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About existence and regularity of positive solutions for a quasilinear Schrödinger equation with singular nonlinearity

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Abstract. We establish the existence of positive solutions for the singular quasilinear Schrödinger equation

$$\begin{cases} -\Delta u - \Delta(u^2)u = h(x)u^{-\gamma} + f(x,u) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $1 < \gamma$, $h \in L^1(\Omega)$ and $h > 0$ almost everywhere in Ω. The function f may change sign on Ω. By using the variational method and some analysis techniques, the necessary and sufficient condition for the existence of a solution is obtained.

Keywords: strong singularity, variational methods, regularity, fibering methods, indefinite nonlinearity.

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1 Introduction

In this paper we study the existence of solution for the following quasilinear Schrödinger equation

$$\begin{cases} -\Delta u - \Delta(u^2)u = h(x)u^{-\gamma} + f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $1 < \gamma$, $h \in L^1(\Omega)$, $h > 0$ almost everywhere (a.e.) in Ω and $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function. We assume that the function f satisfies one of the following conditions:

$(f)_1$ $f(x,s) = b(x)s^p$, where $p \in (0,1), b \in L^\infty(\Omega)$ and $b^+ = \max\{b, 0\} \not\equiv 0$.

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$(f)_2$ $f(x,s) = -b(x)s^{22^*-1}$, where $b \in L^\infty(\Omega)$ and $b \geq 0$ a.e. in Ω.

We say that a function $u \in H_0^1(\Omega)$ is a weak solution (solution, for short) of (P) if $u > 0$ a.e. in Ω, and, for every $\varphi \in H_0^1(\Omega)$,

$$hu^{-\gamma}\varphi \in L^1(\Omega) \tag{1.1}$$

and

$$\int_\Omega [(1+2u^2)\nabla u \nabla \varphi + 2u|\nabla u|^2\varphi] = \int_\Omega h(x)u^{-\gamma}\varphi + \int_\Omega f(x,u)\varphi.$$

Consider the following quasilinear Schrödinger equation

$$-\Delta u - \Delta(u^2)u = g(x,u) \quad \text{in } \Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. When $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, recently, there appeared some works dealing with (1.2), see for example [1,17,18] and its references. In these works the nonlinearity is non-singular, and so the authors were able to combine the dual approach of [4] with classic results of variational methods to prove their main results.

When g is singular, problems of type (1.2) was studied by Do Ó–Moameni [6], Liu–Liu–Zhao [16], Wang [26] and Dos Santos–Figueiredo–Severo [24]. In [6] the authors studied the problem

$$\begin{cases} -\Delta u - \frac{1}{2}\Delta(u^2)u = \lambda|u|^2u - u - u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a ball in $\mathbb{R}^N$ centered at the origin, $0 < \gamma < 1$ and $N \geq 2$. They showed that problem (1.3) has a radially symmetric solution $u \in H_0^1(\Omega)$ for $\lambda \in I$, where I is an open interval.

Liu–Liu–Zhao in [16] considered the problem

$$\begin{cases} -\Delta_s u - \frac{s}{2^{s-1}}\Delta(u^2)u = h(x)u^{-\gamma} + \lambda u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $N \geq 3$, Δ_s is the s-Laplacian operator, $2 < 2s < p+1 < \infty$, $0 < \gamma$ and $h \geq 0$ is a nontrivial measurable function satisfying the following condition: there exist a function $\phi_0 \geq 0$ in $C_0^1(\overline{\Omega})$ and $q > N$ such that $h\phi_0^{-\gamma} \in L^q(\Omega)$. The authors used sub-supersolution method, truncation arguments and variational methods to prove the existence of a $\lambda_* > 0$ such that problem (1.4) has at least two solutions for $\lambda \in (0, \lambda_*)$.

Wang in [26], by using minimax methods and some analysis techniques, showed the existence and uniqueness of solutions to the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = h(x)u^{-\gamma} - u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $N \geq 3$, $\gamma \in (0,1)$, $p \in [2, 22^*]$, $h \in L^{\frac{22^*}{22^*-1+\gamma}}(\Omega)$ and $h > 0$ a.e. in Ω.

In [24], Dos Santos–Figueiredo–Severo studied the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = h(x)u^{-\gamma} + z(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $N \geq 3$, h is a nonnegative function, $\gamma > 0$ is a constant and the nonlinearity $z : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies some conditions. By using sub-supersolution method, truncation arguments and the Mountain Pass Theorem they showed the existence of two solutions. We would like to emphasize that for the authors to use the sub-supersolution method, the following assumption was very important: there exist $\phi_0 \in C^1_0(\overline{\Omega})$, $\phi_0 \geq 0$, and $q > N$ such that $h\phi_0^{-\gamma} \in L^q(\Omega)$. Furthermore, we note that our assumption on the function h is different (see (1.7) below), because it does not guarantee that $hv_0^{-\gamma} \in L^q(\Omega)$ for some $q > N$.

Singular elliptic problems has been studied extensively in recent years, see [5,7,11–14,21–23,25] and the references therein. Especially, Sun in [25] considered the problem

$$\begin{cases} -\Delta u = h(x)u^{-\gamma} + b(x)u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $b \in L^\infty(\Omega)$ is a non-negative function, $0 < p < 1 < \gamma$, $h \in L^1(\Omega)$ and $h > 0$ a.e. in Ω. By using variational methods the author showed that the existence of $H^1_0(\Omega)$–solutions of (1.6) is related to a compatibility hypothesis between on the couple $(h(x), \gamma)$. More precisely, problem (1.6) has a solution in $H^1_0(\Omega)$ if and only if there exists $v_0 \in H^1_0(\Omega)$ such that

$$\int_\Omega h(x)|v_0|^{1-\gamma} < \infty. \tag{1.7}$$

Motivated by above results, our main purpose in this paper is to investigate the existence of $H^1_0(\Omega)$-solutions for problem (P). We shall show that the compatibility condition (1.7) on the couple $(h(x), \gamma)$ is also optimal for the existence of weak solutions to problem (P). Under additional assumption on the function h we show that the solutions of (P) belong to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, and as a consequence we obtain uniqueness of solution.

Before giving our main results, we need an additional assumption. The function $d(x) = d(x, \partial\Omega)$ denotes the distance from a point $x \in \overline{\Omega}$ to the boundary $\partial\Omega$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of $\Omega \subset \mathbb{R}^N$.

We introduce the following assumption:

(bh) $b \geq 0$ a.e. in Ω and there exist constants $c > 0$ and $\beta \in (0,1)$ such that

$$h(x) \leq c d^{\gamma-\beta}(x), \qquad \forall x \in \Omega. \tag{1.8}$$

Our first result is the following.

Theorem 1.1. *If $(f)_1$ holds, then:*

a) problem (P) admits a solution $u \in H^1_0(\Omega)$ if and only if there exists a function $v_0 \in H^1_0(\Omega)$ satisfying (1.7);

b) *under the additional assumption* (bh) *the solution* u *obtained in* $a)$ *belongs to* $C^{1,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0,1)$. *In particular, problem* (P) *has a unique solution in* $H_0^1(\Omega)$.

It is worth pointing out that there are some differences between problems (P) and (1.5). We give one in the following example.

Example 1.2. Let Ω_0 be an open set with $\overline{\Omega}_0 \subset \Omega$ and $\beta, p \in (0,1)$. Then the functions $h(x) = d^{\gamma-\beta}(x), x \in \Omega$ and $f(x,s) = (2\chi_{\overline{\Omega}_0}(x) - 1)s^p, (x,s) \in \Omega \times \mathbb{R}$ satisfy (1.7) and $(f)_1$, respectively (see Remark 3.3). Here we denote by $\chi_{\overline{\Omega}_0}$ the characteristic function of $\overline{\Omega}_0$. We claim that the functions h and f do not satisfy the assumption (h_1) in [24]. To see this let $y \in \partial\Omega$ and $k, s > 0$. Since $\lim_{x\to y} -kh(x) = \lim_{x\to y} -kd^{\gamma-\beta}(x) = 0$, we can find $\epsilon > 0$ such that

$$f(x,s) = -s^p < -kh(x) \quad \text{for every } x \in \{x \in \Omega : |x-y| < \epsilon\} \setminus \overline{\Omega}_0.$$

This proves the claim.

