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Existence of symmetric positive solutions for a singular system with coupled integral boundary conditions

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Abstract. In this paper, we study a class of nonlinear singular system with coupled integral boundary condition. Based on the Guo–Krasnosel'skii fixed point theorem, some new results on the existence of symmetric positive solutions for the coupled singular system are obtained. The impact of the two different parameters on the existence of symmetric positive solutions is also investigated. Finally, an example is then given to demonstrate the applicability of our results.

Keywords: coupled singular system, symmetric positive solutions, fixed point theorem in cones, coupled integral boundary conditions.

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1 Introduction

This paper is concerned with the existence of symmetric positive solutions for the following singular fourth-order boundary value system with coupled integral boundary conditions (BCs)

$$\begin{cases} (\phi_p(u''(t)))'' = \lambda_1 a_1(t) f_1(t, u(t), v(t)), \quad -1 < t < 1, \\ (\phi_p(v''(t)))'' = \lambda_2 a_2(t) f_2(t, u(t), v(t)), \quad -1 < t < 1, \\ u(-1) = u(1) = \displaystyle\int_{-1}^{1} v(s)dA_1(s), \ v(-1) = v(1) = \int_{-1}^{1} u(s)dA_2(s), \\ \phi_p(u''(-1)) = \phi_p(u''(1)) = \displaystyle\int_{-1}^{1} \phi_p(v''(s))dB_1(s), \\ \phi_p(v''(-1)) = \phi_p(v''(1)) = \displaystyle\int_{-1}^{1} \phi_p(u''(s))dB_2(s), \end{cases} \tag{1.1}$$

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where λ_1 and λ_2 are positive parameters, $\phi_p(x) = |x|^{p-2}x, p > 1, \phi_q = \phi_p^{-1}, \frac{1}{p} + \frac{1}{q} = 1$, $f_1, f_2 : [-1,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and $f_1(\cdot, x, y), f_2(\cdot, x, y)$ are symmetric on $[-1,1]$ for all $x, y \in [0,\infty)$, $a_1, a_2 : (-1,1) \to [0,\infty)$ are symmetric on $(-1,1)$ and may be singular at $t = -1$ and/or $t = 1$, and the integrals from (1.1) are Riemann–Stieltjes integrals with a signed measure, that is, A_i, B_i $(i = 1,2)$ are functions of boundary variation on $[-1,1]$. By a symmetric positive solution of the system (1.1), we mean a pair of functions $(u,v) \in (C^2[-1,1] \cap C^4(-1,1)) \times (C^2[-1,1] \cap C^4(-1,1))$ satisfying (1.1), u, v are symmetric and $u(t) > 0$, $v(t) > 0$ for all $t \in [-1,1]$.

Coupled BCs arise from the study of reaction–diffusion equations, Sturm–Liouville problems, mathematical biology and so on. In [1], Asif and Khan studied the following a coupled singular system subject to four-point coupled BCs of the type

$$\begin{cases} -x''(t) = f(t, x(t), y(t)), & t \in (0,1), \\ -y''(t) = g(t, x(t), y(t)), & t \in (0,1), \\ x(0) = 0, \quad x(1) = \alpha y(\xi), \\ y(0) = 0, \quad y(1) = \beta x(\eta), \end{cases} \tag{1.2}$$

where the parameters α, β, ξ, η satisfy $\xi, \eta \in (0,1)$ and $0 < \alpha\beta\xi\eta < 1$, $f, g : (0,1) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and singular at $t = 0$, $t = 1$. The authors obtained at least one positive solution to the system (1.2) by using the Guo–Krasnosel'skii fixed-point theorem. For other recent results concerning the Coupled BCs, we refer the reader to [2, 5, 6, 8, 15].

We notice that a type of symmetric problem has received much attention, for instance, [3, 7, 9, 11–14, 16] and the references therein. At the same time, a class of boundary value problems with integral BCs appeared in heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics. For earlier contributions on problems with Lebesgue integral BCs, we refer the reader to [3, 13, 14, 16] and the more general nonlocal Riemann-Stieltjes integral BCs, we refer the reader to [2, 5, 6, 10] and references therein, such integral BCs are a general type of nonlocal boundary conditions and cover multi-point and integral BCs as special cases. Infante, Minhós, Pietramala [5] gave a general method for dealing with these problems in the important case when $p = 2$. Ma [14] studied the existence of a symmetric positive solution for the following singular fourth-order nonlocal boundary value problem

$$\begin{cases} u''''(t) = h(t) f(t, u(t)), \ \ 0 < t < 1, \\ u(0) = u(1) = \int_0^1 p(s) u(s) ds, \\ u''(0) = u''(1) = \int_0^1 q(s) u''(s) ds, \end{cases}$$

where $p, q \in L^1[0,1]$, $h : (0,1) \to [0,+\infty)$ is continuous, symmetric on $(0,1)$ and may be singular at $t = 0$ and $t = 1$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $f(\cdot, x)$ is symmetric on $[0,1]$ for all $x \in [0,+\infty)$. The existence of at least one symmetric positive solution was obtained by the application of the fixed point index in cones.

In [16], Zhang, Feng, Ge studied fourth-order boundary value problem with integral BCs:

$$\begin{cases} (\phi_p(u''(t)))'' = \omega(t) f(t, u(t)), \ \ 0 < t < 1, \\ u(0) = u(1) = \int_0^1 g(s) u(s) ds, \\ \phi_p(u''(0)) = \phi_p(u''(1)) = \int_0^1 h(s) \phi_p(u''(s)) ds, \end{cases}$$

where $\phi_p(t) = |t|^{p-2}t, p > 1, \phi_q = \phi_p^{-1}, \frac{1}{p} + \frac{1}{q} = 1, \omega, g, h \in L^1[0,1]$ are nonnegative, symmetric on $[0,1]$. $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, $f(1-t,x) = f(t,x)$ for all $(t,x) \in [0,1] \times [0,\infty)$. By using of fixed point theorem in cones, the existence and multiplicity of symmetric positive solutions were obtained, and the nonexistence of a positive solution was also studied.

