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Strong solutions for the steady incompressible MHD equations of non-Newtonian fluids

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Abstract. In this paper we deal with a system of partial differential equations describing a steady motion of an incompressible magnetohydrodynamic fluid, where the extra stress tensor is induced by a potential with p-structure ($p = 2$ corresponds to the Newtonian case). By using a fixed point argument in an appropriate functional setting, we proved the existence and uniqueness of strong solutions for the problem in a smooth domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ under the conditions that the external force is small in a suitable norm.

Keywords: strong solutions, existence and uniqueness, incompressible magnetohydrodynamics, non-Newtonian fluids.

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1 Introduction and main result

Magnetohydrodynamics (MHD) concerns the interaction of electrically conductive fluids and electromagnetic fields. The system of partial differential equations in MHD are basically obtained through the coupling of the dynamical equations of the fluids with the Maxwell's equations which is used to take into account the effect of the Lorentz force due to the magnetic field, it has spanned a very large range of applications [21, 24, 25]. By neglecting the displacement current term, a commonly used simplified MHD system could be described by

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \operatorname{div} \boldsymbol{\tau}(\mathcal{D}\boldsymbol{u}) + \nabla p = \dfrac{1}{\mu}(\nabla \times \boldsymbol{b}) \times \boldsymbol{b} + \boldsymbol{f}, & \text{in } Q_T, \\ \boldsymbol{b}_t + \dfrac{1}{\mu} \operatorname{curl} \left(\dfrac{1}{\sigma} \operatorname{curl} \boldsymbol{b} \right) = \operatorname{curl}(\boldsymbol{u} \times \boldsymbol{b}), & \text{in } Q_T, \\ \operatorname{div} \boldsymbol{u} = 0, \qquad \operatorname{div} \boldsymbol{b} = 0, & \text{in } Q_T, \end{cases} \tag{1.1}$$

where $Q_T = \Omega \times (0, T)$, the unknown functions $\boldsymbol{u} = (u_1(x,t), u_2(x,t), \dots, u_n(x,t))$ denotes the velocity of the fluid, $\boldsymbol{b} = (b_1(x,t), b_2(x,t), \dots, b_n(x,t))$ the magnetic field, $p = p(x,t)$ the pressure and $\boldsymbol{f} = (f_1(x,t), f_2(x,t), \dots, f_n(x,t))$ the external force applied to the fluid.

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Also, $\boldsymbol{\tau} = (\tau_{ij})$ is the stress tensor depending on the strain rate tensor $\mathcal{D}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$, $\mu > 0$ and $\sigma > 0$ denotes the permeability coefficient and the electric conductivity coefficient respectively. For the sake of simplicity, in this work, we take $\mu = 1$ and $\sigma = 1$.

Due to the conventional belief that the Navier–Stokes equations are an accurate model for the motion of incompressible fluids in many practical situations, the majority of the known work have assumed that the stress tensor $\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u})$ is a linear function of the strain rate $\mathcal{D}\boldsymbol{u}$. In this way we obtain the conventional system for MHD, and this classical model has been extensively studied. For instance, Duvaut and Lions [7] established the local existence and uniqueness of a solution in the Sobolev space $H^s(R^N)(s \geq N)$. They also proved the global existence of a solutions to this system with small initial data. Sermange and Temam [28] proved the existence of a unique global solution in the two space dimensions. For the zero magnetic diffusion case, Lin, Xu and Zhang [22] and Xu and Zhang [29] established the global well-posedness in two and three dimensional space, respectively, under the assumption that the initial data are sufficiently close to the equilibrium state. The global existence of smooth solutions was proved by Lei [18] for the ideal MHD with axially symmetric initial datum in $H^s(R^3)$ with $s \geq 2$. For more details, one can also refer [3–5,8,9,11,13–16,23] and the reference cited therein.

In recent years, the flow of non-Newtonian fluids (i.e. the stress tensor $\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u})$ being a non-linear function of $\mathcal{D}\boldsymbol{u}$) has gained much importance in numerous technological applications. Further, the motion of the non-Newtonian fluids in the presence of a magnetic field in different contexts has been studied by several authors (see [2,6,26]). A typical form of the stress tensor $\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u})$ is of some p– structure with $\mathcal{D}\boldsymbol{u}$ which were firstly proposed by Ladyzhenskaya in [19,20]. For the MHD equations of non-Newtonian type (1.1), the known results are limited and here we only recall two results closely related to ours. In case that $\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u}) = |\mathcal{D}\boldsymbol{u}|^{p-2}\mathcal{D}\boldsymbol{u}$ for $p \geq \frac{5}{2}$, Samokhin proved in [27] the existence of weak solutions by using Galerkin method and the monotone theory, which solve the equations in the sense of distributions and satisfy the following energy inequality

$$\sup_{0\leq t\leq T}(\|u(t)\|_2^2 + \|b(t)\|_2^2) + 2\int_0^T (\|\nabla u(t)\|_p^p + \|\nabla b(t)\|_2^2)dt \leq (\|u_0\|_2^2 + \|b_0\|_2^2).$$

Later on, Gunzburger and his collaborators considered (1.1) with $\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u}) = (1 + |\mathcal{D}\boldsymbol{u}|^{p-2})\mathcal{D}\boldsymbol{u}$ for the case of bounded or periodic domains, and they showed the existence and uniqueness of a weak solutions, see [12] for more details.

