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Existence of Peregrine type solutions in fractional reaction–diffusion equations

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Abstract. In this article, we analyze the existence of Peregrine type solutions for the fractional reaction–diffusion equation by applying splitting-type methods. Peregrine type functions have two main characteristics, these are direct sum of functions of periodic type and functions that tend to zero at infinity. Well-posedness results are obtained for each particular characteristic, and for both combined.

Keywords: fractional diffusion, global existence, splitting method.

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1 Introduction

We study the non-autonomous system

$$\partial_t u + \sigma(-\Delta)^\beta u = F(t, u), \tag{1.1}$$

where $u(x,t) \in Z$ for $x \in \mathbb{R}^n$, $t > 0$, $\sigma \geq 0$ and $0 < \beta \leq 1$, $F : \mathbb{R} \times Z \to Z$ a continuous map and Z a Banach space. We consider the initial value problem $u(x, 0) = u_0(x)$.

The aim of this paper is to analyze the existence of solutions for the fractional reaction–diffusion equation by applying splitting methods to functions that have two main characteristics: these are direct sum of functions of periodic type and functions that vanish at infinity. We will call them from now on, "Peregrine type solutions". A similar type of solution is also studied in the context of the non-linear Schrödinger equation, under the name of "Peregrine solitons". These solutions were analyzed in [22], and have multiple applications, for example [5,12,16,17,26]. To achieve our goal, we use recent results concerning global existence on fractional reaction–diffusion equations [6] based in similar numerical splitting techniques [7,13], introduced for other purposes. Fractional reaction–diffusion equations are frequently used on

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many different topics of applied mathematics such as biological models, population dynamics models, nuclear reactor models, just to name a few (see [4, 9, 10] and references therein).

The fractional model captures the faster spreading rates and power law invasion profiles observed in many applications compared to the classical model ($\beta = 1$) characterized by the behavior of the classical semigroup [15]. The main constituent of the model is the fractional Laplacian, described by standard theories of fractional calculus (for a complete survey see [21]). There are many different equivalent definitions of the fractional Laplacian and its properties are well understood (see [8, 14, 18–20, 23, 27]). The non-autonomous non-linear reaction–diffusion equation dynamics were studied by [1, 24] and others, analyzing the stability and evolution of the problem.

The paper is organized as follows: In Section 2 we set notations and preliminary results and in Section 3 we present the main results, primarily focusing on each characteristic of the direct sum separately. Finally, both results are combined to reach the existence of Peregrine type solutions.

2 Notations and preliminaries

We investigate continuous, Banach space valued functions. For a Banach space Z, we define $C_{\mathrm{u}}(\mathbb{R}^d, Z)$ as the set of uniformly continuous and bounded functions on $\mathbb{R}^d$ with values in Z. Defining the norm

$$\|u\|_{\infty,Z} = \sup_{x\in\mathbb{R}^d} |u(x)|_Z,$$

$C_{\mathrm{u}}(\mathbb{R}^d, Z)$ is a Banach space.

It is easy to see that if $g \in L^1(\mathbb{R}^d)$ and $u \in C_u(\mathbb{R}^d, Z)$ the Bochner integral is defined in the following way,

$$(g * u)(x) = \int_{\mathbb{R}^d} g(y)u(x-y)dy$$

This determines an element of $C_{\mathrm{u}}(\mathbb{R}^d, Z)$ and the linear operator $u \mapsto g * u$ is continuous (see [2, 11]). The following results show that the operator $-(-\Delta)^\beta$ defines a continuous contraction semigroup in the Banach space $C_{\mathrm{u}}(\mathbb{R}^d, Z)$. We define the space $C_0(\mathbb{R}^d, Z)$ of functions that converge to 0 when $|x| \to \infty$. The following lemma is a consequence of Lévy–Khintchine formula for infinitely divisible distributions and properties of the Fourier transform.

Lemma 2.1. *Let $0 < \beta \leq 1$ and $g_\beta \in C_0(\mathbb{R}^d)$ such that $\hat{g}_\beta(\xi) = e^{-|\xi|^{2\beta}}$. Then g_β is positive, invariant under rotations of $\mathbb{R}^d$, integrable and*

$$\int_{\mathbb{R}^d} g_\beta(x)dx = 1.$$

Proof. The first statement follows from Theorem 14.14 of [25], the remaining claims are an easy consequence of the definition of $\hat{g}_\beta$. ☐

Based on the previous lemma, we recall some results about Green's function related to the linear operator $\partial_t + \sigma(-\Delta)^\beta$.

