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Blow-up analysis for a doubly nonlinear parabolic system with multi-coupled nonlinearities*

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Abstract

This paper deals with the global existence and the global nonexistence of a doubly nonlinear parabolic system coupled via both nonlinear reaction terms and nonlinear boundary flux. The authors first establish a weak comparison principle, then by constructing various upper and lower solutions, some appropriate conditions for global existence and global nonexistence of solutions are determined respectively.

Keywords: Doubly nonlinear parabolic system; Global existence; Blow up; Multi-coupled; Nonlinearity.

1 Introduction

In this paper, we consider the following problem:

$$(u^{n_1})_t = \Delta_{m_1} u + u^{\alpha_1} v^{p_1}, \quad (v^{n_2})_t = \Delta_{m_2} v + u^{p_2} v^{\beta_1}, \quad x \in \Omega, t > 0, \tag{1.1}$$
$$\nabla_{m_1} u \cdot \nu = u^{\alpha_2} v^{q_1}, \quad \nabla_{m_2} v \cdot \nu = u^{q_2} v^{\beta_2}, \quad x \in \partial\Omega, t > 0, \tag{1.2}$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \overline{\Omega}, \tag{1.3}$$

where $\Delta_k u = \operatorname{div}(|\nabla u|^{k-1}\nabla u) = \sum_{i=1}^{N} (|\nabla u|^{k-1} u_{x_i})_{x_i}$, $\nabla_k u = (|\nabla u|^{k-1} u_{x_1}, \cdots, |\nabla u|^{k-1} u_{x_N})$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $m_i > 1$, $n_i, \alpha_i, \beta_i > 0$, $p_i, q_i \geq 0$, $i = 1, 2$. ν denotes the outer unit normal on the boundary, $u_0(x)$, $v_0(x) \in C^1(\bar{\Omega})$ are positive and satisfy the compatibility conditions.

Parobolic equations like Eq.(1.1) appear in population dynamics, chemical reactions, heat transfer like, for instance, the description of turbulent filtration

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in porous media, the theory of non-Newtonian fluids perturbed by nonlinear terms and forced by rather irregular period in time excitations, the flow of a gas through a porous medium in a turbulent regime or the spread of biological (see [1, 2, 3] and the references given therein). In particular, Eq.(1.1) may be used to describe the nonstationary flows in a porous medium of fluids with a power dependence of the tangential stress on the velocity of displacement under polytropic conditions. In this case, Eq.(1.1) are called the non-Newtonian polytropic filtration equations (see [4]-[8] and the references therein). We refer to [9] for further information on these phenomena. Recently a connection has been revealed with soil science, specifically with flows in reservoirs exhibiting fractured media (see [10]).

Li [11] studied the single parabolic equation with nonlinear boundary condition

$$\begin{aligned} &(u^k)_t = \Delta_p u + u^\alpha, && x \in \Omega, t > 0, \\ &\nabla_p u \cdot \nu = u^\beta, && x \in \partial\Omega, t > 0, \\ &u(x,0) = u_0(x), && x \in \overline{\Omega} \end{aligned} \tag{1.4}$$

with $k, p > 0$, $\alpha, \beta \geq 0$. It is known that the solutions of Eq. (1.4) exist globally if and only if $\alpha \leq k$ and $\beta \leq \min\{k, (k+1)p/(p+1)\}$.

In [12], Li et al. considered the following system with nonlinear boundary conditions

$$\begin{aligned} &(u^{k_1})_t = \Delta_m u, (v^{k_2})_t = \Delta_n v, && x \in \Omega, t > 0, \\ &\nabla_m u \cdot \nu = u^\alpha v^p, \nabla_n v \cdot \nu = u^q v^\beta, && x \in \partial\Omega, t > 0, \\ &u(x,0) = u_0(x), v(x,0) = v_0(x), && x \in \overline{\Omega}. \end{aligned} \tag{1.5}$$

They obtained necessary and sufficient conditions on the global existence of all positive (*weak*) solutions.

In [13], Song and Zheng studied the following quasilinear parabolic system with multi-coupled nonlinearities

$$\begin{aligned} &(u^m)_t = \Delta u + u^{\alpha_1} v^{p_1}, (v^n)_t = \Delta v + u^{q_1} v^{\beta_1}, && x \in \Omega, t > 0, \\ &\frac{\partial u}{\partial \nu} = u^{\alpha_2} v^{p_2}, \frac{\partial v}{\partial \nu} = u^{q_2} v^{\beta_2}, && x \in \partial\Omega, t > 0, \\ &u(x,0) = u_0(x), v(x,0) = v_0(x), && x \in \overline{\Omega} \end{aligned} \tag{1.6}$$

with $m, n > 0$, $\alpha_i, \beta_i, p_i, q_i \geq 0$, $i = 1, 2$. They obtained the necessary and sufficient conditions to the global existence of solutions for $0 < m, n < 1$. They also considered the case of $m, n \geq 1$ and $0 < m < 1, n \geq 1$. However, they only gave some sufficient conditions to the global existence and blowup of solutions.

Motivated by the references cited above, we study the influence of nonlinear reaction terms and nonlinear boundary flux on the existence and nonexistence of global solutions of (1.1) – (1.3). Due to the nonlinear diffusion terms and doubly degeneration for $u = 0$, $|\nabla u| = 0$ or $v = 0$, $|\nabla v| = 0$, we have some new difficulties to be overcome. Noticing that the system (1.1) includes the Newtonian filtration system ($p = 2$) and the non-Newtonian filtration system ($m = 1$) formally, so the method for it should be synthetic. In fact, we can use the methods for the above two systems to deal with it. Then we investigate the global existence or blow-up properties of weak solutions to the problem (1.1) depending on the relations among the parameters $m_1, m_2, n_1, n_2, p_1, p_2, q_1, q_2, \alpha_1, \alpha_2, \beta_1, \beta_2$. Note that (1.1) has nonlinear and nonlocal sources $u^{\alpha_1}v^{p_1}$, $u^{p_2}v^{\beta_1}$ and nonlinear boundary sources $u^{\alpha_2}v^{q_1}, u^{q_2}v^{\beta_2}$, which make the behavior of the solution different from that for that of homogeneous Neumann or Dirichlet boundary value problems. However, it is difficult to use the same methods as that in [13] to get the desired result. To overcome these difficulties, we used some modification of the technique in [12] so that we can handle the nonlinearities. Then, we use some functions to control the nonlocal sources and prove, with the technique in [12], that the control for the nonlocal sources is suitable. Finally we also need to consider the effect of these nonlinear terms in the proof of the global existence(blow-up) property of solutions to (1.1).

Our main results are stated as follows.

Theorem 1.1 *Assume $n_1 < m_1, n_2 < m_2$, then all positive solutions of problem $(1.1)-(1.3)$ exist globally if and only if $\alpha_1 \le n_1$, $\alpha_2 \le n_1$, $\beta_1 \le n_2$, $\beta_2 \le n_2$, $p_1 p_2 \le (n_1-\alpha_1)(n_2-\beta_1)$, $p_1 q_2 \le (n_1-\alpha_1)(n_2-\beta_2)$, $p_2 q_1 \le (n_1-\alpha_2)(n_2-\beta_1)$ and $q_1 q_2 \le (n_1-\alpha_2)(n_2-\beta_2)$.*