Regularity results for singular elliptic equations have been studied in Giacomoni–Schindler–Takáč [8], Giacomoni–Saoudi [9] and Marino–Winkert [19] in the particular context of weak singularity, that is $\gamma \in (0,1)$. Specifically, in [8] the $C^{1,\alpha}(\overline{\Omega})$ regularity is proved. In the present paper, we consider the opposite situation where $\gamma > 1$ (namely, strong singularity) and give conditions on h which guarantee the $C^{1,\alpha}(\overline{\Omega})$ regularity of weak solutions of (P). We observe that due to the difference between the types of singularities, and also due to the structure of problem (P_A) below, the regularity result of [8] can not be applied to prove Theorem 1.1-b).

Now we state our second result.

Theorem 1.3. *Suppose* $(f)_2$ *holds. Then problem* (P) *admits a unique solution* $u \in H_0^1(\Omega)$ *if and only if there exists a function* $v_0 \in H_0^1(\Omega)$ *satisfying* (1.7).

To prove the existence of a solution for problem (P), we use the method of changing variables developed in Colin–Jeanjean [4]. With this approach, the energy functional associated to the new problem has nonhomogeneous terms (see problem (P_A)) and some difficulties arise. For example, the techniques used by the works mentioned above do not apply directly here. In order to deal with these difficulties, we make a careful analysis of the fiber maps associated to the energy functional associated to the new problem and we will approach it in a new way.

We emphasize that Theorem 1.1 extends the main result of Sun [25] (see Theorem 1.2 in [25]), in the sense that we consider the operator $Lu = -\Delta u - \Delta(u^2)u$ instead of the Laplacian operator and the potential b may change sign on Ω. As far as we know, the regularity of solution (and consequently the uniqueness) obtained in Theorem 1.1-b) is new. Also, Theorem 1.3 extends Theorem 1.1 of Wang [26] in the sense that we consider the case $\gamma > 1$.

The paper is organized as follows. In the next section we reformulate problem (P) into a new one which finds its natural setting in the Sobolev space $H_0^1(\Omega)$ and we prove some important lemmas. In section 3, we give the proof of Theorem 1.1. In section 4, we prove Theorem 1.3 and in the Appendix we prove some properties of the positive solutions of problem $-\Delta u - \Delta(u^2)u = h(x)u^{-\gamma} + \lambda b(x)u^p$ in Ω, where the parameter $\lambda \geq 0$ varies.

Notation. Throughout the paper we make use of the following notation:

- c, C denote positive constants, which may vary from line to line.
- $H_0^1(\Omega)$ denotes the Sobolev space equipped with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 dx\right)^2$.

- $L^s(\Omega)$, $1 \le s \le \infty$, denotes the Lebesgue space with the norms $\|u\|_s = \left(\int_\Omega |\nabla u|^s dx\right)^{1/s}$, for $1 \le p < \infty$, $\|u\|_\infty = \inf\{C > 0 : |u(x)| \le C \text{ a.s. in } \Omega\}$.
- For $0 < \alpha \le 1$, $C^{1,\alpha}(\overline{\Omega})$ denotes the space of Hölder functions with exponent α. The norm of $C^{1,\alpha}(\overline{\Omega})$ is denoted by $|\cdot|_{1,\alpha}$.
- We denote by ϕ_1 the L^∞-normalized (that is, $|\phi_1|_\infty = 1$) positive eigenfunction of $(-\Delta, H_0^1(\Omega))$.
- If B is a measurable set in $\mathbb{R}^N$, we denote by χ_B the characteristic function of B.

2 Reformulation of the problem and preliminaries

The natural energy functional corresponding to the problem (P) is the following:

$$J(u) = \frac{1}{2}\int_\Omega (1+2u^2)|\nabla u|^2 + \frac{1}{\gamma - 1}\int_\Omega h(x)|u|^{1-\gamma} - \int_\Omega F(x,u), \qquad u \in D(J),$$

where

$$D(J) = \left\{ u \in H_0^1(\Omega) : \int_\Omega h(x)|u|^{1-\gamma} < \infty \right\}$$

and $F(x,s) = \int_0^s f(x,t)dt$.

However, this functional is not well defined, because $\int_\Omega u^2|\nabla u|^2 dx$ is not finite for all $u \in H_0^1(\Omega)$, hence it is difficult to apply variational methods directly. In order to overcome this difficulty, we use the following change of variables introduced by [4], namely, $v := g^{-1}(u)$, where g is defined by

$$\begin{cases} g'(t) = \frac{1}{(1+2|g(t)|^2)^{\frac{1}{2}}} & \text{in } [0,\infty), \\ g(t) = -g(-t) & \text{in } (-\infty, 0]. \end{cases}$$

We list some properties of g, whose proofs can be found in Liu [15].

Lemma 2.1. *The function g satisfies the following properties:*

(1) *g is uniquely defined, C^∞ and invertible;*

(2) *$g(0) = 0$;*

(3) *$0 < g'(t) \le 1$ for all $t \in \mathbb{R}$;*

(4) *$\frac{1}{2}g(t) \le tg'(t) \le g(t)$ for all $t > 0$;*

(5) *$|g(t)| \le |t|$ for all $t \in \mathbb{R}$;*

(6) *$|g(t)| \le 2^{1/4}|t|^{1/2}$ for all $t \in \mathbb{R}$;*

(7) *$(g(t))^2 - g(t)g'(t)t \ge 0$ for all $t \in \mathbb{R}$;*

(8) *There exists a positive constant C such that $|g(t)| \ge C|t|$ for $|t| \le 1$ and $|g(t)| \ge C|t|^{1/2}$ for $|t| \ge 1$;*

(9) *$g''(t) < 0$ when $t > 0$ and $g''(t) > 0$ when $t < 0$;*

(10) *the functions $(g(t))^{1-\gamma}$ and $(g(t))^{-\gamma}$ are decreasing for all $t > 0$;*

(11) *the function* $(g(t))^p t^{-1}$ *is decreasing for all* $t > 0$;

(12) $|g(t)g'(t)| < 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$.

Proof. We only prove (10) and (11). From $g(t), g'(t) > 0$ for $t > 0$ and $\gamma > 1$, we obtain

$$\left[(g(t))^{1-\gamma}\right]' = (1-\gamma)(g(t))^{-\gamma}g'(t) < 0, \qquad \forall t > 0$$

and

$$[(g(t))^{-\gamma}]' = -\gamma(g(t))^{-\gamma-1}g'(t) < 0, \qquad \forall t > 0,$$

which imply that $(g(t))^{1-\gamma}$ and $(g(t))^{-\gamma}$ are decreasing for all $t > 0$. Therefore, (10) has been proved.

(11) Using the fact that $p < 1$ and (4) we find

$$\begin{aligned}\left[(g(t))^p t^{-1}\right]' &= p(g(t))^{p-1}g'(t)t^{-1} - (g(t))^p t^{-2} \\ &= p(g(t))^{p-1}(g'(t)t)t^{-2} - (g(t))^p t^{-2} \\ &< t^{-2}\left[(g(t))^{p-1}g(t) - (g(t))^p\right] \\ &= 0,\end{aligned}$$

for all $t > 0$. Hence the function $(g(t))^p t^{-1}$ is decreasing for all $t > 0$. The lemma is proved. □

After a change of variable $v = g^{-1}(u)$, we define an associated problem

$$\begin{cases} -\Delta v = [h(x)(g(v))^{-\gamma} + f(x, g(v))]\, g'(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_A}$$

We say that a function $v \in H_0^1(\Omega)$ is a weak solution (solution, for short) of (P_A) if $v > 0$ a.e. in Ω, and, for every $\varphi \in H_0^1(\Omega)$,

$$h(x)(g(v))^{-\gamma}g'(v)\varphi \in L^1(\Omega)$$

and

$$\int_\Omega \nabla v \nabla \varphi = \int_\Omega h(x)(g(v))^{-\gamma}g'(v)\varphi + \int_\Omega f(x, g(v))g'(v)\varphi.$$

It is easy to see that problem (P_A) is equivalent to our problem (P), which takes $u = g(v)$ as its solutions.

The energy functional associated with problem (P_A) is defined as

$$\Phi(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{\gamma-1}\int_\Omega h(x)|g(v)|^{1-\gamma} - \int_\Omega F(x, g(v)), \qquad v \in D(\Phi),$$

if $D(\Phi) \neq \emptyset$, where

$$D(\Phi) = \left\{ v \in H_0^1(\Omega) : \int_\Omega h(x)|g(v)|^{1-\gamma} < \infty \right\}$$

and $F(x, s) = \int_0^s f(x, t)dt$.

We shall justify that Φ is well defined by showing that $D(\Phi) \neq \emptyset$. We first remark that if v_0 satisfies (1.7), then $|v_0|$ satisfies (1.7), too. Hence, without loss of generality we can assume that $v_0 > 0$ a.e. in Ω.