Proposition 2.2. *Let $\sigma > 0$ and $0 < \beta \leq 1$, the function $G_{\sigma,\beta}$ given by*

$$G_{\sigma,\beta}(t,x) = (\sigma t)^{-\frac{d}{2\beta}} g_\beta((\sigma t)^{-\frac{1}{2\beta}} x),$$

verifies

i. $G_{\sigma,\beta}(\cdot,t) > 0$;

ii. $G_{\sigma,\beta}(\cdot,t) \in L^1(\mathbb{R}^d)$ *and*

$$\int_{\mathbb{R}^d} G_{\sigma,\beta}(t,x)dx = 1;$$

iii. $G_{\sigma,\beta}(\cdot,t) * G_{\sigma,\beta}(\cdot,t') = G_{\sigma,\beta}(\cdot,t+t')$, *for* $t,t' > 0$;

iv. $\partial_t G_{\sigma,\beta} + \sigma(-\Delta)^\beta G_{\sigma,\beta} = 0$ *for* $t > 0$.

Proof. The first and second statements are a consequence of the definition of $\hat{g}_\beta$. The third and fourth statements are immediate applying Fourier transform. □

In the following proposition, we have that the linear operator $-\sigma(-\Delta)^\beta$ defines a continuous contraction semigroup in $C_u(\mathbb{R}^d, Z)$.

Proposition 2.3. *For any* $\sigma > 0$ *and* $0 < \beta \leq 1$, *the map* $\mathsf{S} : \mathbb{R}_+ \to \mathcal{B}(C_u(\mathbb{R}^d, Z))$ *defined by* $\mathsf{S}(t)u = G_{\sigma,\beta}(\cdot,t) * u$ *is a continuous contraction semigroup.*

Proof. The proof can be found in [6, Proposition 2.2]. □

Next, we consider integral solutions of the problem (1.1). We say that $u \in C([0,T], C_u(\mathbb{R}^d, Z))$ is a mild solution of (1.1) iff u verifies

$$u(t) = \mathsf{S}(t)u_0 + \int_0^t \mathsf{S}(t-t')F(t',u(t'))dt'. \tag{2.1}$$

A continuous map $F : \mathbb{R}_+ \times Z \to Z$ is called locally Lipschitz if, given $R, T > 0$ there exists $L > 0$ such that if $t \in [0,T]$ and $|z|_Z, |\tilde{z}|_Z \leq R$, then

$$|F(t,z) - F(t,\tilde{z})|_Z \leq L|z - \tilde{z}|_Z.$$

In this case, for any $z_0 \in Z$ there exists a unique (maximal) solution of the Cauchy problem

$$z(t) = z_0 + \int_{t_0}^t F(t', z(t'))dt' \tag{2.2}$$

defined in $[t_0, t_0 + T^*(t_0,z_0))$, with $T^*(t_0,z_0)$ the maximal time of existence. It is easy to see that there exists a nonincreasing function $\mathcal{T} : \mathbb{R}_+^2 \to \mathbb{R}_+$, such that

$$\mathcal{T}(T,R) \leq \inf\{T^*(t_0,z_0) : 0 \leq t_0 \leq T, |z_0|_Z \leq R\}.$$

Also, one of the following alternatives holds:

- $T^*(t_0,z_0) = \infty$;
- $T^*(t_0,z_0) < \infty$ and $|z(t)|_Z \to \infty$ when $t \uparrow t_0 + T^*(t_0,z_0)$.

We can see that $F : \mathbb{R}_+ \times C_u(\mathbb{R}^d, Z) \to C_u(\mathbb{R}^d, Z)$, given by $F(t,u)(x) = F(t,u(x))$ is continuous and locally Lipschitz. Therefore, we can consider problem (2.2) in $C_u(\mathbb{R}^d, Z)$.

We denote by $\mathsf{N} : \mathbb{R} \times \mathbb{R} \times C_u(\mathbb{R}^d, Z) \to C_u(\mathbb{R}^d, Z)$ the flow generated by the integral equation (2.2) as $u(t) = \mathsf{N}(t,t_0,u_0)$, defined for $t_0 \leq t < t_0 + T^*(t_0,u_0)$.

We recall well-known local existence results for evolution equations.

