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On mild solutions to fractional differential equations with nonlocal conditions *

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Abstract: We prove new existence results of mild solutions to fractional differential equations with nonlocal conditions in Banach spaces. The nonlocal item is only assumed to be continuous. This generalizes some recent results in this area.

Keywords: Nonlocal condition; fractional differential equations; strongly continuous semigroup; fixed point theorem.

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1 Introduction

In this paper, we are concerned with the existence of mild solutions for a fractional differential equation with nonlocal conditions of the form:

$$\begin{cases} D^q u(t) = Au(t) + f(t, u(t)), \quad 0 \le t \le T, \\ u(0) = u_0 - g(u), \end{cases} \tag{1.1}$$

where D^q is the Caputo fractional derivatives of order q with $0 < q \le 1$, $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \ge 0$, X a real Banach

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space endowed with the norm $\|\cdot\|$, f and g are appropriate continuous functions to be specified later.

Recently, the fractional differential equations are appropriate models for describing real world problems, which cannot be described using classical integer order differential equations. So, they have been studied by many researchers. And, some recent contributions to the theory of fractional differential equations can be seen in [1–5, 13, 16–18, 20, 21].

On the other hand, the following differential equations with nonlocal conditions have been studied extensively in the literature, since it is demonstrated that the nonlocal problems have better effects in applications than the classical ones.

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \quad 0 \le t \le T, \\ u(0) = u_0 - g(u). \end{cases} \tag{1.2}$$

Many authors developed different techniques and methods to solve the above nonlocal problem. For more details on this topic we refer the reader to [7, 9–12, 14, 15, 19] and references therein.

Naturally, some researchers combined the above two directions and studied the fractional differential equation (1.1) with nonlocal conditions. In [8, 23], the authors studied the existence of mild solutions to equation (1.1) when the nonlocal item g was assumed to be Lipschitz or compact function in different frameworks. In this paper, we study further the existence of mild solutions to nonlocal problem (1.1). By using the ideas in [10, 22], we prove the existence of mild solutions to equation (1.1) without the Lipschitz or compact assumption on the nonlocal item g. Actually, the continuity of g is only assumed and g is completely determined on $[\delta, T]$ for some small $\delta > 0$ or g is continuous in $C([0, T], X)$ with $L^1([0, T], X)$ topology (see Corollaries 3.5-3.7). Our results extend some existing ones in this area.

This paper has three sections. In the next section, we recall some definitions on Caputo fractional derivatives and mild solutions to equation (1.1). In the last section, we establish the existence of mild solutions to equation (1.1) via the techniques developed in [10, 22].

2 Preliminaries

Throughout this paper, let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ be the set of positive integers, real numbers and positive real numbers, respectively. We denote by X a Banach spaces with norm $\|\cdot\|$,

$C([0,T],X)$ the space of all X-valued continuous functions on $[0,T]$ and $L^1([0,T],X)$ the space of X-valued Bochner integrable functions on $[0,T]$ with the norm $\|f\|_{L^1} = \int_0^T \|f(t)\|\,\mathrm{d}t$.

Now, let us recall some basic definitions and results on fractional derivative and fractional differential equations.

Definition 2.1. ([20]) The fractional order integral of the function $f \in L^1([0,T],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s,$$

where Γ is the Gamma function.

Definition 2.2. ([20]) The Riemann-Liouville fractional order derivative of order α of a function f given on the interval $[0,+\infty)$ is defined by

$${}^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\,\mathrm{d}s,$$

where $\alpha \in (n-1,n)$, $n \in \mathbb{N}$.

Definition 2.3. ([20]) The Caputo fractional order derivative of order α of a function f given on the interval $[0,+\infty)$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s,$$

where $\alpha \in (n-1,n)$, $n \in \mathbb{N}$.

If f takes values in Banach space X, the integrals which appear in above three definitions are taken in Bochner's sense.

In this paper, we always suppose that the linear operator $A : D(A) \subset X \to X$ generates a compact strongly continuous semigroup $\{T(t) :\ t \geq 0\}$, i.e., $T(t)$ is compact for any $t > 0$. Moreover, we denote

$$M := \sup_{t\geq 0} \|T(t)\| < \infty.$$

Now, using the probability density function and its Laplace transform developed in [6] (also see [8, 21]), we can give the following definition of mild solutions to equation (1.1).