Inspired and motivated by the above mentioned work and wide applications of coupled BCs in various fields of sciences and engineering, we study the existence of symmetric positive solutions to a singular system (1.1). Of necessity $u'(0) = 0$, $v'(0) = 0$, $u'''(0) = 0$ and $v'''(0) = 0$ for symmetric functions $(u,v) \in (C^2[-1,1] \cap C^4(-1,1)) \times (C^2[-1,1] \cap C^4(-1,1))$, so the problem can be handled by considering the simpler problem (2.1) on $[0,1]$, then using symmetry $u(-t) = u(t)$, $v(-t) = v(t)$ to extend the solution to $[-1,1]$.

Our work presented in this paper has the following new features. First of all, we discuss the system (1.1) subject to coupled BCs with p-Laplacian operators, Riemann–Stieltjes integral BCs are a general type of nonlocal boundary conditions and cover multi-point and integral BCs as special cases, these are different from [3,7,11–14,16]. The second new feature is that the system (1.1) possesses singularity, that is, the nonlinear terms may be singular at $t = -1, 1$. Thirdly, we involve the parameter λ_i $(i = 1,2)$ in the model and obtain the sufficient conditions for the existence of symmetric positive solutions of system (1.1) within certain interval of λ_i $(i = 1,2)$. To the best knowledge of the authors, there is no earlier literature studying the existence of symmetric positive solutions for boundary value system with coupled integral BCs.

The rest of the paper is organized as follows. In Section 2, we present a positive cone, a fixed point theorem which will be used to prove existence of symmetric positive solutions, Green's function for the modified system and some related lemmas. In Section 3, we present main results of the paper and in Section 4 an example is given to illustrate the application of our main results.

2 Preliminaries and lemmas

We recall that the function ω is said to be concave on $[a,b]$ if

$$\omega(\tau t_1 + (1-\tau)t_2) \geq \tau\omega(t_1) + (1-\tau)\omega(t_2), \quad t_1, t_2 \in [a,b], \tau \in (0,1),$$

and the function ω is said to be symmetric on $[-1,1]$ if $\omega(-t) = \omega(t)$, $t \in [-1,1]$.

Remark 2.1. If $(u,v) \in (C^2[-1,1] \cap C^4(-1,1)) \times (C^2[-1,1] \cap C^4(-1,1))$ is a symmetric positive solution of the singular system (1.1), obviously, $u'(0) = 0$, $v'(0) = 0$, $u'''(0) = 0$ and $v'''(0) = 0$ are necessary. So the problem (1.1) can be handled by considering the following simpler problem

$$\begin{cases} (\phi_p(u''(t)))'' = \lambda_1 a_1(t) f_1(t,u(t),v(t)), \quad 0 < t < 1, \\ (\phi_p(v''(t)))'' = \lambda_2 a_2(t) f_2(t,u(t),v(t)), \quad 0 < t < 1, \\ u'(0) = 0,\ u(1) = \displaystyle\int_0^1 v(s)dA_1(s),\ v'(0) = 0,\ v(1) = \displaystyle\int_0^1 u(s)dA_2(s), \\ \phi_p(u'''(0)) = 0,\ \phi_p(u''(1)) = \displaystyle\int_0^1 \phi_p(v''(s))dB_1(s), \\ \phi_p(v'''(0)) = 0,\ \phi_p(v''(1)) = \displaystyle\int_0^1 \phi_p(u''(s))dB_2(s), \end{cases} \tag{2.1}$$

on $[0,1]$, then using symmetry $u(-t) = u(t)$ to extend the solution to $[-1,1]$. In view of the above, we will concentrate our study on the system (2.1).

The basic space used in this paper is $E = C[0,1] \times C[0,1]$. Obviously, the space E is a Banach space if it is endowed with the norm as follows:

$$\|(u,v)\|_1 := \max\{\|u\|, \|v\|\}, \quad \|u\| = \max_{0\le t\le 1} |u(t)|, \quad \|v\| = \max_{0\le t\le 1} |v(t)|$$

for any $(u,v) \in E$.

Set

$$G(t,s) = \begin{cases} 1-t, & 0 \le s \le t \le 1, \\ 1-s, & 0 \le t \le s \le 1, \end{cases}$$

$$\begin{aligned} \kappa_i &= \int_0^1 dA_i(t), \quad & \varrho_i &= \int_0^1 G(t,t)dA_i(t), \quad & \mathcal{G}_i(s) &= \int_0^1 G(t,s)dA_i(t), \quad & i &= 1,2, \\ \tilde{\kappa}_i &= \int_0^1 dB_i(t), \quad & \tilde{\varrho}_i &= \int_0^1 G(t,t)dB_i(t), \quad & \tilde{\mathcal{G}}_i(s) &= \int_0^1 G(t,s)dB_i(t), \quad & i &= 1,2. \end{aligned} \tag{2.2}$$

Obviously,

$$G(t,t)G(s,s) \le G(t,s) \le G(s,s) \text{ or } G(t,t), \quad \forall\, t,s \in [0,1]. \tag{2.3}$$

In the rest of the paper, we make the following assumptions:

($\mathbf{H}_1$) A_i, B_i $(i=1,2)$ are functions of boundary variation on $[-1,1]$, $\Delta_1 = 1 - \kappa_1\kappa_2$, and

$$\kappa_i > 0, \quad \tilde{\kappa}_i > 0, \quad 0 < \kappa_1\kappa_2 < 1, \quad 0 < \tilde{\kappa}_1\tilde{\kappa}_2 < 1, \quad i = 1,2.$$

($\mathbf{H}_2$) $a_1, a_2 : (-1,1) \to [0,\infty)$ are continuous, symmetric on $(-1,1)$ and

$$0 < \int_0^1 G(s,s)a_i(s)ds < \infty, \quad i = 1,2.$$

($\mathbf{H}_3$) $f_1, f_2 : [-1,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and $f_1(\cdot,x,y), f_2(\cdot,x,y)$ are symmetric on $[-1,1]$ for all $x,y \in [0,\infty)$, i.e., f_i satisfy $f_i(-t,x,y) = f_i(t,x,y)$ $(i=1,2)$.