Theorem 2.4. *There exists a function $T^* : C_u(\mathbb{R}^d, Z) \to \mathbb{R}_+$ such that for $u_0 \in C_u(\mathbb{R}^d, Z)$, exists a unique $u \in C([0, T^*(u_0)), C_u(\mathbb{R}^d, Z))$ mild solution of (1.1) with $u(0) = u_0$. Moreover, one of the following alternatives holds:*

- $T^*(u_0) = \infty$;
- $T^*(u_0) < \infty$ *and* $\lim_{t \uparrow T^*(u_0)} \|u(t)\|_{\infty,Z} = \infty$.

Proof. See Theorem 4.3.4 in [11]. □

Proposition 2.5. *Under conditions of the theorem above, we have the following statements:*

1. $T^* : C_u(\mathbb{R}^d, Z) \to \mathbb{R}_+$ *is lower semi-continuous;*
2. *If* $u_{0,n} \to u_0$ *in* $C_u(\mathbb{R}^d, Z)$ *and* $0 < T < T^*(u_0)$, *then* $u_n \to u$ *in the Banach space* $C([0, T], C_u(\mathbb{R}^d, Z))$.

Proof. See Proposition 4.3.7 in [11]. □

3 Peregrine type solutions

In this section, we analyze the existence of Peregrine type solutions for the fractional reaction–diffusion equation by applying splitting methods [6]. Peregrine type functions have two main characteristics: these are direct sum of functions of periodic type and functions that vanish at infinity. As a reference, we consider a solution of the non-linear Schrödinger equation, (Peregrine solitons), which entails these two characteristics. The explicit solution achieved in [22] is:

$$u(x,t) = \left[1 - \frac{4(1+2it)}{1+4x^2+4t^2}\right] e^{i(kx-\omega t)}$$

Well-posedness of the solution is obtained for each particular characteristic, to then combine both results using convergence theorems from [6]. In addition, we observe that the evolution of the periodic part is independent of the part that tends to zero at infinity (Theorem 3.9). For instance, suppose that the non-linearity is autonomous and of polynomial type (as in the Fitzhugh–Nagumo equation, see [3]), such as $F(u) = u^2$. If $u(t) = v(t) + w(t)$, where $v(t)$ is a periodic function and $w(t)$ is a function that vanishes when the spatial variable tends to infinity, then we have

$$F(u) = F(v+w) = (v+w)^2 = v^2 + 2vw + w^2$$

where, v^2 is periodic and $2vw + w^2$ tends to zero. In this specific case we can appreciate the *absorption*, i.e. the vanishing component is imposed in the crossed terms. As $v^2 = F(v)$, we expect that the periodic part of the initial data evolves independently from the rest for the non-linear equation. In this section we obtain general results to which this example refers.

Let $\{\gamma_1, \ldots, \gamma_q\}$ be q linearly independent vectors of $\mathbb{R}^d$ and let Γ be the lattice generated, i.e., $\Gamma = \{n_1\gamma_1 + \cdots + n_q\gamma_q : n_j \in \mathbb{Z}\}$. A function $u \in C_u(\mathbb{R}^d, Z)$ is Γ-periodic if $u(x+\gamma) = u(x)$ for any $\gamma \in \Gamma$. We denote the set of Γ–periodic functions of $C_u(\mathbb{R}^d, Z)$ by $C_u(\mathbb{R}^d/\Gamma, Z)$.

We recall the notation of the space $C_0(\mathbb{R}^d, Z)$ of functions that converge to 0 when $|x| \to \infty$. It is easy to prove the following result.

Proposition 3.1. *$C_u(\mathbb{R}^d/\Gamma, Z), C_0(\mathbb{R}^d, Z) \subset C_u(\mathbb{R}^d, Z)$ are closed subspaces. Moreover, $C_0(\mathbb{R}^d, Z) \cap C_u(\mathbb{R}^d/\Gamma, Z) = \{0\}$.*

Proof. Let $u \in C_u(\mathbb{R}^d/\Gamma, Z)$, we set $x \in \mathbb{R}^d$ and $u(x) = \lim_{|\gamma|\to\infty} u(x+\gamma)$. If $u \in C_0(\mathbb{R}^d, Z)$, then $\lim_{|\gamma|\to\infty} u(x+\gamma) = 0$. Therefore, $u(x) = 0$ for any $x \in \mathbb{R}^d$. □

Lemma 3.2. *Let X be a Banach space and let $X_1, X_2 \subset X$ be closed subspaces such that $X_1 \cap X_2 = \{0\}$, the following statements are equivalent*

i. $X_1 \oplus X_2$ is closed.

ii. The projector $P : X_1 \oplus X_2 \to X_1$ is continuous.

