of $\{u_n\}$. By usual arguments, we can assume that up to a subsequence, there exists $u \in H_0^1(\Omega)$ such that

- $u_n \rightharpoonup u$ in $H_0^1(\Omega)$;
- $|x|^{\frac{\alpha}{p(\alpha)-\varepsilon}} u_n \to |x|^{\frac{\alpha}{p(\alpha)-\varepsilon}} u$ in $L^{p(\alpha)-\varepsilon}(\Omega)$;
- $\cdot u_n \to u$ for almost every $x \in \Omega$.

We now show that the convergence of u_n to u is strong.
First of all, from the above convergence properties, we obtain

$$\left\| u_n |x|^{\frac{\alpha}{p(\alpha)-\varepsilon}} - u|x|^{\frac{\alpha}{p(\alpha)-\varepsilon}} \right\|_{L^{p(\alpha)-\varepsilon}(\Omega)} \to 0, \qquad n \to \infty.$$

As $J'(u_n) \to 0$ and $u_n \rightharpoonup u$, we also have $\langle J'(u_n), u_n - u\rangle \to 0$ and obviously $\langle J'(u), u_n - u\rangle \to 0$. Then, as $n \to \infty$, on the one hand,

$$\langle J'(u_n) - J'(u), u_n - u\rangle \leq |\langle J'(u_n), u_n - u\rangle| + |\langle J'(u), u_n - u\rangle| = o(1).$$

On the other hand,

$$\begin{aligned} &\langle J'(u_n) - J'(u), u_n - u\rangle \\ &\quad = \int_\Omega |\nabla u_n - \nabla u|^2 - \int_\Omega \mu \frac{|u_n - u|^2}{|x|^2} - \int_\Omega |x|^\alpha (|u_n|^{p(\alpha)-2-\varepsilon}u_n - |u|^{p(\alpha)-2-\varepsilon}u)(u_n - u) \\ &\quad = \|u_n - u\|^2 - \int_\Omega |x|^\alpha (|u_n|^{p(\alpha)-2-\varepsilon}u_n - |u|^{p(\alpha)-2-\varepsilon}u)(u_n - u). \end{aligned}$$

We claim $\int_\Omega |x|^\alpha (|u_n|^{p(\alpha)-2-\varepsilon}u_n - |u|^{p(\alpha)-2-\varepsilon}u)(u_n - u) \to 0$.

Indeed, by Hölder's inequality,

$$\begin{aligned}
&\int_\Omega |x|^\alpha |u_n|^{p(\alpha)-2-\varepsilon} u_n(u_n-u) \\
&\le \int_\Omega |x|^\alpha |u_n|^{p(\alpha)-1-\varepsilon}|u_n-u| \\
&= \int_\Omega |x|^{\alpha\cdot\frac{p(\alpha)-1-\varepsilon}{p(\alpha)-\varepsilon}} |u_n|^{p(\alpha)-1-\varepsilon} |x|^{\alpha\cdot\frac{1}{p(\alpha)-\varepsilon}}|u_n-u| \\
&\le \left[\int_\Omega \left(|x|^{\alpha\cdot\frac{p(\alpha)-1-\varepsilon}{p(\alpha)-\varepsilon}} |u_n|^{p(\alpha)-1-\varepsilon}\right)^{\frac{p(\alpha)-\varepsilon}{p(\alpha)-1-\varepsilon}}\right]^{\frac{p(\alpha)-1-\varepsilon}{p(\alpha)-\varepsilon}} \left[\int_\Omega \left(|x|^{\alpha\cdot\frac{1}{p(\alpha)-\varepsilon}}|u_n-u|\right)^{p(\alpha)-\varepsilon}\right]^{\frac{1}{p(\alpha)-\varepsilon}} \qquad (2.4)\\
&= \left(\int_\Omega |x|^\alpha |u_n|^{p(\alpha)-\varepsilon}\right)^{\frac{p(\alpha)-1-\varepsilon}{p(\alpha)-\varepsilon}} \left\| |x|^{\frac{\alpha}{p(\alpha)-\varepsilon}}|u_n-u|\right\|_{L^{p(\alpha)-\varepsilon}(\Omega)} \\
&\le C\|u_n\|^{p(\alpha)-1-\varepsilon}\left\|u_n|x|^{\frac{\alpha}{p(\alpha)-\varepsilon}} - u|x|^{\frac{\alpha}{p(\alpha)-\varepsilon}}\right\|_{L^{p(\alpha)-\varepsilon}(\Omega)} \\
&= o(1).
\end{aligned}$$

By (2.4), similar calculation also gives

$$\int_\Omega |x|^\alpha |u|^{p(\alpha)-2-\varepsilon} u(u_n-u) = o(1). \qquad (2.5)$$

From the above analysis, we obtain

$$o(1) = \langle J'(u_n) - J'(u), u_n - u\rangle = \|u_n-u\|^2 + o(1),$$

which implies $u_n \to u$ in $H_0^1(\Omega)$ and proves that J satisfies $(PS)_c$ condition for every $c \in \mathbb{R}$. □

Lemma 2.7. *The function J admits a $(PS)_c$ sequence in the cone of nonnegative function at the level*

$$c = \inf_{g\in\Gamma}\max_{t\in[0,1]} J(g(t)),$$

where $\Gamma = \{g \in C([0,1], H_0^1(\Omega)) : g(0) = 0, J(g(1)) < 0\}$.

Proof. We next prove that J satisfies all the hypotheses of the mountain pass lemma. Obviously, $J(0) = 0$.

From the Sobolev–Hardy inequality, we obtain

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{p(\alpha)-\varepsilon}\int_\Omega |x|^\alpha |u|^{p(\alpha)-1-\varepsilon}u \\
&\ge \frac{1}{2}\|u\|^2 - C_1\|u\|^{p(\alpha)-\varepsilon}.
\end{aligned}$$

For any α, we can choose ε small enough such that $p(\alpha)-\varepsilon > 2$. From the above analysis,there exist $\rho, e > 0$ such that $J(u) \ge \rho, \forall u \in \{u \in H_0^1(\Omega) : \|u\| = e\}$. Furthermore, for any $u \in H_0^1(\Omega)$,

$$J(tu) = \frac{t^2}{2}\|u\|^2 - \frac{t^{p(\alpha)-\varepsilon}}{p(\alpha)-\varepsilon}\int_\Omega |x|^\alpha |u|^{p(\alpha)-1-\varepsilon}u.$$

We obtain $J(tu) \to -\infty$ as $t \to \infty$. Hence, we can choose $t_0 > 0$ such that $J(t_0u) < 0$. Therefore, by Lemma 2.2, we infer that J admits a $(PS)_c$ sequence at level c, such sequence may be chosen in the set of nonnegative functions because $J(|u|) \le J(u)$ for all $u \in H_0^1(\Omega)$. □

By Lemma 2.6, 2.7 and mountain pass lemma, we get a nonnegative solution $u \in H_0^1(\Omega)$ for (1.1), this solution is positive by the maximum principle.

3 Estimate of the singularity

First, we fix $p = p(\alpha) - 1 - \varepsilon > 0$ in problem (1.1) and study the singularity and radial symmetry of the solution $u_\varepsilon(x) \in H_0^1(\Omega)$. By standard elliptic regularity theory, $u_\varepsilon(x) \in C^2(\Omega\backslash\{0\}) \bigcap C^1(\overline{\Omega}\backslash\{0\})$. Hence the singular point of $u_\varepsilon(x)$ should be the origin.

Suppose that $u_\varepsilon(x) \in H_0^1(\Omega)$ satisfies problem (1.1).