Theorem 1.2 *Assume $n_1 \ge m_1, n_2 \ge m_2$, then all positive solutions of problem $(1.1)-(1.3)$ exist globally if $\alpha_1 \le n_1$, $\alpha_2 \le \dfrac{m_1(n_1+1)}{m_1+1}$, $\beta_1 \le n_2$, $\beta_2 \le \dfrac{m_2(n_2+1)}{m_2+1}$, $p_1 p_2 \le (n_1-\alpha_1)(n_2-\beta_1)$, $p_1 q_2 \le (n_1-\alpha_1)\big(\dfrac{m_2(n_2+1)}{m_2+1}-\beta_2\big)$, $p_2 q_1 \le (n_2-\beta_1)\big(\dfrac{m_1(n_1+1)}{m_1+1}-\alpha_2\big)$ and $q_1 q_2 \le \big(\dfrac{m_1(n_1+1)}{m_1+1}-\alpha_2\big)\big(\dfrac{m_2(n_2+1)}{m_2+1}-\beta_2\big)$. While the solutions will blow up in finite time if at least one of the following conditions holds:*
(a) $\alpha_1 > n_1$;
(b) $\alpha_2 > \dfrac{m_1(n_1+1)}{m_1+1}$;
(c) $\beta_1 > n_2$;
(d) $\beta_2 > \dfrac{m_2(n_2+1)}{m_2+1}$;
(e) $p_1 p_2 > (n_1-\alpha_1)(n_2-\beta_1)$;
(f) $p_1 q_2 > (n_1-\alpha_1)\big(\dfrac{m_2(n_2+1)}{m_2+1}-\beta_2\big)+(n_2-m_2)\big(\dfrac{(n_1-\alpha_1)(n_2+1)}{m_2+1}+\dfrac{2q_2}{m_2}\big)$;
(g) $p_2 q_1 > (n_2-\beta_1)\big(\dfrac{m_1(n_1+1)}{m_1+1}-\alpha_2\big)+(n_1-m_1)\big(\dfrac{(n_2-\beta_1)(n_1+1)}{m_1+1}+\dfrac{2q_1}{m_1}\big)$;
(h) $q_1 q_2 > \big(\dfrac{m_1(n_1+1)}{m_1+1}-\alpha_2\big)\big(\dfrac{m_2(n_2+1)}{m_2+1}-\beta_2\big)$.

Theorem 1.3 *Assume $n_1 < m_1, n_2 \geq m_2$, then all positive solutions of problem* (1.1) – (1.3) *exist globally if* $\alpha_1 \leq n_1$, $\alpha_2 \leq n_1$, $\beta_1 \leq n_2$, $\beta_2 \leq \dfrac{m_2(n_2+1)}{m_2+1}$, $p_1p_2 \leq (n_1-\alpha_1)(n_2-\beta_1)$, $p_1q_2 \leq (n_1-\alpha_1)\big(\dfrac{m_2(n_2+1)}{m_2+1}-\beta_2\big)$, $p_2q_1 \leq (n_1-\alpha_2)(n_2-\beta_1)$ *and* $q_1q_2 \leq (n_1-\alpha_2)\big(\dfrac{m_2(n_2+1)}{m_2+1}-\beta_2\big)$.
While the solutions will blow up in finite time if at least one of the following conditions holds:
(*a*) $\alpha_1 > n_1$;
(*b*) $\alpha_2 > n_1$;
(*c*) $\beta_1 > n_2$;
(*d*) $\beta_2 > \dfrac{m_2(n_2+1)}{m_2+1}$;
(*e*) $p_1p_2 > (n_1-\alpha_1)(n_2-\beta_1)$;
(*f*) $p_1q_2 > (n_1-\alpha_1)\big(\dfrac{m_2(n_2+1)}{m_2+1}-\beta_2\big)+(n_2-m_2)\big(\dfrac{(n_1-\alpha_1)(n_2+1)}{m_2+1}+\dfrac{2q_2}{m_2}\big)$;
(*g*) $p_2q_1 > (n_1-\alpha_2)(n_2-\beta_1)$;
(*h*) $q_1q_2 > (n_1-\alpha_2)\big(\dfrac{m_2(n_2+1)}{m_2+1}-\beta_2\big)$.

Theorem 1.4 *Assume $n_1 \geq m_1, n_2 < m_2$, then all positive solutions of problem* (1.1) – (1.3) *exist globally if* $\alpha_1 \leq n_1$, $\alpha_2 \leq \dfrac{m_1(n_1+1)}{m_1+1}$, $\beta_1 \leq n_2$, $\beta_2 \leq n_2$, $p_1p_2 \leq (n_1-\alpha_1)(n_2-\beta_1)$, $p_1q_2 \leq (n_1-\alpha_1)(n_2-\beta_2)$, $p_2q_1 \leq (n_2-\beta_1)\big(\dfrac{m_1(n_1+1)}{m_1+1}-\alpha_2\big)$ *and* $q_1q_2 \leq \big(\dfrac{m_1(n_1+1)}{m_1+1}-\alpha_2\big)(n_2-\beta_2)$.
While the solutions will blow up in finite time if at least one of the following conditions holds:
(*a*) $\alpha_1 > n_1$;
(*b*) $\alpha_2 > \dfrac{m_1(n_1+1)}{m_1+1}$;
(*c*) $\beta_1 > n_2$;
(*d*) $\beta_2 > n_2$;
(*e*) $p_1p_2 > (n_1-\alpha_1)(n_2-\beta_1)$;
(*f*) $p_1q_2 > (n_1-\alpha_1)(n_2-\beta_2)$;
(*g*) $p_2q_1 > (n_2-\beta_1)\big(\dfrac{m_1(n_1+1)}{m_1+1}-\alpha_2\big)+(n_1-m_1)\big(\dfrac{(n_2-\beta_1)(n_1+1)}{m_1+1}+\dfrac{2q_1}{m_1}\big)$;
(*h*) $q_1q_2 > \big(\dfrac{m_1(n_1+1)}{m_1+1}-\alpha_2\big)(n_2-\beta_2)$.

This paper is organized as follows. Some preliminaries will be given in Section 2. Theorem 1.1-1.4 will be proved in Sections 3-5, respectively.

2 Preliminaries

As it is well known that degenerate and singular equations need not possess classical solutions, we give a precise definition of a weak solution to $(1.1)-(1.3)$.

Definition 2.1 *Let $T > 0$ and $Q_T = \Omega \times (0,t]$. A function $(u(x,t), v(x,t))$ is called a weak upper(or lower) solution of Problem (1.1)-(1.3) in Q_T if all of the following hold:*
(i) $u, v \in L^\infty(0,T; W^{1,\infty}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)) \cap C(\overline{Q_T})$;
(ii) $(u(x,0), v(x,0)) \geq (\leq)(u_0(x), v_0(x))$;
(iii) For any positive two functions $\psi_1(x,t), \psi_2(x,t) \in L^1(0,T; W^{1,2}(\Omega)) \cap L^2(Q_T)$, one has

$$\iint_{Q_T} [(u^{n_1})_t \psi_1 + \nabla_{m_1} u \cdot \nabla \psi_1] dxdt$$
$$\geq (\leq) \int_0^T \int_{\partial\Omega} u^{\alpha_2} v^{q_1} \psi_1 dsdt + \iint_{Q_T} u^{\alpha_1} v^{p_1} \psi_1 dxdt,$$
$$\iint_{Q_T} [(v^{n_2})_t \psi_2 + \nabla_{m_2} v \cdot \nabla \psi_2] dxdt$$
$$\geq (\leq) \int_0^T \int_{\partial\Omega} u^{q_2} v^{\beta_2} \psi_2 dsdt + \iint_{Q_T} u^{p_2} v^{\beta_1} \psi_2 dxdt.$$

In particular, $(u(x,t), v(x,t))$ is called a weak solution of $(1.1) - (1.3)$ if it is both a weak upper and a lower solution. For every $T < \infty$, if $(u(x,t), v(x,t))$ is a solution of (1.1)-(1.3) in Q_T, we say that $(u(x,t), v(x,t))$ is global.

Next we give some preliminary propositions and a fact.