In this paper, in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3), we consider a steady incompressible MHD equations of non-Newtonian fluids described by

$$\begin{cases} -\operatorname{div}\left[2\mu(1+|D\boldsymbol{u}|^2)^{\frac{p-2}{2}}D\boldsymbol{u}\right] + \nabla p = \boldsymbol{f} - \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + (\nabla\times\boldsymbol{b})\times\boldsymbol{b}, & x \in \Omega, \\ -\Delta\boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, & x \in \Omega, \\ \operatorname{div}\boldsymbol{u} = 0, \qquad \operatorname{div}\boldsymbol{b} = 0, & x \in \Omega, \end{cases} \tag{1.2}$$

supplemented by the boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = 0, \quad \boldsymbol{b}\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \qquad (\nabla\times\boldsymbol{b})\times\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{0}, \tag{1.3}$$

where $p > 1$, $\boldsymbol{n}$ is the unit outward normal vector of $\partial\Omega$.

Remark 1.1. Since $\boldsymbol{u}$ and $\boldsymbol{b}$ are divergence free (i.e. $\operatorname{div}\boldsymbol{u} = 0$, $\operatorname{div}\boldsymbol{b} = 0$), an elementary computations leads to the formulas

$$\operatorname{curl}\operatorname{curl}\boldsymbol{b} = -\Delta\boldsymbol{b}, \qquad \operatorname{curl}(\boldsymbol{u}\times\boldsymbol{b}) = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b}. \tag{1.4}$$

The aim of this paper is to prove the existence and uniqueness of strong solutions to system (1.2)–(1.3) under the assumption that the L^q-norm of the external force field $\boldsymbol{f}$ is small in a suitable sense. Our approach is based on regularity results for the Stokes problem and magnetic equation, and a fixed-point argument.

Throughout the paper, for $m \in \mathbb{N}$, the standard Lebesgue spaces are denoted by $\mathbf{L}^q(\Omega)$ and their norms by $\|\cdot\|_q$, the standard Sobolev spaces are denoted by $\mathbf{W}^{m,q}(\Omega)$ and their norms by $\|\cdot\|_{m,q}$. We also denote by $\mathbf{W}_0^{m,q}(\Omega)$ the closure in $\mathbf{W}^{m,q}(\Omega)$ of $C_0^\infty(\Omega)$. $W^{-1,q}(\Omega)$ denotes the dual of $W_0^{1,q}(\Omega)$ and their norms by $\|\cdot\|_{-1,q;\Omega}$. For $x, y \in \mathbb{R}$ we denote $(x,y)^+ = \max\{x,y\}$, $x^+ = \max\{x,0\}$. We introduce the constants

$$S_p := (|p-2|,2)^+, \qquad r_p := \frac{1+(p-3)^+-(p-4)^+}{2}, \qquad \gamma_p := \frac{[(p,3)^+-2]^{(p,3)^+-2}}{[(p,3)^+-1]^{(p,3)^+-1}}. \tag{1.5}$$

We also introduce the space

$$\begin{aligned}
\mathcal{V} &:= \{\boldsymbol{u} \in C_0^\infty(\Omega),\ \operatorname{div}\boldsymbol{u} = 0\};\\
\mathbf{V}_p &:= \{\boldsymbol{u} \in \mathbf{W}_0^{1,p}(\Omega) : \operatorname{div}\boldsymbol{u} = 0\};\\
\mathbf{V}_{m,p} &:= \{\boldsymbol{v} \in \mathbf{W}_0^{1,p}(\Omega) \cap \mathbf{W}^{m,p}(\Omega) : \operatorname{div}\boldsymbol{v} = 0\ \};\\
\mathbf{W} &:= \{\boldsymbol{b} \in \mathbf{W}^{1,2}(\Omega) : \operatorname{div}\boldsymbol{b} = 0\ , \boldsymbol{b}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\}.
\end{aligned}$$

Also, for $q > r > n$ and $\delta > 0$, let us denote by B_δ the convex set defined by

$$B_\delta := \Big\{(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbf{V}_{2,q} \times (\mathbf{W}^{2,r}(\Omega) \cap \mathbf{W}) : C_E\|\nabla\boldsymbol{\xi}\|_{1,q} \le \delta, \quad C_{\widetilde{E}}\|\nabla\boldsymbol{\eta}\|_{1,r} \le \delta\Big\}, \tag{1.6}$$

where C_E is the norm of the embedding of $W^{1,q}(\Omega)$ into $L^\infty(\Omega)$ and $C_{\widetilde{E}}$ is the norm of the embedding of $W^{1,r}(\Omega)$ into $L^\infty(\Omega)$, also C_p denotes the Poincaré constant corresponding to the general Poincaré inequality $\|\cdot\|_s \le C_p\|\nabla(\cdot)\|_s$. We consider the space $\mathbf{V}_{2,q} \times \mathbf{W}^{2,r}(\Omega)$ endowed with the norm

$$\|(\boldsymbol{\xi},\boldsymbol{\eta})\|_{1,q,r} := \max\{\|\nabla\boldsymbol{\xi}\|_{1,q}, \ \|\nabla\boldsymbol{\eta}\|_{1,r}\}.$$

Now, we formulate the main theorem of this paper.