Definition 2.4. A continuous function u is said to be a mild solution of (1.1) if u satisfies

$$u(t) = \int_0^\infty \xi_q(\sigma)T(t^q\sigma)(u_0 - g(u))\,\mathrm{d}\sigma + q\int_0^t\int_0^\infty \sigma(t-s)^{q-1}\xi_q(\sigma)T((t-s)^q\sigma)f(s,u(s))\,\mathrm{d}\sigma\,\mathrm{d}s$$

for $t \in [0,T]$, where ξ_q is a probability density function defined on $(0,\infty)$ such that

$$\int_0^\infty \xi_q(\sigma)\,\mathrm{d}\sigma = 1, \quad \int_0^\infty \sigma^v\xi_q(\sigma)\,\mathrm{d}\sigma = \frac{\Gamma(1+v)}{\Gamma(1+qv)}, \quad v \in [0,1].$$

3 Main Results

Let r be a fixed positive real number. Write

$$B_r := \{x \in X; \|x\| \le r\}.$$

$$W_r := \{u \in C([0,T],X); u(t) \in B_r \text{ for } t \in [0,T]\}.$$

Clearly, B_r, W_r are bounded closed and convex sets. We make the following assumptions.

(H1) $f : [0,T] \times X \to X$ is continuous.

(H2) $g : C([0,T],X) \to X$ is continuous.

(H) The set $g(\overline{conv}QW_r)$ is pre-compact, where $\overline{conv}B$ denotes the convex closed hull of set $B \subseteq C([0,T],X)$.

Remark 3.1. It is easy to see that condition (H) is weaker than the compactness and convexity of g. The same hypothesis can be seen from [10, 22], where the authors considered the existence of mild solutions for semilinear nonlocal problems of integer order when A is a linear, densely defined operator on X which generates a C_0-semigroup. After the proof of our main results, we will give some special types of nonlocal item g which is neither Lipschitz nor compact, but satisfies the condition (H) in the next Corollaries.

Under these assumptions, we can prove the main results in this paper.

Theorem 3.2. *Assume that conditions (H1), (H2) and (H) are satisfied. Then the nonlocal problem (1.1) has at least one mild solution provided that*

$$M[\|u_0\| + \sup_{u\in W_r}\|g(u)\| + \frac{T^q}{\Gamma(1+q)}\sup_{s\in[0,T],u\in W_r}\|f(s,u(s))\|] \le r. \tag{3.1}$$

Proof. For $u \in C([0,T],X)$, from the properties of probability density function ξ_q and condition (H1), it follows that

$$\begin{aligned}
&\int_0^t \int_0^\infty \sigma(t-s)^{q-1}\xi_q(\sigma)T((t-s)^q\sigma)f(s,u(s))\,\mathrm{d}\sigma\,\mathrm{d}s \\
\leq& \frac{M}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\,\mathrm{d}s \sup_{s\in[0,T]}\|f(s,u(s))\| \\
\leq& \frac{MT^q}{q\Gamma(1+q)}\sup_{s\in[0,T]}\|f(s,u(s))\|,
\end{aligned}$$

which means that $\int_0^\infty \sigma(t-s)^{q-1}\xi_q(\sigma)T((t-s)^q\sigma)f(s,u(s))\,\mathrm{d}\sigma$ is Bochner's integrable on $[0,t]$ with respect to $s \in [0,t]$ for all $t \in [0,T]$. Define the mapping Q on W_r by

$$(Qu)(t) = \int_0^\infty \xi_q(\sigma)T(t^q\sigma)(u_0 - g(u))\,\mathrm{d}\sigma + q\int_0^t \int_0^\infty \sigma(t-s)^{q-1}\xi_q(\sigma)T((t-s)^q\sigma)f(s,u(s))\,\mathrm{d}\sigma\,\mathrm{d}s$$

for $t \in [0,T]$. It is easy to see that the fixed point of Q is a mild solution of nonlocal problem (1.1). Subsequently, we will prove that Q has a fixed point by using Schauder's fixed point theorem.