Lemma 2.2. *Assume that (H_1) holds. Then for any $x,y \in L^1(0,1) \cap C(0,1)$, the system of BVPs consisting of the equations*

$$-u''(t) = \phi_p^{-1}(x(t)), \quad -v''(t) = \phi_p^{-1}(y(t)), \quad t \in (0,1) \tag{2.4}$$

and the BCs

$$u'(0) = 0,\ u(1) = \int_0^1 v(t)dA_1(t), \quad v'(0) = 0,\ v(1) = \int_0^1 u(t)dA_2(t) \tag{2.5}$$

has a unique integral representation

$$u(t) = \int_0^1 H_1(t,s)\phi_p^{-1}(x(s))ds + \int_0^1 K_1(s)\phi_p^{-1}(y(s))ds, \tag{2.6}$$

$$v(t) = \int_0^1 H_2(t,s)\phi_p^{-1}(y(s))ds + \int_0^1 K_2(s)\phi_p^{-1}(x(s))ds, \tag{2.7}$$

where

$$H_1(t,s) = G(t,s) + \frac{\kappa_1}{\Delta_1}\mathcal{G}_2(s), \quad K_1(s) = \frac{1}{\Delta_1}\mathcal{G}_1(s), \tag{2.8}$$

$$H_2(t,s) = G(t,s) + \frac{\kappa_2}{\Delta_1}\mathcal{G}_1(s), \quad K_2(s) = \frac{1}{\Delta_1}\mathcal{G}_2(s). \tag{2.9}$$

Proof. Let

$$u(t) = \int_0^1 G(t,s)\phi_p^{-1}(x(s))ds + c_1 + c_3(t-1), \tag{2.10}$$

$$v(t) = \int_0^1 G(t,s)\phi_p^{-1}(y(s))ds + c_2 + c_4(t-1), \tag{2.11}$$

where c_1, c_2, c_3 and c_4 are constants to be determined. Clearly, $u(t)$ and $v(t)$ satisfy (2.4). In the following, we determine c_i $(1 \le i \le 4)$ so that $u(t)$ and $v(t)$ satisfy (2.5). Substituting (2.10) and (2.11) into (2.5), we obtain $c_3 = c_4 = 0$ and

$$c_1 - \kappa_1 c_2 = \int_0^1 \mathcal{G}_1(s)\phi_p^{-1}(y(s))ds, \tag{2.12}$$

$$-\kappa_2 c_1 + c_2 = \int_0^1 \mathcal{G}_2(s)\phi_p^{-1}(x(s))ds. \tag{2.13}$$

Note that

$$\begin{vmatrix} 1 & -\kappa_1 \\ -\kappa_2 & 1 \end{vmatrix} = 1 - \kappa_1\kappa_2 = \Delta_1 \neq 0.$$

Thus, the system (2.12)–(2.13) has a unique solution for c_i $(1 \le i \le 2)$. By the Cramer's rule and simple calculations, it follows that

$$c_1 = \frac{1}{\Delta_1}\left[\int_0^1 \mathcal{G}_1(s)\phi_p^{-1}(y(s))ds + \kappa_1 \int_0^1 \mathcal{G}_2(s)\phi_p^{-1}(x(s))ds\right],$$

$$c_2 = \frac{1}{\Delta_1}\left[\int_0^1 \mathcal{G}_2(s)\phi_p^{-1}(x(s))ds + \kappa_2 \int_0^1 \mathcal{G}_1(s)\phi_p^{-1}(y(s))ds\right].$$

Then from (2.10) and (2.11), it is obvious that (2.6) and (2.7) hold. □

Similar to the proof of Lemma 2.2, we have

Lemma 2.3. *Assume that* (H_1)–(H_3) *hold. Then for any* $u, v \in C[0,1]$, *the system*

$$\begin{cases} -x''(t) = \lambda_1 a_1(t) f_1(t,u(t),v(t)), & t \in (0,1), \\ -y''(t) = \lambda_2 a_2(t) f_2(t,u(t),v(t)), & t \in (0,1), \\ x'(0) = 0,\ x(1) = \int_0^1 y(t)dB_1(t), \\ y'(0) = 0,\ y(1) = \int_0^1 x(t)dB_2(t), \end{cases}$$

has a unique integral representation

$$x(t) = \lambda_1 \int_0^1 \widetilde{H}_1(t,s)a_1(s)f_1(s,u(s),v(s))ds + \lambda_2 \int_0^1 \widetilde{K}_1(s)a_2(s)f_2(s,u(s),v(s))ds,$$

$$y(t) = \lambda_2 \int_0^1 \widetilde{H}_2(t,s)a_2(s)f_2(s,u(s),v(s))ds + \lambda_1 \int_0^1 \widetilde{K}_2(s)a_1(s)f_1(s,u(s),v(s))ds,$$

where $\Delta_2 = 1 - \tilde{\kappa}_1\tilde{\kappa}_2$, *and* $\widetilde{K}_1(s) = \frac{1}{\Delta_2}\widetilde{\mathcal{G}}_1(s)$, $\widetilde{K}_2(s) = \frac{1}{\Delta_2}\widetilde{\mathcal{G}}_2(s)$,

$$\widetilde{H}_1(t,s) = G(t,s) + \frac{\tilde{\kappa}_1}{\Delta_2}\widetilde{\mathcal{G}}_2(s), \quad \widetilde{H}_2(t,s) = G(t,s) + \frac{\tilde{\kappa}_2}{\Delta_2}\widetilde{\mathcal{G}}_1(s).$$