Proof. Since $X_1 \oplus X_2$ is a Banach space, the linear map $\phi : X_1 \times X_2 \to X_1 \oplus X_2$ given by $\phi(x_1, x_2) = x_1 + x_2$ is bijective, and continuous. By the closed graph theorem we have ϕ^{-1} is also a continuous operator. We express the projector as $P = \pi_1\phi^{-1}$ and then P is continuous. On the other hand, $X_1 \oplus X_2 = P^{-1}X_1$, since P continuous and X_1 a closed subspace, $X_1 \oplus X_2$ is closed. □

Lemma 3.3. *The projector $P : C_u(\mathbb{R}^d/\Gamma, Z) \oplus C_0(\mathbb{R}^d, Z) \to C_u(\mathbb{R}^d/\Gamma, Z)$ is continuous.*

Proof. Let $u = v + w \in C_u(\mathbb{R}^d/\Gamma, Z) \oplus C_0(\mathbb{R}^d, Z)$, $v \in C_u(\mathbb{R}^d/\Gamma, Z)$ and $w \in C_0(\mathbb{R}^d, Z)$. For any $x \in \mathbb{R}^d$, we can see that

$$v(x) = \lim_{\substack{|\gamma|\to\infty \\ \gamma\in\Gamma}} v(x+\gamma) = \lim_{\substack{|\gamma|\to\infty \\ \gamma\in\Gamma}} u(x+\gamma),$$

then $|v(x)|_Z \le \|u\|_{\infty,Z}$, which implies $\|v\|_{\infty,Z} = \|Pu\|_{\infty,Z} \le \|u\|_{\infty,Z}$. □

Corollary 3.4. *The direct sum $X_{\Gamma,Z} = C_u(\mathbb{R}^d/\Gamma, Z) \oplus C_0(\mathbb{R}^d, Z)$ is a closed subspace of $C_u(\mathbb{R}^d, Z)$.*

To obtain the existence of solutions in the space $X_{\Gamma,Z}$, we first study each case separately. We analyze the existence of solutions for the case of Γ periodic functions using the translation function.

Given $\gamma \in \mathbb{R}^d$ we define $\mathsf{T}_\gamma : C_u(\mathbb{R}^d, Z) \to C_u(\mathbb{R}^d, Z)$ as $(\mathsf{T}_\gamma u)(x) = u(x+\gamma)$. Since $\mathsf{S}(t)$ is a convolution operator, it is easy to see that $\mathsf{T}_\gamma \mathsf{S}(t) = \mathsf{S}(t)\mathsf{T}_\gamma$. Using that $\mathsf{T}_\gamma F(t,u) = F(t, \mathsf{T}_\gamma u)$ we obtain

$$\mathsf{T}_\gamma u(t) = \mathsf{S}(t)\mathsf{T}_\gamma u_0 + \int_0^t \mathsf{S}(t-t')F(t, \mathsf{T}_\gamma u(t'))dt'.$$

Therefore, $\mathsf{T}_\gamma u$ is the solution of (2.1) with initial data $\mathsf{T}_\gamma u_0$.

Proposition 3.5. *If $u_0 \in C_u(\mathbb{R}^d/\Gamma, Z)$, then the solution u of the equation (2.1) verifies $u(t) \in C_u(\mathbb{R}^d/\Gamma, Z)$ for $0 \le t < T^*(u_0)$.*

Proof. Since $\mathsf{T}_\gamma u_0 = u_0$ for any $\gamma \in \Gamma$, $\mathsf{T}_\gamma u, u$ are solutions with the same initial data. From uniqueness, we have $\mathsf{T}_\gamma u = u$. Therefore, $u(t) \in C_u(\mathbb{R}^d/\Gamma, Z)$. □

We now analyze the existence of solution in the space $C_0(\mathbb{R}^d, Z)$. We first study the linear part.

Lemma 3.6. *If $u \in C_0(\mathbb{R}^d, Z)$, then $\mathsf{S}(t)u \in C_0(\mathbb{R}^d, Z)$ for $t \in \mathbb{R}_+$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with $|x_n| \to \infty$. Then we have

$$|(\mathsf{S}(t)u)(x_n)|_Z \le \int_{\mathbb{R}^d} G_{\sigma,\beta}(t,y)|u(x_n - y)|_Z dy.$$

As $G_{\sigma,\beta}(t,\cdot)|u(x_n - \cdot)|_Z \le G_{\sigma,\beta}(t,\cdot)\|u\|_{\infty,Z}$ and $G_{\sigma,\beta}(t,y)|u(x_n - y)|_Z \to 0$, from dominated convergence theorem we obtain $\lim_{n\to\infty} |(\mathsf{S}(t)u)(x_n)|_Z = 0$. Since $\{x_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence, we have $\mathsf{S}(t)u \in C_0(\mathbb{R}^d, Z)$. □

We now study the non-linear part.

