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Nonautonomous equations and almost reducibility sets

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Abstract. For a nonautonomous differential equation, we consider the almost reducibility property that corresponds to the reduction of the original equation to an autonomous equation via a coordinate change preserving the Lyapunov exponents. In particular, we characterize the class of equations to which a given equation is almost reducible. The proof is based on a characterization of the almost reducibility to an autonomous equation with a diagonal coefficient matrix. We also characterize the notion of almost reducibility for an equation $x' = A(t,\theta)x$ depending continuously on a real parameter θ. In particular, we show that the almost reducibility set is always an $F_{\sigma\delta}$-set and for any $F_{\sigma\delta}$-set containing zero we construct a differential equation with that set as its almost reducibility set.

Keywords: almost reducibility, nonautonomous equations.

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1 Introduction

We first describe the reducibility property and the type of problems considered in the paper. Let $A(t)$ and $B(t)$ be $q \times q$ matrices varying continuously with $t \geq 0$ and consider the linear equations

$$x' = A(t)x \quad \text{and} \quad y' = B(t)y. \tag{1.1}$$

Let $T(t,s)$ and $S(t,s)$ be the corresponding evolution families such that

$$T(t,s)x(s) = x(t) \quad \text{and} \quad S(t,s)y(s) = y(t)$$

for any solutions x and y of the equations in (1.1) and for any $t, s \geq 0$. We say that the equations are *equivalent via a coordinate change* $U(t)$ given by invertible $q \times q$ matrices if

$$U(t)^{-1}T(t,s)U(s) = S(t,s) \quad \text{for all } t, s \geq 0. \tag{1.2}$$

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More generally, one can also consider piecewise continuous functions $A(t)$ and $B(t)$ (see Section 2), in which case the evolution families $T(t,s)$ and $S(t,s)$ are still continuous in (t,s).

In this paper we consider the class of equations that are equivalent to an autonomous equation. Namely, we say that the equation $x' = A(t)x$ is *reducible via a coordinate change* $U(t)$ if it is equivalent to some autonomous equation $y' = By$. Moreover, we say that the equation $x' = A(t)x$ is *almost reducible* if it is equivalent to some autonomous equation via a Lyapunov coordinate change $U(t)$, that is, a coordinate change satisfying

$$\lim_{t\to\infty} \frac{1}{t} \log\|U(t)\| = \lim_{t\to\infty} \frac{1}{t} \log\|U(t)^{-1}\| = 0. \tag{1.3}$$

The Lyapunov coordinate changes are the only coordinate changes that preserve simultaneously the Lyapunov exponents of all sequences of invertible matrices with a finite Lyapunov exponent. More precisely, for each $v \in \mathbb{R}^q$ let

$$\lambda_A(v) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log\|T(t,0)v\|$$

be the Lyapunov exponent associated with the equation $x' = A(t)x$, with the convention that $\log 0 = -\infty$. The Lyapunov exponent $\lambda_B(v)$ for the equation $y' = B(t)y$ is defined similarly. The former statement on the preservation of the Lyapunov exponents means that a coordinate change $U(t)$ is a Lyapunov coordinate change if and only if the evolution families of any two equivalent equations as in (1.1) that satisfy (1.2) also satisfy

$$\lambda_A(U(0)v) = \lambda_B(v) \quad \text{for all } v \in \mathbb{R}^q.$$

This causes that the almost reducibility property occurs naturally whenever we want to reduce the original dynamics to a simpler one without changing the asymptotic behavior given by the Lyapunov exponents.

A first notion of reducibility is due to Lyapunov [5] (see [7] for an English translation). He considered instead bounded coordinate changes with bounded inverses, that is, transformations satisfying

$$\sup_{t\geq 0}\|U(t)\| < +\infty \quad \text{and} \quad \sup_{t\geq 0}\|U(t)^{-1}\| < +\infty. \tag{1.4}$$

We refer the reader to [4,6,8,9] and the references therein for some early results as well as to the book [3] for a global panorama of the area in 1980. While the coordinate changes satisfying (1.4) are appropriate to study uniform Lyapunov stability (because bounded coordinate changes preserve this type of stability), in order to study nonuniform Lyapunov stability it is crucial to consider Lyapunov coordinate changes as in (1.3).

We first give a characterization of the almost reducibility of an equation to an autonomous equation with a diagonal coefficient matrix (see Theorem 2.1).

Theorem 1.1. *For an equation $x' = A(t)x$ on $\mathbb{R}^q$ such that the Lyapunov exponent λ_A is finite on $\mathbb{R}^q \setminus \{0\}$, we have*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} A(s)\, ds = \inf \sum_{j=1}^{q} \lambda_A(v_j)$$

with the infimum taken over all bases $v_1, \ldots, v_q$ for $\mathbb{R}^q$ if and only if the equation is almost reducible to an equation $y' = By$ with B diagonal.

We shall use this result to characterize the autonomous equations to which a given equation is almost reducible (see Theorem 2.2).

Theorem 1.2. *$x' = A(t)x$ is almost reducible to $y' = By$ and $y' = Cy$ if and only if the eigenvalues of B and C, counted with multiplicities and eventually up to a permutation, have the same real parts.*

We also characterize completely the notion of almost reducibility for continuous 1-parameter families of linear differential equations. Namely, we consider equations $x' = A(t,\theta)x$ depending continuously on a real parameter θ. The *almost reducibility set* of this equation is the set of all $\theta \in \mathbb{R}$ for which the equation is almost reducible. We have the following result (see Theorem 3.1).