Proposition 2.1 *(Comparison principle). Assume that u_0, v_0 are positive $C^1(\overline{\Omega})$ functions and (u, v) is any weak solution of (1.1)-(1.3) in Q_T. Also assume that $(\underline{u}, \underline{v}) \geq (\delta, \delta) > 0$ and $(\overline{u}, \overline{v})$ are a lower and an upper solution of $(1.1)-(1.3)$ in Q_T, respectively, with nonlinear boundary flux $(\underline{\lambda u}^{\alpha_2} \underline{v}^{q_1}, \underline{\lambda u}^{q_2} \underline{v}^{\beta_2})$ and $(\overline{\lambda u}^{\alpha_2} \overline{v}^{q_1}, \overline{\lambda u}^{q_2} \overline{v}^{\beta_2})$, and with nonlinear reaction terms $(\underline{u}^{\alpha_1} \underline{v}^{p_1}, \underline{u}^{p_2} \underline{v}^{\beta_1})$ and $(\overline{u}^{\alpha_1} \overline{v}^{p_1}, \overline{u}^{p_2} \overline{v}^{\beta_1})$, where $0 < \underline{\lambda} < 1 < \overline{\lambda}$. Then we have $(\overline{u}, \overline{v}) \geq (u, v) \geq (\underline{u}, \underline{v})$ in Q_T.*

Proof. For small $\sigma > 0$, letting $\psi_\sigma(z) = \min\{1, \max\{z/\sigma, 0\}\}$, $z \in \mathbb{R}$, and setting $\psi_1 = \psi_\sigma(\underline{u} - u)$, according to the definition of solutions and lower solutions, we have

$$\iint_{Q_\tau} [(\underline{u}^{n_1} - u^{n_1})_t \psi_\sigma(\underline{u} - u) + (\nabla_{m_1} \underline{u} - \nabla_{m_1} u) \cdot \nabla \psi_\sigma(\underline{u} - u)] \mathrm{d}x\mathrm{d}t \leq$$
$$\int_0^\tau \int_{\partial\Omega} (\underline{\lambda u}^{\alpha_2} \underline{v}^{q_1} - u^{\alpha_2} v^{q_1}) \psi_\sigma(\underline{u} - u) \mathrm{d}s\mathrm{d}t + \iint_{Q_\tau} (\underline{u}^{\alpha_1} \underline{v}^{p_1} - u^{\alpha_1} v^{p_1}) \psi_\sigma(\underline{u} - u) \mathrm{d}x\mathrm{d}t.$$

Define

$$\chi(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

As in [14], by letting $\sigma \to 0$, we get

$$\iint_{Q_\tau} [(\underline{u}^{n_1} - u^{n_1})_t \chi(\underline{u} - u)\mathrm{d}x\mathrm{d}t \leq \int_0^\tau \int_{\partial\Omega} (\underline{\lambda u}^{\alpha_2} \underline{v}^{q_1} - u^{\alpha_2} v^{q_1})\chi(\underline{u} - u)\mathrm{d}s\mathrm{d}t$$
$$+ \iint_{Q_\tau} (\underline{u}^{\alpha_1} \underline{v}^{p_1} - u^{\alpha_1} v^{p_1})\chi(\underline{u} - u)\mathrm{d}x\mathrm{d}t,$$

that is

$$\begin{aligned} &\int_\Omega (\underline{u}^{n_1} - u^{n_1})_+|_{t=\tau}\mathrm{d}x \\ \leq &\int_0^\tau \int_{\partial\Omega} (\underline{\lambda u}^{\alpha_2} \underline{v}^{q_1} - u^{\alpha_2} v^{q_1})_+ \mathrm{d}s\mathrm{d}t + \iint_{Q_\tau} (\underline{u}^{\alpha_1} \underline{v}^{p_1} - u^{\alpha_1} v^{p_1})_+ \mathrm{d}x\mathrm{d}t \\ \leq &\int_0^\tau \int_{\partial\Omega} [\underline{v}^{q_1} (\underline{\lambda u}^{\alpha_2} - u^{\alpha_2})_+ + u^{\alpha_2}(\underline{v}^{q_1} - v^{q_1})_+]\mathrm{d}s\mathrm{d}t \\ &+ \iint_{Q_\tau} [\underline{v}^{p_1}(\underline{u}^{\alpha_1} - u^{\alpha_1})_+ + u^{\alpha_1}(\underline{v}^{p_1} - v^{p_1})_+]\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2.7}$$

where $W_+ = \max\{W, 0\}$. Since $\underline{\lambda} < 1$, $(0,0) < (\delta, \delta) \leq (\underline{u}(x,0), \underline{v}(x,0)) \leq (u_0(x), v_0(x))$, it follows from the continuity of $\underline{u}$, $\underline{v}$, u and v that there exists a $\tau > 0$ sufficiently small such that

$$\underline{\lambda u}^{\alpha_2} \leq u^{\alpha_2}, \ \underline{v}^{p_1} \leq v^{p_1} \text{ for } (x,t) \in Q_\tau.$$

It follows that

$$\begin{aligned} &\int_\Omega (\underline{u}^{n_1} - u^{n_1})_+|_{t=\tau}\mathrm{d}x \\ \leq &c_1 \iint_{Q_\tau} (\underline{u}^{\alpha_1} - u^{\alpha_1})_+ \mathrm{d}x\mathrm{d}t + c_2 \iint_{Q_\tau} (\underline{v}^{p_1} - v^{p_1})_+ \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.8}$$

Similarly, we have

$$\begin{aligned} &\int_\Omega (\underline{v}^{n_2} - v^{n_2})_+|_{t=\tau}\mathrm{d}x \\ \leq &c_3 \iint_{Q_\tau} (\underline{v}^{\beta_1} - v^{\beta_1})_+ \mathrm{d}x\mathrm{d}t + c_4 \iint_{Q_\tau} (\underline{u}^{p_2} - u^{p_2})_+ \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.9}$$

Now, (2.8) and (2.9) combined with the Gronwall's Lemma show that $(\underline{u}, \underline{v}) \leq (u, v)$ in $\overline{Q}_\tau$.

Define $\tau^* = \sup\{\tau \in [0, T] : (\underline{u}(x,t), \underline{v}(x,t)) \leq (u(x,t), v(x,t))$ for all $(x,t) \in \overline{Q}_\tau\}$. We claim that $\tau^* = T$. Otherwise, from the continuity of $\underline{u}, \underline{v}, u, v$ there exists an $\varepsilon > 0$, such that $\tau^* + \varepsilon < T$, $\underline{\lambda u}^{\alpha_2} \leq u^{\alpha_2}, \underline{v}^{p_1} \leq v^{p_1}$ and $\underline{\lambda v}^{\beta_2} \leq v^{\beta_2}$,

$\underline{\lambda} u^{p_2} \leq u^{p_2}$ for all $t \in [0, \tau^* + \varepsilon]$. By (2.7), (2.8) and (2.9) we have $(\underline{u}, \underline{v}) \leq (u, v)$ on $\overline{Q}_{\tau^*+\varepsilon}$, which contradicts the definition of τ^*. Hence, $(\underline{u}, \underline{v}) \leq (u, v)$ on $\overline{Q}_T$.

Obviously, (δ, δ) is a lower solution of $(1.1) - (1.3)$ in Q_T, where $\delta_0 = \min\{\min_{\overline{\Omega}} u_0(x),\ \min_{\overline{\Omega}} v_0(x)\} > 0$. Therefore, $(u, v) \geq (\delta, \delta) > (0, 0)$ in Q_T. Using this fact, as in the above proof we can proof that $(u, v) \leq (\overline{u}, \overline{v})$ in Q_T.$\square$

For convenience, we denote $\delta = \min\{\min_{\overline{\Omega}} u_0(x), \min_{\overline{\Omega}} v_0(x)\} > 0$ and $0 < \underline{\lambda} < 1 < \overline{\lambda}$, which are fixed constants.

Let $\varphi_k(x)(k = m_1, m_2)$ be the first eigenfunction of

$$-\Delta_k \varphi = \lambda \varphi^k(x) \quad \text{in} \quad \Omega, \quad \varphi_k(x) = 0 \quad \text{on} \quad \partial\Omega, \tag{2.10}$$

with the first eigenvalue λ_k normalized by $\|\varphi_k(x)\|_\infty = 1$, then $\lambda_k > 0$, $\varphi_k(x) > 0$ in Ω and $\varphi_k(x) \in W_0^{1,k+1}(\Omega) \bigcap C^1(\Omega)$ and $\partial\varphi_k(x)/\partial\nu < 0$ on $\partial\Omega$ (see [15]-[17]). Thus there exist some positive constants A_k, B_k, C_k, D_k such that

$$A_k \leq -\frac{\partial \varphi_k(x)}{\partial \nu} \leq B_k, |\nabla\varphi_k(x)| \geq C_k, x \in \partial\Omega; \quad |\nabla\varphi_k(x)| \leq D_k, x \in \overline{\Omega}. \tag{2.11}$$

We have also $|\nabla\varphi_k(x)| \geq E_k$ provided $x \in \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \varepsilon_k\}$ with $E_k = C_k/2$ and some positive constant ε_k. For the fixed ε_k, there exists a positive constant F_k such that $\varphi_k(x) \geq F_k$ if $x \in \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon_k\}$.