We have the following lemma.

Lemma 2.2. *Let v be Lebesgue measurable and suppose that $v > 0$ a.e. in Ω. The following statements are equivalent:*

(a) $\int_\Omega h(x)|v|^{1-\gamma} < \infty$;

(b) $\int_\Omega h(x)(g(v))^{-\gamma} g'(v)v < \infty$;

(c) $\int_\Omega h(x)(g(v))^{1-\gamma} < \infty$.

In particular, if condition (1.7) holds, then $D(\Phi) \neq \emptyset$.

Proof. $(a) \Rightarrow (b)$: First, we decompose Ω as $\Omega = A_1 \cup A_2$, where

$$A_1 = \{x \in \Omega : |v(x)| \le 1\} \text{ and } A_2 = \{x \in \Omega : |v(x)| > 1\}.$$

It is easy to see that

$$h(x)(g(v))^{-\gamma} g'(v)v = h(x)(g(v))^{-\gamma} g'(v)v\chi_{A_1} + h(x)(g(v))^{-\gamma} g'(v)v\chi_{A_2},$$

thus

$$\int_\Omega h(x)(g(v))^{-\gamma} g'(v)v < \infty$$

if and only if

$$h(x)(g(v))^{-\gamma} g'(v)v\chi_{A_1} \in L^1(\Omega) \quad \text{and} \quad h(x)(g(v))^{-\gamma} g'(v)v\chi_{A_2} \in L^1(\Omega). \tag{2.1}$$

Let us show that (2.1) holds, and consequently that $\int_\Omega h(x)(g(v))^{-\gamma} g'(v)v < \infty$. Indeed, by Lemma 2.1 (4), (8) we have

$$\begin{aligned} |h(x)(g(v(x)))^{-\gamma} g'(v(x))v(x)| &\le h(x)(g(v(x)))^{1-\gamma} \\ &\le C^{1-\gamma} h(x) v^{1-\gamma}(x), \qquad \forall x \in A_1 \end{aligned}$$

and

$$\begin{aligned} |h(x)(g(v(x)))^{-\gamma} g'(v(x))v(x)| &\le h(x)(g(v(x)))^{1-\gamma} \\ &\le C^{1-\gamma} h(x) v^{(1-\gamma)/2}(x) \\ &\le C^{1-\gamma} h(x), \qquad \forall x \in A_2, \end{aligned}$$

which shows (2.1), because $h|v|^{1-\gamma}, h \in L^1(\Omega)$.

$(b) \Rightarrow (c)$: By Lemma 2.1 (4) we obtain

$$\int_\Omega h(x)(g(v))^{1-\gamma} = \int_\Omega h(x)(g(v))^{-\gamma} g(v) \le 2\int_\Omega h(x)(g(v))^{-\gamma} g'(v)v < \infty.$$

$(c) \Rightarrow (a)$: From Lemma 2.1 (5) we find

$$\int_\Omega h(x)|v|^{1-\gamma} \le \int_\Omega h(x)(g(v))^{1-\gamma} < \infty.$$

The proof of the lemma is completed. □

From now on we will assume (1.7) and as a consequence, by Lemma 2.2 we obtain $D(J) \neq \emptyset$ and $D(\Phi) \neq \emptyset$. Moreover $D(J) = D(\Phi)$.

The fact that we are looking for positive solutions leads us to introduce the sets

$$V_+ = \left\{ v \in H_0^1(\Omega) \setminus \{0\} : v \geq 0 \right\}$$

and

$$D_+(J) = \{ v \in V_+ : v \in D(J) \}.$$

For each $v \in D_+(J)$ we define the fiber map $\phi_v : (0, \infty) \to \mathbb{R}$ by

$$\phi_v(t) := \Phi(tv) = \frac{t^2}{2} \int_\Omega |\nabla v|^2 + \frac{1}{\gamma - 1} \int_\Omega h(x)(g(tv))^{1-\gamma} - \int_\Omega F(x, g(tv)).$$

In what follows, we will study the main properties of the fiber maps.

Lemma 2.3. *If $v \in D_+(J)$, then $\phi_v \in C^1((0,\infty), \mathbb{R})$.*

Proof. It is clear that $\tilde{\Gamma} \in C^1((0,\infty), \mathbb{R})$, where

$$\tilde{\Gamma}(t) = \frac{t^2}{2} \int_\Omega |\nabla v|^2 - \int_\Omega F(x, g(tv)).$$

Therefore, it is sufficient to show that $\Gamma \in C^1((0,\infty), \mathbb{R})$, where Γ is defined by

$$\Gamma(t) = \int_\Omega h(x)(g(tv))^{1-\gamma}.$$

Let $t > 0$. For every $s > 0$, by the Mean Value Theorem there exists a measurable function $\theta = \theta(s, x) \in (0, 1)$ such that $t + \theta(s, x)s \to t$ as $s \to 0$ and

$$\Gamma(t+s) - \Gamma(t) = (1-\gamma) \int_\Omega h(x)(g((t+\theta s)v))^{-\gamma} g'((t+\theta s)v) s v.$$

Since, by Lemma 2.1(9), (10), the function $g^{-\gamma} g'$ is decreasing on $(0, \infty)$ it follows that

$$(g((t+\theta s)v))^{-\gamma} g'((t+\theta s)v) \leq (g(tv))^{-\gamma} g'(tv) \quad \text{a.e. in } \Omega.$$

Furthermore, as a consequence of Lemma 2.2 we have $h(g(tv))^{-\gamma} g'(tv) v \in L^1(\Omega)$. Hence, applying the Lebesgue's dominated convergence theorem we obtain

$$\Gamma'(t) = \lim_{s \to 0} \frac{\Gamma(t+s) - \Gamma(t)}{s} = (1-\gamma) \int_\Omega h(x)(g(tv))^{-\gamma} g'(tv) v,$$

that is, Γ is differentiable at t. Finally, using Lemma 2.2 and the Lebesgue's dominated convergence theorem we deduce that the function $\Gamma' : (0, \infty) \longrightarrow \mathbb{R}$ defined by

$$\Gamma'(t) = (1-\gamma) \int_\Omega h(x)(g(tv))^{-\gamma} g'(tv) v,$$

is continuous, namely, $\Gamma \in C^1((0,\infty), \mathbb{R})$. The proof is complete. □

Our next result deals with the existence of global minima of ϕ_v, for every $v \in D_+(J)$.

Lemma 2.4. *If $v \in D_+(J)$, then there exists a $t(v) > 0$ such that*

$$\phi_v(t(v)) = \inf_{t > 0} \phi_v(t).$$

Proof. We only give here the proof for the case in which $(f)_1$ holds. The case that $(f)_2$ holds is similar.

First, we claim that

$$\lim_{t\to 0}\phi_v(t) = \infty \quad \text{and} \quad \lim_{t\to\infty}\phi_v(t) = \infty. \tag{2.2}$$

In fact, by Lemma 2.1 (5) we have

$$\int_\Omega h(x)(g(tv))^{1-\gamma}dx \geq t^{1-\gamma}\int_\Omega h(x)|v|^{1-\gamma}$$

and

$$t^{p+1}\int_\Omega |b(x)||v|^{p+1} \geq \left|\int_\Omega b(x)(g(tv))^{p+1}\right| \geq 0,$$

whence

$$\lim_{t\to 0}\int_\Omega h(x)(g(tv))^{1-\gamma}dx = \infty \quad \text{and} \quad \lim_{t\to 0}\int_\Omega b(x)(g(tv))^{p+1} = 0.$$

Since $\gamma > 1$, we deduce from this that $\lim_{t\to 0}\phi_v(t) = \infty$. Moreover, one has

$$\lim_{t\to\infty}\phi_v(t) \geq \lim_{t\to\infty} t^2\left[\|v\|^2 - t^{p-2}\frac{\|b\|_\infty}{p+1}\int_\Omega |v|^{p+1}dx\right] = \infty,$$

that is, $\lim_{t\to\infty}\phi_v(t) = \infty$.

Finally, from the continuity of ϕ_v and (2.2) we deduce that there exists a $t(v) > 0$ such that $\phi_v(t(v)) = \inf_{t>0}\phi_v(t)$. This concludes the proof of the lemma. □

The following pictures give the possible graphs of the fiber maps.