Theorem 1.3. *The almost reducibility set of $x' = A(t,\theta)x$ is an $F_{\sigma\delta}$-set.*

Finally, we establish a partial converse of Theorem 1.3. Namely, we construct a differential equation with given $F_{\sigma\delta}$-set containing zero as its almost reducibility set (see Theorem 4.1).

Theorem 1.4. *Given an integer $q \geq 2$ and an $F_{\sigma\delta}$-set M containing zero, there exists an equation $x' = A(t,\theta)x$ whose almost reducibility set is equal to M. Moreover, given an unbounded nondecreasing function $\rho(t) \geq 0$, we may require that*

$$\|A(t,\theta)\| \leq \rho(t)(1+|\theta|) \quad \text{for all } t \geq 0 \text{ and } \theta \in \mathbb{R}.$$

The proof of Theorem 1.4 is partly inspired by arguments in [1].

2 The notion of almost reducibility

We introduce the notion of almost reducibility for the class of nonautonomous linear equations and we establish some of its basic properties. In particular, we characterize completely the class of autonomous equations to which a given nonautonomous equation is almost reducible.

Let M_q be the set of all $q \times q$ matrices with real entries and let $GL_q \subset M_q$ be the subset of all invertible matrices. Consider a piecewise continuous function $A\colon \mathbb{R}_0^+ \to M_q$. We say that the equation

$$x' = A(t)x \tag{2.1}$$

is *almost reducible* to an equation $x' = Bx$ for some matrix $B \in M_q$ if there exist matrices $U(t) \in GL_q$ for $t \geq 0$ satisfying

$$\lim_{t\to\infty} \frac{1}{t} \log\|U(t)\| = \lim_{t\to\infty} \frac{1}{t} \log\|U(t)^{-1}\| = 0 \tag{2.2}$$

such that

$$U(t)^{-1}T(t,s)U(s) = e^{B(t-s)} \quad \text{for } t,s \geq 0, \tag{2.3}$$

where $T(t,s)$ is the evolution family associated with equation (2.1). This means that we have $T(t,s)x(s) = x(t)$ for any solution $x = x(t)$ of the equation $x' = A(t)x$ and all $t,s \geq 0$. Then we also say that equation (2.1) is *almost reducible*. The family $(U(t))_{t\geq 0}$ is called a *Lyapunov coordinate change*.

We start by describing when a nonautonomous equation is almost reducible to an autonomous equation with a diagonal coefficient matrix. The *Lyapunov exponent* $\lambda\colon \mathbb{R}^q \to [-\infty,+\infty]$ associated with equation (2.1) is defined by

$$\lambda(v) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log\|T(t,0)v\|,$$

with the convention that $\log 0 = -\infty$. We shall always assume that λ takes only finite values on $\mathbb{R}^q \setminus \{0\}$. It follows from the theory of Lyapunov exponents that these finite values are say $\lambda_1 < \cdots < \lambda_p$ for some positive integer $p \le q$ and that the sets

$$E_i = \{v \in \mathbb{R}^q : \lambda(v) \le \lambda_i\}$$

are linear subspaces for $i = 1, \ldots, p$. A basis $v_1, \ldots, v_q$ for $\mathbb{R}^q$ is said to be *normal* (with respect to equation (2.1)) if for each $i = 1, \ldots, p$ some elements of $\{v_1, \ldots, v_q\}$ form a basis for E_i.

Theorem 2.1. *Let $x' = A(t)x$ be an equation on $\mathbb{R}^q$ whose Lyapunov λ takes only finite values on $\mathbb{R}^q \setminus \{0\}$. Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} A(s)\, ds = \sum_{j=1}^q \lambda(v_j) \tag{2.4}$$

for some normal basis $v_1, \ldots, v_q$ for $\mathbb{R}^q$ if and only if the equation $x' = A(t)x$ is almost reducible to an autonomous equation with a diagonal coefficient matrix, whose entries on the diagonal are then necessarily $\lambda(v_1), \ldots, \lambda(v_q)$, up to a permutation.

Proof. Assume first that (2.4) holds for some normal basis $v_1, \ldots, v_q$ for $\mathbb{R}^q$. Let $U(0)$ be the matrix with columns $v_1, \ldots, v_q$ and for each $t > 0$, let

$$U(t) = T(t,0)U(0)\operatorname{diag}\big(e^{-\lambda(v_1)t}, \ldots, e^{-\lambda(v_q)t}\big).$$

Then

$$U(t)^{-1}T(t,s)U(s) = \operatorname{diag}\big(e^{\lambda(v_1)(t-s)}, \ldots, e^{\lambda(v_q)(t-s)}\big),$$

that is, property (2.3) holds taking

$$B = \operatorname{diag}\big(\lambda(v_1), \ldots, \lambda(v_q)\big).$$

In order to show that $(U(t))_{t\ge0}$ is a Lyapunov coordinate change, notice that the columns of $U(t)$ are the vectors

$$T(t,0)v_1e^{-\lambda(v_1)t}, \ldots, T(t,0)v_qe^{-\lambda(v_q)t}.$$

Since

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log\big(\|T(t,0)v_i\|e^{-\lambda(v_i)t}\big) = 0, \tag{2.5}$$

we obtain

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \|U(t)\| \le 0.$$

Now we consider the matrices

$$U(t)^{-1} = \operatorname{diag}\big(e^{\lambda(v_1)t}, \ldots, e^{\lambda(v_q)t}\big)\big(T(t,0)U(0)\big)^{-1}.$$