Proposition 2.2 *Assume $n_1 < m_1, n_2 < m_2$, if one of the following conditions holds: $(1°)$ $\alpha_1 > n_1$; $(2°)$ $\beta_1 > n_2$; $(3°)$ $\alpha_2 > n_1$; $(4°)$ $\beta_2 > n_2$; $(5°)$ $q_1 q_2 > (n_1 - \alpha_2)(n_2 - \beta_2)$. Then the solutions of* $(1.1) - (1.3)$ *blow up in finite time.*

Proof. For $(1°)$ or $(2°)$, without loss of generality, assume $\alpha_1 > n_1$. Consider the single equation

$$\begin{cases} (z^{n_1})_t = \Delta_{m_1} z + \delta^{p_1} z^{\alpha_1}, & (x,t) \in \Omega \times (0,T), \\ \nabla_{m_1} z \cdot \nu = \delta^{q_1} z^{\alpha_2}, & (x,t) \in \partial\Omega \times (0,T), \\ z(x,0) = z_0(x), & (x,t) \in \overline{\Omega}. \end{cases}$$

We know from [11] that z blows up in finite time. Since $v \geq \delta$ by the comparison principle, thus (z, δ) is a subsolution of $(1.1) - (1.3)$ and (u, v) blows up in finite time.

For $(3°)$ or $(4°)$ or $(5°)$, since the solution of the system in [12] is a lower solution of $(1.1) - (1.3)$, in view of the blow up results of [12], under the condition of Proposition 2.2, the solution of $(1.1) - (1.3)$ blows up in finite time. $\square$

The following Proposition $3 - 5$ can be proved in the similar procedure.

Proposition 2.3 *Assume $n_1 \geq m_1$, $n_2 \geq m_2$, if one of the following conditions holds: $(1°)$ $\alpha_1 > n_1$; $(2°)$ $\beta_1 > n_2$; $(3°)$ $\alpha_2 > \dfrac{m_1(n_1+1)}{m_1+1}$; $(4°)$*

$\beta_2 > \dfrac{m_2(n_2+1)}{m_2+1}$; (5°) $q_1q_2 \le \big(\dfrac{m_1(n_1+1)}{m_1+1} - \alpha_2\big)\big(\dfrac{m_2(n_2+1)}{m_2+1} - \beta_2\big)$. *Then the solutions of (1.1)-(1.3) blow up in finite time.*

Proposition 2.4 *Assume $n_1 < m_1, n_2 \ge m_2$, if one of the following conditions holds:* (1°) $\alpha_1 > n_1$; (2°) $\beta_1 > n_2$; (3°) $\alpha_2 > n_1$; (4°) $\beta_2 > \dfrac{m_2(n_2+1)}{m_2+1}$; (5°) $q_1q_2 > (n_1-\alpha_2)\big(\dfrac{m_2(n_2+1)}{m_2+1} - \beta_2\big)$. *Then the solutions of (1.1)-(1.3) blow up in finite time.*

Proposition 2.5 *Assume $n_1 \ge m_1, n_2 < m_2$, if one of the following conditions holds:* (1°) $\alpha_1 > n_1$; (2°) $\beta_1 > n_2$; (3°) $\alpha_2 > \dfrac{m_1(n_1+1)}{m_1+1}$; (4°) $\beta_2 > n_2$; (5°) $q_1q_2 > \big(\dfrac{m_1(n_1+1)}{m_1+1} - \alpha_2\big)(n_2-\beta_2)$. *Then the solutions of (1.1)-(1.3) blow up in finite time.*

At the end of this section, we describe a simple fact without proof.
Fact 1 *Suppose that positive constants A, B, C, D satisfy $AB < CD$, then for any two positive constants a, b, there exist two positive constants l_1, l_2 such that $al_1^C > l_2^A$ and $bl_2^D > l_1^B$.*

3 Proof of the Theorem 1.1

In this section we will divide the proof of Theorem 1.1 into following lemmas.

Lemma 3.1 *Assume $n_1 < m_1, n_2 < m_2$. If $\alpha_1 \le n_1$, $\alpha_2 \le n_1$, $\beta_1 \le n_2$, $\beta_2 \le n_2$, $p_1p_2 \le (n_1-\alpha_1)(n_2-\beta_1)$, $p_1q_2 \le (n_1-\alpha_1)(n_2-\beta_2)$, $p_2q_1 \le (n_1-\alpha_2)(n_2-\beta_1)$ and $q_1q_2 \le (n_1-\alpha_2)(n_2-\beta_2)$, then the solutions of problem (1.1)-(1.3) exist globally.*

Proof. Construct

$$\overline{u}(x,t) = R_1e^{l_1t}\log((1-\varphi_{m_1}(x))e^{(n_1-m_1)l_1t/m_1} + R_2),$$
$$\overline{v}(x,t) = R_3e^{l_2t}\log((1-\varphi_{m_2}(x))e^{(n_2-m_2)l_2t/m_2} + R_2),$$

where $R_1, R_2, R_3, l_1, l_2 > 0$ are to be determined.

For $(x,t) \in \Omega \times R^+$, by direct computation, we have

$$(\overline{u}^{n_1})_t \ge \frac{n_1l_1}{2}R_1^{n_1}(\log R_2)^{n_1}e^{n_1l_1t}, \quad \Delta_{m_1}\overline{u} \le \frac{\lambda_{m_1}R_1^{m_1}e^{n_1l_1t}}{R_2^{m_1}}.$$

Similarly,

$$(\overline{v}^{n_2})_t \geq \frac{n_2 l_2}{2} R_3^{n_2} (\log R_2)^{n_2} e^{n_2 l_2 t}, \quad \Delta_{m_2} \overline{v} \leq \frac{\lambda_{m_2} R_3^{m_2} e^{n_2 l_2 t}}{R_2^{m_2}}.$$

Moreover,

$$\begin{aligned}
\overline{\lambda}\overline{u}^{\alpha_1}\overline{v}^{p_1} &\leq \overline{\lambda} R_1^{\alpha_1} R_3^{p_1} (\log(1+R_2))^{\alpha_1+p_1} e^{(\alpha_1 l_1 + p_1 l_2)t},\\
\overline{\lambda}\overline{u}^{p_2}\overline{v}^{\beta_1} &\leq \overline{\lambda} R_1^{p_2} R_3^{\beta_1} (\log(1+R_2))^{p_2+\beta_1} e^{(p_2 l_1 + \beta_1 l_2)t}.
\end{aligned}$$

By setting $c_{m_1} = C_{m_1}$ if $m_1 \geq 1$, $c_{m_1} = D_{m_1}$ if $m_1 < 1$ and $c_{m_2} = C_{m_2}$ if $m_2 \geq 1$, $c_{m_2} = D_{m_2}$ if $m_2 < 1$, on the boundary, we have

$$\begin{aligned}
\nabla_{m_1}\overline{u}\cdot\nu &\geq \frac{R_1^{m_1} A_{m_1} c_{m_1}^{m_1-1} e^{n_1 l_1 t}}{(1+R_2)^{m_1}}, \quad \overline{\lambda}\overline{u}^{\alpha_2}\overline{v}^{q_1} \leq \overline{\lambda} R_1^{\alpha_2} R_3^{q_1} (\log(1+R_2))^{\alpha_2+q_1} e^{(l_1\alpha_2 + l_2 q_1)t};\\
\nabla_{m_2}\overline{v}\cdot\nu &\geq \frac{R_3^{m_2} A_{m_2} c_{m_2}^{m_2-1} e^{n_2 l_2 t}}{(1+R_2)^{m_2}}, \quad \overline{\lambda}\overline{u}^{q_2}\overline{v}^{\beta_2} \leq \overline{\lambda} R_1^{q_2} R_3^{\beta_2} (\log(1+R_2))^{\beta_2+q_2} e^{(l_1 q_2 + l_2\beta_2)t}.
\end{aligned}$$

and

$$\begin{aligned}
\overline{u}(x,0) &= R_1 \log((1-\varphi_{m_1}(x)) + R_2) \geq R_1 \log R_2,\\
\overline{v}(x,0) &= R_3 \log((1-\varphi_{m_2}(x)) + R_2) \geq R_3 \log R_2.
\end{aligned}$$

Choose R_2 such that $R_2 \log R_2 \geq 2\max\{(m_1-n_1)/m_1, (m_2-n_2)/m_2\}$ and by Fact 1 there exist two positive constants R_1, R_3 such that

$$\begin{aligned}
R_1^{m_1-\alpha_2} &\geq R_3^{q_1}\overline{\lambda}(1+R_2)^{m_1}(A_{m_1}c_{m_1}^{m_1-1})^{-1}(\log(1+R_2))^{\alpha_2+q_1},\\
R_3^{m_2-\beta_2} &\geq R_1^{q_2}\overline{\lambda}(1+R_2)^{m_2}(A_{m_2}c_{m_2}^{m_2-1})^{-1}(\log(1+R_2))^{\beta_2+q_2}.
\end{aligned}$$

Next, choose R_1, R_3 such that $R_1 \log R_2 \geq \|u_0\|_\infty$, $R_3 \log R_2 \geq \|v_0\|_\infty$.