Let $v(x) = |x|^\nu u_\varepsilon(x)$, then

$$-\Delta u = (-\nu^2 - 2\nu + N\nu)|x|^{-\nu-2}v(x) + 2\nu|x|^{-\nu-2}x\nabla v(x) - |x|^{-\nu}\Delta v(x),$$

$$\mu\frac{u}{|x|^2} + |x|^\alpha u^{p(\alpha)-1-\varepsilon} = \mu|x|^{-\nu-2}v(x) + |x|^{\alpha-(p(\alpha)-1-\varepsilon)\nu}v(x)^{p(\alpha)-1-\varepsilon}.$$

From equation in (1.1),

$$\begin{aligned}(-\nu^2 - 2\nu + N\nu)|x|^{-\nu-2}v(x) + 2\nu|x|^{-\nu-2}x\nabla v(x) - |x|^{-\nu}\Delta v(x)&\\ = \mu|x|^{-\nu-2}v(x) + |x|^{\alpha-(p(\alpha)-1-\varepsilon)\nu}v(x)^{p(\alpha)-1-\varepsilon}&.\end{aligned}$$

Multiply both sides of the above equation by $|x|^{-\nu}$, then we get

$$[-\nu^2 + (N-2)\nu]|x|^{-2\nu-2}v(x) - \operatorname{div}(|x|^{-2\nu}\nabla v(x)) = \mu|x|^{-2\nu-2}v(x) + |x|^{\alpha-(p(\alpha)-\varepsilon)\nu}v(x)^{p(\alpha)-1-\varepsilon}.$$

For $\nu = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}$, we obtain

$$\begin{cases} -\operatorname{div}(|x|^{-2\nu}\nabla v) = |x|^{-(p(\alpha)-\varepsilon)\nu+\alpha}v^{p(\alpha)-1-\varepsilon}, & x \in \Omega,\\ v > 0, & x \in \Omega,\\ v = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

By the regularity theory of elliptic equations, $v_\varepsilon(x) \in C^2(\Omega\backslash\{0\}) \bigcap C^1(\overline{\Omega}\backslash\{0\})$. Moreover, we have

Lemma 3.1.

(i) $v(x) \in H_0^1(\Omega, |x|^{-2\nu})$.

(ii) $v(x)$ *is bounded in* Ω.

Proof. (i) For any $u(x) \in H_0^1(\Omega)$ satisfying problem (1.1), by Hardy inequality, we have

$$\begin{aligned}\int_\Omega |x|^{-2\nu}|\nabla v|^2 &= \int_\Omega |x|^{-2\nu}||x|^\nu\nabla u + \nu|x|^{\nu-2}ux|^2\\ &\le 2\left(\int_\Omega |\nabla u|^2 + \nu^2\int_\Omega \frac{u^2}{|x|^2}\right)\\ &\le C.\end{aligned}$$

Hence, we claim $v(x) = |x|^\nu u(x) \in H_0^1(\Omega, |x|^{-2\nu})$.

(ii) From Caffarelli–Kohn–Nirenberg inequality mentioned in Lemma 2.3, we have

$$\left(\int_\Omega |x|^{m_1}|\nabla u|^2\right)^{\frac{1}{2}} \ge C_{m_1,n_1}\left(\int_\Omega |x|^{n_1}|u|^{p(m_1,n_1)}\right)^{\frac{1}{p(m_1,n_1)}}, \qquad \forall u \in H_0^1(\Omega, |x|^{m_1}), \tag{3.2}$$

where

$$m_1 = -2\nu, \qquad n_1 = -(p(\alpha)-\varepsilon)\nu + \alpha, \qquad p(m_1,n_1) = p(\alpha) + \frac{\varepsilon\nu}{\sqrt{\bar{\mu}-\mu}}.$$

Note that

$$\int_\Omega |x|^{m_1}\nabla v\cdot\nabla\varphi = \int_\Omega |x|^{n_1} v^p\varphi, \qquad \forall\varphi\in H_0^1(\Omega,|x|^{m_1}).$$

For $s,l>1$, define $v_l(x)=\min\{v(x),l\}$. Taking $\varphi = v\cdot v_l^{2(s-1)}\in H_0^1(\Omega,|x|^{m_1})$ in the above equation, we have

$$\int_\Omega |x|^{m_1}|\nabla v|^2 v_l^{2(s-1)} + 2(s-1)\int_\Omega |x|^{m_1}|\nabla v_l|^2 v_l^{2(s-1)} = \int_\Omega |x|^{n_1} v^{p+1} v_l^{2(s-1)}.$$

Hence,

$$\begin{aligned}
&\left(\int_\Omega |x|^{n_1}(v\cdot v_l^{s-1})^{p(m_1,n_1)}\right)^{\frac{2}{p(m_1,n_1)}}\\
&\quad\le C_{m_1,n_1}^{-2}\int_\Omega |x|^{m_1}|\nabla(v\cdot v_l^{s-1})|^2\\
&\quad\le 2C_{m_1,n_1}^{-2}\left((s-1)^2\int_\Omega |x|^{m_1}|\nabla v_l|^2 v_l^{2(s-1)} + \int_\Omega |x|^{m_1}|\nabla v|^2 v_l^{2(s-1)}\right)\\
&\quad\le 2C_{m_1,n_1}^{-2}s\int_\Omega |x|^{n_1}v^{p+2s-1}.
\end{aligned} \tag{3.3}$$

From (3.3) and Levi's theorem, we see that $v\in L^{p+2s-1}(\Omega,|x|^{n_1})$ implies $v\in L^{sp(m_1,n_1)}(\Omega,|x|^{n_1})$. For $j=0,1,2,\ldots$, by induction we define

$$\begin{cases} p-1+2s_0 = p(m_1,n_1),\\ p-1+2s_{j+1} = p(m_1,n_1)s_j, \end{cases} \tag{3.4}$$

$$\begin{cases} M_0 = (C\cdot C_{m_1,n_1}^{-2})^{\frac{p(m_1,n_1)}{2}},\\ M_{j+1} = (2C_{m_1,n_1}^{-2}s_jM_j)^{\frac{p(m_1,n_1)}{2}}, \end{cases} \tag{3.5}$$

where C is a fixed number such that $\int_\Omega |x|^{m_1}|\nabla v|^2\le C$.

From (3.4), we see that

$$s_j = \frac{(2^{-1}p(m_1,n_1))^{j+1}(p(m_1,n_1)-p-1)+p-1}{p(m_1,n_1)-2}.$$

From (3.5), similar to the computation in [21], we can see that

$$\exists d>0 \text{ and } d \text{ is independent of } j, \text{such that } M_j\le e^{ds_{j-1}}.$$

Since $2<p+1<p(m_1,n_1)$, it follows that $s_j>1$ for all $j\ge 0$, $s_j\to+\infty$ as $j\to+\infty$.

By (3.3), (3.4) and (3.5),

$$\begin{aligned}
\int_\Omega |x|^{n_1}v^{p+2s_1-1} &\le (2C_{m_1,n_1}^{-2}s_0)^{\frac{p(m_1,n_1)}{2}}\left(\int_\Omega |x|^{n_1}v^{p+2s_0-1}\right)^{\frac{p(m_1,n_1)}{2}}\\
&\le (2C_{m_1,n_1}^{-2}s_0)^{\frac{p(m_1,n_1)}{2}}\left(C^{\frac{p(m_1,n_1)}{2}}C_{m_1,n_1}^{-p(m_1,n_1)}\right)^{\frac{p(m_1,n_1)}{2}}\\
&\le (2C_{m_1,n_1}^{-2}s_0M_0)^{\frac{p(m_1,n_1)}{2}}\\
&\le M_1.
\end{aligned}$$

Similarly,

$$\int_\Omega |x|^{n_1} v^{p+2s_j-1} \le M_j.$$

Hence, by $p+2s_{j+1}-1 = p(m_1,n_1)s_j$, denoting $C(\Omega,n_1) = \max_{x\in\Omega}|x|^{-n_1}$, we obtain

$$\begin{aligned} |v|_{L^{p(m_1,n_1)s_j}(\Omega)} &\le \left(\int_\Omega |v|^{p(m_1,n_1)s_j}|x|^{n_1}\cdot|x|^{-n_1}\right)^{\frac{1}{p(m_1,n_1)s_j}} \\ &\le C(\Omega,n_1)^{\frac{1}{p(m_1,n_1)s_j}} |v|^{\frac{1}{p(m_1,n_1)s_j}}_{L^{p(m_1,n_1)s_j}(\Omega,|x|^{n_1})} \\ &\le C(\Omega,n_1)^{\frac{1}{p(m_1,n_1)s_j}} M_{j+1}^{\frac{1}{p(m_1,n_1)s_j}} \\ &\le C(\Omega,n_1)^{\frac{1}{p(m_1,n_1)s_j}} e^{\frac{d}{p(m_1,n_1)}}. \end{aligned}$$

Taking limit on each side of the above inequality and using $s_j \to +\infty$, as $j \to +\infty$, we have

$$|v|_{L^\infty(\Omega)} \le e^{\frac{d}{p(m_1,n_1)}},$$

which implies the conclusion. □

From Lemma 3.1, we can see that $v(x) = |x|^\nu u(x)$ is bounded form above in Ω. For the lower bound of $v(x) = |x|^\nu u(x)$, we have

Lemma 3.2. *Suppose that $u(x) \in H_0^1(\Omega)$ satisfies problem (1.1) and $0 \le \mu < \overline{\mu}$, then for any $B_\rho \subset\subset \Omega$ there exists a $C(\rho) > 0$, such that*

$$u(x) \ge C(\rho)|x|^{-\nu}, \qquad \forall x \in B_\rho \subset\subset \Omega.$$

Proof. Let $f(x) = \min\{|x|^\alpha u^{p(\alpha)-1-\varepsilon}(x), l\}$ with $l > 0$, then $f \in L^\infty(\Omega)$.