Lemma 3.7. *Let $u_0, \tilde{u}_0 \in C_u(\mathbb{R}^d, Z)$, if $u_0 - \tilde{u}_0 \in C_0(\mathbb{R}^d, Z)$, then $\mathsf{N}(t, t_0, u_0) - \mathsf{N}(t, t_0, \tilde{u}_0) \in C_0(\mathbb{R}^d, Z)$ for $0 \le t < \min\{T^*(u_0), T^*(\tilde{u}_0)\}$.*

Proof. Let $u(t) = \mathsf{N}(t, t_0, u_0)$ and $\tilde{u}(t) = \mathsf{N}(t, t_0, \tilde{u}_0)$, for any $x \in \mathbb{R}^d$ we have

$$\begin{aligned}|u(x,t) - \tilde{u}(x,t)|_Z &\le |u_0(x) - \tilde{u}_0(x)|_Z + \int_0^t |F(t', u(x,t')) - F(t', \tilde{u}(x,t'))|_Z dt' \\ &\le |u_0(x) - \tilde{u}_0(x)|_Z + L\int_0^t |u(x,t') - \tilde{u}(x,t')|_Z dt'.\end{aligned}$$

From Gronwall's lemma, we obtain the inequality $|u(x,t) - \tilde{u}(x,t)|_Z \le e^{Lt}|u_0(x) - \tilde{u}_0(x)|_Z$. Given $\varepsilon > 0$, there exists $r > 0$ such that $|u_0(x) - \tilde{u}_0(x)|_Z < \varepsilon e^{-Lt}$ for $|x| > r$, then $|u(x,t) - \tilde{u}(x,t)|_Z < \varepsilon$, which implies $u(t) - \tilde{u}(t) \in C_0(\mathbb{R}^d, Z)$. □

For the next proposition, we recall results from [6], based in numerical splitting techniques [7, 13] for evolution equations. These are used to prove the convergence of the approximate solution, that is constructed by the time-splitting of the linear and the non-linear component.

Proposition 3.8. *Let $u_0, \tilde{u}_0 \in C_u(\mathbb{R}^d, Z)$, such that $u_0 - \tilde{u}_0 \in C_0(\mathbb{R}^d, Z)$ and let $u, \tilde{u}$ be the corresponding solutions of (2.1). For any $0 \le t < \min\{T^*(u_0), T^*(\tilde{u}_0)\}$, it is verified $u(t) - \tilde{u}(t) \in C_0(\mathbb{R}^d, Z)$.*

Proof. For $t \in [0, \min\{T^*(u_0), T^*(\tilde{u}_0)\})$, let $n \in \mathbb{N}$, $h = t/n$ and $\{U_{h,k}\}_{0\le k\le n}$, $\{\tilde{U}_{h,k}\}_{0\le k\le n}$ sequences defined in terms of a recurrence, in the following way.

Let $\{U_{h,k}\}_{0\le k\le n}$, $\{V_{h,k}\}_{1\le k\le n}$ be the sequences given by $U_{h,0} = u_0$,

$$V_{h,k+1} = \mathsf{S}(h)U_{h,k}, \tag{3.1a}$$

$$U_{h,k+1} = \mathsf{N}(kh + h, kh + h/2, V_{h,k+1}), \quad k = 0, \dots, n-1. \tag{3.1b}$$

We claim that $U_{h,k} - \tilde{U}_{h,k} \in C_0(\mathbb{R}^d, Z)$ for $k = 0, \dots, n$. Clearly, the assertion is true for $k = 0$. If $U_{h,k-1} - \tilde{U}_{h,k-1} \in C_0(\mathbb{R}^d, Z)$, from Lemma 3.7, we have $\mathsf{N}(kh, kh - h/2, V_{h,k-1}) - \mathsf{N}(kh, kh - h/2, \tilde{V}_{h,k-1}) \in C_0(\mathbb{R}^d, Z)$. Using Lemma 3.6, we can see that

$$V_{h,k} - \tilde{V}_{h,k} = S(h)(\mathsf{N}(kh, kh - h/2, V_{h,k-1}) - \mathsf{N}(kh, kh - h/2, \tilde{V}_{h,k-1})) \in C_0(\mathbb{R}^d, Z).$$

We now recall Proposition 4.2 and Theorem 4.2 from [6] that assures us that $U_{h,n} \to u(t)$ and $\tilde{U}_{h,n} \to \tilde{u}(t)$ when $n \to \infty$.