Since the conditions of this lemma, there exist positive constants l_1, l_2 satisfying $n_1 l_1 \geq \alpha_1 l_1 + p_1 l_2, n_2 l_2 \geq p_2 l_1 + \beta_1 l_2, n_1 l_1 \geq \alpha_2 l_1 + q_1 l_2, n_2 l_2 \geq q_2 l_1 + \beta_2 l_2$ and

$$\begin{aligned}
l_1 &\geq \frac{2\lambda_{m_1} R_1^{m_1-n_1}}{n_1(\log R_2)^{n_1} R_2^{m_1}} + \frac{2\overline{\lambda}{R_1}^{\alpha_1}{R_3}^{p_1}(\log(1+R_2))^{\alpha_1+p_1}}{n_1(R_1\log R_2)^{n_1}},\\
l_2 &\geq \frac{2\lambda_{m_2} R_3^{m_2-n_2}}{n_2(\log R_2)^{n_2} R_2^{m_2}} + \frac{2\overline{\lambda}{R_1}^{p_2}{R_3}^{\beta_1}(\log(1+R_2))^{p_2+\beta_1}}{n_2(R_3\log R_2)^{n_2}}.
\end{aligned}$$

Thus, $(\overline{u}, \overline{v})$ is an upper solution of (1.1) − (1.3), which means that the solutions of (1.1) − (1.3) are global. □

Lemma 3.2 *Suppose* $\alpha_1 \leq n_1$, $\beta_1 \leq n_2$, $p_1 p_2 > (n_1-\alpha_1)(n_2-\beta_1)$, *then all positive solutions of problem* (1.1) − (1.3) *blow up in finite time.*

Proof. Considering the following ordinary differential system

$$\begin{cases} (w^{n_1})_t =w^{\alpha_1} z^{p_1}, (z^{n_2})_t = w^{p_2} z^{\beta_1}, t > 0, \\ w(x,0) =\delta > 0, z(x,0) = \delta > 0. \end{cases} \tag{3.12}$$

Let $y(t)$ be the solution of the problem

$$\begin{cases} \dfrac{dy}{dt} =\varepsilon_1 y^{\sigma}, t > 0, \\ y(0) =\varepsilon_2, \end{cases}$$

where
$\varepsilon_1 = \min\{1, \frac{n_1(n_2+p_1-\beta_1)}{n_2(n_1+p_2-\alpha_1)}\}, \varepsilon_2 = \min\{\delta^{n_1}, \delta^{\frac{n_1(n_2+p_1-\beta_1)}{n_1+p_2-\alpha_1}}\}, \sigma = \frac{\alpha_1}{n_1} + \frac{p_1(n_1+p_2-\alpha_1)}{n_1(n_2+p_1-\beta_1)}$.
By the assumption, we have $\sigma > 1$ and hence $y(t)$ blows up in finite time.

Let $(\underline{w}, \underline{z}) = (y^{\frac{1}{n_1}}, y^{\frac{n_1+p_2-\alpha_1}{n_1(n_2+p_1-\beta_1)}})$, it can be verified that $(\underline{w}, \underline{z})$ is a lower solution of (3.12). Set $(\underline{u}, \underline{v}) = (w, z)$, then $(\underline{u}, \underline{v})$ is a subsolution of $(1.1)-(1.3)$. Therefore the solution (u, v) of $(1.1) - (1.3)$ blows up in finite time. □

Lemma 3.3 *Assume $n_2 < m_2$, if $\alpha_1 \le n_1$, $\beta_2 \le n_2$ and $p_1q_2 > (n_1 - \alpha_1)(n_2 - \beta_2)$, then the solutions of problem $(1.1) - (1.3)$ blow up in finite time.*

Proof. We prove this lemma by dividing into following two subcases:
(i) $(n_1 - \alpha_1)(n_2 - \beta_2) < p_1q_2 < (m_2 - n_2)q_2 + (n_1 - \alpha_1)(m_2 - \beta_2)$;
(ii) $p_1q_2 \ge (m_2 - n_2)q_2 + (n_1 - \alpha_1)(m_2 - \beta_2)$.

Subcase (i). Construct

$$\underline{u} = (b - ct)^{-l_2}, \underline{v} = ((b - ct)^{-l_1} + ah^{1+1/m_2}(x))^{\theta} = w^{\theta},$$

where $h(x) = \Sigma_{i=1}^{N} x_i + Nd + 1, d = \max\{|x| \mid x \in \overline{\Omega}\}$ and

$$\begin{aligned} l_1 =& \frac{(m_2 - n_2)q_2 + (n_1 - \alpha_1)(m_2 - \beta_2) - p_1q_2}{m_2\Big(p_1q_2 - (n_1 - \alpha_1)(n_2 - \beta_2)\Big)}, \quad l_2 = \frac{p_1 + n_2 - \beta_2}{p_1q_2 - (n_1 - \alpha_1)(n_2 - \beta_2)}, \\ a =& \min\{\underline{\lambda}^{1/m_2}(\theta^{m_2}(1 + 1/m_2)^{m_2} N^{m_2/2}(2Nd + 1)2^{m_2(\theta-1)})^{-1/m_2}, \\ &b^{-l_1}(2Nd + 1)^{-1-1/m_2}\}, \quad \theta = \frac{1 + m_2 l_1}{l_1(m_2 - n_2)}, \\ b =& \max\{\delta^{-1/l_2}, (\frac{1}{2}\delta^{1/\theta})^{-1/l_1}\}, \\ c =& \min\{\underline{\lambda}(n_1 l_2)^{-1}, (n_2 l_1)^{-1} a^{m_2} \theta^{m_2-1}(1 + 1/m_2)^{m_2} N^{(m_2+1)/2}\}. \end{aligned}$$

For $(x, t) \in \Omega \times (0, b/c)$, we can get

$$(\underline{u}^{n_1})_t = cl_2 n_1 (b - ct)^{-l_2 n_1 - 1}, \quad \underline{\lambda} \underline{u}^{\alpha_1} \underline{v}^{p_1} \ge \underline{\lambda}(b - ct)^{-l_1 p_1 \theta - l_2 \alpha_1},$$

Similarly,

$$\begin{aligned} &(\underline{v}^{n_2})_t \le cl_1 n_2 \theta (b - ct)^{-l_1-1} w^{\theta n_2 - 1} \le cl_1 n_2 \theta w^{\theta n_2 + 1/l_1}, \\ &\Delta_{m_2} \underline{v} \ge (a\theta(1 + 1/m_2))^{m_2} N^{(m_2+1)/2} w^{m_2(\theta-1)}. \end{aligned}$$

On the other hand, on the boundry, we have

$$\begin{aligned}
\nabla_{m_2}\underline{v}\cdot\nu \leq &(a\theta(1+\frac{1}{m_2}))^{m_2}N^{m_2/2}(2Nd+1)2^{m_2(\theta-1)}(b-ct)^{-m_2(\theta-1)l_1},\\
\underline{\lambda u}^{q_2}\underline{v}^{\beta_2}\geq &\underline{\lambda}(b-ct)^{-l_2q_2-l_1\beta_2\theta}.
\end{aligned}$$

Moreover, it is easy to see that $\underline{u}(x,0)\leq\delta\leq u_0(x)$, $\underline{v}(x,0)\leq\delta\leq v_0(x)$, so $(\underline{u},\underline{v})$ is a subsolution of (1.1) − (1.3), which blows up in finite time.

Subcase (ii). For $p_1q_2\geq(m_2-n_2)q_2+(n_1-\alpha_1)(m_2-\beta_2)$, choose $p_0<p_1$, such that $(n_1-\alpha_1)(n_2-\beta_2)<p_0q_2<(m_2-n_2)q_2+(n_1-\alpha_1)(m_2-\beta_2)$ and $\underline{v}^{p_1}\geq\underline{v}^{p_0}$.