Let $u_1 \ge 0$ and $u_1 \in H_0^1(\Omega)$ be the solution of the following linear problem

$$\begin{cases} -\Delta u_1 = \mu \dfrac{u_1}{|x|^2} + f, & x \in \Omega, \\ u_1 = 0, & x \in \partial\Omega. \end{cases} \tag{3.6}$$

Set $U = u - u_1$, then $U \in H_0^1(\Omega)$ and U satisfies the following problem

$$\begin{cases} -\Delta U = \mu \dfrac{U}{|x|^2} + g, & x \in \Omega, \\ U = 0, & x \in \partial\Omega, \end{cases} \tag{3.7}$$

where $g \ge 0$ and $0 \le \mu < \overline{\mu} = (\frac{N-2}{2})^2$.

From Lemma 2.4, there exist solutions for problem (3.6) and (3.7). From the Hardy inequality and the comparison principle proved in [15], we obtain that u is a super-solution of problem (3.6) and $0 \le u_1 \le u$. Actually we can prove this as follows. Multiplying $U^- := \max\{0, -U(x)\}$ on both side of equation in (3.7) and integrating by parts, we have

$$-\int_\Omega |\nabla U^-|^2 = -\int_\Omega \mu \frac{(U^-)^2}{|x|^2} + \int_\Omega gU^-.$$

It follows that $U^- = 0$ in Ω and hence $U \ge 0$.

By Lemma 3.1, there exists a constant $C_1 > 0$ such that $0 \le u_1(x) \le u(x) \le C_1|x|^{-\nu}$. So it suffices to prove the result for u_1.

Since $u_1 \not\equiv 0, u_1 \ge 0$ and $-\Delta u_1 \ge 0$ in Ω, there exists $\delta > 0$ such that for sufficiently small $\rho > 0$ it holds that $u_1 \ge \delta$ for $\forall x \in B_{2\rho}$. Choose $C(\rho) \ge 0$ satisfying $C(\rho)|x|^{-\nu} \le \delta$ for $|x| = \rho$ and set $\omega = (u_1 - C|x|^{-\nu})^-$. By $\int_{B_\rho} |\nabla |x|^{-\nu}|^2 < \infty$ and $u_1 \in H_0^1(B_\rho)$, we have $\omega \in H_0^1(B_\rho)$.

From (3.6) and the fact that $|x|^{-\nu}$ is the solution of equation $-\Delta u - \mu\frac{u}{|x|^2} = 0$, the linear combination of u_1 and $|x|^{-\nu}$ is the solution of $-\Delta u = \mu\frac{u}{|x|^2} + f$. Hence,

$$-\Delta(u_1 - C|x|^{-\nu}) = \mu\frac{(u_1 - C|x|^{-\nu})}{|x|^2} + f.$$

Multiply ω on both side of the above equation and integrate by part, we obtain

$$-\int_{B_\rho} |\nabla\omega|^2 + \int_{B_\rho} \mu\frac{\omega^2}{|x|^2} = \int_{B_\rho} f\omega \ge 0.$$

Since $0 \le \mu < \bar{\mu}$, it follows that $\omega = 0$.

Another proof of $\omega = 0$: It only need to prove that $-\int_{B_\rho} |\nabla\omega|^2 + \int_{B_\rho} \mu\frac{\omega^2}{|x|^2}\omega \ge 0$. Otherwise,

$$\begin{aligned} 0 &> -\int_{B_\rho} |\nabla\omega|^2 + \int_{B_\rho} \mu\frac{\omega^2}{|x|^2} \\ &= \int_{B_\rho} \nabla(u_1 - C|x|^{-\nu})\cdot\nabla\omega - \int_{B_\rho} \frac{\mu}{|x|^2}(u_1 - C|x|^{-\nu})\omega \\ &= \int_{B_\rho} f\omega - C\Big(\int_{B_\rho} \nabla|x|^{-\nu}\cdot\nabla\omega - \int_{B_\rho} \frac{\mu}{|x|^2}|x|^{-\nu}\omega\Big) \\ &= \int_{B_\rho} f\omega + \frac{C\nu}{\rho^{\nu+1}}\int_{\partial B_\rho} \omega \\ &> \frac{C\nu}{\rho^{\nu+1}}\int_{\partial B_\rho} \omega \\ &\ge 0. \end{aligned}$$

This is a contradiction and we are done. □

Proposition 3.3. *Suppose that $u(x) \in H_0^1(\Omega)$ satisfies problem (1.1) and $0 \le \mu < \overline{\mu}$. Then for any $\Omega' \subset\subset \Omega$ there exists two positive constants C_1 and C_2, such that*

$$\begin{cases} u(x)|x|^\nu \ge C_1, & \forall x \in \Omega' \subset\subset \Omega. \\ u(x)|x|^\nu \le C_2, & \forall x \in \Omega. \end{cases} \tag{3.8}$$

Next, we use Lemma 2.5 and Proposition 3.3 to prove that the solution is radially symmetric with $\Omega = B_R$.

Theorem 3.4. *Suppose that $-2 < \alpha < 0$ and $p(\alpha) = \frac{2(N+\alpha)}{N-2}$. Then the solution of problem (1.1) is radially symmetric.*

Proof. Using the previous notations, we only need to show that $v(x)$ is radially symmetric in Ω. By the regularity theory of elliptic equations, we have $v(x) \in C^2(B_R\backslash\{0\}) \cap C^1(\overline{B_R}\backslash\{0\})$. Next, we have to prove that $v(x)$ satisfies Lemma 2.5. From (3.1), we obtain

$$\partial_i(|x|^{-2\nu}\partial_i v) + |x|^{-(p(\alpha)-\varepsilon)\nu+\alpha} v^{p(\alpha)-1-\varepsilon} = 0.$$

Hence, $v(x)$ satisfies Lemma 2.5 when $b = -2\nu$,$a = -(p(\alpha) - \varepsilon)\nu + \alpha$, $q = p(\alpha) - 1 - \varepsilon$ and $K = 1$. □

4 Some basic estimates

Set $r = |x|$. Let $v(r) = |x|^\nu u_\varepsilon(x)$. Then $v(r)$ satisfies

$$\begin{cases} v'' + \dfrac{N-1-2\nu}{r} v' + \dfrac{1}{r^{(p(\alpha)-2-\varepsilon)\nu-\alpha}} v^{p(\alpha)-1-\varepsilon} = 0, \\ v(r) > 0, \text{ for } 0 < r < R, \\ v(R) = 0. \end{cases} \tag{4.1}$$

Let $t = \left(\frac{N-2\nu-2}{r}\right)^{N-2\nu-2}$ and $y(t) = (N-2\nu-2)^{-g(\alpha,\varepsilon)} v(r)$, where $g(\alpha,\varepsilon) = \frac{(p(\alpha)-2-\varepsilon)\nu-\alpha}{p(\alpha)-2-\varepsilon}$. Then problem (4.1) can be rewritten as

$$\begin{cases} y''(t) = -t^{-k(\alpha,\varepsilon)} y^{p(\alpha)-1-\varepsilon}, \\ y(t) > 0, \text{ for } T < t < \infty, \\ y(T) = 0, \end{cases} \tag{4.2}$$

where $k(\alpha,\varepsilon) = \frac{2m+\alpha}{m-1} - \frac{(p(\alpha)-2-\varepsilon)\nu}{m-1}$, $m = 1 + 2\sqrt{\bar{\mu} - \mu} = N - 2\nu - 1$, $T = \left(\frac{m-1}{R}\right)^{m-1}$, $p(\alpha) - 1 = 2k(\alpha,\varepsilon) - 3 - \frac{2\nu\varepsilon}{m-1}$.

In order to simplify the expression, we will always replace $k(\alpha,\varepsilon)$ with k in the sequel.