Theorem 1.2. *Assume that $q > r > n$, $p > 1$, $\mu > 0$, and let $\boldsymbol{f} \in \mathbf{L}^q(\Omega)$. There exist positive constant $\overline{C} = \overline{C}(C_0, C_p, C_E, C_{\widetilde{E}}, C_{-1}, c_2)$ such that if*

$$\overline{C}\left[\Big(1+\frac{1}{\mu}\Big)\frac{\overline{C}\|\boldsymbol{f}\|_q}{\mu} + S_p\Big(\frac{\overline{C}\|\boldsymbol{f}\|_q}{\mu}\Big)^{2r_p}\Big(1+\frac{\overline{C}\|\boldsymbol{f}\|_q}{\mu}\Big)^{(p-4)^+}\right] < \frac{1}{4^{(p-2,1)^+}}, \tag{1.7}$$

then, problem (1.2)–(1.3) has a unique strong solution $(\boldsymbol{u},\boldsymbol{b}) \in \mathbf{V}_{2,q} \times \mathbf{W}^{2,r}(\Omega)$.

Remark 1.3. As usual, the pressure π has disappeared from the notion of solution. Actually, the pressure may be recovered by de Rham Theorem at least in $L^2(\Omega)$, such that the triple $(\boldsymbol{u},\pi,\boldsymbol{b})$ satisfies equations (1.2)–(1.3) almost everywhere (see [11]).

The rest of our paper is organized as follows: in Section 2, we review some known results and Section 3 is devoted to proving the main theorem to problem (1.2)–(1.3).

2 Preliminary lemmas

In this section, we recall some basic facts which will be used later.

Lemma 2.1 ([10, Theorem 6.1, pp. 225]). *Let $m \geq -1$ be an integer and let Ω be a bounded domain in $\mathbb{R}^n$ $(n = 2, 3)$ with boundary $\partial\Omega$ of class C^k with $k = (m+2, 2)^+$. Then for any $\boldsymbol{\psi} \in \mathbf{W}^{m,\rho}(\Omega)$, the following system*

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{\psi}, & x \in \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, & x \in \Omega, \\ \boldsymbol{u}|_{\partial\Omega} = 0, \end{cases}$$

admits a unique solution $[\boldsymbol{u}, \pi] \in \mathbf{W}^{m+2,\rho}(\Omega) \times W^{m+1,\rho}(\Omega)$. Moreover, the following estimate holds

$$\|\nabla \boldsymbol{u}\|_{m+1,\rho} + \|\pi\|_{m+1,\rho/\mathbb{R}} \leq C_m \|\boldsymbol{\psi}\|_{m,\rho},$$

where $C_m = C_m(n, \rho, \Omega)$ is a positive constant.

Lemma 2.2 ([1]). *Let r_p, γ_p are given by (1.5) and let $G : \mathbb{R}^+ \to \mathbb{R}$ be defined by*

$$G(\delta) = A\delta^2 - \delta + E\delta\mathcal{H}(\delta) + D,$$

where A, E, D are positive constants and $\mathcal{H}(x) = x^{2r_p}(1+x)^{(p-4)^+}$. Thus, if the following assertion holds

$$AD + ED^{2r_p}(1+D)^{(p-4)^+} \leq \gamma_p,$$

then G possesses at least one root δ_0. Moreover, $\delta_0 > D$ and for every $\beta \in [1, 2]$ the following estimate holds

$$\frac{\beta-1}{\beta}\delta_0 + \frac{2-\beta}{\beta}A\delta_0^2 + \frac{2r_p+1-\beta}{\beta}E\delta_0\mathcal{H}(\delta_0) + \frac{E(p-4)^+}{\beta}\delta_0^{2r_p+2}(1+\delta_0)^{(p-4)^+-1} \leq D.$$

Lemma 2.3 ([17]). *Let X and Y be Banach spaces such that X is reflexive and $X \hookrightarrow Y$. Let B be a non-empty, closed, convex and bounded subset of X and let $T : B \to B$ be a mapping such that*

$$\|T(u) - T(v)\|_Y \leq K\|u - v\|_Y, \qquad \forall\, u, v \in B \ \ (0 < K < 1),$$

then T has a unique fixed point in B.