Consider the problem

$$\begin{aligned}
&(w^{n_1})_t=\Delta_{m_1}w+w^{\alpha_1}z^{p_0},(z^{n_2})_t=\Delta_{m_2}z+w^{p_2}z^{\beta_1}, && x\in\Omega,t>0,\\
&\nabla_{m_1}w\cdot\nu=w^{\alpha_2}z^{q_1},\nabla_{m_2}z\cdot\nu=w^{q_2}z^{\beta_2}, && x\in\partial\Omega,t>0,\\
&w(x,0)=w_0(x),z(x,0)=z_0(x), && x\in\overline{\Omega}.
\end{aligned}$$

We know from the Subcase (i) that (w,z) blows up in finite time, so the solutions of (1.1) − (1.3) blow up in finite time. □

Lemma 3.4 *Assume $n_1<m_1$. If $\alpha_1\leq n_1$, $\beta_2\leq n_2$ and $p_2q_1>(n_1-\alpha_2)(n_2-\beta_1)$, then the solutions of problem (1.1)-(1.3) blow up in finite time.*

Proof. We can prove this lemma in the similar way as that of lemma 3.3. □

We get the proof of Theorem 1.1 by combining Proposition 2 and Lemma 3.1–3.4.

4 Proof of the Theorem 1.2

In this section we will divide the proof of Theorem 1.2 into following lemmas.

Lemma 4.1 *Suppose $n_1\geq m_1,n_2\geq m_2$. If $\alpha_1\leq n_1$, $\alpha_2\leq\frac{m_1(n_1+1)}{m_1+1}$, $\beta_1\leq n_2$, $\beta_2\leq\frac{m_2(n_2+1)}{m_2+1}$, $p_1p_2\leq(n_1-\alpha_1)(n_2-\beta_1)$, $p_1q_2\leq(n_1-\alpha_1)\big(\frac{m_2(n_2+1)}{m_2+1}-\beta_2\big)$, $p_2q_1\leq(n_2-\beta_1)\big(\frac{m_1(n_1+1)}{m_1+1}-\alpha_2\big)$ and $q_1q_2\leq\big(\frac{m_1(n_1+1)}{m_1+1}-\alpha_2\big)\big(\frac{m_2(n_2+1)}{m_2+1}-\beta_2\big)$, then the solutions of problem (1.1)-(1.3) exist globally .*

Proof. Construct

$$\begin{aligned}
\overline{u}(x,t)=&e^{l_1t}(M+\overline{\lambda}^{\frac{1}{m_1}}e^{-L_1\varphi_{m_1}(x)e^{(n_1-m_1)l_1t/(m_1+1)}}(2M)^{\frac{q_1+\alpha_2}{m_1}}L_1^{-1}(A_{m_1}c_{m_1}^{m_1-1})^{-\frac{1}{m_1}})\\
\triangleq&e^{l_1t}w,\\
\overline{v}(x,t)=&e^{l_2t}(M+\overline{\lambda}^{\frac{1}{m_2}}e^{-L_2\varphi_{m_2}(x)e^{(n_2-m_2)l_2t/(m_2+1)}}(2M)^{\frac{q_2+\beta_2}{m_2}}L_2^{-1}(A_{m_2}c_{m_2}^{m_2-1})^{-\frac{1}{m_2}})\\
\triangleq&e^{l_2t}z,
\end{aligned}$$

where $c_{m_1} = C_{m_1}$ if $m_1 \geq 1$, $c_{m_1} = D_{m_1}$ if $m_1 < 1$ and $c_{m_2} = C_{m_2}$ if $m_2 \geq 1$, $c_{m_2} = D_{m_2}$ if $m_2 < 1$, $\varphi_{m_i}(x), A_{m_i}, C_{m_i}, D_{m_i}$, $i = 1, 2$. are defined in (2.10) and (2.11), l_1, l_2 are positive constants to be determined, $M = \max\{1, \|u_0\|_\infty, \|v_0\|_\infty\}$ and

$$\begin{aligned}
L_1 = \overline{\lambda}^{1/m_1} \max\{ & \frac{n_1 - m_1}{m_1 + 1} 2^{(q_1+\alpha_2+m_1)/m_1} M^{(q_1+\alpha_2-m_1)/m_1} (A_{m_1} c_{m_1}^{m_1-1})^{-1/m_1}, \\
& 2^{(q_1+\alpha_2)/m_1} M^{(q_1+\alpha_2-m_1)/m_1} (A_{m_1} c_{m_1}^{m_1-1})^{-1/m_1} \}, \\
L_2 = \overline{\lambda}^{1/m_2} \max\{ & \frac{n_2 - m_2}{m_2 + 1} 2^{(q_2+\beta_2+m_2)/m_2} M^{(q_2+\beta_2-m_2)/m_2} (A_{m_2} c_{m_2}^{m_2-1})^{-1/m_2}, \\
& 2^{(q_2+\beta_2)/m_2} M^{(q_2+\beta_2-m_2)/m_2} (A_{m_2} c_{m_2}^{m_2-1})^{-1/m_2} \}.
\end{aligned}$$

We know that $-L_1\varphi_{m_1}(x) e^{(n_1-m_1)l_1 t/(m_1+1)} e^{-L_1\varphi_{m_1}(x) e^{(n_1-m_1)l_1 t/(m_1+1)}} \geq -e^{-1}$ for any $y > 0$. Thus for $(x, t) \in \Omega \times R^+$, a simple computation shows

$$\begin{aligned}
(\overline{u}^{n_1})_t = & n_1 l_1 e^{n_1 l_1 t} w^{n_1} + n_1 e^{n_1 l_1 t} w^{n_1 - 1} \overline{\lambda}^{\frac{1}{m_1}} (2M)^{(q_1+\alpha_2)/m_1} L_1^{-1} (A_{m_1} c_{m_1}^{m_1-1})^{-\frac{1}{m_1}} \\
& \times \frac{(n_1 - m_1) l_1}{m_1 + 1} (-L_1 \varphi_{m_1}(x)) e^{(n_1-m_1)l_1 t/(m_1+1)} e^{-L_1\varphi_{m_1}(x) e^{(n_1-m_1)l_1 t/(m_1+1)}} \\
\geq & \frac{1}{2} n_1 l_1 e^{n_1 l_1 t}.
\end{aligned}$$

In addition,

$$\begin{aligned}
\Delta_{m_1} \overline{u} \leq & \overline{\lambda}(\lambda_{m_1} + L_1 m_1 D_{m_1}^{m_1+1})(2M)^{q_1+\alpha_2} (A_{m_1} c_{m_1}^{m_1-1})^{-1} e^{n_1 l_1 t}, \\
\overline{\lambda}\overline{u}^{\alpha_1}\overline{v}^{p_1} \leq & \overline{\lambda}(2M)^{p_1+\alpha_1} e^{(\alpha_1 l_1 + p_1 l_2)t}.
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
(\overline{v}^{n_2})_t \geq & \frac{1}{2} n_2 l_2 e^{n_2 l_2 t}, \quad \overline{\lambda}\overline{u}^{p_2}\overline{v}^{\beta_1} \leq \overline{\lambda}(2M)^{p_2+\beta_1} e^{(p_2 l_1 + \beta_1 l_2)t}, \\
\Delta_{m_2} \overline{v} \leq & \overline{\lambda}(\lambda_{m_2} + L_2 m_2 D_{m_2}^{m_2+1})(2M)^{q_2+\beta_2} (A_{m_2} c_{m_2}^{m_2-1})^{-1} e^{n_2 l_2 t}.
\end{aligned}$$

Moreover, on the boundary, we have

$$\begin{aligned}
\nabla_{m_1} \overline{u} \cdot \nu \geq & \overline{\lambda}(2M)^{q_1+\alpha_2} e^{m_1(n_1+1)l_1 t/(m_1+1)}, \quad \overline{\lambda}\overline{u}^{\alpha_2}\overline{v}^{q_1} \leq \overline{\lambda}(2M)^{q_1+\alpha_2} e^{(\alpha_2 l_1 + q_1 l_2)t}; \\
\nabla_{m_2} \overline{v} \cdot \nu \geq & \overline{\lambda}(2M)^{q_2+\beta_2} e^{m_2(n_2+1)l_2 t/(m_2+1)}, \quad \overline{\lambda}\overline{u}^{q_2}\overline{v}^{\beta_2} \leq \overline{\lambda}(2M)^{q_2+\beta_2} e^{(q_2 l_1 + \beta_2 l_2)t}.
\end{aligned}$$