We have

$$(T(t,0)U(0))^{-1} = C(t)/\det(T(t,0)U(0))$$

for some matrices $C(t)$ with (i,j) entry given by $(-1)^{i+j}\Delta^{ji}(t)$, where $\Delta^{ji}(t)$ is the determinant of the matrix obtained from $T(t,0)U(0)$ erasing its jth line and ith column. Then

$$U(t)^{-1} = D(t)\frac{\exp \sum_{j=1}^q \lambda(v_j)t}{\det(T(t,0)U(0))}, \tag{2.6}$$

where

$$D(t) = \operatorname{diag}\big(e^{-\sum_{j\neq 1}\lambda(v_j)t}, \dots, e^{-\sum_{j\neq q}\lambda(v_j)(m-1)}\big)C(t).$$

By Liouville's theorem we have

$$\det T(t,0) = \exp \int_0^t \operatorname{tr} A(s)\, ds \tag{2.7}$$

and so it follows from (2.4) that

$$\lim_{t\to\infty} \frac{1}{t} \log \det T(t,0) = \sum_{j=1}^{q} \lambda(v_j).$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{\exp \sum_{j=1}^{q} \lambda(v_j)t}{|\det(T(t,0)U(0))|} = 0. \tag{2.8}$$

The (i,j) entry of $D(t)$ is given by $(-1)^{i+j}\bar{\Delta}^{ji}(t)$, where $\bar{\Delta}^{ji}(t)$ is the determinant of the matrix obtained from $T(t,0)U(0)$ dividing each kth column by $e^{\lambda(v_k)t}$ and then erasing the jth line and the ith column. It follows from (2.5) that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log |\bar{\Delta}^{ji}(t)| \leq 0 \quad \text{and so} \quad \overline{\lim_{t\to\infty}} \frac{1}{t} \log \|D(t)\| \leq 0.$$

Therefore, by (2.6) and (2.8), we obtain

$$\underline{\lim_{t\to\infty}} \frac{1}{t} \log\big(\|U(t)^{-1}\|^{-1}\big) = -\overline{\lim_{t\to\infty}} \frac{1}{t} \log \|U(t)^{-1}\| \geq 0,$$

which shows that $(U(t))_{t\geq 0}$ is a Lyapunov coordinate change.

Now assume that the equation $x' = A(t)x$ is almost reducible to an autonomous equation with a diagonal coefficient matrix, that is,

$$U(t)^{-1}T(t,s)U(s) = \operatorname{diag}\big(e^{a_1(t-s)}, \dots, e^{a_q(t-s)}\big) \tag{2.9}$$

for some matrices $U(t) \in GL_q$, for $t \geq 0$, satisfying (2.2) and some numbers $a_1, \dots, a_q \in \mathbb{R}$. Let $v_1, \dots, v_q$ be the columns of $U(0)$. Then

$$\|U(t)^{-1}T(t,0)v_i\| = e^{a_i t}.$$

By (2.2), this implies that the basis $v_1, \dots, v_q$ is normal with $\lambda(v_i) = a_i$ for $i = 1, \dots, q$. Moreover, again by (2.9), we have

$$\det(U(t)^{-1}) \det T(t,0) \det U(0) = e^{\sum_{j=1}^{q} \lambda(v_i)t}. \tag{2.10}$$

Since $\det U(t)$ is a sum of products of the entries of $U(t)$, by (2.2) we have

$$\lim_{t\to\infty} \frac{1}{t} \log|\det U(t)| = 0$$

and so it follows from (2.7) and (2.10), that identity (2.4) holds. □

We use Theorem 2.1 to characterize the class of autonomous equations to which an equation $x' = A(t)x$ is almost reducible.

Theorem 2.2. *Let $x' = A(t)x$ be an equation on $\mathbb{R}^q$ that is almost reducible to an equation $x' = Bx$. Then the equation $x' = A(t)x$ is almost reducible to an equation $x' = Cx$ if and only if the eigenvalues $\lambda_i(B)$ and $\lambda_i(C)$, respectively, of B and C counted with multiplicities, satisfy*

$$\operatorname{Re}\lambda_i(B) = \operatorname{Re}\lambda_i(C) \quad \textit{for } i = 1, \dots, q,$$

eventually up to a permutation.

Proof. First assume that the equation $x' = A(t)x$ is almost reducible to both $x' = Bx$ and $x' = Cx$. Consider Lyapunov coordinate changes $(U(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ such that

$$U(t)^{-1}T(t,s)U(s) = e^{B(t-s)} \quad \text{and} \quad V(t)^{-1}T(t,s)V(s) = e^{C(t-s)}$$

for $t, s \geq 0$. Then

$$W(t)^{-1}e^{B(t-s)}W(s) = e^{C(t-s)}$$

for $t, s \geq 0$, where the matrices $W(t) = U(t)^{-1}V(t)$ form again a Lyapunov coordinate change. It follows readily from the identity

$$W(t)^{-1}e^{Bt}W(0) = e^{Ct}$$

that the Lyapunov exponents λ^B and λ^C associated, respectively, with the equations $x' = Bx$ and $x' = Cx$ satisfy

$$\lambda^B(W(0)v) = \lambda^C(v) \quad \text{for all } v \in \mathbb{R}^q. \tag{2.11}$$

The values of λ^B and λ^C are, respectively, $\operatorname{Re}\lambda_i(B)$ and $\operatorname{Re}\lambda_i(C)$ for $i = 1, \dots, q$, counted with their multiplicities and so it follows readily from (2.11) that

$$\operatorname{Re}\lambda_i(B) = \operatorname{Re}\lambda_i(C) \quad \text{for } i = 1, \dots, q, \tag{2.12}$$

eventually up to a permutation.