3 Proof of Theorem 1.2

Our proof relies on a Banach fixed point theorem. Toward this aim, we first reformulate the problem as follows

$$\begin{cases} -\mu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} - \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + (\nabla \times \boldsymbol{b}) \times \boldsymbol{b} + \operatorname{div}[2\mu\sigma(|D\boldsymbol{u}|^2)D\boldsymbol{u}], & x \in \Omega, \\ -\Delta \boldsymbol{b} = (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{b}, & x \in \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, \qquad \operatorname{div} \boldsymbol{b} = 0, & x \in \Omega, \\ \boldsymbol{u}|_{\partial\Omega} = 0, \qquad \boldsymbol{b} \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \qquad (\nabla \times \boldsymbol{b}) \times \boldsymbol{n}|_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

where $\sigma(x) = (1+x)^{\frac{p-2}{2}} - 1$.

Given $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{V}_{2,q} \times \mathbf{W}^{2,r}(\Omega)$, we consider the following problem

$$\begin{cases} -\mu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} - \operatorname{div}(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) + (\nabla \times \boldsymbol{\eta}) \times \boldsymbol{\eta} + \operatorname{div}[2\mu\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi}], & x \in \Omega, \\ -\Delta \boldsymbol{b} = (\boldsymbol{\eta} \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{\eta}, & x \in \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, \qquad \operatorname{div} \boldsymbol{b} = 0, & x \in \Omega, \\ \boldsymbol{u}|_{\partial\Omega} = 0, \qquad \boldsymbol{b} \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \qquad (\nabla \times \boldsymbol{b}) \times \boldsymbol{n}|_{\partial\Omega} = 0. \end{cases} \tag{3.2}$$

From Lemma 2.1 and Proposition 2.30 in [11], there exists a unique solution $(\boldsymbol{u}, \boldsymbol{b}) \in \mathbf{V}_{2,q} \times \mathbf{W}^{2,r}(\Omega)$ to (3.2). We define the mapping

$$T : (\boldsymbol{\xi}, \boldsymbol{\eta}) \to (\boldsymbol{u}, \boldsymbol{b}).$$

Our purpose now is to prove that $T_{B_{\delta_0}}$ is a contraction from B_{δ_0} to itself for some $\delta_0 > 0$. Here B_{δ_0} is the closed ball defined in (1.6).

Proposition 3.1. *Let $q > r > n$, $p > 1$, $\mu > 0$, and let $\boldsymbol{f} \in \mathbf{L}^q(\Omega)$. There exists a positive constant $M_1 = M_1(C_0, C_p, C_E, C_{\widetilde{E}})$ such that if*

$$\frac{M_1^2 \|\boldsymbol{f}\|_q}{\mu^2} + M_1 S_p \left(\frac{M_1 \|\boldsymbol{f}\|_q}{\mu}\right)^{2r_p} \left(1 + \frac{M_1 \|\boldsymbol{f}\|_q}{\mu}\right)^{(p-4)^+} \leq \gamma_p, \tag{3.3}$$

then $T(B_{\delta_0}) \subseteq B_{\delta_0}$ for some $\delta_0 > 0$.

Proof. Let $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in B_\delta$. From Lemma 2.1, $\boldsymbol{u} \in \mathbf{V}_{2,q}$ and it satisfies

$$\|\nabla \boldsymbol{u}\|_{1,q} \leq \frac{C_0}{\mu}\big(\|\boldsymbol{f}\|_q + \|\boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi}\|_q + \|(\nabla \times \boldsymbol{\eta}) \times \boldsymbol{\eta}\|_q + \|\operatorname{div}[2\mu\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi}]\|_q\big). \tag{3.4}$$

Notice that

$$\begin{aligned} \|(\nabla \times \boldsymbol{\eta}) \times \boldsymbol{\eta}\|_q &\leq \|\boldsymbol{\eta}\|_\infty \|\nabla \boldsymbol{\eta}\|_q \leq C_{\widetilde{E}} \|\boldsymbol{\eta}\|_{1,r} \|\nabla \boldsymbol{\eta}\|_{1,r} \\ &\leq \delta(C_p + 1)\|\nabla \boldsymbol{\eta}\|_r \leq \delta(C_p + 1)\|\nabla \boldsymbol{\eta}\|_{1,r} \\ &\leq \frac{(C_p + 1)}{C_{\widetilde{E}}}\delta^2, \end{aligned} \tag{3.5}$$

reasoning as in [1], we could obtain

$$\|\boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi}\|_q + \|\operatorname{div}[2\mu\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi}]\|_q \leq \frac{C_p}{C_E}\delta^2 + \frac{4\mu S_p}{C_E}\delta\mathcal{H}(\delta). \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we get

$$\|\nabla \boldsymbol{u}\|_{1,q} \leq \frac{M_1}{\mu}\big(\|\boldsymbol{f}\|_q + \delta^2 + \mu S_p \delta \mathcal{H}(\delta)\big),$$

where $M_1 = C_0 \max\{1, \frac{C_p}{C_E} + \frac{(C_p+1)}{C_{\widetilde{E}}}, \frac{4}{C_E}\}$.