Firstly, we prove that the mapping Q is continuous on $C([0,T],X)$. For this purpose, let $\{u_n\}_{n=1}^{+\infty}$ be a sequence in $C([0,T],X)$ with $\lim_{n\to\infty} u_n = u$ in $C([0,T],X)$. By the continuity of f, we deduce that $f(s,u_n(s))$ converges to $f(s,u(s))$ in X uniformly for $s \in [0,T]$, it follows that

$$\begin{aligned}
&\|(Qu_n)(t) - (Qu)(t)\| \\
\leq& M\Big[\|g(u_n) - g(u)\| + \frac{q}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|f(s,u_n(s)) - f(s,u(s))\|\,\mathrm{d}s\Big] \\
\leq& M\Big[\|g(u_n) - g(u)\| + \frac{T^q}{\Gamma(1+q)}\sup_{s\in[0,T]}\|f(s,u_n(s)) - f(s,u(s))\|\Big].
\end{aligned}$$

Then by the continuity of g and f, we get $\lim_{n\to\infty} Qu_n = Qu$ in $C([0,T],X)$, which implies that the mapping Q is continuous on $C([0,T],X)$.

Secondly, we claim that $QW_r \subseteq W_r$. In fact, for any $u \in W_r$, by (3.1), we have

$$\|(Qu)(t)\| \leq M\Big[\|u_0\| + \sup_{u\in W_r}\|g(u)\| + \frac{T^q}{\Gamma(1+q)}\sup_{s\in[0,T],u\in W_r}\|f(s,u(s))\|\Big] \leq r,$$

i.e., Q maps W_r into itself.

Thirdly, we show that there exists a set $W \subseteq W_r$ such that $Q : W \to W$ is a compact mapping. For this purpose, let

$$Qu = Q^1 u + Q^2 u$$

with

$$(Q^1 u)(t) = \int_0^\infty \xi_q(\sigma) T(t^q \sigma)(u_0 - g(u))\, \mathrm{d}\sigma,$$

$$(Q^2 u)(t) = q \int_0^t \int_0^\infty \sigma (t-s)^{q-1} \xi_q(\sigma) T((t-s)^q \sigma) f(s, u(s))\, \mathrm{d}\sigma\, \mathrm{d}s$$

for $t \in [0, T]$.

For $t \in (0, T]$ and $\delta > 0$, set

$$(Q_\delta^1 u)(t) = T(t^q \delta) \int_\delta^\infty \xi_q(\sigma) T(t^q \sigma - t^q \delta)(u_0 - g(u))\, \mathrm{d}\sigma, \quad u \in W_r.$$

By (3.1) and the compactness of $T(t)$, $t > 0$, we deduce that $Q_\delta^1 W_r(t)$ is relatively compact in X for any $\delta > 0$. Moreover, we have

$$\begin{aligned}
&\|(Q^1 u)(t) - (Q_\delta^1 u)(t)\| \\
\leq &\| \int_0^\infty \xi_q(\sigma) T(t^q \sigma)(u_0 - g(u))\, \mathrm{d}\sigma - \int_\delta^\infty \xi_q(\sigma) T(t^q \sigma)(u_0 - g(u))\, \mathrm{d}\sigma \| \\
\leq &M(\|u_0\| + \sup_{u \in W_r} \|g(u)\|) \int_0^\delta \xi_q(\sigma)\, \mathrm{d}\sigma \\
\to &0,
\end{aligned}$$

as $\delta \to 0$, which implies that $Q^1 W_r(t)$ is relatively compact in X for every $t \in (0, T]$ since there are a family of relatively compact sets arbitrarily close to it. Next, we prove that $Q^1 W_r$ is equicontinuous on $[\eta, T]$ for any small positive number η. For $u \in W_r$ and $\eta \leq t_1 < t_2 \leq T$, there exist positive numbers δ and N such that