First we give

Lemma 4.1. *Let $y(t)$ be a solution of problem (4.2), then there exists a positive number $\gamma < \infty$ such that*

$$\lim_{t\to\infty} y'(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} y(t) = \gamma.$$

Proof. By Proposition 3.3, it is obvious that $y(t)$ is bounded in $[T,\infty)$. From (4.2), we know $y''(t) < 0$ for all $t > T$, so $y'(t)$ decreases strictly in $t \in (T,\infty)$. Hence

$$y'(t) \to c \quad \text{x0as } t \to +\infty.$$

If $c > 0$, we can deduce $y(t) \to +\infty$ when $t \to +\infty$. Similarly, when $c < 0$, we have $y(t) \to -\infty$ when $t \to +\infty$. However, the boundedness of $y(t)$ leads to the contradiction. Hence, $\lim_{t\to\infty} y'(t) = 0$ and $\lim_{t\to\infty} y(t) = \gamma$. □

Remark 4.2.

(i) From Lemma 4.1, if we define $v(0) = \lim_{r\to 0} v(r) = (N-2\nu-2)^{g(\alpha,\varepsilon)}\gamma$, then $v(r) \in C[0,R]$. Furthermore, $v'(r) < 0$ for all $r \in (0,R]$.

(ii) $y'(t) > 0$ for all $t > T$ and $y'(t) \sim \frac{1}{k-1} t^{1-k} \gamma^{p(\alpha)-1-\varepsilon}$ as $t \to \infty$.

Next, we consider

$$\begin{cases} y''(t) + t^{-k} y^{p(\alpha)-1-\varepsilon} = 0, \quad t < \infty, \\ \lim\limits_{t\to\infty} y(t) = \gamma, \end{cases} \tag{4.3}$$

where $\gamma > 0$.

Since $k > 2$, it follows from [2] that problem (4.3) has a unique solution which will be denoted by $y(t,\gamma)$ for every $\gamma > 0$. Define

$$T(\gamma) = \inf\{t > 0 : y(t,\gamma) > 0 \text{ on } (t,\infty)\}. \tag{4.4}$$

From Lemma 4.1, we have $\lim_{t\to\infty} w(s) = 1$, where $w(s) = \frac{y(t)}{\gamma}$. Hence,

$$T(1) = \inf\{s > 0 : w(s,1) > 0 \text{ on } (s,\infty)\}.$$

Set $t = \gamma^{\frac{p(\alpha)-2-\varepsilon}{k-2}} s$, then

$$w''(s) = -s^{-k} w^{p(\alpha)-1-\varepsilon}(s).$$

So, we have

$$T(\gamma) = \gamma^{\frac{p(\alpha)-2-\varepsilon}{k-2}} T(1).$$

By Lemma 5.1 in Section 5, $T(1) > 0$. Thus for every $\gamma > 0$, $T(\gamma) > 0$.

Hence, for any $T > 0$ and given $\varepsilon > 0$ small, there exists a unique γ such that problem (4.3) has a solution $y(t,\gamma)$ such that $\gamma > 0$, $T(\gamma) > 0$.

Remark 4.3. From the above analysis, when Ω is a ball centered at the origin, we conclude that the solution to problem (1.1) is unique.

Now we give an upper and lower bound for $y(t,\gamma)$.

Lemma 4.4. *Suppose $\varepsilon > 0$, then*

$$y(t,\gamma) < z(t,\gamma), \quad \textit{for } T(\gamma) \le t < \infty, \tag{4.5}$$

where

$$z(t,\gamma) = \gamma \left(1 + \frac{1}{k-1}\frac{\gamma^{p(\alpha)-2-\varepsilon}}{t^{k-2}}\right)^{-\frac{1}{k-2}}.$$

Proof. Since

$$(y' t^{k-1} y^{1-k})' = -(k-1) t^{k-2} y^{-k} H_1(t),$$

where

$$H_1(t) = t(y')^2 - yy' + \frac{1}{k-1} t^{1-k} y^{p+1}$$

and $y'(t) \sim \frac{1}{k-1} t^{1-k} \gamma^{p(\alpha)-1-\varepsilon}$ (see Remark 4.2), we have $\lim_{t\to\infty} H_1(t) = 0$.

By $H_1'(t) = \frac{1}{k-1} t^{1-k} y'(t)(p-2k+3) y^p$, we have

$$H_1'(t) < 0 \quad \text{for } \forall t \in [T,\infty).$$

Hence $H_1(t)$ decreases strictly on $[T,\infty)$. In combination with $\lim_{t\to\infty} H_1(t) = 0$, we can obtain $H_1(t) > 0$ on (T,∞) which implies $(y' t^{k-1} y^{1-k})' < 0$.

Integrating $(y' t^{k-1} y^{1-k})' < 0$ from $t > T$ to $t = \infty$, we deduce

$$y^{1-k} y'(t) > \frac{1}{k-1} \gamma^{p-k+1} t^{1-k}, \quad \text{for } T < t < \infty.$$

Integrating the above equation again from $t > T$ to $t = \infty$, we deduce

$$y^{2-k}(t) > \frac{1}{k-1} \gamma^{p-k+1} t^{2-k} + \gamma^{2-k}, \quad \text{for } T < t < \infty,$$

which implies the conclusion. □

Remark 4.5. The function $z(t,\gamma)$ is the solution of the following problem

$$\begin{cases} z''(t) + t^{-k}\gamma^{-(\frac{2\nu}{m-1}+1)\varepsilon} z^{2k-3} = 0, \quad 0 < t < \infty, \\ \lim_{t\to\infty} z(t,\gamma) = \gamma. \end{cases} \tag{4.6}$$

In the sequel, $z(t,\gamma)$ plays an important role.

Set

$$T_{\alpha,\varepsilon} = \frac{\gamma^{\frac{p(\alpha)-2-\varepsilon}{k-2}}}{k_1(\alpha,\varepsilon)} = \frac{\gamma^{2-\frac{m-1-2\nu}{(m-1)(k-2)}\varepsilon}}{k_1(\alpha,\varepsilon)}, \tag{4.7}$$

where $k_1(\alpha,\varepsilon) = (k-1)^{\frac{1}{k-2}}$.

Then for any $\beta > 0$, direct computation gives

$$z(\beta T_{\alpha,\varepsilon},\gamma) = C_{\alpha,\beta,\varepsilon}\gamma, \tag{4.8}$$

where $C_{\alpha,\beta,\varepsilon} = \frac{\beta}{(1+\beta^{k-2})^{\frac{1}{k-2}}}$.

Lemma 4.6. *Let $\beta > 0$ and $\varepsilon > 0$, then for every $t \geq \beta T_{\alpha,\varepsilon}$,*

$$y(t,\gamma) \geq z(t,\gamma)(1 - d_{\alpha,\beta,\varepsilon}\varepsilon),$$

where

$$d_{\alpha,\beta,\varepsilon} = \frac{(1 - C_{\alpha,\beta,\varepsilon})(1 + 2\nu/(m-1))}{C_{\alpha,\beta,\varepsilon}^{2+(1+2\nu/(m-1))\varepsilon}}.$$

Proof. Integrating (4.3) twice, we have

$$y(t,\gamma) = \gamma - \int_t^\infty (s-t)s^{-k} y^{2k-3-(1+2\nu/(m-1))\varepsilon}(s,\gamma)ds.$$

Hence, by Lemma 4.4, we obtain

$$y(t,\gamma) > \gamma - \int_t^\infty (s-t)s^{-k} z^{2k-3-(1+2\nu/(m-1))\varepsilon}(s,\gamma)ds.$$

Similarly, integrate (4.6) for z twice, then

$$z(t,\gamma) = \gamma - \int_t^\infty (s-t)s^{-k}\gamma^{-(1+2\nu/(m-1))\varepsilon} z^{2k-3}(s,\gamma)ds.$$

Hence

$$y(t,\gamma) > z(t,\gamma) - \int_t^\infty (s-t)s^{-k} z^{2k-3}(s,\gamma)(z^{-(1+2\nu/(m-1))\varepsilon} - \gamma^{-(1+2\nu/(m-1))\varepsilon})ds. \tag{4.9}$$

By the mean value theorem, we deduce

$$|z^{-(1+2\nu/(m-1))\varepsilon} - \gamma^{-(1+2\nu/(m-1))\varepsilon}| = (1+2\nu/(m-1))\varepsilon\theta^{-1-(1+2\nu/(m-1))\theta}|z(s,\gamma) - \gamma|,$$

where $z(s,\gamma) \leq \theta \leq \gamma$.