On the other hand, by Proposition 2.30 in [11], there exists a constant $c_1 > 0$ such that

$$\begin{aligned} \|\nabla \boldsymbol{b}\|_{1,r} &\leq c_1 \left[\|\boldsymbol{\eta} \cdot \nabla \boldsymbol{\xi}\|_r + \|\boldsymbol{\xi} \cdot \nabla \boldsymbol{\eta}\|_r\right] \\ &\leq c_1 \left[C_{\widetilde{E}} \|\boldsymbol{\eta}\|_{1,r} \|\nabla \boldsymbol{\xi}\|_{1,q} + C_E \|\boldsymbol{\xi}\|_{1,q} \|\nabla \boldsymbol{\eta}\|_{1,r}\right] \\ &\leq c_1 \left[C_{\widetilde{E}}(C_p + 1)\|\nabla \boldsymbol{\eta}\|_r \|\nabla \boldsymbol{\xi}\|_{1,q} + C_E(C_p + 1)\|\nabla \boldsymbol{\xi}\|_q \|\nabla \boldsymbol{\eta}\|_{1,r}\right] \\ &\leq c_1 \left[C_{\widetilde{E}}(C_p + 1)\|\nabla \boldsymbol{\eta}\|_{1,r} \frac{\delta}{C_E} + C_E(C_p + 1)\|\nabla \boldsymbol{\xi}\|_{1,q} \frac{\delta}{C_{\widetilde{E}}}\right] \\ &\leq c_1 \left[\frac{(C_p + 1)}{C_E}\delta^2 + \frac{(C_p + 1)}{C_{\widetilde{E}}}\delta^2\right] \\ &\leq 2M_2\delta^2, \end{aligned} \tag{3.7}$$

where $M_2 = c_1 \max\{\frac{(C_p+1)}{C_E}, \frac{(C_p+1)}{C_{\widetilde{E}}}\}$. In order to ensure that $T(B_\delta) \subseteq B_\delta$, it is enough to show that

$$\begin{aligned} \|\nabla \boldsymbol{u}\|_{1,q} &\le \frac{M_1}{\mu}\big(\|\boldsymbol{f}\|_q + \delta^2 + \mu S_p \delta \mathcal{H}(\delta)\big) \le \delta, \\ \|\nabla \boldsymbol{b}\|_{1,r} &\le 2M_2\delta^2 \le \delta. \end{aligned} \tag{3.8}$$

Using Lemma 2.2 with $A = \frac{M_1}{\mu}$, $E = M_1 S_p$ and $D = \frac{M_1\|\boldsymbol{f}\|_q}{\mu}$, there exists $\delta_1 > \frac{M_1\|\boldsymbol{f}\|_q}{\mu}$ such that

$$\frac{M_1}{\mu}\big(\|\boldsymbol{f}\|_q + \delta_1^2 + \mu S_p \delta_1 \mathcal{H}(\delta_1)\big) \le \delta_1,$$

provided that

$$AD + ED^{2r_p}(1+D)^{(p-4)^+} \le \gamma_p,$$

which holds from the hypothesis (3.3). Also, it holds ($\beta = 2$ in Lemma 2.2) that

$$\delta_1 \le \frac{2M_1\|\boldsymbol{f}\|_q}{\mu}.$$

On the other hand, we reformulate the inequality $(3.8)_2$ as

$$2M_2\delta^2 - \delta \le 0. \tag{3.9}$$

Due to

$$\Delta = 1 > 0,$$

we deduce that for some δ, the inequality (3.9) is valid.

Take the constant D to satisfy $\delta^- < D < 2D < \delta^+$, where

$$\delta^\pm = \frac{1}{4M_2} \pm \sqrt{1} = \frac{1 \pm 4M_2}{4M_2}.$$

Moreover, given that for every $\delta \in [\delta^-, \delta^+]$, the inequality (3.9) is valid, we can choose $\delta_2 \in (\delta^-, D)$ such that

$$2M_2\delta_2^2 \le \delta_2.$$

In conclusion, we obtain

$$\delta_2 < \frac{M_1\|\boldsymbol{f}\|_q}{\mu} < \delta_1 \le \frac{2M_1\|\boldsymbol{f}\|_q}{\mu}.$$

Thus, taking $\delta_0 = \delta_1$ we obtain that $T(B_{\delta_0}) \subseteq B_{\delta_0}$. □

Proposition 3.2. *There is a positive constant $m = m(C_{-1}, C_p, c_2, C_E, C_{\widetilde{E}})$ such that if*

$$m\left[\left(1+\frac{1}{\mu}\right)\frac{M_1\|\boldsymbol{f}\|_q}{\mu} + S_p\left(\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{2r_p}\left(1+\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{(p-4)^+}\right] < \frac{1}{4^{(p-2,1)^+}}, \tag{3.10}$$

then $T : B_{\delta_0} \to B_{\delta_0}$ is a contraction in $\mathbf{W}_0^{1,q}(\Omega) \times \mathbf{W}^{1,r}(\Omega)$.