$$\begin{aligned}
&\|(Q^1 u)(t_2) - (Q^1 u)(t_1)\| \\
\leq &\int_0^\delta \xi_q(\sigma) \|[T(t_2^q \sigma) - T(t_1^q \sigma)](u_0 - g(u))\|\, \mathrm{d}\sigma + \int_\delta^N \xi_q(\sigma) \|[T(t_2^q \sigma) - T(t_1^q \sigma)](u_0 - g(u))\|\, \mathrm{d}\sigma \\
&+ \int_N^\infty \xi_q(\sigma) \|[T(t_2^q \sigma) - T(t_1^q \sigma)](u_0 - g(u))\|\, \mathrm{d}\sigma \\
\leq &\int_\delta^N \xi_q(\sigma) \|T(t_2^q \sigma) - T(t_1^q \sigma)\| \cdot \|(u_0 - g(u))\|\, \mathrm{d}\sigma + 2M(\|u_0\| + \sup_{u \in W_r} \|g(u)\|) \int_0^\delta \xi_q(\sigma)\, \mathrm{d}\sigma \\
&+ 2M(\|u_0\| + \sup_{u \in W_r} \|g(u)\|) \int_N^\infty \xi_q(\sigma)\, \mathrm{d}\sigma.
\end{aligned}$$

Now, as $T(\cdot)$ is compact, $T(t)$ is operator norm continuous for $t > 0$. Thus $T(t)$ is operator norm continuous uniformly for $t \in [\eta^q\delta, T^qN]$. Combining this with the absolute continuity of $\xi_q(\cdot)$ on $[0,\infty)$, it follows that Q^1W_r is equicontinuous on $[\eta, T]$.

For $Q^2 : W_r \to C([0,T], X)$, we claim that it is a compact mapping. In fact, $Q^2W_r(0)$ is relatively compact. For $t \in (0,T]$, let $\delta \in (0,t)$ and define a mapping on W_r by

$$\begin{aligned}(Q^2_\delta u)(t) =& q\int_0^{t-\delta}\int_\delta^\infty \sigma(t-s)^{q-1}\xi_q(\sigma)T((t-s)^q\sigma)f(s,u(s))\,\mathrm{d}\sigma\,\mathrm{d}s\\ =& qT(\delta^q\delta)\int_0^{t-\delta}\int_\delta^\infty \sigma(t-s)^{q-1}\xi_q(\sigma)T((t-s)^q\sigma-\delta^q\delta)f(s,u(s))\,\mathrm{d}\sigma\,\mathrm{d}s\end{aligned}$$

for $u \in W_r$. We get that $Q^2_\delta W_r(t)$ is relatively compact for any $\delta \in (0,t)$ since $T(\delta^q\delta)$ is compact. Moreover, for $u \in W_r$, we obtain

$$\begin{aligned}&\|(Q^2u)(t) - (Q^2_\delta u)(t)\|\\ \le& q\int_{t-\delta}^{t}\int_0^\infty \sigma(t-s)^{q-1}\xi_q(\sigma)\|T((t-s)^q\sigma)f(s,u(s))\|\,\mathrm{d}\sigma\,\mathrm{d}s\\ &+ q\int_0^{t-\delta}\int_0^\delta \sigma(t-s)^{q-1}\xi_q(\sigma)\|T((t-s)^q\sigma)f(s,u(s))\|\,\mathrm{d}\sigma\,\mathrm{d}s\\ \le& M[\frac{\delta^q}{\Gamma(1+q)} + T^q\int_0^\delta \sigma\xi_q(\sigma)\,\mathrm{d}\sigma]\sup_{s\in[0,T],u\in W_r}\|f(s,u(s))\|\\ \to& 0,\end{aligned}$$

as $\delta \to 0$, which implies that $Q^2W_r(t)$ is relatively compact in X for every $t \in (0,T]$ since there are a family of relatively compact sets arbitrarily close to it. Next, we prove that Q^2W_r is equicontinuous on $[0,T]$. For $u \in W_r$ and $0 \le t_1 < t_2 \le T$, we have