Hence, using (4.8), if $\beta T_{\alpha,\varepsilon} \leq t < \infty$, we have

$$\begin{aligned} &|z^{-(1+2\nu/(m-1))\varepsilon} - \gamma^{-(1+2\nu/(m-1))\varepsilon}| \\ &\quad \leq (1+2\nu/(m-1))\varepsilon(C_{\alpha,\beta,\varepsilon}\gamma)^{-1-(1+2\nu/(m-1))\varepsilon}\gamma \\ &\quad \leq (1+2\nu/(m-1))\varepsilon C_{\alpha,\beta,\varepsilon}^{-1-(1+2\nu/(m-1))\varepsilon}\gamma^{-(1+2\nu/(m-1))\varepsilon}. \end{aligned}$$

Using this bound in (4.9), if $t \geq \beta T_{\alpha,\varepsilon}$,

$$\begin{aligned} y(t,\gamma) &> z(t,\gamma) - (1+2\nu/(m-1))\varepsilon C_{\alpha,\beta,\varepsilon}^{-1-(1+2\nu/(m-1))\varepsilon} \\ &\quad \times \int_t^\infty (s-t)s^{-k}\gamma^{-(1+2\nu/(m-1))\varepsilon} z^{2k-3}(s,\gamma)ds \\ &= z(t,\gamma) + (1+2\nu/(m-1))\varepsilon C_{\alpha,\beta,\varepsilon}^{-1-(1+2\nu/(m-1))\varepsilon}(z(t,\gamma)-\gamma). \end{aligned} \tag{4.10}$$

On the other hand, by (4.8), if $t \geq \beta T_{\alpha,\varepsilon}$,

$$\gamma = C_{\alpha,\beta,\varepsilon}^{-1} z(\beta T_{\alpha,\varepsilon},\gamma) \leq C_{\alpha,\beta,\varepsilon}^{-1} z(t,\gamma).$$

So we can deduce from (4.10) that

$$\begin{aligned} y(t,\gamma) &> z(t,\gamma)(1 + \frac{(1+2\nu/(m-1))}{C_{\alpha,\beta,\varepsilon}^{1+(1+2\nu/(m-1))\varepsilon}}(1 - \frac{1}{C_{\alpha,\beta,\varepsilon}})\varepsilon) \\ &= z(t,\gamma)(1 - d_{\alpha,\beta,\varepsilon}\varepsilon), \end{aligned}$$

which is the bound we want to prove. □

Now we return to problem (4.2). We fix T and denote the solution by $y(t)$. Then

$$\gamma(\varepsilon) = \lim_{t\to\infty} y(t)$$

and $\gamma(\varepsilon)$ depends on ε. The next lemma tells us the asymptotic behavior of $\gamma(\varepsilon)$ as $\varepsilon \to 0$.

Lemma 4.7.

$$\lim_{\varepsilon\to 0} \gamma(\varepsilon) = \infty$$

Proof. By contradiction, we can assume there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ as $n \to \infty$, and a number $M > 0$ such that $\gamma(\varepsilon_n) \leq M$ for all n. Then we can choose a number $\beta_1 > 0$ satisfying

$$\beta_1 T_{\alpha,\varepsilon_n} = \beta_1 k_1^{-1}\gamma(\varepsilon_n)^{2-\frac{m-1+2\nu}{(m-1)(k-2)}\varepsilon} \leq \beta_1(\tfrac{n-2}{n+\alpha})^{\frac{n-2}{\alpha+2}} M^2 + o(1) \leq T \quad \text{for large } n.$$

So by Lemma 4.6, we have

$$z(t,\gamma(\varepsilon_n))(1 - d_{\alpha,\beta,\varepsilon_n}\varepsilon_n) < z(T,\gamma(\varepsilon_n))(1 - d_{\alpha,\beta,\varepsilon_n}\varepsilon_n) \leq y(T,\gamma(\varepsilon_n)) = 0,$$

for $0 < t < \beta_1 T_{\alpha,\varepsilon_n}$ and large n, which is impossible. □

Finally, we give two formulae to use later. Define incomplete Beta function

$$B(\varsigma,P,Q) = \int_\varsigma^\infty x^{P-1}(1+x)^{-P-Q}dx,$$

where P and Q are positive parameters. It is well-known that

$$B(0,P,Q) = \frac{\Gamma(P)\Gamma(Q)}{\Gamma(P+Q)}. \tag{4.11}$$

Lemma 4.8. *Suppose $k > 2, p = 2k-3-(1+\frac{2\nu}{m-1})\varepsilon$ and ε small. Then*

$$(i) \qquad \int_t^\infty s^{-k}z^p(s,\gamma)ds = k_1(\alpha,\varepsilon)k_2(\alpha,\varepsilon)\gamma^{-1+\varphi(\alpha,\varepsilon)}B(\tau(\alpha,\varepsilon),1-\varphi(\alpha,\varepsilon),k_2(\alpha,\varepsilon)),$$

$$(ii) \qquad \int_t^\infty s^{-k}z^{p+1}(s,\gamma)ds = k_1(\alpha,\varepsilon)k_2(\alpha,\varepsilon)\gamma^{\varphi(\alpha,\varepsilon)}B(\tau(\alpha,\varepsilon),k_2(\alpha,\varepsilon)-\varphi(\alpha,\varepsilon),k_2(\alpha,\varepsilon)),$$

where $\varphi(\alpha,\varepsilon) = \frac{m-1+2\nu}{(m-1)(k-2)}\varepsilon$, $k_1(\alpha,\varepsilon) = (k-1)^{\frac{1}{k-2}}$, $k_2(\alpha,\varepsilon) = \frac{k-1}{k-2}$, $\tau(\alpha,\varepsilon) = (\frac{t}{T_{\alpha,\varepsilon}})^{k-2}$.

Proof. (i) Insert the expression

$$z(t,\gamma) = \gamma \left(1 + \frac{1}{k-1} \frac{\gamma^{p(\alpha)-2-\varepsilon}}{t^{k-2}} \right)^{-\frac{1}{k-2}}$$

into the integral

$$\begin{aligned} \int_t^\infty s^{-k} z^p(s,\gamma) ds &= \gamma^p \int_t^\infty s^{p-k} \left(s^{k-2} + \frac{1}{k-1} \gamma^{p-1} \right)^{-\frac{p}{k-2}} ds \\ &= \gamma^p \int_t^\infty s^{p-k} (s^{k-2} + T_{\alpha,\varepsilon}^{k-2})^{-\frac{p}{k-2}} ds \end{aligned} \tag{4.12}$$

and by routine calculus, we can get the result as follows.

By making the change of variable $x = (\frac{s}{T_{\alpha,\varepsilon}})^{k-2}$, we can write (4.12) as

$$\begin{aligned} \int_t^\infty s^{-k} z^p(s,\gamma) ds &= \gamma^p \int_t^\infty s^{p-k} (s^{k-2} + T_{\alpha,\varepsilon}^{k-2})^{-\frac{p}{k-2}} ds \\ &= \frac{\gamma^p}{k-2} T_{\alpha,\varepsilon}^{1-k} \int_{(\frac{t}{T_{\alpha,\varepsilon}})^{k-2}}^\infty x^{P-1} (1+x)^{-P-Q} dx, \end{aligned}$$

where $P = \frac{p-k-1}{k-2}$ and $Q = \frac{k-1}{k-2}$.

Since

$$\frac{\gamma^p}{k-2} T_{\alpha,\varepsilon}^{1-k} = k_1(\alpha,\varepsilon) k_2(\alpha,\varepsilon) \gamma^{-1+\varphi(\alpha,\varepsilon)},$$

we have

$$\int_t^\infty s^{-k} z^p(s,\gamma) ds = k_1(\alpha,\varepsilon) k_2(\alpha,\varepsilon) \gamma^{-1+\varphi(\alpha,\varepsilon)} B(\tau(\alpha,\varepsilon), 1-\varphi(\alpha,\varepsilon), k_2(\alpha,\varepsilon)),$$

where $\varphi(\alpha,\varepsilon) = \frac{m-1+2\nu}{(m-1)(k-2)}\varepsilon$, $k_1(\alpha,\varepsilon) = (k-1)^{\frac{1}{k-2}}$, $k_2(\alpha,\varepsilon) = \frac{k-1}{k-2}$, $\tau(\alpha,\varepsilon) = (\frac{t}{T_{\alpha,\varepsilon}})^{k-2}$.