Proof. Let $(\boldsymbol{\xi},\boldsymbol{\eta})$, $(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}}) \in B_{\delta_0}$ and let $(\boldsymbol{u},\boldsymbol{b})$, $(\hat{\boldsymbol{u}},\hat{\boldsymbol{b}})$ be their respective images under T. Then, from (3.2) we obtain

$$\begin{cases} -\mu\Delta(\boldsymbol{u}-\hat{\boldsymbol{u}}) + \nabla(p-\hat{p}) = \boldsymbol{F}, & x \in \Omega, \\ -\Delta(\boldsymbol{b}-\hat{\boldsymbol{b}}) = \boldsymbol{G}, & x \in \Omega, \\ \operatorname{div}(\boldsymbol{u}-\hat{\boldsymbol{u}}) = 0, \qquad \operatorname{div}(\boldsymbol{b}-\hat{\boldsymbol{b}}) = 0, & x \in \Omega, \\ (\boldsymbol{u}-\hat{\boldsymbol{u}})|_{\partial\Omega} = 0, \qquad (\boldsymbol{b}-\hat{\boldsymbol{b}})\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \qquad (\nabla\times(\boldsymbol{b}-\hat{\boldsymbol{b}}))\times\boldsymbol{n}|_{\partial\Omega} = 0, & \end{cases}$$

where

$$\boldsymbol{F} := \operatorname{div}(\hat{\boldsymbol{\xi}}\otimes\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}\otimes\boldsymbol{\xi}) + (\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta} - (\nabla\times\hat{\boldsymbol{\eta}})\times\hat{\boldsymbol{\eta}} + 2\mu\operatorname{div}[\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi} - \sigma(|D\hat{\boldsymbol{\xi}}|^2)D\hat{\boldsymbol{\xi}}],$$
$$\boldsymbol{G} := (\boldsymbol{\eta}\cdot\nabla)\boldsymbol{\xi} - (\hat{\boldsymbol{\eta}}\cdot\nabla)\hat{\boldsymbol{\xi}} + (\hat{\boldsymbol{\xi}}\cdot\nabla)\hat{\boldsymbol{\eta}} - (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\eta}.$$

Applying Lemma 2.1 with $\boldsymbol{\psi} = \boldsymbol{F}$ we obtain

$$\begin{aligned} \|\nabla(\boldsymbol{u}-\hat{\boldsymbol{u}})\|_q \leq \frac{C_{-1}}{\mu}(&\|\operatorname{div}(\hat{\boldsymbol{\xi}}\otimes\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}\otimes\boldsymbol{\xi})\|_{-1,q} + \|(\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta} - (\nabla\times\hat{\boldsymbol{\eta}})\times\hat{\boldsymbol{\eta}}\|_{-1,q} \\ &+ 2\mu\|\operatorname{div}[\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi} - \sigma(|D\hat{\boldsymbol{\xi}}|^2)D\hat{\boldsymbol{\xi}}]\|_{-1,q}). \end{aligned} \tag{3.11}$$

Notice that

$$\begin{aligned} &\|(\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta} - (\nabla\times\hat{\boldsymbol{\eta}})\times\hat{\boldsymbol{\eta}}\|_{-1,q} \\ &\quad\leq \|(\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta} - (\nabla\times\hat{\boldsymbol{\eta}})\times\hat{\boldsymbol{\eta}}\|_r \\ &\quad= \|(\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta} - (\nabla\times\hat{\boldsymbol{\eta}})\times\boldsymbol{\eta} + (\nabla\times\hat{\boldsymbol{\eta}})\times\boldsymbol{\eta} - (\nabla\times\hat{\boldsymbol{\eta}})\times\hat{\boldsymbol{\eta}}\|_r \\ &\quad\leq \|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\|\boldsymbol{\eta}\|_\infty + \|\nabla\hat{\boldsymbol{\eta}}\|_r\|\boldsymbol{\eta}-\hat{\boldsymbol{\eta}}\|_\infty \\ &\quad\leq C_{\widetilde{E}}\|\boldsymbol{\eta}\|_{1,r}\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r + \|\nabla\hat{\boldsymbol{\eta}}\|_{1,r}C_{\widetilde{E}}\|\boldsymbol{\eta}-\hat{\boldsymbol{\eta}}\|_{1,r} \\ &\quad\leq C_{\widetilde{E}}(C_p+1)\|\nabla\boldsymbol{\eta}\|_r\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r + \delta_0(C_p+1)\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r \\ &\quad\leq C_{\widetilde{E}}(C_p+1)\|\nabla\boldsymbol{\eta}\|_{1,r}\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r + \delta_0(C_p+1)\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r \\ &\quad\leq \delta_0(C_p+1)\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r + \delta_0(C_p+1)\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r, \\ &\quad= 2\delta_0(C_p+1)\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r, \end{aligned} \tag{3.12}$$

reasoning as in [1], we obtain

$$\begin{aligned} \|\operatorname{div}(\hat{\boldsymbol{\xi}}\otimes\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}\otimes\boldsymbol{\xi})\|_{-1,q} &\leq C\|(\hat{\boldsymbol{\xi}}\otimes\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}\otimes\boldsymbol{\xi})\|_q \\ &\leq CC_p(C_p^q+1)^{\frac{1}{q}}\delta_0\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q, \end{aligned} \tag{3.13}$$

$$\begin{aligned} 2\mu\|\operatorname{div}[\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi} - \sigma(|D\hat{\boldsymbol{\xi}}|^2)D\hat{\boldsymbol{\xi}}]\|_{-1,q} &\leq C\mu\|[\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi} - \sigma(|D\hat{\boldsymbol{\xi}}|^2)D\hat{\boldsymbol{\xi}}]\|_q \\ &\leq C\mu S_p\mathcal{H}(2\delta_0)\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q. \end{aligned} \tag{3.14}$$

