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Homogeneous Herz spaces with variable exponents and regularity results

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Abstract. In this paper we deal with the second order divergence form operators $\mathcal{L}$ with coefficients satisfying the vanishing mean oscillation property and we prove some regularity results for a solution to $\mathcal{L}u = \operatorname{div} f$, where f belongs to homogeneous Herz spaces with variable exponents $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}$.

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1 Introduction

Throughout the paper let us assume that Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, with a sufficiently smooth boundary, namely $\partial\Omega \in C^{1,1}$, and let us also consider the divergence form elliptic equation

$$\mathcal{L}u := \sum_{i,j=1}^{n} (a_{ij} u_{x_i})_{x_j} = \operatorname{div} f, \quad \text{a.e. in } \Omega. \tag{1.1}$$

Problems related to divergence form elliptic equations have a long history. The first studies deal with the following problem:

$$\begin{cases} Lu = -\operatorname{div} A\nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω, as before, is a bounded open subset of $\mathbb{R}^n$ and $A = A(x) = (a_{ij}(x))$ is a $n \times n$ matrix of real-valued, measurable functions that satisfies the ellipticity condition

$$\lambda|\xi|^2 \leq \langle A\xi, \xi\rangle \leq \Lambda|\xi|^2, \qquad 0 < \lambda < \Lambda, \quad \xi \in \mathbb{R}^n.$$

If A is continuous and $\partial\Omega \in C^{2,\alpha}$ then, the classical L^p theory is treated in Gilbarg and Trudinger [12]. Miranda [18] showed that if $n \geq 3$, $\partial\Omega \in C^3$ and $A \in W^{1,n}(\Omega)$, then any weak solution of $Lu = F$, $f \in L^q(\Omega)$, $q \geq 2$, is a strong solution and

$$\|D^2 u\|_{L^2(\Omega)} \leq C(\|f\|_{L^q(\Omega)} + \|u\|_{L^1(\Omega)}).$$

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In the context of non-divergence form elliptic operators, a similar problem was considered by Chiarenza and Franciosi [5]. They produced that if $n \geq 3$, Ω is bounded and $\partial\Omega \in C^2$, then the non-divergence form equation $\mathrm{tr}\,(AD^2u) = f$, with $f \in L^2(\Omega)$ and A in a suitable vanishing Morrey space, has a unique solution u satisfying $\|u\|_{W^{2,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}$. This result was generalized by Chiarenza, Frasca and Longo [7], who showed that if $f \in L^p(\Omega)$, $1 < p < \infty$, then the same equation has a unique solution satisfying $\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}$. These results were further generalized by Vitanza [25,26].

Divergence form equations of the form $\mathrm{div}\, A\nabla u = \mathrm{div}\, F$ were considered by Di Fazio [11] who used the methods in [6,7]. In [11] the author obtained some regularity results in the framework of L^p spaces in the case $a_{ij} \in VMO$ (see Section 3 for definitions). Furthermore, Ragusa in [20,21] extended the results by Di Fazio studying the interior $L^{p,\lambda}$-regularity under the same assumptions on the coefficients. As a consequence of the $L^{p,\lambda}$-theory, Ragusa obtained some $C^{(0,\alpha)}$-regularity properties for a solution of the Dirichlet problem associated to a divergence form elliptic equation.

We say that a function $u \in W^1\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}$ is a solution of (1.1) if

$$\int_\Omega \sum_{i,j=1}^n a_{ij}u_{x_i}\chi_{x_j}\,\mathrm{d}x = -\int_\Omega \sum_{i=1}^n f_i\chi_{x_i}\,\mathrm{d}x, \qquad \forall\chi \in C_0^\infty(\Omega).$$

In the last years there has been an increasing interest in the study of functional spaces with variable exponents; many authors deal with the boundedness of integral operators in such spaces and this speculation is of independent interest. However, it is also interesting the applications of the boundedness properties of singular integral operators to the new regularity theory of partial differential equations, possibly with discontinuous coefficients.

This scientific note is a first step in the study of regularity properties of solutions to divergence form elliptic equations with discontinuous coefficients in the context of homogeneous Herz spaces with two variable exponents.

Precisely, the goal of this paper is to prove that a solution of (1.1) satisfies some regularity properties, being $f = (f_1, \ldots, f_n)$ such that, for every $i = 1, \ldots, n$, f_i belongs to the homogeneous Herz space $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}$ for suitable constant α and functions p, q and the coefficients a_{ij} belonging to $VMO \cap L^\infty(\Omega)$.