$$\begin{aligned}&\|(Q^2u)(t_2) - (Q^2u)(t_1)\|\\ =& \|q\int_{t_1}^{t_2}\int_0^\infty \sigma(t_2-s)^{q-1}\xi_q(\sigma)T((t_2-s)^q\sigma)f(s,u(s))\,\mathrm{d}\sigma\,\mathrm{d}s\\ &+ q\int_0^{t_1}\int_0^\infty \sigma(t_2-s)^{q-1}\xi_q(\sigma)T((t_2-s)^q\sigma)f(s,u(s))\,\mathrm{d}\sigma\,\mathrm{d}s\\ &- q\int_0^{t_1}\int_0^\infty \sigma(t_1-s)^{q-1}\xi_q(\sigma)T((t_2-s)^q\sigma)f(s,u(s))\,\mathrm{d}\sigma\,\mathrm{d}s\\ &+ q\int_0^{t_1}\int_0^\infty \sigma(t_1-s)^{q-1}\xi_q(\sigma)T((t_2-s)^q\sigma)f(s,u(s))\,\mathrm{d}\sigma\,\mathrm{d}s\\ &- q\int_0^{t_1}\int_0^\infty \sigma(t_1-s)^{q-1}\xi_q(\sigma)T((t_1-s)^q\sigma)f(s,u(s))\,\mathrm{d}\sigma\,\mathrm{d}s\|,\end{aligned}$$

which implies that

$$
\begin{aligned}
&\|(Q^2u)(t_2) - (Q^2u)(t_1)\| \\
\leq &q \int_{t_1}^{t_2} \int_0^\infty \sigma(t_2 - s)^{q-1} \xi_q(\sigma) \|T((t_2 - s)^q \sigma) f(s, u(s))\| \, \mathrm{d}\sigma \, \mathrm{d}s \\
&+ q \int_0^{t_1} \int_0^\infty \sigma[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] \xi_q(\sigma) \|T((t_2 - s)^q \sigma) f(s, u(s))\| \, \mathrm{d}\sigma \, \mathrm{d}s \\
&+ q \int_0^{t_1} \int_0^\infty \sigma(t_1 - s)^{q-1} \xi_q(\sigma) \|[T((t_2 - s)^q \sigma) - T((t_1 - s)^q \sigma)] f(s, u(s))\| \, \mathrm{d}\sigma \, \mathrm{d}s \\
\leq &\frac{M(t_2 - t_1)^q}{\Gamma(1+q)} \sup_{s \in [0,T], u \in W_r} \|f(s, u(s))\| + \frac{M[t_1^q - t_2^q + (t_2 - t_1)^q]}{\Gamma(1+q)} \sup_{s \in [0,T], u \in W_r} \|f(s, u(s))\| \\
&+ q \int_0^{t_1} \int_0^\infty \sigma(t_1 - s)^{q-1} \xi_q(\sigma) \|[T((t_2 - s)^q \sigma) - T((t_1 - s)^q \sigma)] f(s, u(s))\| \, \mathrm{d}\sigma \, \mathrm{d}s.
\end{aligned} \tag{3.2}
$$

For the last expression of the right side of the above inequality, if $t_1 = 0$, then it equals to zero; if $t_1 > 0$, then there exist positive numbers δ and N such that

$$
\begin{aligned}
&q \int_0^{t_1} \int_0^\infty \sigma(t_1 - s)^{q-1} \xi_q(\sigma) \|[T((t_2 - s)^q \sigma) - T((t_1 - s)^q \sigma)] f(s, u(s))\| \, \mathrm{d}\sigma \, \mathrm{d}s \\
= &q \int_0^{t_1 - \delta} \int_\delta^N \sigma(t_1 - s)^{q-1} \xi_q(\sigma) \|[T((t_2 - s)^q \sigma) - T((t_1 - s)^q \sigma)] f(s, u(s))\| \, \mathrm{d}\sigma \, \mathrm{d}s \\
&+ q \int_0^{t_1 - \delta} \int_0^\delta \sigma(t_1 - s)^{q-1} \xi_q(\sigma) \|[T((t_2 - s)^q \sigma) - T((t_1 - s)^q \sigma)] f(s, u(s))\| \, \mathrm{d}\sigma \, \mathrm{d}s \\
&+ q \int_0^{t_1 - \delta} \int_N^\infty \sigma(t_1 - s)^{q-1} \xi_q(\sigma) \|[T((t_2 - s)^q \sigma) - T((t_1 - s)^q \sigma)] f(s, u(s))\| \, \mathrm{d}\sigma \, \mathrm{d}s \\
&+ q \int_{t_1 - \delta}^{t_1} \int_0^\infty \sigma(t_1 - s)^{q-1} \xi_q(\sigma) \|[T((t_2 - s)^q \sigma) - T((t_1 - s)^q \sigma)] f(s, u(s))\| \, \mathrm{d}\sigma \, \mathrm{d}s \\
\leq &q \int_0^{t_1 - \delta} \int_\delta^N \sigma(t_1 - s)^{q-1} \xi_q(\sigma) \|T((t_2 - s)^q \sigma) - T((t_1 - s)^q \sigma)\| \cdot \|f(s, u(s))\| \, \mathrm{d}\sigma \, \mathrm{d}s \\
&+ 2MT^q \Big(\int_0^\delta \sigma \xi_q(\sigma) \, \mathrm{d}\sigma + \int_N^\infty \sigma \xi_q(\sigma) \, \mathrm{d}\sigma \Big) \sup_{s \in [0,T], u \in W_r} \|f(s, u(s))\| \\
&+ \frac{2M}{\Gamma(1+q)} \int_{t_1 - \delta}^{t_1} (t_1 - s)^{q-1} \, \mathrm{d}s \sup_{s \in [0,T], u \in W_r} \|f(s, u(s))\|.
\end{aligned} \tag{3.3}
$$