(ii) In a similar way as in (i). □

We end this section by giving

$$\begin{aligned} &\lim_{\varepsilon\to 0} k = \frac{2N-2+\alpha}{N-2} \triangleq k_0, \qquad \lim_{\varepsilon\to 0} k_1(\alpha,\varepsilon) = (\frac{N+\alpha}{N-2})^{\frac{N-2}{\alpha+2}} \triangleq k_1, \\ &\lim_{\varepsilon\to 0} k_2(\alpha,\varepsilon) = \frac{N+\alpha}{\alpha+2} \triangleq k_2, \qquad \lim_{\varepsilon\to 0} C_{\alpha,\beta,\varepsilon} = \frac{\beta}{(1+\beta^{\frac{\alpha+2}{N-2}})^{\frac{N-2}{\alpha+2}}} \triangleq C_{\alpha,\beta}, \\ &\lim_{\varepsilon\to 0} d_{\alpha,\beta,\varepsilon} = \frac{(1-c_{\alpha,\beta})(1+2\nu/(m-1))}{c_{\alpha,\beta}^2} \triangleq d_{\alpha,\beta}, \qquad \lim_{\varepsilon\to 0} \varphi(\alpha,\varepsilon) = \lim_{\varepsilon\to 0} \tau(\alpha,\varepsilon) = 0. \end{aligned} \tag{4.13}$$

5 Proof of the main results

Note that if $u_\varepsilon(x)$ is a solution of problem (1.1) when $\Omega = B_R$, then from the previous analysis, we know that

$$\lim_{|x|\to 0} u_\varepsilon(x)|x|^\nu = (N-2\nu-2)^{g(\alpha,\varepsilon)} \gamma(\varepsilon),$$

where $g(\alpha,\varepsilon) = \frac{(p(\alpha)-2-\varepsilon)\nu-\alpha}{p(\alpha)-2-\varepsilon}$ and $R = (m-1)T^{-1/(m-1)}$. Thus we need to understand how $\gamma(\varepsilon)$ tends to infinity as $\varepsilon \to 0$.

We define the following Pohozaev functional introduced from [1] and [22],

$$H(t) = ty'^2 - yy' + 2t^{1-k}\frac{y^{p+1}}{p+1}, \tag{5.1}$$

where

$$p = 2k - 3 - \left(1 + \frac{2\nu}{m-1}\right)\varepsilon = p(\alpha) - 1 - \varepsilon.$$

If $y(t)$ solves problem (4.3), then

$$H'(t) = -\frac{(1+2\nu/(m-1))\varepsilon}{p+1}t^{-k}y^{p+1} \tag{5.2}$$

and $y'(t) = O(t^{1-k})$ as $t \to \infty$ (see Remark 4.2). Hence

$$\lim_{t\to\infty} H(t) = 0.$$

Since $H(T) = Ty'^2(T)$, integrating (5.2) from $t > T$ to $t = \infty$, we obtain

$$Ty'^2(T) = \frac{(1+2\nu/(m-1))\varepsilon}{p+1}\int_T^\infty t^{-k}y^{p+1}(t)dt. \tag{5.3}$$

This equation is crucial for us to obtain the desired results.

Lemma 5.1. *Let $T(\gamma)$ be defined as (4.4), then $T(1) > 0$.*

Proof. By Lemma 4.2, $y(t,1) \le z(t,1)$ for $t \ge T(1)$. Suppose in contrast that $T(1) = 0$, then

$$y'(0,1) \le z'(0,1) = k_1^{k-1}(\alpha,\varepsilon).$$

So

$$y(t,1) \le k_1^{k-1}(\alpha,\varepsilon)t, \qquad t \ge 0,$$

which means $H(0) = 0$.

On the other hand, combination of (5.2) and the fact $\lim_{t\to\infty} H(t) = 0$ yields $H(t) > 0$ for $T(1) \le t < \infty$. This is a contradiction and our conclusion follows. □

Lemma 5.2. $\lim_{\varepsilon\to 0}\gamma^{1-\varphi(\alpha,\varepsilon)}y'(T) = k_1$, *where $\gamma = \gamma(\varepsilon)$.*

Proof. Integrating equation (4.2) over (T,∞), we derive

$$y'(T) = \int_T^\infty t^{-k}y^p(t)dt < \int_T^\infty t^{-k}z^p(t)dt. \tag{5.4}$$

Hence, by Lemma 4.8 (i) and Lemma 4.4, as $\varepsilon \to 0$,

$$\gamma^{1-\varphi(\alpha,\varepsilon)}y'(T) \le k_1(\alpha,\varepsilon)k_2(\alpha,\varepsilon)B\left(\left(\frac{T}{T_{\alpha,\varepsilon}}\right)^{k-2}, 1-\varphi(\alpha,\varepsilon), k_2(\alpha,\varepsilon)\right) \to k_1k_2B(0,1,k_2).$$

By (4.11) and the fact that $\Gamma(x+1) = x\Gamma(x)$, we deduce

$$k_1k_2B(0,1,k_2) = \frac{k_1k_2}{k_2} = k_1.$$

Therefore

$$\limsup_{\varepsilon\to 0} \gamma^{1-\varphi(\alpha,\varepsilon)} y'(T) \le k_1. \tag{5.5}$$

Next, we shall show that for any $\delta > 0$,

$$\liminf_{\varepsilon\to 0} \gamma^{1-\varphi(\alpha,\varepsilon)} y'(T) \ge k_1 - \delta, \tag{5.6}$$

which completes the proof of this lemma.

For a given $\beta > 0$, by (4.2) and Lemma 4.7, we can choose $\varepsilon > 0$ so small that $\beta T_{\alpha,\varepsilon} > T$. Thus (5.4) can be written as

$$\begin{aligned}\gamma^{1-\varphi(\alpha,\varepsilon)} y'(T) &= \gamma^{1-\varphi(\alpha,\varepsilon)} \int_T^{\beta T_{\alpha,\varepsilon}} t^{-k} y^p(t)dt + \gamma^{1-\varphi(\alpha,\varepsilon)} \int_{\beta T_{\alpha,\varepsilon}}^{\infty} t^{-k} y^p(t)dt \\ &= G_1(\alpha,\beta,\varepsilon) + G_2(\alpha,\beta,\varepsilon).\end{aligned} \tag{5.7}$$

Because $z(t) \le \left(\frac{\gamma}{T_{\alpha,\varepsilon}}\right)t$ for all $t > 0$, using Lemma 4.4, we have

$$\begin{aligned}G_1(\alpha,\beta,\varepsilon) &\le \gamma^{1-\varphi(\alpha,\varepsilon)} \left(\frac{\gamma}{T_{\alpha,\varepsilon}}\right)^p \int_T^{\beta T_{\alpha,\varepsilon}} t^{p-k} dt \\ &< \gamma^{1-\varphi(\alpha,\varepsilon)} \left(\frac{\gamma}{T_{\alpha,\varepsilon}}\right)^p \frac{(\beta T_{\alpha,\varepsilon})^{p-k+1}}{p-k+1} \\ &= \frac{k_1^{k-1}(\alpha,\varepsilon)}{(k-2)(1-\varphi(\alpha,\varepsilon))} \beta^{(k-2)(1-\varphi(\alpha,\varepsilon))}.\end{aligned} \tag{5.8}$$

On the other hand, by Lemma 4.3 and (i) of Lemma 4.8, for $\varepsilon > 0$ small,

$$\begin{aligned}G_2(\alpha,\beta,\varepsilon) &> \gamma^{1-\varphi(\alpha,\varepsilon)} (1-d_{\alpha,\beta,\varepsilon}\varepsilon)^p \int_{\beta T_{\alpha,\varepsilon}}^{\infty} t^{-k} z^p(t)dt \\ &= (1-d_{\alpha,\beta,\varepsilon}\varepsilon)^p k_1(\alpha,\varepsilon) k_2(\alpha,\varepsilon) B(\beta^{k-2}, 1-\varphi(\alpha,\varepsilon), k_2(\alpha,\varepsilon)).\end{aligned} \tag{5.9}$$

Combining (5.7), (5.8) and (5.9), we derive

$$\liminf_{\varepsilon\to 0} \gamma^{1-\varphi(\alpha,\varepsilon)} y'(T) \ge k_1 k_2 B(\beta^{k_0-2}, 1, k_2) - L_1 \beta^{k_0-2},$$

where $L_1 = \lim_{\varepsilon\to 0} \frac{k_1^{k-1}(\alpha,\varepsilon)}{(k-2)(1-\varphi(\alpha,\varepsilon))}$.