The functional class where the coefficients of the principal part belong is defined in the classical paper [24] by Sarason as a proper subspace of the John–Nirenberg space BMO [13] whose BMO over a ball vanishes as the radius of the ball goes to zero.

It is worth pointing out that preparatory to the study of the desired regularity properties is the boundedness of singular integral operators with Calderón–Zygmund kernels and their commutators.

In order to prove our regularity results we need similar results in the framework of Herz spaces with variable exponents and use the technique adopted in [6,7].

In the next section we collect some definitions on Lebesgue spaces with variable exponent and homogeneous Herz spaces with two variable exponents. In Section 3 we give a brief exposition of two fundamental assumptions on the coefficients of the differential operator under consideration. Namely, we introduce the John–Nirenberg class of function with bounded mean oscillation and the Sarason class of functions with vanishing mean oscillation. In Section 4 we show some technical tools concerning the boundedness of fractional integral operators and commutators having variable kernels in the framework of variable exponent

Herz spaces. In Section 5 we prove the regularity, in variable exponent Herz spaces, for the first order derivatives of the solutions to elliptic equations in divergence form.

2 Homogeneous Herz spaces with variable exponent

Let Ω be a measurable set in $\mathbb{R}^n$. We firstly recall the definition of the Lebesgue spaces with variable exponent. For a deeper discussion of Lebesgue spaces with variable exponent we refer the reader to [9]. We recall [15, 16, 22, 23] for recent developments and applications of nonstandard functional classes.

Definition 2.1. Let $p(\cdot) : \Omega \to [1, \infty[$ be a measurable function. Let us set the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as follows

$$L^{p(\cdot)}(\Omega) = \left\{ f : \Omega \to \mathbb{R}^n : f \text{ is measurable and } \int_\Omega |f(x)|^{p(x)} dx < +\infty \text{ for some constant } \eta > 0 \right\}.$$

and the space $L^{p(\cdot)}_{\text{loc}}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(\Omega) = \left\{ f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset \Omega \right\}.$$

The Lebesgue spaces $L^{p(\cdot)}(\Omega)$ is a Banach space respect to the Luxemburg–Nakano norm defined by

$$\|f\|_{L^{p(x)}(\Omega)} = \inf \left\{ \eta > 0 : \int_\Omega \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx \leq 1 \right\}.$$

We would like to point out that if the function $p(x) = p_0$ is a constant function, then $L^{p(\cdot)}(\mathbb{R}^n)$ is $L^{p_0}(\mathbb{R}^n)$. This implies that the Lebesgue spaces with variable exponent generalize the usual Lebesgue spaces. Moreover, we observe that $L^{p(\cdot)}(\mathbb{R}^n)$ have many properties in common with the classical Lebesgue spaces.

Throughout this paper we set

$$p_- = \operatorname{ess\,inf} \{p(x) : x \in \Omega\}, \qquad p_+ = \operatorname{ess\,sup} \{p(x) : x \in \Omega\}$$

and denote by $\mathcal{P}(\Omega)$ the set of all measurable functions $p(x)$ satisfying $p_- > 1$ and $p_+ < \infty$, $\mathcal{P}_0(\Omega)$ the set of all measurable functions $p(\cdot)$ such that $p_- > 0$ and $p_+ < \infty$.

Remark 2.2. Given a function $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, let us set the space $L^{p(\cdot)}(\mathbb{R}^n)$ as

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f \;\middle|\; |f|^{p_0} \in L^{q(\cdot)}(\mathbb{R}^n) \text{ for some } p_0 \;:\; 0 < p_0 < p_- \text{ and } q(x) = \frac{p(x)}{p_0} \right\}$$

and we consider the following quasinorm on this space

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \| \, |f|^{p_0} \, \|^{1/p_0}_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Let us recall the Hardy–Littlewood maximal operator

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy,$$

where B ranges in the class of the spheres of $\mathbb{R}^n$. Let us denote by $\mathcal{B}(\Omega)$ the set of all functions $p(\cdot) \in \mathcal{P}(\Omega)$ such that M is bounded on $L^{p(\cdot)}(\Omega)$.