As $C_0(\mathbb{R}^d, Z)$ is closed and $U_{h,n} - \tilde{U}_{h,n} \to u(t) - \tilde{u}(t)$, we obtain the result. □

In the following theorem, we prove the existence of solutions in $X_{\Gamma,Z}$, but also the *absorption* mentioned in the introduction concerning the evolution of the initial condition component in the space $C_0(\mathbb{R}^d, Z)$.

Theorem 3.9. *For any $u_0 \in X_{\Gamma,Z}$, the solution u of the equation* (2.1) *satisfies $u(t) \in X_{\Gamma,Z}$ for $0 \le t < T^*(u_0)$. Moreover, if $u_0 = v_0 + w_0$ with $v_0 \in C_u(\mathbb{R}^d/\Gamma, Z)$ and $w_0 \in C_0(\mathbb{R}^d, Z)$, then $u(t) = v(t) + w(t)$, where v is the solution of* (2.1) *with initial data v_0 and w is the solution of*

$$w(t) = \mathsf{S}(t)w_0 + \int_0^t \mathsf{S}(t-t')\left(F(t, v(t') + w(t')) - F(t, v(t'))\right) dt'.$$

Proof. As $u_0 \in X_{\Gamma,Z} \subset C_u(\mathbb{R}^d, Z)$, by Theorem 2.4 we have $u(t) \in C_u(\mathbb{R}^d, Z)$ with maximal time of existence $T^*(u_0)$. We observe that as $v_0 \in C_u(\mathbb{R}^d/\Gamma, Z)$ then by Proposition 3.5 we know that $v(t) \in C_u(\mathbb{R}^d/\Gamma, Z)$ with maximal time of existence $T^*(v_0)$. We define $w(t) = u(t) - v(t)$. By hypothesis, we have $w_0 = w(0) = u(0) - v(0) = u_0 - v_0 \in C_0(\mathbb{R}^d, Z)$ therefore, by Proposition 3.8 we know that $w(t) \in C_0(\mathbb{R}^d, Z)$. Then, we obtain $u(t) = v(t) + w(t) \in X_{\Gamma,Z}$, where $v(t) \in C_u(\mathbb{R}^d/\Gamma, Z)$ and $w(t) \in C_0(\mathbb{R}^d, Z)$ in the interval $[0, T_{min})$ where $T_{min} = \min\{T(u_0), T(v_0)\}$. For $T^*(v_0) \ge T^*(u_0)$, we have the result.

Suppose that $T^*(v_0) < T^*(u_0)$.

Let $T \in (0, T^*(u_0))$ and $M = \max_{0 \le t \le T} \|u(t)\|_{\infty,Z}$. We define $\mathcal{T} = \{t \in [0, T] : u(t) \notin X_{\Gamma,Z}\}$, that is, the times for which we have $u(t) \notin X_{\Gamma,Z}$. Suppose that $\mathcal{T} \neq \varnothing$. Then there exists $t_1 = \inf \mathcal{T}$.

Clearly, $t_1 = 0$ is not possible because we have already seen that $u(t) \in X_{\Gamma,Z}$, in the interval $[0, T^*(v_0))$. In the same way, if $t_1 > 0$ and additionally $t_1 < T^*(v_0)$ we have $u(t) \in X_{\Gamma,Z}$ and that is a contradiction. We analyze the remaining case, $t_1 > 0$ and $T > t_1 > T^*(v_0)$.

We observe that, by Theorem 2.4 we obtain that $\lim_{t \to T^*(v_0)} \|v(t)\|_{\infty,Z} = +\infty$ but on the other hand, by Lemma 3.3 we have $\|v(t)\|_{\infty,Z} \le \|P\|_{\infty,Z} \|u(t)\|_{\infty,Z} \le \|P\|_{\infty,Z} M$ that is, the norm $v(t)$ is bounded for $t \in [0, T^*(v_0)) \subset [0, T]$, which is a contradiction.

So we finally have that $u(t) \in X_{\Gamma,Z}$ for $t \in [0, T^*(u_0))$. □

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