Hence, given any $\delta > 0$, we can choose $\beta > 0$ such that (5.6) holds. This completes the proof. □

Lemma 5.3. $\lim_{\varepsilon\to 0} \gamma^{1-\varphi(\alpha,\varepsilon)} \int_T^{\infty} t^{-k} y^{p+1}(t, \gamma(\varepsilon))dt = k_1 k_2 [\Gamma(k_2)]^2/\Gamma(2k_2)$, *where* $\gamma = \gamma(\varepsilon)$.

Proof. By Lemma 4.4 and Lemma 4.8(ii), as $\varepsilon \to 0$, we deduce

$$\begin{aligned}\gamma^{-\varphi(\alpha,\varepsilon)} \int_T^{\infty} t^{-k} y^{p+1}(t,\gamma)dt &\le \gamma^{-\varphi(\alpha,\varepsilon)} \int_T^{\infty} t^{-k} z^{p+1}(t,\gamma)dt \\ &= k_1(\alpha,\varepsilon) k_2(\alpha,\varepsilon) B\left(\left(\frac{T}{T_{\alpha,\varepsilon}}\right)^{k-2}, k_2(\alpha,\varepsilon)-\varphi(\alpha,\varepsilon), k_2(\alpha,\varepsilon)\right) \\ &\to k_1 k_2 B(0, k_2, k_2) \\ &= k_1 k_2 [\Gamma(k_2)]^2/\Gamma(2k_2).\end{aligned} \tag{5.10}$$

Hence,

$$\limsup_{\varepsilon\to 0}\gamma^{-\varphi(\alpha,\varepsilon)}\int_T^\infty t^{-k}y^{p+1}(t,\gamma)dt \le k_1k_2[\Gamma(k_2)]^2/\Gamma(2k_2).$$

Next, we shall show that for any $\delta > 0$,

$$\liminf_{\varepsilon\to 0}\gamma^{-\varphi(\alpha,\varepsilon)}\int_T^\infty t^{-k}y^{p+1}(t,\gamma)dt \ge k_1k_2[\Gamma(k_2)]^2/\Gamma(2k_2)-\delta.$$

which completes the proof of this lemma.

For a given $\beta > 0$, by (4.2) and Lemma 4.7, we can choose a sufficiently small ε such that $\beta T_{\alpha,\varepsilon} > T$. Thus (5.10) can be written as

$$\begin{aligned}\gamma^{-\varphi(\alpha,\varepsilon)}&\int_T^\infty t^{-k}y^{p+1}(t,\gamma)dt\\ &=\gamma^{-\varphi(\alpha,\varepsilon)}\int_T^{\beta T_{\alpha,\varepsilon}} t^{-k}y^{p+1}(t,\gamma)dt+\gamma^{-\varphi(\alpha,\varepsilon)}\int_{\beta T_{\alpha,\varepsilon}}^\infty t^{-k}y^{p+1}(t,\gamma)dt\\ &=G_3(\alpha,\beta,\varepsilon)+G_4(\alpha,\beta,\varepsilon).\end{aligned}\tag{5.11}$$

Because $z(t) \le (\frac{\gamma}{T_{\alpha,\varepsilon}})t$ for all $t > 0$, using Lemma 4.4, we have

$$\begin{aligned}G_3(\alpha,\beta,\varepsilon) &< \gamma^{-\varphi(\alpha,\varepsilon)}\int_T^{\beta T_{\alpha,\varepsilon}} t^{-k}z^{p+1}(t,\gamma)dt\\ &\le \gamma^{-\varphi(\alpha,\varepsilon)}\int_T^{\beta T_{\alpha,\varepsilon}} t^{-k}\left(\frac{\gamma}{T_{\alpha,\varepsilon}}t\right)^{p+1}dt\\ &=\frac{k_1(\alpha,\varepsilon)}{k-1-\varepsilon}\beta^{k-1-\varepsilon}.\end{aligned}\tag{5.12}$$

On the other hand, by Lemma 4.3 and (ii) of Lemma 4.8, for $\varepsilon > 0$ small,

$$\begin{aligned}G_4(\alpha,\beta,\varepsilon) &> \gamma^{-\varphi(\alpha,\varepsilon)}(1-d_{\alpha,\beta,\varepsilon}\varepsilon)^{p+1}\int_{\beta T_{\alpha,\varepsilon}}^\infty t^{-k}z^{p+1}(t,\gamma)dt\\ &=(1-d_{\alpha,\beta,\varepsilon}\varepsilon)^{p+1}k_1(\alpha,\varepsilon)k_2(\alpha,\varepsilon)B(\beta^{k-2},k_2(\alpha,\varepsilon)-\varphi(\alpha,\varepsilon),k_2(\alpha,\varepsilon)).\end{aligned}\tag{5.13}$$

Combining (5.11), (5.12) and (5.13), we derive

$$\liminf_{\varepsilon\to 0}\gamma^{-\varphi(\alpha,\varepsilon)}\int_T^\infty t^{-k}y^{p+1}(t,\gamma)dt \ge k_1k_2B(\beta^{k_0-2},k_2,k_2)-L_2\beta^{k_0-1}.\tag{5.14}$$

where $L_2=\lim_{\varepsilon\to 0}\frac{k_1(\alpha,\varepsilon)}{k-1-\varepsilon}$.

Hence, given any $\delta > 0$, we can choose $\beta > 0$ such that this conclusion is tenable. □

Now we are ready to analyze the behavior of $\gamma(\varepsilon)$ as $\varepsilon \to 0$.

Theorem 5.4. *Let $y(t)$ be the solution of problem (4.2) and denote*

$$\gamma(\varepsilon)=\lim_{t\to\infty}y(t).$$

Then

$$\lim_{\varepsilon\to 0}\varepsilon\gamma^2(\varepsilon)=\frac{4(N+\alpha)\sqrt{\bar{\mu}-\mu}}{(N-2)^2}\frac{k_1}{k_2}\frac{\Gamma(2k_2)}{[\Gamma(k_2)]^2}T,$$

where k_1 and k_2 are defined by (4.13).

Proof. Noting (5.3), we have

$$\left(1+\frac{2\nu}{m-1}\right)\varepsilon\gamma^{2-\varphi(\alpha,\varepsilon)} = (p+1)T\frac{[\gamma^{1-\varphi(\alpha,\varepsilon)}y'(T)]^2}{\gamma^{-\varphi(\alpha,\varepsilon)}\int_T^\infty t^{-k}y^{p+1}(t)dt}. \tag{5.15}$$

From (5.15), Lemma 5.2 and Lemma 5.3, we have

$$\lim_{\varepsilon\to 0}\varepsilon\gamma^{2-\varphi(\alpha,\varepsilon)}(\varepsilon) = \frac{4(N+\alpha)\sqrt{\bar{\mu}-\mu}}{(N-2)^2}\frac{k_1}{k_2}\frac{\Gamma(2k_2)}{[\Gamma(k_2)]^2}T. \tag{5.16}$$

The exponent $2-\varphi(\alpha,\varepsilon)$ in (5.16) may be replaced by 2, because

$$\lim_{\varepsilon\to 0}\gamma(\varepsilon)^{\varphi(\alpha,\varepsilon)} = 1. \tag{5.17}$$

To see this, note that (5.16) implies that

$$\gamma(\varepsilon)^{2-\varphi(\alpha,\varepsilon)} < \frac{C}{\varepsilon},$$

for small ε and some constant C. Therefore

$$\ln\gamma(\varepsilon)^{\varphi(\alpha,\varepsilon)} = \varphi(\alpha,\varepsilon)\ln\gamma(\varepsilon) < \frac{\varphi(\alpha,\varepsilon)}{2-\varphi(\alpha,\varepsilon)}\ln\frac{C}{\varepsilon}.$$