Now assume that property (2.12) holds, eventually up to a permutation. Again, the values of the Lyapunov exponents λ^B and λ^C are, respectively, $\operatorname{Re}\lambda_i(B)$ and $\operatorname{Re}\lambda_i(C)$ for $i = 1, \dots, q$, counted with their multiplicities. Therefore, condition (2.4) holds for the differential equations $x' = Bx$ and $x' = Cx$. By Theorem 2.1, there exist Lyapunov coordinate changes $(\bar{U}(t))_{t\geq 0}$ and $(\bar{V}(t))_{t\geq 0}$ such that

$$\bar{U}(t)^{-1}e^{B(t-s)}\bar{U}(s) = \operatorname{diag}\big(\operatorname{Re}\lambda_1(B), \dots, \operatorname{Re}\lambda_q(B)\big)^{t-s}$$

and

$$\bar{V}(t)^{-1}e^{C(t-s)}\bar{V}(s) = \operatorname{diag}\big(\operatorname{Re}\lambda_1(C), \dots, \operatorname{Re}\lambda_q(C)\big)^{t-s}$$

for $t \geq 0$. By (2.12), we obtain

$$\bar{U}(t)^{-1}e^{B(t-s)}\bar{U}(s) = \bar{V}(t)^{-1}e^{C(t-s)}\bar{V}(s)$$

for $t \geq 0$ and so

$$W(t)^{-1}T(t,s)W(s) = e^{C(t-s)}$$

for $t, s \geq 0$, where

$$W(t) = U(t)\bar{U}(t)\bar{V}(t)^{-1}$$

for each $t \geq 0$. Since $(W(t))_{t\geq 0}$ is a Lyapunov coordinate change, we conclude that $x' = A(t)x$ is almost reducible to the equation $x' = Cx$. □

3 Characterization of almost reducibility sets

In this section we give a characterization of the almost reducibility sets of a differential equation $x' = A(t,\theta)x$ depending on a real parameter θ. Namely, we show that any such set is an $F_{\sigma\delta}$-set. More precisely, let $\mathcal{M}$ be the set of all equations $x' = A(t,\theta)x$ such that the map

$$\mathbb{R}_0^+ \times \mathbb{R} \ni (t,\theta) \mapsto A(t,\theta) \in M_q$$

is piecewise continuous in t and continuous in θ. We denote by $T_\theta(t,s)$ the corresponding evolution family. The *almost reducibility set* of an equation $x' = A(t,\theta)x$ is the set of all $\theta \in \mathbb{R}$ for which the equation is almost reducible.

Theorem 3.1. *The almost reducibility set of any equation $x' = A(t,\theta)x$ in $\mathcal{M}$ is an $F_{\sigma\delta}$-set.*

Proof. Let M be the almost reducibility set of the equation. For each $n \in \mathbb{N}$ and $\varepsilon > 0$, we define a function $g_{n,\varepsilon}\colon M_q \times GL_q \times \mathbb{R} \to [0,n]$ by

$$g_{n,\varepsilon}(B,C,\theta) = \sup_{t\geq 0} \min\{n, h_t(B,C,\theta)\},$$

where

$$f_t(B,C,\theta) = \max\big\{\|e^{Bt}CT_\theta(0,t)\|e^{-\varepsilon t}, \|T_\theta(t,0)C^{-1}e^{-Bt}\|e^{-\varepsilon t}\big\}.$$

The function $g_{n,\varepsilon}$ is lower semicontinuous in (B,C,θ) since the functions

$$\|e^{Bt}CT_\theta(0,t)\|e^{-\varepsilon t} \quad \text{and} \quad \|T_\theta(t,0)C^{-1}e^{-Bt}\|e^{-\varepsilon t}$$

are continuous (in view of the continuous dependence of a solution on a parameter) and the supremum of any number of continuous functions is lower semicontinuous. Therefore, the set

$$D_{n,\varepsilon} = g_{n,\varepsilon}^{-1}(-\infty, n/2]$$

is closed for each $n \in \mathbb{N}$ and $\varepsilon > 0$.

Lemma 3.2. *The equation $x' = A(t,\theta)x$ is almost reducible to the equation $x' = Bx$ if and only if there exists $C \in GL_q$ such that for each $\varepsilon > 0$ we have*

$$g_{n,\varepsilon}(B,C,\theta) \leq n/2 \quad \textit{for some } n \in \mathbb{N}. \tag{3.1}$$

Proof of the lemma. First assume that the equation $x' = A(t)x$ is almost reducible to the equation $x' = Bx$. Then there exists a Lyapunov coordinate change $(U(t))_{t\geq 0}$ satisfying (2.3). By property (2.2), for each $\varepsilon > 0$ we have

$$-\varepsilon < -\frac{1}{t}\log\|U(t)^{-1}\| \leq \frac{1}{t}\log\|U(t)\| < \varepsilon$$

for any sufficiently large t and so there exists $c = c(\varepsilon) > 0$ such that

$$c^{-1}e^{-\varepsilon t} < \|U(t)^{-1}\|^{-1} \leq \|U(t)\| < ce^{\varepsilon t} \tag{3.2}$$

for all $t \geq 0$. Now take $C = U(0)^{-1}$. By (2.3) with $s = 0$ we have

$$U(t) = T_\theta(t,0)C^{-1}e^{-Bt} \quad \text{and} \quad U(t)^{-1} = e^{Bt}CT_\theta(0,t).$$

Hence, it follows readily from (3.2) that

$$\sup_{t\geq 0}\big(\|e^{Bt}CT_\theta(0,t)\|e^{-\varepsilon t}+\|T_\theta(t,0)C^{-1}e^{-Bt}\|e^{-\varepsilon t}\big)<\infty$$

and so property (3.1) holds.