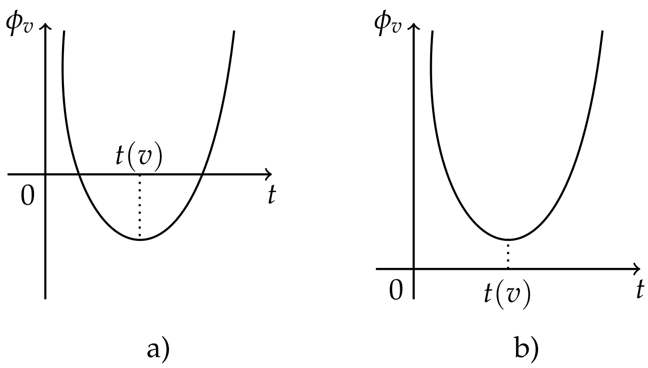

Figure 2.1: Possible graphs of the fiber maps.

Motivated by [25], we define the following constraint sets

$$\mathcal{N}_1 = \left\{v \in V_+ : \|v\|^2 - \int_\Omega f(x, g(v))g'(v)v \geq \int_\Omega h(x)(g(v))^{-\gamma}g'(v)v\right\}$$

and

$$\mathcal{N}_2 = \left\{v \in V_+ : \|v\|^2 - \int_\Omega f(x, g(v))g'(v)v = \int_\Omega h(x)(g(v))^{-\gamma}g'(v)v\right\}.$$

Observe that if v is a solution of (P_A) then $v \in \mathcal{N}_2$ and $\mathcal{N}_2 \subset \mathcal{N}_1$.

It should be noted that for $\gamma > 1$, $\mathcal{N}_2$ is not closed as usual (certainly not weakly closed).

We prove that every function in $D_+(J)$ may be projected on the set $\mathcal{N}_2$. In particular, $\mathcal{N}_1 \neq \emptyset$.

Lemma 2.5. *For any $v \in D_+(J)$ we have $t(v)v \in \mathcal{N}_2$.*

Proof. From Lemma 2.4 we infer that $t(v)$ is a global minimum of ϕ_v and hence, by Lemma 2.3 one has $\phi_v'(t(v)) = 0$. Thus

$$\begin{aligned} 0 &= t(v)\phi_v'(t(v)) \\ &= \|t(v)v\|^2 - \int_\Omega h(x)(g(t(v)v))^{-\gamma}g'(t(v)v)(t(v)v) - \int_\Omega f(x,t(v)v)g'(t(v)v)(t(v)v) = 0, \end{aligned}$$

namely, $t(v)v \in \mathcal{N}_2 \subset \mathcal{N}_1$. The proof is complete. □

We end this section with the following lemmas, which will be used to prove the regularity of the solutions.

Lemma 2.6. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Let $u \in L^1_{loc}(\Omega)$ and assume that, for some $k \geq 0$, u satisfies, in the sense of distributions,*

$$\begin{cases} -\Delta u + ku \geq 0 & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega. \end{cases}$$

Then either $u \equiv 0$, or there exists $\epsilon > 0$ such that $u(x) \geq \epsilon d(x, \partial\Omega)$, $x \in \Omega$.

Proof. See Brezis–Nirenberg [3, Theorem 3]. □

Lemma 2.7. *Let $a \in L^1(\Omega)$ and suppose that there exist constants $\delta \in (0,1)$ and $C > 0$ such that $|a(x)| \leq C\phi_1^{-\delta}(x)$, for a.e. $x \in \Omega$. Then, the problem*

$$\begin{cases} -\Delta u = a & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in H^1_0(\Omega)$. Furthermore, there exist constants $\alpha \in (0,1)$ and $M > 0$ depending only on C, α, Ω such that $u \in C^{1,\alpha}(\overline{\Omega})$ and $|u|_{1,\alpha} < M$.

Proof. See Hai [11, Lemma 2.1, Remark 2.2]. □

Remark 2.8. For future use we recall that there exist constants $l_1, l_2 > 0$ such that

$$l_1 d(x, \partial\Omega) \leq \phi_1(x) \leq l_2 d(x, \partial\Omega), \qquad x \in \Omega,$$

where ϕ_1 is the first eigenfunction of $(-\Delta, H^1_0(\Omega))$.

Lemma 2.9. *Let $\psi_j : \Omega \times (0,\infty) \longrightarrow [0,\infty)$, $j = 1,2$ are measurable functions such that*

$$\psi_1(x,s) \leq \psi_2(x,s) \quad \text{for all } (x,s) \in \Omega \times (0,\infty),$$

and for each $x \in \Omega$, the function $s \longmapsto \psi_1(x,s)s^{-1}$ is decreasing on $(0,\infty)$. Furthermore let $u, v \in H^1(\Omega)$, with $u \in L^\infty(\Omega)$, $u > 0$, $v > 0$ on Ω are such that

$$-\Delta u \leq \psi_1(x,u) \quad \text{and} \quad -\Delta v \geq \psi_2(x,v) \quad \text{on } \Omega.$$

If $u \leq v$ on $\partial\Omega$ and $\psi_1(\cdot,u)$ (or $\psi_2(\cdot,u)$) belongs to $L^1(\Omega)$, then $u \leq v$ on Ω.

Proof. See Mohammed [20, Theorem 4.1]. □

3 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. First, we shall show the existence of a global minimum of Φ on $\mathcal{N}_1$. For this purpose, we need the following lemma.

Lemma 3.1. *The set $\mathcal{N}_1$ is not empty and the functional Φ is coercive on $\mathcal{N}_1$.*

Proof. Since (1.7) holds, Lemmas 2.2 and 2.5 imply $\mathcal{N}_1 \neq \emptyset$. We now show that Φ is coercive on $\mathcal{N}_1$. Indeed, for every $v \in \mathcal{N}_1$,

$$\begin{aligned}\Phi(v) &= \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{\gamma-1}\int_\Omega h(x)(g(v))^{1-\gamma} - \frac{1}{p+1}\int_\Omega b(x)(g(v))^{p+1} \\ &\geq \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\|b\|_\infty}{p+1}\int_\Omega (g(v))^{p+1},\end{aligned}$$

and from Lemma 2.1 (5) and Sobolev embedding we obtain

$$\Phi(v) \geq \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\|b\|_\infty}{p+1}\int_\Omega |v|^{p+1} \geq \frac{\|v\|^2}{2} - C\frac{\|v\|^{p+1}}{p+1},$$

for some constant $C > 0$. Since $p \in (0,1)$ one infers that Φ is coercive on $\mathcal{N}_1$. □

As an immediate consequence of Lemma 3.1, we can deduce that

$$J_1 = \inf_{v\in\mathcal{N}_1} \Phi(v) \quad \text{and} \quad J_2 = \inf_{v\in\mathcal{N}_2} \Phi(v)$$

are well defined with $J_1, J_2 \in \mathbb{R}$ and $J_2 \geq J_1$.

We now prove that the infimum of Φ on $\mathcal{N}_1$ is attained.

Lemma 3.2. *There exists $v \in \mathcal{N}_2$ such that $J_1 = \Phi(v) = J_2$.*

Proof. Let $\{v_n\} \subset \mathcal{N}_1$ be a minimizing sequence for Φ. From Lemma 3.1 the sequence $\{v_n\} \subset \mathcal{N}_1$ is bounded and then, up to subsequences, there exists $v \in H_0^1(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v & \text{in } H_0^1(\Omega), \\ v_n \longrightarrow v & \text{in } L^s(\Omega) \text{ for all } s \in (0, 2^*), \\ v_n \longrightarrow v & \text{a.e. in } \Omega. \end{cases}$$

Since $v_n > 0$ a.e. in Ω, we have $v \geq 0$ a.e. in Ω, that is, $v \in V_+$. From the definition of $\mathcal{N}_1$ and Lemma 2.1 (3), (4), (5) it follows that for some constant C one has

$$\begin{aligned}\frac{1}{2}\int_\Omega h(x)(g(v_n))^{1-\gamma} &\leq \int_\Omega h(x)(g(v_n))^{-\gamma} g'(v_n)v_n \\ &\leq \|v_n\|^2 - \int_\Omega b(x)(g(v_n))^p g'(v_n)v_n \\ &\leq \|v_n\|^2 + \int_\Omega |b(x)||v_n|^{p+1} \\ &\leq \|v_n\|^2 + c||v_n||^{p+1} \\ &\leq C.\end{aligned}$$

Therefore, using Fatou's lemma we get $\int_\Omega \theta(x) \le C < \infty$, where

$$\theta(x) = \begin{cases} h(x)(g(v(x)))^{1-\gamma}, & \text{if } v(x) \neq 0 \\ \infty, & \text{if } v(x) = 0. \end{cases}$$