Thus, combining the above inequalities (3.2) (3.3) with the norm continuity of $T(t)$ uniformly on $[\delta^q \delta, T^q N]$ and the absolute continuity of integrals, we obtain the equicontinuity of $Q^2 W_r$ on $[0, T]$. Therefore, $Q^2 : W_r \to C([0, T], X)$ is a compact mapping by the Arzela-Ascoli theorem.

In summary, we have proven that $QW_r(t)$ is relatively compact for every $t \in (0, T]$ and QW_r is equicontinuous on $[\eta, T]$ for any small positive number η.

Now, let $W = \overline{conv}QW_r$, we get that W is a bounded closed and convex subset of $C([0,T],X)$ and $QW \subseteq W$. It is easy to see that $QW(t)$ is relatively compact in X for every $t \in (0,T]$ and QW is equicontinuous on $[\eta, T]$ for any small positive number η. Moreover, we know that $g(W) = g(\overline{conv}QW_r)$ is pre-compact due to the condition (H).

Thus, we claim that $Q : W \to W$ is a compact mapping. In fact, it is easy to see that $Q^1W(0) = \int_0^\infty \xi_q(\sigma)T(t^q\sigma)(u_0 - g(W))\,\mathrm{d}\sigma$ is relatively compact since $g(W) = g(\overline{conv}QW_r)$ is pre-compact. It remains to prove that Q^1W is equicontinuous on $[0,T]$. For that, let $u \in W$ and $0 \le t_1 < t_2 \le T$, there exists positive number N such that

$$\begin{aligned}
&\|(Q^1u)(t_2) - (Q^1u)(t_1)\| \\
\le& \int_0^N \xi_q(\sigma)\|[T(t_2^q\sigma) - T(t_1^q\sigma)](u_0 - g(u))\|\,\mathrm{d}\sigma + \int_N^\infty \xi_q(\sigma)\|[T(t_2^q\sigma) - T(t_1^q\sigma)](u_0 - g(u))\|\,\mathrm{d}\sigma \\
\le& \int_0^N \xi_q(\sigma)\|[T(t_2^q\sigma) - T(t_1^q\sigma)](u_0 - g(u))\|\,\mathrm{d}\sigma + 2M(\|u_0\| + \sup_{u\in W}\|g(u)\|)\int_N^\infty \xi_q(\sigma)\,\mathrm{d}\sigma.
\end{aligned}$$

In view of the compactness of $g(W)$ and the strong continuity of $T(t)$ on $[0, T^qN]$, we obtain the equicontinuity of Q^1W on $[0,T]$. Thus, $Q^1 : W \to C([0,T],X)$ is a compact mapping by the Arzela-Ascoli theorem, and hence $Q : W \to W$ is also a compact mapping. Now, Schauder's fixed point theorem implies that Q has a fixed point on W, which gives rise to a mild solution of nonlocal problem (1.1). □

The following theorem is a direct consequence of Theorem 3.2.

Theorem 3.3. *Assume that conditions (H1), (H2) and (H) are satisfied for each $r > 0$. If*

$$\frac{\|g(u)\|}{\|u\|} \to 0, \quad \|u\| \to \infty, \tag{3.4}$$

$$\frac{\|f(t,x)\|}{\|x\|} \to 0, \quad \|x\| \to \infty \tag{3.5}$$

for all $t \in [0,T]$, then the nonlocal problem (1.1) has at least one mild solution.