Definition 2.3. Let us consider $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$. The mixed Lebesgue sequence space with variable exponent $\ell^{q(\cdot)}(L^{p(\cdot)})$ is the set of all sequences $\{f_j\}_{j\in\mathbb{N}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|\{f_j\}_{j\in\mathbb{N}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{\eta > 0 : Q_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{\frac{f_j}{\zeta}\right\}_{j\in\mathbb{N}}\right) \le 1\right\} < \infty,$$

where

$$Q_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j\in\mathbb{N}}) = \sum_{j=0}^{\infty} \inf\left\{\zeta_j > 0 : \int_{\mathbb{R}^n}\left(\frac{|f_j(x)|}{\zeta_j^{\frac{1}{q(x)}}}\right)^{p(x)} \mathrm{d}x \le 1\right\}.$$

We observe that for $q_+ < \infty$, we obtain

$$Q_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j\in\mathbb{N}}) = \sum_{j=0}^{\infty} \||f_j|^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $C_k = B_k \setminus B_{k-1}$, $\chi_k := \chi_{C_k}$, $k \in \mathbb{Z}$.

Definition 2.4. Let $\alpha \in \mathbb{R}^n$, $q(\cdot), p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}$ is defined as follows:

$$\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)} = \left\{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}} < \infty\right\},$$

where

$$\begin{aligned}\|f\|_{\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}} &= \|\{2^{k\alpha}|f\chi_k|\}_{k\in\mathbb{N}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &= \inf\left\{\eta > 0 : \sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \le 1\right\}.\end{aligned}$$

It is easy to see that $\dot{K}^{0,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$.

In the sequel we use the following result.

Lemma 2.5 ([10]). *Let $p_h \in \mathcal{B}(\mathbb{R}^n)$ for $h = 1, 2$, then there exist constants $0 < t_{h1}, t_{h2} < 1$ and $C > 0$ such that for all balls $B \subset \mathbb{R}^n$ and all measurable subset $R \subset B$,*

$$\frac{\|\chi_R\|_{L^{p_h(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_h(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|R|}{|B|}\right)^{t_{h1}}, \qquad \frac{\|\chi_R\|_{L^{p'_h(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'_h(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|R|}{|B|}\right)^{t_{h2}},$$

where with $p'_h(\cdot)$ is the conjugate exponent function defined as

$$\frac{1}{p_h(x)} + \frac{1}{p'_h(x)} = 1, \qquad x \in \mathbb{R}^n.$$

3 Calderón–Zygmund operators, *BMO* and *VMO* spaces

In the sequel we make use of Calderón–Zygmund operators and their commutators (see e.g. [3,4]).

Definition 3.1. Let T be a bounded linear operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. We say that T is a standard operator if it satisfies the following conditions:

- T extends to a bounded linear operator on $L^2(\mathbb{R}^n)$,
- there exists a function $K(x,y)$ defined on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ such that

$$|K(x,y)| \leq \frac{C}{|x-y|^n},$$

  where $C > 0$,
- $\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)f(y)g(x)\,\mathrm{d}x\,\mathrm{d}y$, for $f, g \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \varnothing$.

An operator T is called a γ-Calderón–Zygmund operator if K is a kernel satisfying

$$|K(x,y) - K(z,y)| \leq C\frac{|x-z|^\gamma}{|x-y|^{n+\gamma}},$$

$$|K(y,x) - K(y,z)| \leq C\frac{|x-z|^\gamma}{|x-y|^{n+\gamma}},$$

if $|x-z| < \frac{1}{2}|x-y|$ for some $\gamma \in]0,1]$.

The commutator of the Calderón–Zygmund operator is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

In 1983, Journé proved that a γ-Calderón–Zygmund operator is bounded on $L^p(\mathbb{R}^n)$ (see [14]). Coifman, Rochberg and Weiss in [8] proved that the commutator $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in]0,1[$.

Kováčik and J. Rákosník in [17] introduced Lebesgue spaces and Sobolev spaces with variable exponent. For a recent treatment of Lebesgue spaces with variable exponent, we refer the reader to [9].

In the last decades, there was an increasing interest in the study of functional spaces having variable exponent thanks to the wide variety of applications, for instance, in fluid dynamics and differential equations.

In particular, Herz spaces play an important role in harmonic analysis. In this paper, we apply to the theory of regularity of solutions to partial differential equations the main results contained in [2] where the authors deal with the boundedness of Calderón–Zygmund operator and their commutator on Herz spaces with two variable exponents $p(\cdot)$, $q(\cdot)$.

In order to develop a satisfactory theory of regularity of solutions to linear elliptic differential equations, following the pioneering scientific note [6], we assume that the coefficients of the differential operators under consideration belong to the Sarason class of functions having vanishing mean oscillation. According to this requirement on the coefficients, we point out that the coefficients could be discontinuous.