Since $g(0) = 0$ (by Lemma 2.1 (2)) and $\int_\Omega \theta(x) < \infty$, it follows that $v > 0$ a.e. in Ω. Thus, using Fatou's lemma again, we obtain

$$0 < \int_\Omega h(x)(g(v))^{-\gamma} g'(v) v \le C$$

and this jointly with Lemma 2.2 imply that $v \in D_+(J)$. As a consequence, Lemmas 2.4 and 2.5 apply yielding a global minimum $t(v) > 0$ such that $\phi_v(t(v)) = \inf_{t>0} \phi_v(t)$ and $t(v)v \in \mathcal{N}_2$. Furthermore, we have

$$\begin{aligned}
J_1 &= \lim_{n\to\infty} \Phi(v_n) = \liminf_{n\to\infty} \Phi(v_n) \\
&= \liminf_{n\to\infty} \left[\frac{1}{2} \int_\Omega |\nabla v_n|^2 + \frac{1}{\gamma-1} \int_\Omega h(x) g(v_n)^{1-\gamma} - \frac{1}{p+1} \int_\Omega b(x)(g(v_n))^{p+1} \right] \\
&\ge \liminf_{n\to\infty} \left[\frac{1}{2} \int_\Omega |\nabla v_n|^2 \right] + \liminf_{n\to\infty} \left[\frac{1}{\gamma-1} \int_\Omega h(x)(g(v_n))^{1-\gamma} \right] - \frac{1}{p+1} \int_\Omega b(x)(g(v))^{p+1} \\
&\ge \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{\gamma-1} \int_\Omega h(x)(g(v))^{1-\gamma} - \frac{1}{p+1} \int_\Omega b(x)(g(v))^{p+1} = \phi_v(1) \\
&\ge \phi_v(t(v)) = \Phi(t(v)v) \\
&\ge J_2 \\
&\ge J_1.
\end{aligned}$$

Hence

$$J_1 = \phi_v(1) = \Phi(v) = J_2,$$

that is, $\phi_v(1) = \phi_v(t(v)) = \inf_{t>0} \phi_v(t)$. This implies $\phi_v'(1) = 0$ and consequently $v \in \mathcal{N}_2 \subset \mathcal{N}_1$. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. a) Necessity. Suppose that $u \in H_0^1(\Omega)$ is a solution of (P), by taking $\varphi = u$ in (1.1), we have

$$\int_\Omega h(x)|u|^{1-\gamma} < \infty.$$

Sufficiency. Let v be the global minimum obtained in Lemma 3.2. We will prove that v is a solution of (P_A). Let $\varphi \in H_0^1(\Omega)$, $\varphi \ge 0$. Applying Lemma 2.1 (10) we find

$$\int_\Omega h(x)(g(v+\epsilon\varphi))^{1-\gamma} \le \int_\Omega h(x)(g(v))^{1-\gamma} < \infty \qquad \forall \epsilon > 0,$$

namely, $v + \epsilon\varphi \in D_+(J)$ for every $\epsilon > 0$. Then, from Lemmas 2.4 and 2.5 there exists a $t(\epsilon) > 0$ such that $\phi_{v+\epsilon\varphi}(t(\epsilon)) = \inf_{t>0} \phi_{v+\epsilon\varphi}(t)$ and $t(\epsilon)(v+\epsilon\varphi) \in \mathcal{N}_2$. Therefore

$$\Phi(v+\epsilon\varphi) = \phi_{v+\epsilon\varphi}(1) \ge \phi_{v+\epsilon\varphi}(t(\epsilon)) = \Phi(t(\epsilon)(v+\epsilon\varphi)) \ge J_2 = \Phi(v),$$

that is,

$$\int_\Omega \frac{h(x)(g(v+\epsilon\varphi))^{1-\gamma} - h(x)(g(v))^{1-\gamma}}{1-\gamma} \\ \leq \frac{\|v+\epsilon\varphi\|^2 - \|v\|^2}{2} - \int_\Omega \frac{b(x)(g(v+\epsilon\varphi))^{p+1} - b(x)(g(v))^{p+1}}{p+1}.$$

Thus, dividing both sides of the above inequality by $\epsilon > 0$, passing to the limit inferior as $\epsilon \longrightarrow 0$ and using Fatou's Lemma, we have

$$\int_\Omega h(x)(g(v))^{-\gamma}g'(v)\varphi = \int_\Omega \liminf \frac{h(x)(g(v+\epsilon\varphi))^{1-\gamma} - h(x)(g(v))^{1-\gamma}}{1-\gamma} \\ \leq \int_\Omega \nabla v\nabla\varphi - \int_\Omega b(x)(g(v))^p g'(v)\varphi. \tag{3.1}$$

Finally, we can use an argument inspired by Graham–Eagle [10] to show that v is a solution of (P_A). Since $v \in \mathcal{N}_2$, one has

$$\|v\|^2 - \int_\Omega b(x)(g(v))^p g'(v)v - \int_\Omega h(x)(g(v))^{-\gamma}g'(v)v = 0.$$

For arbitrary $\varphi \in H_0^1(\Omega)$ and $\epsilon > 0$, set $\Psi = (v+\epsilon\varphi)^+$ and

$$\Omega_1^\epsilon = \{x \in \Omega : b(x) < 0 \text{ and } v(x) + \epsilon\varphi(x) < 0\}.$$

Then, inserting Ψ into (3.1) and using $v \in \mathcal{N}_2$, we obtain that

$$\begin{aligned}
0 &\leq \int_\Omega \nabla v\nabla\Psi - \int_\Omega b(x)(g(v))^p g'(v)\Psi - \int_\Omega h(x)(g(v))^{-\gamma}g'(v)\Psi \\
&= \int_{[v+\epsilon\varphi\geq 0]} \nabla v\nabla(v+\epsilon\varphi) - b(x)(g(v))^p g'(v)(v+\epsilon\varphi) - h(x)(g(v))^{-\gamma}g'(v)(v+\epsilon\varphi) \\
&= \left(\int_\Omega - \int_{[v+\epsilon\varphi<0]}\right) \nabla v\nabla(v+\epsilon\varphi) - b(x)(g(v))^p g'(v)(v+\epsilon\varphi) - h(x)(g(v))^{-\gamma}g'(v)(v+\epsilon\varphi) \\
&= \|v\|^2 - \int_\Omega b(x)(g(v))^p g'(v)v - \int_\Omega h(x)(g(v))^{-\gamma}g'(v)v \\
&\quad + \epsilon\left[\int_\Omega \nabla v\nabla\varphi - b(x)(g(v))^p g'(v)\varphi - h(x)(g(v))^{-\gamma}g'(v)\varphi\right] \\
&\quad - \int_{[v+\epsilon\varphi<0]} \nabla v\nabla(v+\epsilon\varphi) - b(x)(g(v))^p g'(v)(v+\epsilon\varphi) - h(x)(g(v))^{-\gamma}g'(v)(v+\epsilon\varphi) \\
&\leq \epsilon\left[\int_\Omega \nabla v\nabla\varphi - b(x)(g(v))^p g'(v)\varphi - h(x)(g(u))^{-\gamma}g'(v)\varphi\right] \\
&\quad - \epsilon\int_{[v+\epsilon\varphi<0]} \nabla v\nabla\varphi + \epsilon\int_{\Omega_1^\epsilon} b(x)(g(v))^p g'(v)\varphi.
\end{aligned}$$

Since the measure of the domains of integration $[v+\epsilon\varphi < 0]$ and Ω_1^ϵ tends to zero as $\epsilon \to 0$, we then divide the above expression by $\epsilon > 0$ to obtain

$$0 \leq \int_\Omega \nabla v\nabla\varphi - b(x)(g(v))^p g'(v)\varphi - h(x)(g(v))^{-\gamma}g'(v)\varphi,$$

as $\epsilon \to 0$. Replacing φ by $-\varphi$ we conclude:

$$\int_\Omega \nabla v\nabla\varphi - b(x)(g(v))^p g'(v)\varphi - h(x)(g(v))^{-\gamma}g'(v)\varphi = 0, \qquad \forall\varphi \in H_0^1(\Omega),$$

and therefore v is a solution of (P_A). This means that $u = g(v)$ is a solution of problem (P). We complete the proof of a).