This means that

$$\ln\gamma(\varepsilon)^{\varphi(\alpha,\varepsilon)} \to 0 \quad \text{as } \varepsilon\to 0$$

and (5.17) follows. □

Proof of Theorem 1.2. If $y(t)$ is the solution of (4.2), then

$$v(x) = (N-2\nu-2)^{g(\alpha,\varepsilon)}y((m-1)^{m-1}|x|^{1-m})$$

is the solution of problem (3.1) in B_R with $R=(m-1)T^{-1/(m-1)}$ and

$$u_\varepsilon(x) = |x|^{-\nu}v(x) = (N-2\nu-2)^{g(\alpha,\varepsilon)}|x|^{-\nu}y((m-1)^{m-1}|x|^{1-m}).$$

Therefore, Theorem 5.4 yields

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{|x|\to 0}\varepsilon u_\varepsilon^2|x|^{2\nu} &= \lim_{\varepsilon\to 0}(N-2\nu-2)^{2g(\alpha,\varepsilon)}\varepsilon\gamma^2(\varepsilon)\\ &= 2(\alpha+2)(2\sqrt{\bar{\mu}-\mu})^{\frac{2N+\alpha-2}{\alpha+2}}(N+\alpha)^{\frac{N-2}{\alpha+2}}(N-2)^{-\frac{2\alpha+N+2}{\alpha+2}}\frac{\Gamma(\frac{2(N+2)}{\alpha+2})}{\left[\Gamma(\frac{N+\alpha}{\alpha+2})\right]^2}\frac{1}{R^{2\sqrt{\bar{\mu}-\mu}}},\end{aligned}$$

which is the content of Theorem 1.2. □

Before proving Theorem 1.3, we first give two lemmas.
As a first observation, we note from Lemma 4.4 that

$$y(t,\gamma) < z(t,\gamma) < k_1 t\gamma^{-1+\varphi(\alpha,\varepsilon)} \quad \text{for } t>T.$$

Hence, by Theorem 5.4, for every fixed $t>T$, we have

$$y(t,\gamma(\varepsilon)) = O(\varepsilon^{\frac{1}{2}}) \quad \text{as } \varepsilon\to 0.$$

If we allow t to tend to infinity as $\varepsilon\to 0$, we obtain the following upper bound.

Lemma 5.5. *For every $M > 0$ and $\xi \in (0, \frac{1}{2})$,*

$$\lim_{\varepsilon\to 0} \sup\{y(t, \gamma(\varepsilon)) : T < t < M\varepsilon^{-\xi}\} = 0.$$

To obtain information about the limiting form of $y(t, \gamma(\varepsilon))$ as $\varepsilon \to 0$, we are led by Lemma 5.2 to multiply y as the weight factor $\gamma^{1-\varphi(\alpha,\varepsilon)}$, because

$$\lim_{\varepsilon\to 0} \gamma^{1-\varphi(\alpha,\varepsilon)} y'(T) = k_1. \tag{5.18}$$

In the next lemma, we show that (5.18) continues to be true for values of $t > T$, provided

$$t = O(\gamma^{\sigma}),$$

where σ may be any number less than 2.

Lemma 5.6. *Let $M > 0$ and $0 < \sigma < 2$. Then*

$$\lim_{\varepsilon\to 0} \sup\{|\gamma^{1-\varphi(\alpha,\varepsilon)} y'(t) - k_1| : T < t < M\gamma^{\sigma}\} = 0.$$

Proof. By Lemma 5.2, and the concavity of y,

$$\limsup_{\varepsilon\to 0} \gamma^{1-\varphi(\alpha,\varepsilon)} y(t, \gamma) \le k_1, \qquad \forall t \ge T. \tag{5.19}$$

To get a lower bound on y', we also use the concave property of y. For $\forall t \ge T$ and for $t_0 > t$, we have

$$y'(t_0) > \frac{y(t_0) - y(t)}{t_0 - t} > \frac{1}{t_0}\{y(t_0) - y(t)\}.$$

Hence, by Lemma 4.4,

$$\gamma^{1-\varphi(\alpha,\varepsilon)} y'(t) > \frac{\gamma^{1-\varphi(\alpha,\varepsilon)} y(t_0)}{t_0} - \frac{\gamma^{1-\varphi(\alpha,\varepsilon)} z(t)}{t} \cdot \frac{t}{t_0}. \tag{5.20}$$

We assume that $t = O(\gamma^{\sigma})$ and $0 < \sigma < 2$, so it is possible for us to substitute $\beta T_{\alpha,\varepsilon}$, $\beta > 0$ for t_0. Hence, for $\gamma \to \infty$,

$$\frac{t}{\beta T_{\alpha,\varepsilon}} \to 0. \tag{5.21}$$

By Lemma 4.3,

$$\begin{aligned} \frac{\gamma^{1-\varphi(\alpha,\varepsilon)} y(\beta T_{\alpha,\varepsilon})}{\beta T_{\alpha,\varepsilon}} &\ge \frac{\gamma^{1-\varphi(\alpha,\varepsilon)} z(\beta T_{\alpha,\varepsilon})}{\beta T_{\alpha,\varepsilon}} (1 - d_{\alpha,\beta,\varepsilon}\varepsilon) \\ &= k_1 (1 + \beta^{k-2})^{-\frac{1}{k-2}} (1 - d_{\alpha,\beta,\varepsilon}\varepsilon). \end{aligned} \tag{5.22}$$

Thus, from (5.20)–(5.22), we conclude that

$$\liminf_{\varepsilon\to 0} \gamma^{1-\varphi(\alpha,\varepsilon)} y(t, \gamma) \ge k_1 - \delta(\beta),$$

where $\delta(\beta) \to 0$ as $\beta \to 0$. Because we can choose β small enough, this means

$$\liminf_{\varepsilon\to 0} \gamma^{1-\varphi(\alpha,\varepsilon)} y(t, \gamma) \ge k_1. \tag{5.23}$$

By (5.19) and (5.22), we obtain the desired result. □

Proof of Theorem 1.3. By the concavity of $y(t)$, we deduce

$$y'(t) \le \frac{y(t)-y(T)}{t-T} \le y'(T), \qquad t \ge T. \tag{5.24}$$

So, there exists a $\theta \in [T,t]$ such that

$$y(t) = y'(\theta)(t-T). \tag{5.25}$$

Combining Lemma 5.5, (5.25) and noting that $\lim_{\varepsilon\to 0}\gamma(\varepsilon)^{\varphi(\alpha,\varepsilon)} = 1$, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^{-\frac{1}{2}}y(t) &= \lim_{\varepsilon\to 0}\varepsilon^{-\frac{1}{2}}\gamma^{-1+\varphi(\alpha,\varepsilon)}\lim_{\varepsilon\to 0}\gamma^{1-\varphi(\alpha,\varepsilon)}y(t)\\ &= [A(k_1,k_2,T)]^{-\frac{1}{2}}\lim_{\varepsilon\to 0}\gamma^{1-\varphi(\alpha,\varepsilon)}y'(\theta)(t-T)\\ &= k_1[A(k_1,k_2,T)]^{-\frac{1}{2}}(t-T),\end{aligned} \tag{5.26}$$

where $A(k_1,k_2,T) = \frac{4(N+\alpha)\sqrt{\bar{\mu}-\mu}}{(N-2)^2}\frac{k_1}{k_2}\frac{\Gamma(2k_2)}{[\Gamma(k_2)]^2}T$,
and the convergence is uniform on bounded intervals.

For the solution $u_\varepsilon(x)$ of problem (1.1), (5.26) means that as $\varepsilon \to 0$

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^{-\frac{1}{2}}u_{\varepsilon(x)} &= \lim_{\varepsilon\to 0}|x|^{-\nu}(N-2\nu-2)^{g(\alpha,\varepsilon)}k_1[A(k_1,k_2,T)]^{-\frac{1}{2}}(t-T)\\ &= \frac{1}{2}(\alpha+2)^{-\frac{1}{2}}(2\sqrt{\bar{\mu}-\mu})^{\frac{2N-\alpha-6}{2\alpha+4}}(N+\alpha)^{\frac{N-2}{2\alpha+4}}(N-2)^{\frac{2\alpha-N+6}{2\alpha+4}}R^{\sqrt{\bar{\mu}-\mu}}\frac{\Gamma(\frac{N+\alpha}{\alpha+2})}{\left[\Gamma(\frac{2(N+\alpha)}{\alpha+2})\right]^{\frac{1}{2}}}\\ &\quad\times\left(\frac{1}{|x|^{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}} - \frac{1}{|x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}|R|^{2\sqrt{\bar{\mu}-\mu}}}\right).\end{aligned}$$

Hence, we obtain the desired result. □

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