Lemma 2.4. *Assume that* (H_1) *holds. Then the functions* $H_i(t,s)$, $\widetilde{H}_i(t,s)$, $K_i(s)$, $\widetilde{K}_i(s)$ $(i=1,2)$ *are continuous and*

$$H_i(t,s)>0,\quad \widetilde{H}_i(t,s)>0,\quad K_i(s)>0,\quad \widetilde{K}_i(s)>0,\qquad \text{for } t,s\in[0,1),\ i=1,2,$$

$$H_i(t,s)\ge 0,\quad \widetilde{H}_i(t,s)\ge 0,\quad K_i(s)\ge 0,\quad \widetilde{K}_i(s)\ge 0,\qquad \text{for } t,s\in[0,1],\ i=1,2.$$

Proof. It follows from (2.3), Lemmas 2.2 and 2.3 that the results of Lemma 2.4 are true. □

Lemma 2.5. *Assume that* (H_1) *holds. For all* $t,s\in[0,1]$*, we have*

$$\frac{\kappa_i\varrho_j}{\Delta_1}G(s,s)\le H_i(t,s)\le\frac{1}{\Delta_1}G(s,s),\quad i=1,2,\ i+j=3, \tag{2.14}$$

$$\frac{\varrho_i}{\Delta_1}G(s,s)\le K_i(s)\le\frac{\kappa_i}{\Delta_1}G(s,s),\quad i=1,2, \tag{2.15}$$

$$\frac{\tilde\kappa_i\tilde\varrho_j}{\Delta_2}G(s,s)\le \widetilde{H}_i(t,s)\le\frac{1}{\Delta_2}G(s,s),\quad i=1,2,\ i+j=3, \tag{2.16}$$

$$\frac{\tilde\varrho_i}{\Delta_2}G(s,s)\le \widetilde{K}_i(s)\le\frac{\tilde\kappa_i}{\Delta_2}G(s,s),\quad i=1,2, \tag{2.17}$$

where κ_i, $\tilde\kappa_i$, ϱ_i *and* $\tilde\varrho_i(i=1,2)$ *are defined by* (2.2).

Proof. First, we will show that (2.14) is true. By (2.3), the first equalities of (2.8) and (2.9), we obtain

$$\begin{aligned}
H_i(t,s)&=G(t,s)+\frac{\kappa_i}{\Delta_1}\mathcal{G}_j(s)\le G(s,s)+\frac{\kappa_i}{\Delta_1}\int_0^1 G(t,s)dA_j(t)\\
&\le G(s,s)+\frac{\kappa_i}{\Delta_1}\int_0^1 dA_j(t)\cdot G(s,s)=\frac{\Delta_1+\kappa_i\kappa_j}{\Delta_1}G(s,s)\\
&=\frac{1}{\Delta_1}G(s,s),\quad t,s\in[0,1],\ i=1,2,i+j=3.
\end{aligned}$$

On the other hand, by (2.3), the first equalities of (2.8) and (2.9), we also have

$$\begin{aligned}
H_i(t,s)&=G(t,s)+\frac{\kappa_i}{\Delta_1}\mathcal{G}_j(s)\ge\frac{\kappa_i}{\Delta_1}\mathcal{G}_j(s)\\
&\ge\frac{\kappa_i}{\Delta_1}\int_0^1 G(t,t)G(s,s)dA_j(t)\\
&=\frac{\kappa_i\varrho_j}{\Delta_1}G(s,s),\quad t,s\in[0,1],\ i=1,2,\ i+j=3.
\end{aligned}$$

Next we show that (2.15) holds. In fact, using (2.3), the second equalities of (2.8) and (2.9), we get

$$K_i(s)=\frac{1}{\Delta_1}\int_0^1 G(t,s)dA_i(t)\le\frac{1}{\Delta_1}\int_0^1 G(s,s)dA_i(t)=\frac{\kappa_i}{\Delta_1}G(s,s),\quad s\in[0,1],\ i=1,2.$$

On the other hand, by (2.3), the second equalities of (2.8) and (2.9), we also have

$$K_i(s)=\frac{1}{\Delta_1}\int_0^1 G(t,s)dA_i(t)\ge\frac{1}{\Delta_1}\int_0^1 G(t,t)G(s,s)dA_i(t)=\frac{\varrho_i}{\Delta_1}G(s,s),\quad s\in[0,1],\ i=1,2.$$

Similar to the proof of (2.14) and (2.15), we obtain (2.16) and (2.17). □

Remark 2.6. From Lemma 2.5, for $t, s \in [0,1]$ we have

$$\nu G(s,s) \le H_i(t,s) \le \mu G(s,s), \quad \nu G(s,s) \le K_i(s) \le \mu G(s,s), \quad i = 1,2,$$
$$\tilde{\nu} G(s,s) \le \widetilde{H}_i(t,s) \le \tilde{\mu} G(s,s), \quad \tilde{\nu} G(s,s) \le \widetilde{K}_i(s) \le \tilde{\mu} G(s,s), \quad i = 1,2,$$

where

$$\mu = \frac{\max\{1, \kappa_1, \kappa_2\}}{\Delta_1}, \qquad \nu = \frac{\min\{\kappa_1\varrho_2, \kappa_2\varrho_1, \varrho_1, \varrho_2\}}{\Delta_1},$$
$$\tilde{\mu} = \frac{\max\{1, \tilde{\kappa}_1, \tilde{\kappa}_2\}}{\Delta_2}, \qquad \tilde{\nu} = \frac{\min\{\tilde{\kappa}_1\tilde{\varrho}_2, \tilde{\kappa}_2\tilde{\varrho}_1, \tilde{\varrho}_1, \tilde{\varrho}_2\}}{\Delta_2}.$$

Denote $C^+[0,1] = \{u \in C[0,1] : u(t) \ge 0, 0 \le t \le 1\}$. Let

$$K = \Big\{(u,v) \in C^+[0,1] \times C^+[0,1] : u, v \text{ are concave on } [0,1],$$
$$\min_{t\in[0,1]} u(t) \ge \gamma \|(u,v)\|_1, \min_{t\in[0,1]} v(t) \ge \gamma \|(u,v)\|_1\Big\},$$

where $\gamma := \frac{\nu \phi_p^{-1}(\tilde{\nu})}{\mu \phi_p^{-1}(\tilde{\mu})}$. Clearly $0 < \gamma < 1$. It is easy to see that K is a cone of E. For any real constant $r > 0$, define $K_r = \{(u,v) : (u,v) \in K, \|(u,v)\|_1 < r\}$ and $\partial K_r = \{(u,v) : (u,v) \in K, \|(u,v)\|_1 = r\}$.