b) Suppose that v is a solution of (P_A). We will show that $v \in C^{1,\alpha}(\overline{\Omega})$ and hence, as $g \in C^\infty$ we get $u = g(v) \in C^{1,\alpha}(\overline{\Omega})$. Since $v \not\equiv 0$ satisfies in the sense of distributions

$$\begin{cases} -\Delta v \geq 0 & \text{in } \Omega, \\ v \geq 0 & \text{in } \Omega, \end{cases}$$

we can apply Lemma 2.6 yielding a $\epsilon > 0$ such that

$$v(x) \geq \epsilon d(x, \partial\Omega), \qquad x \in \Omega,$$

$$\epsilon d(x, \partial\Omega) < 1, \qquad x \in \Omega. \tag{3.2}$$

Then, by (1.8) and Lemma 2.1 (3), (8), (10) there exist constants $c, C > 0$ and $\beta \in (0,1)$ such that

$$\begin{aligned} |h(x)(g(v))^{-\gamma} g'(v)| &\leq h(x)(g(\epsilon d(x,\partial\Omega)))^{-\gamma} \leq h(x)C(\epsilon d(x,\partial\Omega))^{-\gamma} \\ &\leq Ccd^{\gamma-\beta}(x,\partial\Omega)d^{-\gamma}(x,\partial\Omega) \\ &= Cd^{-\beta}(x,\partial\Omega) \\ &\leq C\phi_1^{-\beta}(x) \end{aligned} \tag{3.3}$$

for every $x \in \Omega$, and hence $h(g(v))^{-\gamma} g'(v) \in L^1(\Omega)$. Thus, by Lemma 2.7 there exists a solution $\Psi_1 \in C^{1,\alpha_1}(\overline{\Omega})$, for some $\alpha_1 \in (0,1)$, of the problem

$$\begin{cases} -\Delta w = h(x)(g(v))^{-\gamma} g'(v) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Next, we prove that the problem

$$\begin{cases} -\Delta w = b(x)(g(v))^p g'(v) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

has a unique solution $\Psi_2 \in C^{1,\alpha_2}(\overline{\Omega})$, for some $\alpha_2 \in (0,1)$.

Let $\delta := 1 - p \in (0,1)$. From (3.2) and Lemma 2.1 (8), (12) we have

$$|b(x)g^p(v(x))g'(v(x))| \leq \|b\|_\infty g^{-\delta}(v(x))(g(v(x))g'(v(x))) \leq C\phi_1^{-\delta}(x),$$

that is,

$$|b(x)g^p(v(x))g'(v(x))| \leq C\phi_1^{-\delta}(x),$$

for every $x \in \Omega$ and some constant $C > 0$. Therefore, by Lemma 2.7 problem (3.4) has a unique solution $\Psi_2 \in C^{1,\alpha_2}(\overline{\Omega})$, for some $\alpha_2 \in (0,1)$.

We claim that $v = \Psi_1 + \Psi_2$. Indeed, using the fact that Ψ_1, Ψ_2 and v are solutions, we find

$$\int_\Omega \nabla v \nabla \varphi = \int_\Omega \left[h(x)(g(v))^{-\gamma} g'(v) + b(x)(g(v))^p g'(v) \right] \varphi = \int_\Omega \nabla(\Psi_1 + \Psi_2)\nabla\varphi,$$

for every $\varphi \in H_0^1(\Omega)$. Therefore, $v = \Psi_1 + \Psi_2$, and then $v \in C^{1,\alpha}(\overline{\Omega})$, where $\alpha := \min\{\alpha_1, \alpha_2\} \in (0,1)$. Thus, the claim follows, and consequently $u = g(v) \in C^{1,\alpha}(\overline{\Omega})$ showing the regularity of the solutions of (P).

Finally, we show the uniqueness of solution to (P). For this purpose, we show the uniqueness of solution to (P_A). Let v_1 and v_2 be two solutions of (P_A). We will prove that $v_1 \le v_2$ in Ω. First, let us set

$$j(x,s) := h(x)(g(s))^{-\gamma} g'(s) + b(x)(g(s))^p g'(s).$$

Fix $x \in \Omega$. According to Lemma 2.1 (9), (10), (11), the function $s \longmapsto j(x,s)s^{-1}$ is decreasing on $(0,\infty)$. Moreover, from (3.3) one has

$$0 \le j(x,v_i) \le C\phi_1^{-\beta}(x) + b(x)(g(v_i(x)))^p g'(v_i(x)), \qquad x \in \Omega,$$

hence $j(x,v_i) \in L^1(\Omega)$ for $i = 1,2$. Thus, we can use Lemma 2.9 with $\psi_i = j$ $(i = 1,2)$, $u = v_1$ and $v = v_2$ to get $v_1 \le v_2$ in Ω. Similarly we get $v_2 \le v_1$ in Ω, thus $v_1 = v_2$. This concludes the proof of the theorem.

Remark 3.3. If (1.8) holds, then problem (P) has a solution. Indeed, choose $v_0 = \phi_1 \in H_0^1(\Omega)$. From Remark 2.8 and (1.8) we have $h|\phi_1|^{1-\gamma} \le c l_1^{\beta-\gamma} |\phi_1|^{1-\beta} \in L^1(\Omega)$. Theorem 1.1 a) then guarantees the existence of a solution of (P).

4 Proof of Theorem 1.3

In this section, we assume $(f)_2$, that is, $f(x,s) = -b(x)s^{22^*-1}$ with $0 \le b \in L^\infty(\Omega)$ and $b \not\equiv 0$. Since the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact, the proof of Lemma 3.2 can not be applied directly here. In order to overcome this difficulty, we use the Brezis–Lieb Theorem (see [2]).

Now, we have

$$\Phi(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{\gamma - 1}\int_\Omega h(x)(g(v))^{1-\gamma} + \frac{1}{22^*}\int_\Omega b(x)(g(v))^{22^*},$$

for $v \in D(J)$. From (1.7) and Lemmas 2.2 and 2.5, one has $\mathcal{N}_1 \neq \emptyset$.

We will show the following.

Lemma 4.1. *The functional Φ is coercive on $\mathcal{N}_1$*

Proof. For every $v \in \mathcal{N}_1$, we have $\Phi(v) \ge \frac{1}{2}\|v\|^2$ and hence, Φ is coercive on $\mathcal{N}_1$. □

As an immediate consequence of Lemma 4.1, we can deduce that

$$J_1 = \inf_{v \in \mathcal{N}_1} \Phi(v) \quad \text{and} \quad J_2 = \inf_{v \in \mathcal{N}_2} \Phi(v)$$

are well defined with $J_1, J_2 \in \mathbb{R}$ and $J_2 \ge J_1$.

Next, we prove the following lemma.

Lemma 4.2. *There exists $v \in \mathcal{N}_2$ such that $J_1 = \Phi(v) = J_2$.*

Proof. Let $\{v_n\} \subset \mathcal{N}_1$ be a minimizing sequence for Φ. From Lemma 4.1 the sequence $\{v_n\} \subset \mathcal{N}_1$ is bounded in $H_0^1(\Omega)$, so in $L^{2^*}(\Omega)$ too, and then, up to subsequences, there exists $v \in H_0^1(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v & \text{in } H_0^1(\Omega), \\ v_n \longrightarrow v & \text{in } L^s(\Omega) \text{ for all } s \in (0, 2^*), \\ v_n \longrightarrow v & \text{a.s. in } \Omega. \end{cases}$$

As a consequence, by Lemma 2.1 (6), there exists a constant $C > 0$ such that

$$\int_\Omega b(x)(g(v_n))^{22^*} = \int_\Omega \left[b^{\frac{1}{2^*}}\right]^{2^*} \left[(g(v_n))^2\right]^{2^*} \leq \|b\|_\infty K_0^{22^*} \int_\Omega |v_n|^{2^*} \leq C.$$

Moreover, $b(x)(g(v_n))^{22^*} \longrightarrow b(x)(g(v))^{22^*}$ a.s. in Ω. Hence, by virtue of the Brezis–Lieb Theorem (see [2]) it follows that

$$\begin{aligned} \int_\Omega b(x)(g(v_n))^{22^*} &= \int_\Omega b(x)(g(v))^{22^*} + \int_\Omega b(x)|(g(v_n))^{22^*} - (g(v))^{22^*}| + o(1) \\ &\geq \int_\Omega b(x)(g(v))^{22^*} + o(1). \end{aligned} \tag{4.1}$$

We can repeat the arguments used in Lemma 3.2 to prove the following.

- $v > 0$ a.e. in Ω and $\int_\Omega h(x)(g(v))^{-\gamma} g'(v) v < \infty$;
- there exists $t(v) > 0$ such that $t(v)v \in \mathcal{N}_2$.