Now assume that there exists $C\in GL_q$ satisfying (3.1) for each $\varepsilon>0$. Then there exists $n\in\mathbb{N}$ such that

$$\max\big\{\|e^{Bt}CT_\theta(0,t)\|e^{-\varepsilon t},\|T_\theta(t,0)C^{-1}e^{-Bt}\|e^{-\varepsilon t}\big\}\leq n/2$$

for all $t\geq 0$ and so

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log\|e^{Bt}CT_\theta(0,t)\|\leq 0 \tag{3.3}$$

and

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log\|T_\theta(t,0)C^{-1}e^{-Bt}\|\leq 0. \tag{3.4}$$

Finally, let

$$U(t)=T_\theta(t,0)C^{-1}e^{-Bt}\quad\text{for } t\geq 0.$$

Note that $U(0)=C^{-1}$. Therefore,

$$\begin{aligned}e^{B(t-s)}&=e^{Bt}e^{-Bs}\\&=U(t)^{-1}T_\theta(t,0)U(0)\big(U(0)^{-1}T_\theta(0,s)U(s)\big)\\&=U(t)^{-1}T_\theta(t,s)U(s).\end{aligned}$$

Moreover, since

$$U(t)^{-1}=e^{Bt}CT_\theta(0,t),$$

it follows readily from (3.3) and (3.4) that

$$0\leq\underline{\lim_{t\to\infty}}\frac{1}{t}\log(\|U(t)^{-1}\|^{-1})\leq\overline{\lim_{t\to\infty}}\frac{1}{t}\log\|U(t)\|\leq 0$$

and so condition (2.2) also holds. □

By Lemma 3.2, the equation $x'=A(t,\theta)x$ is almost reducible if and only if there exist $B\in M_q$ and $C\in GL_q$ such that

$$(B,C,\theta)\in D_\varepsilon:=\bigcup_{n\in\mathbb{N}}D_{n,\varepsilon}$$

for each $\varepsilon>0$. Therefore, the almost reducibility set is

$$M=\bigcap_{\varepsilon>0}\pi(D_\varepsilon),$$

where $\pi\colon M_q\times GL_q\times\mathbb{R}\to\mathbb{R}$ is the projection onto the third component. For $k\in\mathbb{N}$ let

$$E_k=\big\{(B,C,\theta)\in M_q\times GL_q\times\mathbb{R}:\|B\|\leq k,\ k^{-1}\leq|\det C|\leq k,\ |\theta|\leq k\big\}.$$

Then each set $D_{n,\varepsilon}\cap E_k$ is compact and

$$D_\varepsilon=\bigcup_{n\in\mathbb{N}}D_{n,\varepsilon}=\bigcup_{n,k\in\mathbb{N}}(D_{n,\varepsilon}\cap E_k).$$

Therefore,

$$M=\bigcap_{\varepsilon>0}\pi(D_\varepsilon)=\bigcap_{p\in\mathbb{N}}\bigcup_{n,k\in\mathbb{N}}\pi(D_{n,1/p}\cap E_k)$$

and since the map π is continuous, each set $\pi(D_{n,1/p}\cap E_k)$ is compact. This shows that the almost reducibility set M is an $F_{\sigma\delta}$-set. □

4 Construction of families of equations

We also construct (as explicitly as possible) a differential equation in $\mathcal{M}$ with a given $F_{\sigma\delta}$-set containing zero as its almost reducibility set.

Theorem 4.1. *Given an integer $q \geq 2$ and an $F_{\sigma\delta}$-set M containing zero, there exists an equation $x' = A(t,\theta)x$ in $\mathcal{M}$ whose almost reducibility set is equal to M. Moreover, given an unbounded nondecreasing function $\rho(t) \geq 0$, we may require that*

$$\|A(t,\theta)\| \leq \rho(t)(1+|\theta|) \quad \textit{for all } t \geq 0 \textit{ and } \theta \in \mathbb{R}.$$

Proof. We start by describing some auxiliary notions that will be used in the proof. Given $a,b,c,\theta \in \mathbb{R}$, we consider the 2×2 matrices

$$B(u,\theta) = \begin{pmatrix} a\theta & c(1-\theta)+b\theta \\ -c(1-\theta)-b\theta & -a\theta \end{pmatrix}, \tag{4.1}$$

where $u = (a,b,c)$ and

$$\nu = \nu(u,\theta) = \sqrt{(a^2-(b-c)^2)\theta^2 - 2c(b-c)\theta - c^2}. \tag{4.2}$$

Then $B(u,\theta)$ has eigenvalues $\pm\nu$. Given $r,s \in \mathbb{R}$ with $rs > 0$ and $d \in \mathbb{R}^+$, we define

$$a = d(s-r), \quad b = d(2rs-r-s), \quad c = 2drs. \tag{4.3}$$

Then

$$a^2 - (b-c)^2 = -4d^2rs < 0$$

and one can show that $\theta \in [r,s]$ if and only if

$$P(u,\theta) := (a^2-(b-c)^2)\theta^2 - 2c(b-c)\theta - c^2 \geq 0. \tag{4.4}$$

Since M is an $F_{\sigma\delta}$-set containing zero, one can write

$$\mathbb{R}\setminus M = \bigcup_{w\in\mathbb{N}} H^w, \quad \text{where } H^w = \bigcap_{i\in\mathbb{N}} U_i^w$$

for some nonempty open sets $U_i^w \subset \mathbb{R}\setminus\{0\}$ satisfying $U_{i+1}^w \subset U_i^w$ for each $w,i \in \mathbb{N}$. Moreover, $U_i^w = \bigcup_{m\in\mathbb{N}} I_{im}^w$ for some nonempty open finite intervals $I_{im}^w \subset \mathbb{R}\setminus\{0\}$ with the property that each $\theta \in U_i^w$ belongs to at most two intervals I_{im}^w (for each $w,i \in \mathbb{N}$).