Employing Lemmas 2.2 and 2.3, the system (2.1) can be expressed as

$$\begin{cases} u(t) = \displaystyle\int_0^1 H_1(t,s)\phi_p^{-1}\Big(\lambda_1 \int_0^1 \widetilde{H}_1(s,\tau)a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau \\ \qquad +\lambda_2 \displaystyle\int_0^1 \widetilde{K}_1(\tau)a_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau\Big)\, ds \\ \qquad + \displaystyle\int_0^1 K_1(s)\phi_p^{-1}\Big(\lambda_2 \int_0^1 \widetilde{H}_2(s,\tau)a_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau \\ \qquad +\lambda_1 \displaystyle\int_0^1 \widetilde{K}_2(\tau)a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau\Big)\, ds, \\ v(t) = \displaystyle\int_0^1 H_2(t,s)\phi_p^{-1}\Big(\lambda_2 \int_0^1 \widetilde{H}_2(s,\tau)a_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau \\ \qquad +\lambda_1 \displaystyle\int_0^1 \widetilde{K}_2(\tau)a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau\Big)\, ds \\ \qquad + \displaystyle\int_0^1 K_2(s)\phi_p^{-1}\Big(\lambda_1 \int_0^1 \widetilde{H}_1(s,\tau)a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau \\ \qquad +\lambda_2 \displaystyle\int_0^1 \widetilde{K}_1(\tau)a_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau\Big)\, ds. \end{cases} \tag{2.18}$$

By a solution of the system (2.1), we mean a solution of the corresponding system of integral equations (2.18). Defined an operator $T : K \to K$ by $T(u,v) = (T_1(u,v), T_2(u,v))$, where operators $T_i : K \to K$ are defined by

$$
\begin{aligned}
T_i(u,v)(t) = \int_0^1 H_i(t,s)\phi_p^{-1}\Big(\lambda_i \int_0^1 \widetilde{H}_i(s,\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau \\
+\lambda_j \int_0^1 \widetilde{K}_i(\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau\Big)\,ds \\
+ \int_0^1 K_i(s)\phi_p^{-1}\Big(\lambda_j \int_0^1 \widetilde{H}_j(s,\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau \\
+\lambda_i \int_0^1 \widetilde{K}_j(\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau\Big)\,ds, \quad t \in [0,1], i = 1,2, i+j = 3.
\end{aligned} \tag{2.19}
$$

Clearly, $(u,v) \in K$ is a fixed point of T if and only if (u,v) is a solution of system (2.1).

Lemma 2.7. *Assume that* (H_1)–(H_3) *hold. Then* $T : K \to K$ *is well defined. Furthermore,* $T : K \to K$ *is a completely continuous operator.*

Proof. For any fixed $(u,v) \in K$, there exists a constant $r > 0$ such that $\|(u,v)\| \le r$. Thus, for any $t \in [0,1]$, it follows from (2.19) and Remark 2.6 that

$$
\begin{aligned}
T_i(u,v)(t) = &\int_0^1 H_i(t,s)\phi_p^{-1}\Big(\lambda_i \int_0^1 \widetilde{H}_i(s,\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_j \int_0^1 \widetilde{K}_i(\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau\Big)\,ds \\
&+ \int_0^1 K_i(s)\phi_p^{-1}\Big(\lambda_j \int_0^1 \widetilde{H}_j(s,\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_i \int_0^1 \widetilde{K}_j(\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau\Big)\,ds \\
\le &\,\mu \int_0^1 G(s,s)\phi_p^{-1}\Big(\lambda_i\tilde{\mu} \int_0^1 G(\tau,\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_j\tilde{\mu} \int_0^1 G(\tau,\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau\Big)\,ds \\
&+ \mu \int_0^1 G(s,s)\phi_p^{-1}\Big(\lambda_j\tilde{\mu} \int_0^1 G(\tau,\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_i\tilde{\mu} \int_0^1 G(\tau,\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau\Big)\,ds \\
\le &\,2\mu\phi_p^{-1}(\tilde{\mu}M) \int_0^1 G(s,s)\phi_p^{-1}\Big(\lambda_1 \int_0^1 G(\tau,\tau)a_1(\tau)d\tau + \lambda_2 \int_0^1 G(\tau,\tau)a_2(\tau)d\tau\Big) \\
< &\,\infty, \quad i = 1,2,\ i+j = 3,
\end{aligned}
$$

where

$$
M = \max_{(t,x,y)\in[0,1]\times[0,r]\times[0,r]} f_1(t,x,y) + \max_{(t,x,y)\in[0,1]\times[0,r]\times[0,r]} f_2(t,x,y).
$$

Thus $T : K \to K$ is well defined.

For all $(u,v) \in K$, by (2.19) we have

$$
\begin{aligned}
(T_i(u,v))''(t) = &-\phi_p^{-1}(\lambda_i \int_0^1 \widetilde{H}_i(s,\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau \\
&+ \lambda_j \int_0^1 \widetilde{K}_i(\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau \le 0,
\end{aligned}
$$

which implies that $T_i(u,v)$ is concave on $[0,1]$. Further, by (2.19), Lemmas 2.2 and 2.3 we obtain $T_i(u,v)(0)\geq 0$, $T_i(u,v)(1)\geq 0$. It follows that $T_i(u,v)(t)\geq 0$ for $t\in[0,1]$.