Then, by (4.1) and the Fatou's lemma we find

$$\begin{aligned} J_1 &= \lim \Phi(v_n) \\ &= \liminf \left[\frac{1}{2}\int_\Omega |\nabla v_n|^2 + \frac{1}{\gamma - 1}\int_\Omega h(x)(g(v_n))^{1-\gamma} + \frac{1}{22^*}\int_\Omega b(x)(g(v_n))^{22^*}\right] \\ &\geq \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{\gamma - 1}\int_\Omega h(x)(g(v))^{1-\gamma} + \frac{1}{22^*}\int_\Omega b(x)(g(v))^{22^*} \\ &= \phi_v(1) \\ &\geq \phi_v(t(v)) = \Phi(t(v)v) \geq J_2 \geq J_1. \end{aligned}$$

Hence

$$J_1 = \phi_v(1) = \Phi(v) = J_2,$$

that is, $\phi_v(1) = \phi_v(t(v)) = \inf_{t>0} \phi_v(t)$. This implies $\phi_v'(1) = 0$ and consequently $v \in \mathcal{N}_2 \subset \mathcal{N}_1$. This ends the proof. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Necessity. Repeating the argument used to prove the corresponding claim in Theorem 1.1 *a*), the result follows.

Sufficiency. Let v be the global minimum obtained in Lemma 4.2. We will prove that v is a solution of (P_A). Let $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$ and $\epsilon > 0$. We can repeat the arguments used in Theorem 1.1 *a*) to prove the following.

- $h(\cdot)(g(v + \epsilon\varphi))^{1-\gamma} \in L^1(\Omega)$;

- there exists a $t(\epsilon) > 0$ such that $\phi_{v+\epsilon\varphi}(t(\epsilon)) = \inf_{t>0} \phi_{v+\epsilon\varphi}(t)$ and $t(\epsilon)(v+\epsilon\varphi) \in \mathcal{N}_2$;
- $\int_\Omega h(x)(g(v))^{-\gamma}g'(v)\varphi \le \int_\Omega \nabla v\nabla\varphi + \int_\Omega b(x)(g(v))^{22^*-1}g'(v)\varphi$.

From this information, as in Theorem 1.1 *a*), we can apply an argument inspired by Graham-Eagle [10] to get

$$\begin{aligned} 0 &\le \|v\|^2 + \int_\Omega b(x)(g(v))^{22^*-1}g'(v)v - \int_\Omega h(x)(g(v))^{-\gamma}g'(v)v \\ &\quad + \epsilon\left[\int_\Omega \nabla v\nabla\varphi + b(x)(g(v))^{22^*-1}g'(v)\varphi - h(x)(g(v))^{-\gamma}g'(v)\varphi\right] \\ &\quad - \int_{[v+\epsilon\varphi<0]} \nabla v\nabla(v+\epsilon\varphi) + b(x)(g(v))^{22^*-1}g'(v)(v+\epsilon\varphi) - h(x)(g(v))^{-\gamma}g'(v)(v+\epsilon\varphi) \\ &\le \epsilon\left[\int_\Omega \nabla v\nabla\varphi + b(x)(g(v))^{22^*-1}g'(v)\varphi - h(x)(g(v))^{-\gamma}g'(v)\varphi\right] \\ &\quad - \epsilon\int_{[v+\epsilon\varphi<0]} \nabla v\nabla\varphi + b(x)(g(v))^{22^*-1}g'(v)\varphi, \end{aligned}$$

for every $\varphi \in H_0^1(\Omega)$.

Since the measure of the domain of integration $[v+\epsilon\varphi<0]$ tends to zero as $\epsilon \to 0$, we then divide the above expression by $\epsilon > 0$ to obtain

$$0 \le \int_\Omega \nabla v\nabla\varphi - b(x)(g(v))^p g'(v)\varphi - h(x)(g(v))^{-\gamma}g'(v)\varphi,$$

as $\epsilon \to 0$. Replacing φ by $-\varphi$ we conclude:

$$\int_\Omega \nabla v\nabla\varphi - b(x)(g(v))^p g'(v)\varphi - h(x)(g(v))^{-\gamma}g'(v)\varphi = 0, \qquad \forall\varphi \in H_0^1(\Omega),$$

and therefore v is a solution of (P_A). This means that $u = g(v)$ is a solution of problem (P).

Finally, we show the uniqueness of solution to (P). For this purpose, we show the uniqueness of solution to (P_A). Let v_1 and v_2 be two solutions of (P_A). We will prove that $v_1 = v_2$ in Ω. First, let us set

$$j(x,t) = -b(x)(g(t))^{22^*-1}g'(t) + h(x)(g(t))^{-\gamma}g'(t),$$

for $x \in \Omega$ and $t > 0$. Note that $j(.,t)$ is decreasing by virtue of Lemma 2.1 (9), (10). Thus,

$$\|v_1 - v_2\|^2 = \int_\Omega (j(x,v_1) - j(x,v_2))(v_1 - v_2) < 0,$$

which yields $v_1 = v_2$. Hence, problem (P_A) has a unique solution. The proof of the theorem is complete.

Appendix A

Consider the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = h(x)u^{-\gamma} + \lambda b(x)u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P_\lambda)$$

where $\lambda \geq 0, 0 < p < 1, 0 \leq b \in L^\infty(\Omega)$ and $b \not\equiv 0$.

This appendix is devoted to the study of some properties of the solutions of (P_λ). From now on we assume (1.7) holds. Therefore, by Theorem 1.1 problem (P_λ) has a solution, which we denote by u_λ.

The main result of this appendix is stated next.

Theorem A.1. *The following properties are valid:*

a) $u_\lambda \geq u_0$ *in* Ω *for every* $\lambda > 0$.

b) $u_\lambda \longrightarrow u_0$ *in* $H_0^1(\Omega)$ *as* $\lambda \longrightarrow 0$.

In order to prove Theorem A.1, we consider the problem

$$\begin{cases} -\Delta v = h(x)(g(v))^{-\gamma} g'(v) + \lambda b(x)(g(v))^p g'(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{D_λ}$$

and denote by $v_\lambda = g^{-1}(u_\lambda)$ the solution obtained in the proof of Theorem 1.1.

Let Φ_λ the energy functional associated to (D_λ). For each $\lambda \geq 0$, let us set

$$\mathcal{N}_\lambda = \left\{ v \in V_+ : ||v||^2 - \int_\Omega \lambda b(g(v))^p g'(v) v \geq \int_\Omega h(x)(g(v))^{-\gamma} g'(v) v \right\}.$$

We can now state the key lemma for proving Theorem A.1.

Lemma A.2. *The following properties hold true:*

a) $v_\lambda \geq v_0$ *in* Ω.

b) $v_\lambda \longrightarrow v_0$ *in* $H_0^1(\Omega)$ *as* $\lambda \longrightarrow 0$.

c) $\lim_{\lambda \to 0} \Phi_\lambda(v_\lambda) = \Phi_0(v_0) > 0$.

d) If (1.8) *holds, then the function* $[0, \infty) \ni \lambda \longmapsto \Phi_\lambda(v_\lambda)$ *is continuous and decreasing.*

Proof. *a)* Using the fact that v_0 and v_λ are solutions of (D_0) and (D_λ), respectively, and Lemma 2.1 (9), (10) we have

$$\begin{aligned} -\|(v_\lambda - v_0)^-\|^2 &= \int_\Omega ((g(v_\lambda))^{-\gamma} g'(v_\lambda) - (g(v_0))^{-\gamma} g'(v_0) + \lambda b(x)(g(v_\lambda))^p g'(v_\lambda))(v_\lambda - v_0)^- \\ &\geq \int_\Omega ((g(v_\lambda))^{-\gamma} g'(v_\lambda) - (g(v_0))^{-\gamma} g'(v_0))(v_\lambda - v_0)^- \\ &= \int_{\{v_\lambda < v_0\}} ((g(v_\lambda))^{-\gamma} g'(v_\lambda) - (g(v_0))^{-\gamma} g'(v_0))(v_\lambda - v_0)^- \geq 0. \end{aligned}$$

As a consequence one has $\|(v_\lambda - v_0)^-\| = 0$, which implies $v_\lambda \geq v_0$ in Ω.

b) Let $\{\lambda_n\} \subset (0, \infty)$ be a sequence such that $\lambda_n \to 0$ and denote by $v_{\lambda_n} = v_n$. We claim that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. Indeed, since $\{v_n\} \subset \mathcal{N}_{\lambda_n}$ it follows that

$$\|v_n\|^2 = \int_\Omega h(x)(g(v_n))^{-\gamma} g'(v_n) v_n + \lambda_n \int_\Omega b(g(v_n))^p g'(v_n) v_n.$$

Thus, from Lemma 2.1 (4), (5), (10) and $v_n \geq v_0$ in Ω we get

$$\begin{aligned}\|v_n\|^2 &\leq \int_\Omega h(x)(g(v_n))^{1-\gamma} + \lambda_n \int_\Omega b(x)(g(v_n))^{p+1} \\ &\leq \int_\Omega h(x)(g(v_0))^{1-\gamma} + \lambda_n \int_\Omega b(x)|v_n|^{p+1} \\ &\leq \int_\Omega h(x)(g(v_0))^{1-\gamma} + \lambda_n C\|v_n\|^{p+1}\end{aligned}$$

and hence $\{v_n\}$ is bounded in $H_0^1(\Omega)$, because $0 < p < 1$.