We still need an additional decomposition. For each interval $I_{im}^w = (\alpha,\beta)$, we consider the sequence $(c_l)_{l\in\mathbb{Z}}$ defined recursively as follows. Take $c_0 = (\alpha+\beta)/2$. For each $l \in \mathbb{N}$, let

$$c_{2l} = \frac{c_{2l-2}+\beta}{2}, \quad c_{-2l} = \frac{c_{-2l+2}+\alpha}{2}$$

and

$$c_{2l-1} = \frac{c_{2l-2}+c_{2l}}{2}, \quad c_{-2l+1} = \frac{c_{-2l+2}+c_{-2l}}{2}.$$

We define $J_{iml}^w = [c_l, c_{l+2}]$ for $l \in \mathbb{Z}$ and so

$$I_{im}^w = \bigcup_{l\in\mathbb{Z}} J_{iml}^w.$$

Note that each point $\theta \in U_i^w$ belongs to at most three intervals J_{iml}^w (for each $w, i, m \in \mathbb{N}$). Moreover, given $\theta \in I_{im}^w$, there exists $l = l(\theta) \in \mathbb{Z}$ with $\theta \in J_{iml}^w$ such that θ is at least at a distance $|J_{iml}^w|/6$ from each endpoint of J_{iml}^w (where $|I|$ denotes the length of the interval I).

Now let $\iota\colon \mathbb{N} \to \mathbb{N}^3 \times \mathbb{Z}$ be a bijection. Writing $J_{iml}^w = [r, s]$ and $\eta = \iota^{-1}(w, i, m, l)$, we consider the unique $d = d(\eta) \in \mathbb{R}^+$ such that

$$\max_{\theta \in \mathbb{R}} P(u(\eta), \theta) = d^2 rs(r-s)^2 = \frac{1}{w}, \tag{4.5}$$

with $u(\eta) = (a, b, c)$ given by (4.3). Then

$$P(u(\eta), \theta) \geq \frac{5}{9} d^2 rs(r-s)^2 = \frac{5}{9w} \quad \text{for } \theta \in \left[r + \frac{s-r}{6}, s - \frac{s-r}{6}\right]. \tag{4.6}$$

Consider the function $\sigma(t) = \min\{\rho(t), t\}$ for $t \geq 0$. Moreover, consider a strictly increasing sequence of positive integers $(\ell_j)_{j\in\mathbb{N}}$ such that $\ell_1 = 1$,

$$\frac{\ell_{3j-2}}{\ell_{3j-1}} \sum_{i=1}^{j-1} \sigma(\ell_{3i-1}) < \frac{1}{j}, \quad \frac{\ell_{3j-1}}{\ell_{3j}} < \frac{1}{j}, \quad \ell_{3j+1} = 2\ell_{3j} - \ell_{3j-1} \tag{4.7}$$

and

$$\sigma(\ell_{3j-1}) \geq 2\kappa \|u(j)\| \tag{4.8}$$

for all $j \in \mathbb{N}$, where $\kappa > 0$ is fixed constant such that

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| \leq \kappa \|(a, b, c, d)\| \quad \text{for any } a, b, c, d \in \mathbb{R}.$$

Finally, let $\Delta_j = [\ell_j, \ell_{j+1})$ for each $j \in \mathbb{N}$ and define

$$A(t, \theta) = \begin{cases} B(u(j), \theta) & \text{if } t \in \Delta_{3j} \text{ for some } j \in \mathbb{N}, \\ -B(u(j), \theta) & \text{if } t \in \Delta_{3j-1} \text{ for some } j \in \mathbb{N}, \\ \mathrm{Id} & \text{if } t \in \Delta_{3j-2} \text{ for some } j \in \mathbb{N}. \end{cases}$$

By (4.1) together with (4.8), we obtain

$$\begin{aligned} \|A(t, \theta)\| &\leq \|B(u(j), \theta)\| \leq \kappa \|u(j)\| \\ &\leq \sigma(\ell_{3j-1})(1 + |\theta|) \\ &\leq \sigma(t)(1 + |\theta|) \leq \rho(t)(1 + |\theta|). \end{aligned}$$

Lemma 4.2. *$x' = A(t, \theta)x$ is not almost irreducible for $\theta \in \mathbb{R} \setminus M$.*