On the other hand, for $(u,v)\in K$, $t\in[0,1]$, using (2.19) and Remark 2.6, we obtain

$$\begin{aligned}
T_i(u,v)(t) \leq{}& \mu\int_0^1 G(s,s)\phi_p^{-1}\bigg(\lambda_i\tilde{\mu}\int_0^1 G(\tau,\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_j\tilde{\mu}\int_0^1 G(\tau,\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau\bigg)\,ds \\
&+\mu\int_0^1 G(s,s)\phi_p^{-1}\bigg(\lambda_j\tilde{\mu}\int_0^1 G(\tau,\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_i\tilde{\mu}\int_0^1 G(\tau,\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau\bigg)\,ds \\
\leq{}& 2\mu\phi_p^{-1}(\tilde{\mu})\int_0^1 G(s,s)\phi_p^{-1}\bigg(\lambda_i\int_0^1 G(\tau,\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_j\int_0^1 G(\tau,\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau\bigg)\,ds, \quad i=1,2,\ i+j=3,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|T_i(u,v)\| \leq{}& 2\mu\phi_p^{-1}(\tilde{\mu})\int_0^1 G(s,s)\phi_p^{-1}\bigg(\lambda_i\int_0^1 G(\tau,\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_j\int_0^1 G(\tau,\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau\bigg)\,ds, \quad i=1,2,\ i+j=3.
\end{aligned}\tag{2.20}$$

Also, for $(u,v)\in K$, $t\in[0,1]$, using Remark 2.6, (2.19) and (2.20), we have

$$\begin{aligned}
T_1(u,v)(t) ={}& \int_0^1 H_1(t,s)\phi_p^{-1}\bigg(\lambda_1\int_0^1 \widetilde{H}_1(s,\tau)a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_2\int_0^1 \widetilde{K}_1(\tau)a_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau\bigg)\,ds \\
&+\int_0^1 K_1(s)\phi_p^{-1}\bigg(\lambda_2\int_0^1 \widetilde{H}_2(s,\tau)a_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_1\int_0^1 \widetilde{K}_2(\tau)a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau\bigg)\,ds \\
\geq{}& \nu\int_0^1 G(s,s)\phi_p^{-1}\bigg(\lambda_1\tilde{\nu}\int_0^1 G(\tau,\tau)a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_2\tilde{\nu}\int_0^1 G(\tau,\tau)a_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau\bigg)\,ds \\
&+\nu\int_0^1 G(s,s)\phi_p^{-1}\bigg(\lambda_2\tilde{\nu}\int_0^1 G(\tau,\tau)a_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_1\tilde{\nu}\int_0^1 G(\tau,\tau)a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau\bigg)\,ds \\
\geq{}& 2\nu\phi_p^{-1}(\tilde{\nu})\int_0^1 G(s,s)\phi_p^{-1}\bigg(\lambda_i\int_0^1 G(\tau,\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_j\int_0^1 G(\tau,\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau\bigg)\,ds \\
\geq{}& \gamma\|T_i(u,v)\|, \quad i=1,2,\ i+j=3.
\end{aligned}$$

This implies that

$$\min_{t\in[0,1]} T_1(u,v)(t) \geq \gamma \|(T_1(u,v), T_2(u,v))\|_1.$$

In the same way as above, we can prove that

$$\min_{t\in[0,1]} T_2(u,v)(t) \geq \gamma \|(T_1(u,v), T_2(u,v))\|_1.$$

Hence, $T(K) \subset K$.

Next, we prove that $T : K \to K$ is completely continuous. For any natural number n, we set

$$a_{in}(t) = \begin{cases} a_i(t), & 0 \leq t \leq \dfrac{n-1}{n}, \\ \inf\limits_{1-\frac{1}{n}\leq s<t} a_i(s), & 1-\dfrac{1}{n} \leq t \leq 1, \end{cases}$$

$i = 1,2$. Then $a_{1n}, a_{2n} : [0,1] \to [0,+\infty)$ are continuous and $a_{1n}(t) \leq a_1(t), a_{2n}(t) \leq a_2(t)$, $t \in (0,1)$. Let

$$\begin{aligned} T_{in}(u,v)(t) = &\int_0^1 H_i(t,s)\phi_p^{-1}\left(\lambda_i \int_0^1 \widetilde{H}_i(s,\tau)a_{in}(\tau)f_i(\tau,u(\tau),v(\tau))d\tau\right. \\ &\left.+\lambda_j \int_0^1 \widetilde{K}_i(\tau)a_{jn}(\tau)f_j(\tau,u(\tau),v(\tau))d\tau\right) ds \\ &+ \int_0^1 K_i(s)\phi_p^{-1}\left(\lambda_j \int_0^1 \widetilde{H}_j(s,\tau)a_{jn}(\tau)f_j(\tau,u(\tau),v(\tau))d\tau\right. \\ &\left.+\lambda_i \int_0^1 \widetilde{K}_j(\tau)a_{in}(\tau)f_i(\tau,u(\tau),v(\tau))d\tau\right) ds, \end{aligned}$$

$i = 1,2,\ i+j = 3$. Similar to [14], by the approximating theorem of completely continuous operators, we can prove $T : K \to K$ is a completely continuous operator. □

Lemma 2.8 ([4]). *Let X be a real Banach space, P be a cone in X. Assume that Ω_1 and Ω_2 are two bounded open sets of X with $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Tx\| \leq \|x\|, \quad x \in P \cap \partial\Omega_1$ and $\|Tx\| \geq \|x\|, \quad x \in P \cap \partial\Omega_2$, or

(ii) $\|Tx\| \geq \|x\|, \quad x \in P \cap \partial\Omega_1$ and $\|Tx\| \leq \|x\|, \quad x \in P \cap \partial\Omega_2$.