First of all, we recall the definition of *BMO* space, due to John and Nirenberg (see [13]).

Definition 3.2. We define the space $BMO(\mathbb{R}^n)$ of functions having bounded mean oscillation as

$$BMO(\mathbb{R}^n) = \{b \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|b\|_* < \infty\},$$

where

$$\|b\|_* = \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |b(x) - b_B| \, \mathrm{d}x,$$

where B ranges in the class of the balls in $\mathbb{R}^n$ and b_B stands for the integral average of the function b over the sphere B.

Let us consider in the next definition a proper subset of the space BMO, studied by Sarason (see [24]).

Definition 3.3. We define the space $VMO(\mathbb{R}^n)$ of functions having vanishing mean oscillation as

$$VMO(\mathbb{R}^n) = \left\{ b \in BMO(\mathbb{R}^n) : \lim_{r \to 0^+} \gamma_b(r) = 0 \right\},$$

where

$$\gamma_b(r) = \sup_{\rho \le r} \frac{1}{|B_\rho|} \int_{B_\rho} |b(x) - b_{B_\rho}| \, \mathrm{d}x$$

and B_ρ varies in the class of ball in $\mathbb{R}^n$ having radius ρ. We say γ_b the VMO-modulus of the function b.

In a similar way, we can define the spaces $BMO(\Omega)$ and $VMO(\Omega)$ of functions defined on a domain $\Omega \subset \mathbb{R}^n$, replacing B and B_ρ by the intersections of the respective balls with Ω.

It is worth pointing out that using the classical Poincaré inequality, it follows that $W^{1,n}(\mathbb{R}^n) \subset VMO$ and, further on, $W^{\theta,n/\theta}(\mathbb{R}^n) \subset VMO$ for $0 < \theta < 1$ as shows the function $f_\alpha(x) = |\log|x||^\alpha$ for $0 < \alpha < 1$. Straightforward calculations yield that $f_\alpha \in VMO$ for every $\alpha \in (0,1)$, $f_\alpha \in W^{1,n}$ for $\alpha \in \left(0, 1 - \frac{1}{n}\right)$, while $f_\alpha \notin W^{1,n}$ for $\alpha \in \left[1 - \frac{1}{n}, 1\right)$.

4 Fractional integral operators

In this section we state some useful results concerning the Calderón–Zygmund integral operators on homogeneous Herz spaces with variable exponents. For the proofs we refer the reader to [2].

Theorem 4.1. *Suppose that $p_1 \in \mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \ge (q_1)_+$. If $-nt_{12} < \alpha < nt_{11}$, with t_{11}, t_{12} as in Lemma 2.5, then the operator T is bounded from $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$.*

Theorem 4.2. *Let $b \in BMO(\mathbb{R}^n)$. Suppose that $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \ge (q_1)_+$. If $-nt_{12} < \alpha < nt_{11}$, with t_{11}, t_{12} as in Lemma 2.5, then the commutator $[b, T]$ is bounded from $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$.*

In the sequel we use the above theorems with $q_1 = q_2$.

Next, we recall from [1] a useful result dealing with the boundedness of a particular fractional integral operator that play an important role in the forthcoming study of the regularity properties for solutions to equation (1.1).

In [1] the authors study several boundedness properties of fractional integral operators having variable kernel and their commutators in the framework of variable exponent Herz spaces. In the sequel let us denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$.

Following [1], let $0 < \mu < n$ and let $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(\mathbb{S}^{n-1})$, $r \geq 1$, be homogeneous of degree zero on $\mathbb{R}^n$. If

1. for any $x, z \in \mathbb{R}^n$, we have $\Omega(x, \lambda z) = \Omega(x, z)$,
2. is finite the norm

$$\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(\mathbb{S}^{n-1})} := \sup_{x\in\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |\Omega(x, z')|^r \, d\sigma(z') \right)^{\frac{1}{r}},$$

we define the fractional integral operator with variable kernel $T_{\Omega,\mu}$ by

$$T_{\Omega,\mu} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} f(y) \, dy.$$

In [1] the reader find a general boundedness result for $T_{\Omega,\mu}$.