Remark 3.4. It is easy to see that if there exist constants $L_1, L_2 > 0$ and $\alpha, \beta \in [0,1)$ such that

$$\|g(u)\| \le L_1(1 + \|u\|)^\alpha,$$

$$\|f(t,x)\| \le L_2(1 + \|x\|)^\beta,$$

then conditions (3.4) and (3.5) are satisfied.

Next, we will give special types of nonlocal item g which is neither Lipschitz nor compact, but satisfies the condition (H).

We give the following assumptions.

(H3) $g : C([0,T], X) \to X$ is a continuous mapping which maps W_r into a bounded set, and there is a $\delta = \delta(r) \in (0,T)$ such that $g(u) = g(v)$ for any $u,\ v \in W_r$ with $u(s) = v(s)$, $s \in [\delta, T]$.

(H4) $g : (C([0,T], X), \|\cdot\|_{L^1}) \to X$ is continuous.

Corollary 3.5. *Assume that conditions (H1)-(H3) are satisfied. Then the nonlocal problem (1.1) has at least one mild solution on* $[0,T]$ *provided that*

$$M[\|u_0\| + \sup_{u\in W_r} \|g(u)\| + \frac{T^q}{\Gamma(1+q)} \sup_{s\in[0,T],u\in W_r} \|f(s,u(s))\|] \le r.$$

Proof. Let $(QW_r)_\delta = \{u \in C([0,T], X); u(t) = v(t)$ for $t \in [\delta, T],\ u(t) = u(\delta)$ for $t \in [0,\delta)$, where $v \in QW_r\}$. From the proof of Theorem 3.2, we know that $(QW_r)_\delta$ is pre-compact in $C([0,T], X)$. Moreover, by condition (H3), $g(\overline{conv}QW_r) = g(\overline{conv}(QW_r)_\delta)$ is also pre-compact in $C([0,T], X)$. Thus, all the hypotheses in Theorem 3.2 are satisfied. Therefore, there is at least one mild solution of nonlocal problem (1.1). □

Corollary 3.6. *Let conditions (H1) and (H2) be satisfied. Suppose that* $g(u) = \sum_{j=1}^{p} c_j u(t_j)$, *where* c_j *are given positive constants, and* $0 < t_1 < t_2 < \cdots < t_p \le T$. *Then the nonlocal problem (1.1) has at least one mild solution on* $[0,T]$ *provided that*

$$M[\|u_0\| + \sum_{j=1}^{p} c_j r + \frac{T^q}{\Gamma(1+q)} \sup_{s\in[0,T],u\in W_r} \|f(s,u(s))\|] \le r.$$

Proof. It is easy to see that the mapping g with $g(u) = \sum_{j=1}^{p} c_j u(t_j)$ satisfies condition (H3). And all the conditions in Corollary 3.5 are satisfied. So the conclusion holds. □

Corollary 3.7. *Assume that conditions (H1), (H2) and (H4) are satisfied. Then the nonlocal problem (1.1) has at least one mild solution on* $[0,T]$ *provided that*

$$M[\|u_0\| + \sup_{u\in W_r} \|g(u)\| + \frac{T^q}{\Gamma(1+q)} \sup_{s\in[0,T],u\in W_r} \|f(s,u(s))\|] \le r.$$

Proof. According to Theorem 3.2, we should only to prove that the hypothesis (H) is satisfied.

For arbitrary $\epsilon > 0$, there exists $0 < \delta < T$ such that $\int_0^\delta \|u(s)\|\,\mathrm{d}s < \epsilon$ for all $u \in QW_r$. Let $(QW_r)_\delta = \{u \in C([0,T],X); u(t) = v(t) \text{ for } t \in [\delta, T],\ u(t) = u(\delta) \text{ for } t \in [0,\delta), \text{ where } v \in QW_r\}$. From the proof of Theorem 3.2, we know that $(QW_r)_\delta$ is pre-compact in $C([0,T],X)$, which implies that $(QW_r)_\delta$ is pre-compact in $L^1([0,T],X)$. Thus, QW_r is pre-compact in $L^1([0,T],X)$ as it has an ϵ-net $(QW_r)_\delta$. By condition $g : (C([0,T],X), \|\cdot\|_{L^1}) \to X$ is continuous and $\overline{conv}QW_r \subseteq (L)\overline{conv}QW_r$, it follows that condition (H) is satisfied, where $(L)\overline{conv}B$ denotes the convex and closed hull of B in $L^1([0,T],X)$. Therefore, the nonlocal problem (1.1) has at least one mild solution on $[0,T]$. ☐