Then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main results

Denote

$$\begin{aligned} L_1 &= \max\left\{2\mu\phi_p^{-1}\left(\tilde{\mu}\int_0^1 G(\tau,\tau)a_1(\tau)d\tau\right), 2\mu\phi_p^{-1}\left(\tilde{\mu}\int_0^1 G(\tau,\tau)a_2(\tau)d\tau\right)\right\}, \\ L_2 &= \min\left\{2\nu\phi_p^{-1}\left(\tilde{\nu}\int_0^1 G(\tau,\tau)a_1(\tau)d\tau\right), 2\nu\phi_p^{-1}\left(\tilde{\nu}\int_0^1 G(\tau,\tau)a_2(\tau)d\tau\right)\right\}, \\ M_i &= \max_{t\in[0,1]} f_i(t,1,1), \quad m_i = \min_{t\in[0,1]} f_i(t,1,1), \quad i = 1,2. \end{aligned}$$

And, we also suppose

(H_4) $f_i(t,x,y)$ is nondecreasing in x and nonincreasing in y, and there exist $\xi_i, \eta_i \in [0,1)$ such that

$$c^{\xi_i} f_i(t,x,y) \le f_i(t,cx,y), \quad f_i(t,x,cy) \le c^{-\eta_i} f_i(t,x,y), \quad \forall x,y>0,\ c \in (0,1),\ i=1,2.$$

(H_5) $f_i(t,x,y)$ is nonincreasing in x and nondecreasing in y, and there exist $\xi_i, \eta_i \in [0,1)$ such that

$$c^{\xi_i} f_i(t,x,y) \le f_i(t,x,cy), \quad f_i(t,cx,y) \le c^{-\eta_i} f_i(t,x,y), \quad \forall x,y>0, c \in (0,1),\ i=1,2.$$

Remark 3.1. (H_4) implies that

$$f_i(t,cx,y) \le c^{\xi_i} f_i(t,x,y), \quad f_i(t,x,cy) \le c^{\eta_i} f_i(t,x,y), \quad \forall x,y>0,\ c>1,\ i=1,2.$$

Remark 3.2. (H_5) implies that

$$f_i(t,x,cy) \le c^{\xi_i} f_i(t,x,y), \quad f_i(t,x,y) \le c^{\eta_i} f_i(t,cx,y), \quad \forall x,y>0,\ c>1,\ i=1,2.$$

Theorem 3.3. *Assume that $f_i(t,1,1) \neq 0$, $t \in [0,1]$ and (H_1)–(H_4) hold. Then for any $0 < r_0 < 1 < R_0 < \infty$, the system (1.1) has at least one symmetric positive solution for*

$$\lambda_i \in \left(\frac{r_0^{p-1-\xi_i}}{m_i \gamma^{\xi_i} \phi_p(L_2)}, \frac{R_0^{p-1-\xi_i} \gamma^{\eta_i}}{M_i \phi_p(L_1)} \right), \quad i=1,2 \tag{3.1}$$

provided that $M_i \phi_p(L_1) r_0^{p-1-\xi_i} < m_i \phi_p(L_2) R_0^{p-1-\xi_i} \gamma^{\xi_i+\eta_i}$ $(i=1,2)$.

Proof. For any $(u,v) \in \partial K_{r_0}$, by the definition of $\|\cdot\|$, we have

$$\begin{aligned} r_0 = \|(u,v)\|_1 \ge \|u\| \ge u(t) \ge \gamma \|(u,v)\|_1 = \gamma r_0, \\ r_0 = \|(u,v)\|_1 \ge \|v\| \ge v(t) \ge \gamma \|(u,v)\|_1 = \gamma r_0. \end{aligned} \tag{3.2}$$

Suppose that λ_i satisfy (3.1). So, for any $(u,v) \in \partial K_{r_0}$, by (H_4) and (3.2), we have

$$\begin{aligned} T_i(u,v)(t) \ge{}& \nu \int_0^1 G(s,s) \phi_p^{-1} \left(\lambda_i \tilde{\nu} \int_0^1 G(\tau,\tau) a_i(\tau) f_i(\tau, \gamma r_0, 1) d\tau \right. \\ &\left. + \lambda_j \tilde{\nu} \int_0^1 G(\tau,\tau) a_j(\tau) f_j(\tau, \gamma r_0, 1)) d\tau \right) ds \\ &+ \nu \int_0^1 G(s,s) \phi_p^{-1} \left(\lambda_j \tilde{\nu} \int_0^1 G(\tau,\tau) a_j(\tau) f_j(\tau, \gamma r_0, 1) d\tau \right. \\ &\left. + \lambda_i \tilde{\nu} \int_0^1 G(\tau,\tau) a_i(\tau) f_i(\tau, \gamma r_0, 1) d\tau \right) ds \\ \ge{}& \nu \left[\phi_p^{-1} \left(\lambda_i \gamma^{\xi_i} r_0^{\xi_i} \tilde{\nu} \int_0^1 G(\tau,\tau) a_i(\tau) f_i(\tau,1,1) d\tau \right) \right. \\ &\left. + \phi_p^{-1} \left(\lambda_j \gamma^{\xi_j} r_0^{\xi_j} \tilde{\nu} \int_0^1 G(\tau,\tau) a_j(\tau) f_j(\tau,1,1) d\tau \right) \right] \\ \ge{}& \nu \left[\phi_p^{-1} \left(\lambda_i \gamma^{\xi_i} r_0^{\xi_i} m_i \tilde{\nu} \int_0^1 G(\tau,\tau) a_i(\tau) d\tau \right) + \phi_p^{-1} \left(\lambda_j \gamma^{\xi_j} r_0^{\xi_j} m_j \tilde{\nu} \int_0^1 G(\tau,\tau) a_j(\tau) d\tau \right) \right] \\ \ge{}& \phi_p^{-1}(\lambda_i \gamma^{\xi_i} r_0^{\xi_i} m_i) \frac{L_2}{2} + \phi_p^{-1}(\lambda_j \gamma^{\xi_j} r_0^{\xi_j} m_j) \frac{L_2}{2} \ge r_0, \quad t \in [0,1],\ i=1,2,\ i+j=3, \end{aligned}$$

which implies that

$$\|T(u,v)\|_1 = \max\{\|T_1(u,v)\|, \|T_2(u,v)\|\} \geq r_0 = \|(u,v)\|_1, \quad (u,v) \in \partial K_{r_0}. \tag{3.3}$$