In the sequel we use, in particular, the integral operator above with $\Omega \equiv 1$ and $\mu = 1$, then we consider

$$T_{1,1} f(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}} \, dy.$$

Theorem 4.3. *Let $1 - nt_{11} < \alpha < nt_{12}$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $(q_2)_- \geq (q_1)_+$. Let $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ such that $(p_1)_+ \leq n$ and define the variable exponent $p_2(\cdot)$ by $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{1}{n}$. Then, the operator $T_{1,1}$ is bounded from $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$.*

5 Regularity for solutions to partial differential equations

In this section we are concerned with the divergence form elliptic equation

$$\mathcal{L}u := \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j} = \operatorname{div} f(x) \tag{5.1}$$

in a bounded open set $\Omega \subset \mathbb{R}^n$ ($n \geq 3$), where:

(H1) $f = (f_1, f_2, \dots, f_n) \in \dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}$;

(H2) $a_{ij} \in L^\infty(\Omega) \cap VMO$, for every $i, j = 1, \dots, n$;

(H3) $a_{ij}(x) = a_{ji}(x)$ for every $i, j = 1, \dots, n$ and for a.a. $x \in \Omega$;

(H4) $\exists \sigma > 0 : \sigma^{-1}|\lambda|^2 \leq a_{ij}(x)\lambda_i\lambda_j \leq \sigma|\lambda|^2$, for every $\lambda \in \mathbb{R}^n$ and a.e. $x \in \Omega$.

We say that a function $u \in W^1\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}$ is a solution of (5.1) if

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij} u_{x_i} \chi_{x_j} \, dx = -\int_\Omega \sum_{i=1}^{n} f_i \chi_{x_i} \, dx, \qquad \forall \chi \in C_0^\infty(\Omega).$$

We set

$$\Gamma(x,t) = \frac{1}{n(2-n)\omega_n \sqrt{\det\{a_{ij}(x)\}}} \left(\sum_{i,j=1}^{n} A_{ij}(x) t_i t_j \right)^{\frac{2-n}{2}},$$

$$\Gamma_i(x,t) = \frac{\partial \Gamma(x,t)}{\partial t_i}, \qquad \Gamma_{ij}(x,t) = \frac{\partial^2 \Gamma(x,t)}{\partial t_i \partial t_j},$$

$$M = \max_{i,j=1,\dots,n} \max_{|\alpha|\le 2n} \left\| \frac{\partial^\alpha \Gamma_{ij}(x,t)}{\partial t^\alpha} \right\|_{L^\infty(\Omega\times\Sigma)}$$

for a.a. $x \in \Omega$ and for every $t \in \mathbb{R}^n \setminus \{0\}$ where A_{ij} stand for the entries of the inverse matrix of the matrix $\{a_{ij}(x)\}_{i,j=1,\dots,n}$, and ω_n is the measure of the unit ball in $\mathbb{R}^n$.

It is well known that $\Gamma_{ij}(x,t)$ are Calderón–Zygmund kernels in the t variable.

Let $r, R \in \mathbb{R}^+$, $r < R$, and $\phi \in C_0^\infty(\mathbb{R})$ be a standard cut-off function such that for every $B_R \subset \Omega$,

$$\phi(x) = 1 \quad \text{in } B_r, \qquad \phi(x) = 0 \quad \forall x \notin B_R.$$

Then, if u is a solution of (1.1) and $v = \phi u$ we have

$$L(v) = \operatorname{div} G + g$$

where we set

$$G = \phi f + u A \nabla \phi$$
$$g = \langle A\nabla u, \nabla\phi\rangle - \langle f, \nabla \phi\rangle.$$

Let us make use of the integral representation formula for the first derivatives of a solution of (5.1), proved in [19].

Lemma 5.1. *Let, for every $i = 1,\dots,n$, $a_{ij} \in L^\infty \cap VMO(\mathbb{R}^n)$ satisfy* (H3), (H4), *let u be a solution of* (1.1) *and let ϕ, g and G be defined as above.*

Then, for every $i = 1,\dots,n$ we have

$$(\phi u)_{x_i}(x) = \sum_{h,j=1}^{n} P.V. \int_{B_R} \Gamma_{ij}(x, x-y)\{(a_{jh}(x) - a_{jh}(y))(\phi u)_{x_h}(y) - G_j(y)\}\, \mathrm{d}y$$
$$- \int_{B_R} \Gamma_i(x, x-y) g(y)\, \mathrm{d}y + \sum_{h=1}^{n} c_{ih}(x) G_h(x), \qquad \forall x \in B_R,$$

setting $c_{ih} = \int_{|t|=1} \Gamma_i(x,t) t_h \, \mathrm{d}\sigma_t$.