Remark 3.8. *Our results extend some recent ones about the fractional differential equations with nonlocal conditions, since neither the Lipschitz continuity nor the compactness assumption on the nonlocal item is required.*

References

[1] R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, On the concept of solutions for fractional differential equations with uncertainty, Nonlinear Anal. TMA 72 (2010), 2859-2862.

[2] M. Benchohra, J. R. Graef and S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, Appl. Anal. 87 (7) (2008), 851-863.

[3] M. Benchohra, S. Hamani and S. K. Ntouyas, boundary value problems for differential equations with fractional order, Surv. Math. Appl. 3 (2008), 1-12.

[4] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, J. Math. Anal. Appl. 338 (2008), 1340-1350.

[5] M. Benchohra, B. A. Slimani, Existence and uniqueness of solutions to impulsive fractional differential equations, Electronic J. Diff. Eqns, 2009(10) (2009), 1-11.

[6] M. EL-Borai, Some probability densities and fundamental solutions of fractional evolution equations, Chaos, Solitons and Fractals 14 (2002), 433-440.

[7] L. Byszewski, V. Lakshmikantham, Theorem about the existence and uniqueness of solutions of a nonlocal Cauchy problem in a Banach space, Appl. Anal. 40 (1990), 11–19.

[8] T. Diagana, G. M. Mophou, G. M. N'Guérékata, On the existence of mild solutions to some semilinear fractional integro-differential equations, Electron. J. Qual. Theory Differ. Equ. 58 (2010) 1-17.

[9] Z. Fan, Existence of nondensely defined evolution equations with nonlocal conditions, Nonlinear Anal. 70 (2009), 3829–3836.

[10] Z. Fan, Impulsive problems for semilinear differential equations with nonlocal conditions, Nonlinear Anal. 72 (2010) 1104–1109.

[11] Z. Fan, Gang Li, Existence results for semilinear differential equations with nonlocal and impulsive conditions, J. Functional Analysis 258 (2010) 1709-1727.

[12] E. Hernández, J. S. dos Santos, K. A. G. Azevedo, Existence of solutions for a class of abstract differential equations with nonlocal conditions, Nonlinear Anal. TMA 74(7) (2011), 2624-2634.

[13] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, Theory of Fractional Dynamic Systems, Cambridge Scientific Publishers, 2009.

[14] J. Liang, J. H. Liu, T. J. Xiao, Nonlocal impulsive problems for nonlinear differential equations in Banach spaces, Math. Comput. Model. 49 (2009), 798–804.

[15] J. Liang, Z. Fan, Nonlocal impulsive Cauchy problems for evolution equations, Advances in Diff. Eqns. (2011), Article ID 784161, 1-17.

[16] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.

[17] G. M. Mophou, G. M. N'Guérékata, Existence of mild solutions of some semilinear neutral fractional functional evolution equations with infinite delay, Appl. Math. Comput. 216 (2010), 61-69.

[18] G. M. Mophou, G. M. N'Guérékata, A note on a semilinear fractional differential equation of neutral type with infinite delay, Advances in Diff. Eqns. (2010), Article ID 674630, 1-8.

[19] S. Ntouyas, P. Tsamatos, Global existence for semilinear evolution equations with nonlocal conditions, J. Math. Anal. Appl. 210 (1997), 679–687.

[20] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[21] J. Wang, Y. Zhou, A class of fractional evolution equations and optimal controls, Nonlinear Anal. RWA 12 (2011) 262-272.

[22] X. Xue, Existence of semilinear differential equations with nonlocal initial conditions, Acta Math. Sinica. 23(6) (2007), 983–988.

[23] Y. Zhou, F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, Nonlinear Anal. RWA 11 (2010) 4465-4475.

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