From (3.11)–(3.14) we obtain

$$\|\nabla(\boldsymbol{u}-\hat{\boldsymbol{u}})\|_q \leq M_3\left[\frac{2\delta_0}{\mu} + S_p\mathcal{H}(2\delta_0)\right]\max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q, \|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\}, \tag{3.15}$$

where $M_3 = C_{-1}\max\{CC_p(C_p^q+1)^{\frac{1}{q}}, 2(C_p+1), C\}$.

On the other hand, again by Proposition 2.30 in [11], there exists a constant $c_2 > 0$ such that

$$
\begin{aligned}
\|\nabla(\boldsymbol{b}-\hat{\boldsymbol{b}})\|_r &\le \|\nabla(\boldsymbol{b}-\hat{\boldsymbol{b}})\|_{1,r} \\
&\le c_2\big[\|(\boldsymbol{\eta}\cdot\nabla)\boldsymbol{\xi}-(\hat{\boldsymbol{\eta}}\cdot\nabla)\hat{\boldsymbol{\xi}}\|_r+\|(\hat{\boldsymbol{\xi}}\cdot\nabla)\hat{\boldsymbol{\eta}}-(\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\eta}\|_r\big] \\
&= c_2\Big[\|(\boldsymbol{\eta}\cdot\nabla)\boldsymbol{\xi}-(\hat{\boldsymbol{\eta}}\cdot\nabla)\boldsymbol{\xi}+(\hat{\boldsymbol{\eta}}\cdot\nabla)\boldsymbol{\xi}-(\hat{\boldsymbol{\eta}}\cdot\nabla)\hat{\boldsymbol{\xi}}\|_r \\
&\quad+\|(\hat{\boldsymbol{\xi}}\cdot\nabla)\hat{\boldsymbol{\eta}}-(\hat{\boldsymbol{\xi}}\cdot\nabla)\boldsymbol{\eta}+(\hat{\boldsymbol{\xi}}\cdot\nabla)\boldsymbol{\eta}-(\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\eta}\|_r\Big] \\
&\le c_2\Big[\|\boldsymbol{\eta}-\hat{\boldsymbol{\eta}}\|_\infty\|\nabla\boldsymbol{\xi}\|_r+\|\hat{\boldsymbol{\eta}}\|_\infty\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_r \\
&\quad+\|\hat{\boldsymbol{\xi}}\|_\infty\|\nabla(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r+\|\hat{\boldsymbol{\xi}}-\boldsymbol{\xi}\|_\infty\|\nabla\boldsymbol{\eta}\|_r\Big] \\
&\le c_2\Big[C_{\widetilde{E}}\|\boldsymbol{\eta}-\hat{\boldsymbol{\eta}}\|_{1,r}\|\nabla\boldsymbol{\xi}\|_r+C_{\widetilde{E}}\|\hat{\boldsymbol{\eta}}\|_{1,r}\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_r \\
&\quad+C_E\|\hat{\boldsymbol{\xi}}\|_{1,q}\|\nabla(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r+C_E\|\hat{\boldsymbol{\xi}}-\boldsymbol{\xi}\|_{1,q}\|\nabla\boldsymbol{\eta}\|_r\Big] \\
&\le c_2\Big[C_{\widetilde{E}}(C_p+1)\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\|\nabla\boldsymbol{\xi}\|_{1,q}+C_{\widetilde{E}}(C_p+1)\|\nabla\hat{\boldsymbol{\eta}}\|_r\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q \\
&\quad+C_E(C_p+1)\|\nabla\hat{\boldsymbol{\xi}}\|_q\|\nabla(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r+C_E(C_p+1)\|\nabla(\hat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q\|\nabla\boldsymbol{\eta}\|_r\Big] \\
&\le c_2\Big[C_{\widetilde{E}}(C_p+1)\|\nabla\boldsymbol{\xi}\|_{1,q}\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r+C_{\widetilde{E}}(C_p+1)\|\nabla\hat{\boldsymbol{\eta}}\|_{1,r}\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q \\
&\quad+C_E(C_p+1)\|\nabla\hat{\boldsymbol{\xi}}\|_{1,q}\|\nabla(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r+C_E(C_p+1)\|\nabla\boldsymbol{\eta}\|_{1,r}\|\nabla(\hat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q\Big] \\
&\le c_2\left[\frac{C_{\widetilde{E}}(C_p+1)}{C_E}\delta_0\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r+(C_p+1)\delta_0\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q\right. \\
&\quad\left.+(C_p+1)\delta_0\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r+\frac{C_E(C_p+1)}{C_{\widetilde{E}}}\delta_0\|\nabla(\hat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q\right] \\
&\le 4M_4\delta_0\max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q,\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\},
\end{aligned}
\tag{3.16}
$$

where $M_4 = c_2\max\{\frac{C_{\widetilde{E}}(C_p+1)}{C_E},(C_p+1),\frac{C_E(C_p+1)}{C_{\widetilde{E}}}\}$.