We are ready to prove our main result.

Theorem 5.2. *Let $\max\{-nt_{12}, 1-nt_{11}\} < \alpha < \min\{nt_{12}, nt_{11}\}$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $(q_2)_- \ge (q_1)_+$. Let $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ such that $(p_1)_+ \le n$ and define the variable exponent $p_2(\cdot)$ by $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{1}{n}$. Let u be a solution of* (1.1) *and let us assume that conditions* (H1)–(H4) *hold. Then, for every compact set $E \subset \Omega$, there exists a positive constant c depending on n, p, q_1, q_2,* $\operatorname{dist}(K, \partial\Omega)$ *such that*

$$\|\partial_{x_i} u\|_{\dot K^{\alpha, q_2(\cdot)}_{p(\cdot)}(E)} \le c \left(\|f\|_{\dot K^{\alpha, q_1(\cdot)}_{p(\cdot)}(E)} + \|u\|_{\dot K^{\alpha, q_1(\cdot)}_{p(\cdot)}(E)} + \|\partial_{x_i} u\|_{\dot K^{\alpha, q_1(\cdot)}_{p(\cdot)}(E)} \right), \qquad \forall i = 1,\dots,n.$$

Proof. Let $E \subset \Omega$ be a compact set. Using the representation formula and the boundedness results, we gain

$$\begin{aligned}\|\partial_{x_h}(\phi u)\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(E)} &\leq \|C[a_{ij},\phi]\partial_{x_h}(u\phi)\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(E)} + \|KG\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(E)} \\ &\quad + \|T_{11}g\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(E)} + \|G\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(E)} \\ &\leq c\|a\|_*\|\partial_{x_h}(u\phi)\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(E)} + \|G\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(E)} + \|g\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(E)} \\ &\quad + \|G\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(E)},\end{aligned} \tag{5.2}$$

where the norm $\|a\|_*$ is taken in the set B_R.

From the hypothesis $(q_2)_- \geq (q_1)_+$, we get $\frac{q_2(\cdot)}{q_1(\cdot)} \in \mathcal{P}(\mathbb{R}^n)$ and $\frac{q_2(\cdot)}{q_1(\cdot)} \geq 1$. Then, for any $f \in \dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}$, we get

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\leq \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q_1(\cdot)} \right\|^{p_k}_{L^{\frac{p(\cdot)}{q_1(\cdot)}}} \\ &\leq \left\{ \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q_1(\cdot)} \right\|^{p_k}_{L^{\frac{p(\cdot)}{q_1(\cdot)}}} \right\}^{p_*} \leq 1,\end{aligned}$$

where

$$p_k = \begin{cases} \left(\frac{q_2(\cdot)}{q_1(\cdot)}\right)_-, & \frac{2^{k\alpha}|f\chi_k|}{\eta} \leq 1, \\ \left(\frac{q_2(\cdot)}{q_1(\cdot)}\right)_+, & \frac{2^{k\alpha}|f\chi_k|}{\eta} > 1, \end{cases} \qquad p_* = \begin{cases} \min_{k\in\mathbb{N}} p_k, & \sum_{k=0}^{\infty} a_k \leq 1, \\ \min_{k\in\mathbb{N}} p_k, & \sum_{k=0}^{\infty} a_k > 1. \end{cases}$$

This implies that $\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)} \subset \dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}$. From this embedding we obtain a refinement of the inequality (5.2) Taking into account that $a \in VMO$, we can choose the radius R of the ball B_R such that $c\|a\|_* < \frac{1}{2}$. This remark allows us to write

$$\begin{aligned}&\|\partial_{x_h}(\phi u)\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(E)} \\ &\quad\leq c\|a\|_*\|\partial_{x_h}(u\phi)\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(E)} + \|G\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(E)} + \|g\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(E)} \\ &\quad= c\|a\|_*\|\partial_{x_h}(u\phi)\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(E)} + \|\phi f + uA\nabla\phi\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(E)} + \|\langle A\nabla u, \nabla\phi\rangle - \langle f, \nabla\phi\rangle\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(E)} \\ &\quad\leq c\Big(\|\partial_{x_h}(u\phi)\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(E)} + \|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(E)} + \|u\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(E)} + \|\partial_{x_i}u\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(E)} + \|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(E)}\Big).\end{aligned}$$

From the last inequality, we easily obtain the desired estimate. □

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