Therefore, there exists $\psi \in H_0^1(\Omega), \psi \geq 0$ such that, up to a subsequence, we have

$$\begin{cases} v_n \rightharpoonup \psi & \text{in } H_0^1(\Omega), \\ v_n \to \psi & \text{in } L^s(\Omega) \text{ for all } s \in (0, 2^*), \\ v_n \to \psi & \text{a.s. in } \Omega. \end{cases}$$

As in the proof of Lemma 3.2, we derive that $\psi > 0$ in Ω. This implies that

$$h(x)(g(v_n))^{-\gamma} g'(v_n)(v_n - \psi) \to 0 \quad \text{a.s. in } \Omega,$$

and by virtue of Lemma 2.1 (4), (9), (10) and $v_n \geq v_0$ in Ω one finds

$$\begin{aligned}|h(x)(g(v_n))^{-\gamma} g'(v_n)(v_n - \psi)| &\leq h(x)(g(v_n))^{1-\gamma} + h(x)(g(v_n))^{-\gamma} g'(v_n)\psi \\ &\leq h(x)(g(v_0))^{1-\gamma} + h(x)(g(v_0))^{-\gamma} g'(v_0)\psi,\end{aligned}$$

where

$$h(x)(g(v_0))^{1-\gamma} + h(x)(g(v_0))^{-\gamma} g'(v_0)\psi \in L^1(\Omega),$$

because v_0 is a solution of (D_0). Hence, by the Lebesgue's dominated convergence theorem we get

$$\int_\Omega h(x)(g(v_n))^{-\gamma} g'(v_n)(v_n - \psi) \longrightarrow 0. \tag{A.1}$$

As a consequence of (A.1) we have

$$\begin{aligned}\lim_{n\to\infty}(v_n, v_n - \psi) &= \lim_{n\to\infty} \int_\Omega \nabla v_n \nabla(v_n - \psi) = \\ &= \lim_{n\to\infty}\left[\int_\Omega h(x)(g(v_n))^{-\gamma} g'(v_n)(v_n - \psi) + \lambda_n \int_\Omega b(x)(g(v_n))^p g'(v_n)(v_n - \psi)\right] \\ &= 0,\end{aligned}$$

and since $v_n \rightharpoonup \psi$, it follows that

$$\lim_{n\to\infty}\|v_n - \psi\|^2 = \lim_{n\to\infty}(v_n, v_n - \psi) - \lim_{n\to\infty}(\psi, v_n - \psi) = 0,$$

namely, $v_n \longrightarrow \psi$ in $H_0^1(\Omega)$ as $n \to \infty$.

To end the proof of b), it is sufficient to show that $\psi = v_0$. Indeed, because v_n is a solution of (D_{λ_n}) one has

$$\int_\Omega \nabla v_n \nabla \varphi = \int_\Omega h(x)(g(v_n))^{-\gamma} g'(v_n)\varphi + \lambda_n \int_\Omega b(x)(g(v_n))^p g'(v_n)\varphi, \tag{A.2}$$

for all $\varphi \in H_0^1(\Omega)$. Moreover, from $v_0 \leq v_n$ in Ω and Lemma 2.1 (9), (10) we find

$$h(x)(g(v_n))^{-\gamma}g'(v_n)\varphi \longrightarrow h(x)(g(\psi))^{-\gamma}g'(\psi)\varphi \quad \text{a.s. in } \Omega,$$

and

$$|h(x)(g(v_n))^{-\gamma}g'(v_n)\varphi| \leq h(x)(g(v_0))^{-\gamma}g'(v_0)\varphi.$$

Therefore, letting $n \to \infty$ in (A.2), and by using Lebesgue's dominated convergence theorem we obtain

$$\int_\Omega \nabla\psi\nabla\varphi = \int_\Omega h(x)(g(\psi))^{-\gamma}g'(\psi)\varphi,$$

for every $\varphi \in H_0^1(\Omega)$. This means that ψ is a solution of (D_0), and by uniqueness of solutions of (D_0) we deduce that $\psi = v_0$. This ends the proof of b).

c) From a) and b) it follows that $v_\lambda \geq v_0$ for all $\lambda > 0$ and $v_\lambda \longrightarrow v_0$ in $H_0^1(\Omega)$ as $\lambda \to 0$. Thus, reasoning as in b), and by using Lebesgue's dominated convergence theorem we get $\lim_{\lambda\to 0}\Phi_\lambda(v_\lambda) = \Phi_0(v_0)$.

d) We can argue as in b) to show that the function is continuous. In order to prove that it is decreasing, let $0 \leq \lambda < \mu$. Then,

$$\Phi_\lambda(v_\lambda) > \Phi_\mu(v_\lambda) \geq \Phi_\mu(t_\mu(v_\lambda)v_\lambda) \geq \Phi_\mu(v_\mu),$$

that is, the function $[0,\infty) \ni \lambda \longmapsto \Phi_\lambda(v_\lambda)$ is decreasing. We complete the proof of the lemma. □

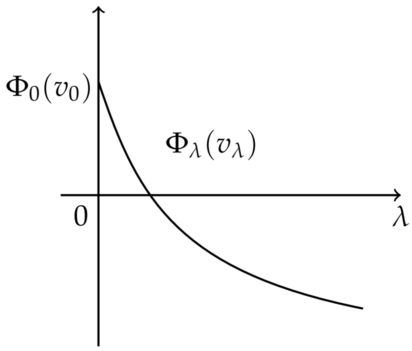

Figure A.1: Graph of function $[0,\infty) \ni \lambda \longmapsto \Phi_\lambda(v_\lambda)$.

We are now in a position to prove Theorem A.1.

Proof of Theorem A.1. a) Let $u_\lambda = g(v_\lambda)$ and $u_0 = g(v_0)$. By Lemma A.2 a) we have $v_\lambda \geq v_0$ in Ω, for every $\lambda \geq 0$. So, by virtue of Lemma 2.1 (3) we find

$$u_\lambda = g(v_\lambda) \geq g(v_0) = u_0 \text{ in } \Omega.$$

This finishes the proof of a).

b) We first observe that $\nabla u_\lambda = g'(v_\lambda)\nabla v_\lambda$, for each $\lambda \geq 0$. Then, as a consequence of the inequality $(x+y)^2 \leq 2(x^2+y^2)$, for $x, y \geq 0$, and Lemma 2.1(3) we get

$$\begin{aligned}
\int_\Omega |\nabla u_\lambda - \nabla u_0|^2 &= \int_\Omega |g'(v_\lambda)\nabla v_\lambda - g'(v_0)\nabla v_0|^2 \\
&\leq \int_\Omega (g'(v_\lambda)|\nabla v_\lambda - \nabla v_0| + |g'(v_\lambda) - g'(v_0)||\nabla v_0|)^2 \\
&\leq 2\int_\Omega (g'(v_\lambda))^2|\nabla v_\lambda - \nabla v_0|^2 + 2\int_\Omega |g'(v_\lambda) - g'(v_0)|^2|\nabla v_0|^2 \\
&\leq 2\int_\Omega |\nabla v_\lambda - \nabla v_0|^2 + 2\int_\Omega |g'(v_\lambda) - g'(v_0)|^2|\nabla v_0|^2.
\end{aligned}$$

Hence, it is sufficient to prove that

$$\int_\Omega |\nabla v_\lambda - \nabla v_0|^2 \longrightarrow 0 \quad \text{and} \quad \int_\Omega |g'(v_\lambda) - g'(v_0)|^2|\nabla v_0|^2 \longrightarrow 0, \quad \text{as } \lambda \to 0.$$

We already know (see Lemma A.2 b)) that $\int_\Omega |\nabla v_\lambda - \nabla v_0|^2 \longrightarrow 0$ as $\lambda \to 0$. Moreover, as $g'(t) \leq 1$ for every $t \geq 0$, we can apply Lebesgue's dominated convergence to infer that

$$\int_\Omega |g'(v_\lambda) - g'(v_0)|^2|\nabla v_0|^2 \longrightarrow 0 \quad \text{as } \lambda \to 0.$$

This completes the proof of Theorem A.1.

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