Proof of the lemma. Take $w \in \mathbb{N}$ such that $\theta \in U_i^w$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\theta \in I_{im}^w$. Moreover, let $l = l(\theta) \in \mathbb{Z}$ be the integer introduced before (4.5) and write $\iota^{-1}(w, i, m, l) = r_i$. For each $j \in \Delta_{3r_i-1} \cup \Delta_{3r_i}$ the matrices $\pm B(u(3r_i), \theta)$ have real eigenvalues. Denoting their (common) top eigenvalue by ν_i, it follows readily from (4.2) and (4.4) together with (4.5) and (4.6) that

$$\frac{1}{2\sqrt{w}} \leq \nu_i \leq \frac{1}{\sqrt{w}}.$$

Denoting by $T_\theta(t, s)$ the evolution family associated with the equation $x' = A(t, \theta)x$, we have $T_\theta(\ell_{3r_i-1}, 0) = \mathrm{Id}$ and so

$$T_\theta(\ell_{3r_i}, 0) = T_\theta(\ell_{3r_i}, \ell_{3r_i-1}).$$

Therefore,

$$\begin{aligned}\|T_\theta(\ell_{3r_i},0)\| &= \|T_\theta(\ell_{3r_i},\ell_{3r_i-1})\| \\ &= \left\|e^{(\ell_{3r_i}-\ell_{3r_i-1})B(u(3r_i),\theta)}\right\| \\ &\geq e^{\nu_i(\ell_{3r_i}-\ell_{3r_i-1})} \\ &\geq \exp\left(\frac{\ell_{3r_i}-\ell_{3r_i-1}}{2\sqrt{w}}\right)\end{aligned}$$

which in view of (4.7) gives

$$\varlimsup_{i\to\infty}\frac{1}{\ell_{3r_i}}\log\|T_\theta(\ell_{3r_i},0)\| \geq \varlimsup_{i\to\infty}\frac{1}{2\sqrt{w}}\left(1-\frac{\ell_{3r_i-1}}{\ell_{3r_i}}\right) = \frac{1}{2\sqrt{w}} > 0. \tag{4.9}$$

Now we assume that the equation $x' = A(t,\theta)x$ is almost reducible to an equation $x' = Bx$. Then there exist matrices $U(t)$ satisfying (2.2) and (2.3). Since $T_\theta(\ell_{3r_i-1},0) = \mathrm{Id}$, we have

$$e^{B\ell_{3r_i-1}} = U(\ell_{3r_i-1})^{-1}U(0)$$

and

$$e^{-B\ell_{3r_i-1}} = U(0)^{-1}U(\ell_{3r_i-1})$$

for all $i \in \mathbb{N}$. Therefore, for each $\varepsilon > 0$ there exists $c_0 = c_0(\varepsilon) > 0$ such that

$$\max\left\{\|e^{B\ell_{3r_i-1}}\|, \|e^{-B\ell_{3r_i-1}}\|\right\} \leq c_0 e^{\varepsilon\ell_{3r_i-1}}$$

for all $i \in \mathbb{N}$. Since $\ell_{3r_i-1} \to \infty$ when $i \to \infty$ and ε is arbitrary, all eigenvalues of B have real part equal to 0 and so

$$\|e^{Bt}\| \leq c_1(1+|t|) \quad \text{for some } c_1 > 0 \text{ and any } t \geq 0.$$

On the other hand, by (2.3) we have

$$T_\theta(t,0) = U(t)e^{Bt}U(0)^{-1}$$

and so

$$\begin{aligned}\|T_\theta(\ell_{3r_i},0)\| &\leq \|U(0)^{-1}\|\cdot\|U(\ell_{3r_i})\|\cdot\|e^{B\ell_{3r_i}}\| \\ &\leq c_1(1+|\ell_{3r_i}|)\|U(0)^{-1}\|\cdot\|U(\ell_{3r_i})\|.\end{aligned}$$

Finally, taking into account that $U(t)$ satisfies (2.2) we obtain

$$\varlimsup_{i\to\infty}\frac{1}{\ell_{3r_i}}\log\|T_\theta(\ell_{3r_i},0)\| \leq 0,$$

which contradicts (4.9). This shows that the equation $x' = A(t,\theta)x$ is not almost reducible. □

Lemma 4.3. *$x' = A(t,\theta)x$ is almost reducible for $\theta \in M$.*

Proof of the lemma. Since $\theta \notin H^w$ for every $w \in \mathbb{N}$, θ belongs to at most finitely many sets U_i^w, $i \in \mathbb{N}$ (because $U_{i+1}^w \subset U_i^w$ for each $w, i \in \mathbb{N}$) and since each element of U_i^w belongs to at most two intervals I_{im}^w with $m \in \mathbb{N}$ and to at most three closed intervals J_{iml}^w with $l \in \mathbb{Z}$, we conclude that θ belongs to finitely many closed intervals J_{iml}^w with $i, m \in \mathbb{N}$ and $l \in \mathbb{Z}$ for each $w \in \mathbb{N}$. This implies that for each $w \in \mathbb{N}$ there exists $N = N_w \in \mathbb{N}$ such that for $\eta \geq N$

with $\iota(\eta) = (w_\eta, i_\eta, m_\eta, l_\eta)$ we have $\theta \notin J^{w_\eta}_{i_\eta m_\eta l_\eta}$ and so also $P(u(\eta), \theta) < 0$ whenever $w_\eta \le w$. In particular, for $\eta \ge N$ with $w_\eta \le w$ we have $\nu = i\bar\nu$ with $\bar\nu \in \mathbb{R}$ and so

$$\|e^{B(u(\eta),\theta)t}\| \le 1 + 2\|B(u(\eta),\theta)\| \cdot |t|$$

(see for example [2, p. 65]). For the values of $\eta \ge N$ with $w_\eta > w$, in view of (4.5) we have

$$P(u(\eta),\theta) \le \frac{1}{w_\eta} \le \frac{1}{w+1}.$$

Take $w \in \mathbb{N}$. If $\eta \ge N$, then

$$\begin{aligned} \|T_\theta(t, \ell_{3\eta-1})\| &\le \left\|e^{B(u(\eta),\theta)(t-\ell_{3\eta-1})}\right\| \\ &\le \big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)(t-\ell_{3\eta-1})\big) \exp\left(\frac{t-\ell_{3\eta-1}}{\sqrt{w+1}}\right) \\ &\le \big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)t\big) \exp\left(\frac{t-\ell_{3\eta-1}}{\sqrt{w+1}}\right) \end{aligned} \tag{4.10}$$

for $t \in \Delta_{3\eta-1} \cup \Delta_{3\eta}$. Now take

$$t \in \Delta_{3\eta-1} \cup \Delta_{3\eta} \cup \Delta_{3\eta+1} \quad \text{with } \eta \in \mathbb{N}.$$