Since the conditions of the lemma, there exist a positive constant l_1, l_2 large such that

$$\begin{aligned}
n_1 l_1 \geq \alpha_1 l_1 + p_1 l_2, \frac{m_1(n_1+1) l_1}{m_1 + 1} \geq \alpha_2 l_1 + q_1 l_2, \\
n_2 l_2 \geq p_2 l_1 + \beta_1 l_2, \frac{m_2(n_2+1) l_2}{m_2 + 1} \geq \beta_2 l_2 + q_2 l_1,
\end{aligned}$$

and

$$l_1 \geq 2\overline{\lambda}(\lambda_{m_1} + L_1 m_1 D_{m_1}^{m_1+1})(2M)^{q_1+\alpha_2}(n_1 A_{m_1} c_{m_1}^{m_1-1})^{-1} + \frac{2\overline{\lambda}}{n_1}(2M)^{p_1+\alpha_1},$$
$$l_2 \geq 2\overline{\lambda}(\lambda_{m_2} + L_2 m_2 D_{m_2}^{m_2+1})(2M)^{q_2+\beta_2}(n_2 A_{m_2} c_{m_2}^{m_2-1})^{-1} + \frac{2\overline{\lambda}}{n_2}(2M)^{p_2+\beta_1}.$$

Thus, $(\overline{u}, \overline{v})$ is a global upper solution of (1.1) – (1.3). The global existence of solution to (1.1) – (1.3) follows from the comparison principle. □

Lemma 4.2 *Suppose $n_2 \geq m_2$. If $\alpha_1 \leq n_1$, $\beta_2 \leq \dfrac{m_2(n_2+1)}{m_2+1}$, $p_1 q_2 > (n_1-\alpha_1)(\dfrac{m_2(n_2+1)}{m_2+1} - \beta_2) + (n_2-m_2)(\dfrac{(n_1-\alpha_1)(n_2+1)}{m_2+1} + \dfrac{2q_2}{m_2})$, then all positive solutions of problem (1.1)-(1.3) blow up in finite time.*

Proof. Set

$$\begin{aligned}\underline{u} =& \delta[(1-ct)^2 + a^2\varphi_{m_1}^2(x)]^{-k} \triangleq \delta A^{-k},\\ \underline{v} =& \delta[(1-ct) + a\varphi_{m_2}(x)]^{-l} \triangleq \delta B^{-l}, 0 < t < 1/c,\end{aligned}$$

where

$$\begin{aligned}k =& \frac{m_2(n_2+1)(p_1+2) - 2\beta_2(m_2+1)}{2(m_2+1)(p_1q_2 + n_1\beta_2 - \alpha_1\beta_2) - 2m_2(n_1-\alpha_1)(n_2+1)},\\ l =& \frac{2q_2(m_2+1) + m_2(n_1-\alpha_1)(n_2+1)}{(m_2+1)(p_1q_2 + n_1\beta_2 - \alpha_1\beta_2) - m_2(n_1-\alpha_1)(n_2+1)},\\ a =& \min\{\frac{1}{2}, \underline{\lambda}^{1/m_2} l^{-1} \delta^{(q_2+\beta_2-m_2)/m_2}(B_{m_2} D_{m_2}^{m_2-1})^{-1/m_2}\},\\ c =& \min\{\frac{\underline{\lambda}\delta^{p_1+\alpha_1-n_1}}{2kn_1(\sqrt{2})^{lp_1}}, \frac{\lambda_{m_2}}{n_2} l^{m_2-1}\delta^{m_2-n_2}(aF_{m_2})^{m_2+ln_2+1}(\frac{2}{3})^{(l+1)m_2},\\ & \frac{(l+1)m_2}{n_2} l^{m_2-1}\delta^{m_2-n_2} a^{m_2+ln_2+2} E_{m_2}^{m_2+1} F_{m_2}^{ln_2+1}(\frac{2}{3})^{(l+1)m_2+1},\\ & \frac{(l+1)m_2}{n_2} l^{m_2-1}\delta^{m_2-n_2} a^{m_2+1} E_{m_2}^{m_2+1}, 1\},\end{aligned}$$

and $\varphi_{m_1}(x), \varphi_{m_2}(x)$ is defined in (2.10) and (2.11). And obviously, $\underline{u}(x,0) \leq \delta \leq u_0(x)$, $\underline{v}(x,0) \leq \delta \leq v_0(x)$. By simple computation, for $(x,t) \in \Omega \times (0, 1/c)$, we can get

$$(\underline{u}^{n_1})_t \leq 2\delta^{n_1} k n_1 c A^{-kn_1-1}, \qquad \underline{\lambda}\underline{u}^{\alpha_1}\underline{v}^{p_1} \geq \underline{\lambda}\delta^{\alpha_1+p_1} 2^{-lp_1/2} A^{-k\alpha_1 - lp_1/2}.$$

Meanwhile,

$$\begin{aligned}(\underline{v}^{n_2})_t =& \delta^{n_2} lcn_2 B^{-ln_2-1},\\ \Delta_{m_2}\underline{v} =& \lambda_{m_2}(a\delta l)^{m_2}\varphi_{m_2}^{m_2} B^{(-l-1)m_2} + am_2(a\delta l)^{m_2}(l+1)|\nabla\varphi_{m_2}|^{m_2+1} B^{(-l-1)m_2-1}.\end{aligned}$$

If $x \in \{x \in \Omega : dist(x, \partial\Omega) > \varepsilon_{m_2}\}, 0 < t < 1/c$ or $x \in \{x \in \Omega : dist(x, \partial\Omega) \leq \varepsilon_{m_2}\}, 0 < t \leq 1/2c$, then

$$(\underline{v}^{n_2})_t \leq \delta^{n_2} lcn_2 (aF_{m_2})^{-ln_2-1},$$

$$\Delta_{m_2}\underline{v} \geq \max\{\lambda_{m_2}(a\delta l)^{m_2} F_{m_2}^{m_2}(\frac{2}{3})^{(l+1)m_2}, am_2(a\delta l)^{m_2}(l+1)E_{m_2}^{m_2+1}(\frac{2}{3})^{(l+1)m_2+1}\}.$$

If $x \in \{x \in \Omega : dist(x, \partial\Omega) \leq \varepsilon_{m_2}\}$, $1/2c < t < 1/c$, then

$$(\underline{v}^{n_2})_t \leq \delta^{n_2} lcn_2 B^{-ln_2-1}, \qquad \Delta_{m_2}\underline{v} \geq am_2(a\delta l)^{m_2}(l+1)E_{m_2}^{m_2+1}B^{(-l-1)m_2-1}.$$

So we have $(\underline{v}^{n_2})_t \leq \Delta_{m_2}\underline{v}$, for $(x, t) \in \Omega \times (0, 1/c)$. In addition,

$$\begin{aligned}
\nabla_{m_2}\underline{v} \cdot \nu =& (a\delta l)^{m_2}|\nabla\varphi_{m_2}|^{m_2-1}(-\frac{\partial\varphi_{m_2}}{\partial\nu})(1-ct)^{(-l-1)m_2} \\
\leq& (a\delta l)^{m_2} B_{m_2} D_{m_2}^{m_2-1}(1-ct)^{(-l-1)m_2(n_2+1)/(m_2+1)}, \\
\underline{\lambda u}^{q_2}\underline{v}^{\beta_2} \geq& \underline{\lambda}\delta^{q_2+\beta_2}(1-ct)^{-2kq_2-l\beta_2}
\end{aligned}$$

for $(x, t) \in \partial\Omega \times (0, 1/c)$.