Combining (3.15) and (3.16), we deduce that

$$
\begin{aligned}
&\max\{\|\nabla(\boldsymbol{u}-\hat{\boldsymbol{u}})\|_q,\|\nabla(\boldsymbol{b}-\hat{\boldsymbol{b}})\|_r\} \\
&\qquad\le\left(\frac{2M_3\delta_0}{\mu}+4M_4\delta_0+M_3S_p\mathcal{H}(2\delta_0)\right)\cdot\max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q,\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\}.
\end{aligned}
$$

From here, and taking into account that $\delta_0 \le \frac{2M_1\|\boldsymbol{f}\|_q}{\mu}$, $\mathcal{H}$ is nondecreasing, $\mathcal{H}(4y)\le 4^{(p-2,1)^+}\mathcal{H}(y)$ and defining $m=\max\{2M_3,4M_4\}$, we get

$$
\begin{aligned}
&\max\{\|\nabla(\boldsymbol{u}-\hat{\boldsymbol{u}})\|_q,\|\nabla(\boldsymbol{b}-\hat{\boldsymbol{b}})\|_r\} \\
&\quad\le m\left[\frac{\delta_0}{\mu}+\delta_0+S_p\mathcal{H}(2\delta_0)\right]\max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q,\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\} \\
&\quad\le m\left[\frac{2M_1\|\boldsymbol{f}\|_q}{\mu^2}+\frac{2M_1\|\boldsymbol{f}\|_q}{\mu}+S_p4^{(p-2,1)^+}\mathcal{H}\left(\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)\right] \\
&\qquad\cdot\max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q,\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\}
\end{aligned}
$$

$$
\begin{aligned}
&= m\left[\left(1+\frac{1}{\mu}\right)\frac{2M_1\|\boldsymbol{f}\|_q}{\mu} + 4^{(p-2,1)^+}S_p\left(\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{2r_p}\left(1+\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{(p-4)^+}\right] \\
&\quad \cdot \max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q, \|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\} \\
&\leq 4^{(p-2,1)^+} m\left[\left(1+\frac{1}{\mu}\right)\frac{M_1\|\boldsymbol{f}\|_q}{\mu} + S_p\left(\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{2r_p}\left(1+\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{(p-4)^+}\right] \\
&\quad \cdot \max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q, \|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\}. \qquad (3.17)
\end{aligned}
$$

Considering the space $Y := \mathbf{W}_0^{1,q}(\Omega) \times \mathbf{W}^{1,r}(\Omega)$, with norm $\max\{\|\nabla \cdot \|_q, \|\nabla \cdot \|_r\}$, the inequality (3.17) implies that

$$
\begin{aligned}
\|T\left(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}\right) - T\left(\boldsymbol{\xi}, \boldsymbol{\eta}\right)\|_Y \leq 4^{(p-2,1)^+} m &\left[\left(1+\frac{1}{\mu}\right)\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right. \\
&\left. + S_p\left(\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{2r_p}\left(1+\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{(p-4)^+}\right] \left\|\left(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}\right) - (\boldsymbol{\xi}, \boldsymbol{\eta})\right\|_Y.
\end{aligned}
$$

From which and hypothesis (3.10), we obtain $T : B_{\delta_0} \to B_{\delta_0}$ is a contraction in $\mathbf{W}_0^{1,q}(\Omega) \times \mathbf{W}^{1,r}(\Omega)$. □

Proof of Theorem 1.2. Notice that for $p \leq 3$, $\gamma_p = 1/4 = 1/4^{(p-2,1)^+}$ and for $p > 3$, $\gamma_p > 1/4^{(p-2,1)^+}$. Thus, by taking $\overline{C} = (M_1, m)^+$ and because of (1.7) implies (3.3) and (3.10), Propositions 3.1 and Propositions 3.2 yield that the mapping $T : B_{\delta_0} \to B_{\delta_0}$ is a contraction in $\mathbf{W}_0^{1,q}(\Omega) \times \mathbf{W}^{1,r}(\Omega)$.

Applying Lemma 2.3 with $X = \mathbf{V}_{2,q} \times \mathbf{W}^{2,r}(\Omega)$, $Y = \mathbf{W}_0^{1,q}(\Omega) \times \mathbf{W}^{1,r}(\Omega)$ and $B = B_{\delta_0}$, we could obtain that T has a unique fixed point in B_{δ_0} and this implies the original problem (1.2)–(1.3) has a unique strong solution $(\boldsymbol{u}, \boldsymbol{b}) \in \mathbf{V}_{2,q} \times \mathbf{W}^{2,r}(\Omega)$.

The proof of Theorem 1.2 is finished. □

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