Since $A(t,\theta) = \mathrm{Id}$ for $t \in \Delta_{3\eta-2}$ and

$$T_\theta(t, \ell_{3N-1}) = T_\theta(t, \ell_{3\eta-1}) \prod_{i=N}^{\eta-1} T_\theta(\ell_{3i+1}, \ell_{3i-1}),$$

using (4.10) we obtain

$$\begin{aligned} \|T_\theta(t, \ell_{3N-1})\| &\le \|T_\theta(t, \ell_{3\eta-1})\| \prod_{i=N}^{\eta-1} \|T_\theta(\ell_{3i+1}, \ell_{3i-1})\| \\ &\le \big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)t\big) \prod_{i=N}^{\eta-1} \big(1 + 2\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big) \\ &\quad \times \exp\left(\frac{1}{\sqrt{w+1}}\left((t-\ell_{3\eta-1}) + \sum_{i=N}^{\eta-1}(\ell_{3i+1} - \ell_{3i-1})\right)\right) \\ &\le \big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)t\big) \prod_{i=1}^{\eta-1} \big(1 + 2\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big) \\ &\quad \times \exp\left(\frac{t - \ell_{3\eta-1} + \ell_{3\eta-2} - \ell_{3N-1}}{\sqrt{w+1}}\right) \\ &\le \big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)t\big) \prod_{i=1}^{\eta-1} \big(1 + 2\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big) \exp\frac{t}{\sqrt{w+1}}. \end{aligned}$$

Then

$$\begin{aligned} \overline{\lim_{t\to\infty}} \frac{1}{t} \log\|T_\theta(t, \ell_{3N-1})\| &\le \frac{1}{\sqrt{w+1}} + \overline{\lim_{t\to\infty}} \frac{1}{t} \log\big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)t\big) \\ &\quad + \overline{\lim_{t\to\infty}} \frac{1}{t} \sum_{i=1}^{\eta-1} \log\big(1 + 2\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big). \end{aligned} \tag{4.11}$$

Since

$$\sigma(\ell_{3\eta-1}) = \min\{\rho(\ell_{3\eta-1}), \ell_{3\eta-1}\} \le \ell_{3\eta-1} \le t,$$

we obtain

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log\big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)t\big) = 0. \tag{4.12}$$

Moreover, since $\log(1+x) \le x$ for all $x \ge 0$, it follows from (4.7) that

$$\begin{aligned}\frac{1}{t}\sum_{i=1}^{\eta-1} \log\big(1 + 2\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big) &\le \frac{2}{\ell_{3\eta-1}} \sum_{i=1}^{\eta-1} \big(\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big) \\ &\le \frac{2(1+|\theta|)\ell_{3\eta-2}}{\ell_{3\eta-1}} \sum_{i=1}^{\eta-1} \sigma(\ell_{3i-1}) \\ &\le \frac{2(1+|\theta|)}{\eta}.\end{aligned}$$

Therefore,

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \sum_{i=1}^{\eta-1} \log\big(1 + 2\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big) = 0 \tag{4.13}$$

since $\eta \to \infty$ when $t \to \infty$. By (4.12) and (4.13), it follows from (4.11) that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log\|T_\theta(t, \ell_{3N-1})\| \le \frac{1}{\sqrt{w+1}}$$

for any $w \in \mathbb{N}$ and so

$$\overline{\lim_{t\to\infty}} \frac{1}{n} \log\|T_\theta(t, 0)\| \le 0.$$

One can also show that

$$\overline{\lim_{t\to\infty}} \frac{1}{n} \log\|T_\theta(0, t)\| \le 0$$

interchanging $B(u, \theta)$ with $-B(u, \theta)$ in the definition of $A(t, \theta)$. This implies that

$$\begin{aligned}\underline{\lim_{t\to\infty}} \frac{1}{t} \log\|T_\theta(t, 0)\| &\ge \underline{\lim_{t\to\infty}} \frac{1}{t} \log\big(\|T_\theta(0, t)\|^{-1}\big) \\ &= -\overline{\lim_{t\to\infty}} \frac{1}{t} \log\|T_\theta(0, t)\| \ge 0\end{aligned}$$

and so

$$\lim_{t\to\infty} \frac{1}{t} \log\|T(t, 0)^{\pm 1}\| = 0.$$

For $U_\theta(t) = T_\theta(t, 0)$ we have

$$U_\theta(t)^{-1} T_\theta(t, 0) U_\theta(t) = T_\theta(0, t) T_\theta(t, 0) = \mathrm{Id}.$$

So, identity (2.3) holds with $B = 0$. This shows that the differential equation $x' = A(t, \theta)x$ is almost reducible. □

In order to construct an equation $x' = \tilde{A}(t, \theta)x$ on $\mathbb{R}^q$ with almost reducibility set M for $q > 2$, it suffices to take

$$\tilde{A}(t, \theta) = \mathrm{diag}(A(t, \theta), 0).$$

This concludes the proof of the theorem. □

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