It is easy to check that $(\underline{u}, \underline{v})$ is a subsolution of $(1.1) - (1.3)$, which blows up in finite time. □

Remark 4.1 *Obviously, the proof of Lemma 4.2 needs not the relation of n_1 and m_1.*

Lemma 4.3 *Suppose $n_1 \geq m_1$. If $\beta_1 \leq n_2$, $\alpha_2 \leq \frac{m_1(n_1+1)}{m_1+1}$, $p_2 q_1 > (n_2 - \beta_1)\big(\frac{m_1(n_1+1)}{m_1+1} - \alpha_2\big) + (n_1 - m_1)\big(\frac{(n_2-\beta_1)(n_1+1)}{m_1+1} + \frac{2q_1}{m_1}\big)$, then all positive solutions of problem (1.1)-(1.3) blow up in finite time .*

Proof. This lemma can be proved by the similar method as that of lemma 4.2. □

It follows from Proposition 2.3 and lemma 3.2, 4.1-4.3 that Theorem 1.2 is true.

5 Proof of the Theorem 1.3 and 1.4

In this section we will divide the proof of Theorem 1.3 into following lemmas.

Lemma 5.1 *Assume $n_1 < m_1, n_2 \geq m_2$. If $\alpha_1 \leq n_1$, $\alpha_2 \leq n_1$, $\beta_1 \leq n_2$, $\beta_2 \leq \frac{m_2(n_2+1)}{m_2+1}$, $p_1p_2 \leq (n_1-\alpha_1)(n_2-\beta_1)$, $p_1q_2 \leq (n_1-\alpha_1)\big(\frac{m_2(n_2+1)}{m_2+1}-\beta_2\big)$, $p_2q_1 \leq (n_1-\alpha_2)(n_2-\beta_1)$ and $q_1q_2 \leq (n_1-\alpha_2)\big(\frac{m_2(n_2+1)}{m_2+1} - \beta_2\big)$, then all positive solutions of problem (1.1)-(1.3) exist globally.*

Proof. Take

$$\begin{aligned}\overline{u}(x,t) =&R_1 e^{l_1 t}\log((1-\varphi_{m_1}(x))e^{(n_1-m_1)l_1 t/m_1}+R_2),\\ \overline{v}(x,t) =&e^{l_2 t}\Big(M+\overline{\lambda}^{1/m_2}e^{-L\varphi_{m_2}(x)e^{(n_2-m_2)l_2 t/(m_2+1)}}\\ &\times(2M)^{(n_2+1)/(m_2+1)}L^{-1}(A_{m_2}c_{m_2}^{m_2-1})^{-1/m_2}\Big),\end{aligned}$$

where $c_{m_2}=C_{m_2}$ if $m_2\geq 1$, $c_{m_2}=D_{m_2}$ if $m_2<1$, R_2 satisfying $R_2\log R_2\geq 2(m_1-n_1)/m_1$ and constants R_1, M, L, l_1, l_2 are to be determined.

By performing direct calculations, for $(x,t)\in\Omega\times R^+$,

$$\begin{aligned}(\overline{u}^{n_1})_t \geq&\frac{n_1 l_1}{2}R_1^{n_1}e^{n_1 l_1 t}(\log((1-\varphi_{m_1}(x))e^{(n_1-m_1)l_1 t/m_1}+R_2))^{n_1}\\ \geq&\frac{n_1 l_1}{2}R_1^{n_1}(\log R_2)^{n_1}e^{n_1 l_1 t},\\ \Delta_{m_1}\overline{u}\leq&\frac{\lambda_{m_1}R_1^{m_1}e^{n_1 l_1 t}}{R_2^{m_1}},\\ (\overline{v}^{n_2})_t\geq&\frac{1}{2}n_2 l_2 e^{n_2 l_2 t},\\ \Delta_{m_2}\overline{v}\leq&\overline{\lambda}(\lambda_{m_2}+Lm_2 D_{m_2}^{m_2+1})(2M)^{m_2(n_2+1)/(m_2+1)}(A_{m_2}c_{m_2}^{m_2-1})^{-1}e^{n_2 l_2 t}.\end{aligned}$$

In addition,

$$\begin{aligned}\overline{\lambda}\overline{u}^{\alpha_1}\overline{v}^{p_1}\leq&\ \overline{\lambda}(2M)^{p_1}(R_1\log(1+R_2))^{\alpha_1}e^{(\alpha_1 l_1+p_1 l_2)t},\\ \overline{\lambda}\overline{u}^{p_2}\overline{v}^{\beta_1}\leq&\ \overline{\lambda}(2M)^{\beta_1}(R_1\log(1+R_2))^{p_2}e^{(p_2 l_1+\beta_1 l_2)t}.\end{aligned}$$

By setting $c_{m_1}=C_{m_1}$, if $m_1\geq 1, c_{m_1}=D_{m_1}$, if $m_1<1$, on the boundary, we have that

$$\begin{aligned}\nabla_{m_1}\overline{u}\cdot\nu\geq&\frac{R_1^{m_1}c_{m_1}^{m_1-1}A_{m_1}}{(1+R_2)^{m_1}}e^{n_1 l_1 t},\\ \overline{\lambda}\overline{u}^{\alpha_2}\overline{v}^{q_1}\leq&\overline{\lambda}(R_1\log(1+R_2))^{\alpha_2}(2M)^{q_1}e^{(\alpha_2 l_1+q_1 l_2)t};\\ \nabla_{m_2}\overline{v}\cdot\nu\geq&\overline{\lambda}(2M)^{m_2(n_2+1)/(m_2+1)}e^{m_2(n_2+1)l_2 t/(m_2+1)},\\ \overline{\lambda}\overline{u}^{q_2}\overline{v}^{\beta_2}\leq&\overline{\lambda}(R_1\log(1+R_2))^{q_2}(2M)^{\beta_2}e^{(q_2 l_1+\beta_2 l_2)t}.\end{aligned}$$

There exist two positive constants R_1, M such that $R_1\log R_2\geq\max\{1,\|u_0\|_\infty\}$, $M\geq\max\{1,\|v_0\|_\infty\}$ and by Fact 1 such that

$$\begin{aligned}&R_1^{m_1-\alpha_2}\geq\overline{\lambda}(2M)^{q_1}(\log(1+R_2))^{\alpha_2}(1+R_2)^{m_1}(A_{m_1}c_{m_1}^{m_1-1})^{-1},\\ &(2M)^{m_2(n_2+1)/(m_2+1)-\beta_2}\geq R_1^{q_2}(\log(1+R_2))^{q_2}.\end{aligned}$$

Set

$$\begin{aligned}L=\overline{\lambda}^{1/m_2}\max\{&\frac{n_2-m_2}{m_2+1}2^{(n_2+m_2+2)/(m_2+1)}M^{(n_2-m_2)/(m_2+1)}(A_{m_2}c_{m_2}^{m_2-1})^{-1/m_2},\\ &2^{(n_2+1)/(m_2+1)}M^{(n_2-m_2)/(m_2+1)}(A_{m_2}c_{m_2}^{m_2-1})^{-1/m_2}\}.\end{aligned}$$

On the other hand , there exist positive constants l_1, l_2 large such that

$$n_1 l_1 \geq \alpha_1 l_1 + p_1 l_2, n_2 l_2 \geq p_2 l_1 + \beta_1 l_2,$$
$$n_1 l_1 \geq \alpha_2 l_1 + q_1 l_2, \frac{m_2(n_2+1)}{m_2+1} l_2 \geq q_2 l_1 + \beta_2 l_2$$

and

$$\begin{aligned} l_1 \geq & \frac{2\lambda_{m_1} R_1^{m_1-n_1}}{n_1 R_2^{m_1} (\log R_2)^{n_1}} + \frac{2\overline{\lambda}(2M)^{p_1}(R_1 \log(1+R_2))^{\alpha_1}}{n_1 (R_1 \log R_2)^{n_1}}, \\ l_2 \geq & 2\overline{\lambda}(\lambda_{m_2} + L m_2 D_{m_2}^{m_2+1})(2M)^{m_2(n_2+1)/(m_2+1)}(n_2 A_{m_2} c_{m_2}^{m_2-1})^{-1} \\ & + \frac{2\overline{\lambda}}{n_2}(R_1 \log(1+R_2))^{p_2}(2M)^{\beta_1}. \end{aligned}$$

We can see that $(\overline{u}, \overline{v})$ is an upper solution of $(1.1)-(1.3)$. Thus the solutions of $(1.1)-(1.3)$ are global. □

By combining Proposition 2.4 and Lemma 3.2, 3.4, 4.2 and 5.1 that Theorem 1.3 is true.

In a similar way to the proof of Theorem 1.3, we have Theorem 1.4.

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