On the other hand, for any $(u,v) \in \partial K_{R_0}$, $t \in [0,1]$, similar to (3.2), we have

$$\begin{aligned} R_0 = \|(u,v)\|_1 \geq u(t) \geq \gamma\|(u,v)\|_1 = \gamma R_0 > \gamma, \\ R_0 = \|(u,v)\|_1 \geq v(t) \geq \gamma\|(u,v)\|_1 = \gamma R_0 > \gamma. \end{aligned} \tag{3.4}$$

By (3.1), (3.4), Remarks 2.6 and 3.1, for any $t \in [0,1]$, we have

$$\begin{aligned} T_i(u,v)(t) \leq{}& \mu \int_0^1 G(s,s)\phi_p^{-1}\Big(\lambda_i\tilde{\mu}\int_0^1 G(\tau,\tau)a_i(\tau)f_i(\tau,R_0,\gamma)d\tau \\ &+\lambda_j\tilde{\mu}\int_0^1 G(\tau,\tau)a_j(\tau)f_j(\tau,R_0,\gamma)d\tau\Big)ds \\ &+\mu\int_0^1 G(s,s)\phi_p^{-1}\Big(\lambda_j\tilde{\mu}\int_0^1 G(\tau,\tau)a_j(\tau)f_j(\tau,R_0,\gamma)d\tau \\ &+\lambda_i\tilde{\mu}\int_0^1 G(\tau,\tau)a_i(\tau)f_i(\tau,R_0,\gamma)d\tau\Big)ds \\ \leq{}& \mu\Big[\phi_p^{-1}\Big(\lambda_i\tilde{\mu}R_0^{\xi_i}\gamma^{-\eta_i}\int_0^1 G(\tau,\tau)a_i(\tau)f_i(\tau,1,1)d\tau\Big) \\ &+\phi_p^{-1}\Big(\lambda_j\tilde{\mu}R_0^{\xi_i}\gamma^{-\eta_j}\int_0^1 G(\tau,\tau)a_j(\tau)f_j(\tau,1,1)d\tau\Big)\Big] \\ \leq{}& \mu\Big[\phi_p^{-1}\Big(\lambda_i\tilde{\mu}R_0^{\xi_i}\gamma^{-\eta_i}M_i\int_0^1 G(\tau,\tau)a_i(\tau)d\tau\Big) \\ &+\phi_p^{-1}\Big(\lambda_j\tilde{\mu}R_0^{\xi_i}\gamma^{-\eta_j}M_j\int_0^1 G(\tau,\tau)a_j(\tau)d\tau\Big)\Big] \\ \leq{}& \phi_p^{-1}(\lambda_i R_0^{\xi_i}\gamma^{-\eta_i}M_i)\frac{L_1}{2} + \phi_p^{-1}(\lambda_j R_0^{\xi_i}\gamma^{-\eta_j}M_j)\frac{L_1}{2} \leq R_0, \quad i=1,2,\ i+j=3. \end{aligned}$$

Therefore, we have

$$\|T(u,v)\|_1 = \max\{\|T_1(u,v)\|, \|T_2(u,v)\|\} \leq R_0 = \|(u,v)\|_1, \quad \forall\ (u,v) \in \partial K_{R_0}. \tag{3.5}$$

It follows from (3.3), (3.5) and Lemma 2.8 that for any $\lambda_i \in \big(\frac{r_0^{p-1-\xi_i}}{m_i\gamma^{\xi_i}\phi_p(L_2)}, \frac{R_0^{p-1-\xi_i}\gamma^{\eta_i}}{M_i\phi_p(L_1)}\big)$ $(i=1,2)$, T has a fixed point $(u_0,v_0) \in K_{R_0} \setminus \bar{K}_{r_0}$ with $r_0 \leq \|(u_0,v_0)\|_1 \leq R_0$. Moreover, (u_0,v_0) is positive. In fact, from $\|(u_0,v_0)\|_1 \geq r_0 > 0$, by construction of the cone K, we have

$$\min_{t\in[0,1]} u_0(t) \geq \gamma\|(u_0,v_0)\|_1 > 0,$$

which implies that $u_0(t) > 0$ for all $t \in [0,1]$. Similarly, we also have $v_0(t) > 0$ for all $t \in [0,1]$. Hence, (u_0,v_0) is a positive solution of the system (2.1). Let

$$u^*(t) = \begin{cases} u_0(-t), & -1 \leq t < 0, \\ u_0(t), & 0 \leq t \leq 1, \end{cases} \qquad v^*(t) = \begin{cases} v_0(-t), & -1 \leq t < 0, \\ v_0(t), & 0 \leq t \leq 1. \end{cases}$$

By Remark 2.1 we know that (u^*,v^*) is the desired symmetric positive solution for the system (1.1). $\square$

Similar to the proof of Theorem 3.3, we have the following result.

Theorem 3.4. *Assume that* $f_i(t,1,1) \neq 0$, $t \in [0,1]$, (H_1)–(H_3) *and* (H_5) *hold. Then for any* $0 < r_0 < 1 < R_0 < \infty$, *the system* (1.1) *has at least one symmetric positive solution for*

$$\lambda_i \in \left(\frac{r_0^{p-1-\xi_i}}{m_i \gamma^{\xi_i} \phi_p(L_2)}, \frac{R_0^{p-1-\xi_i} \gamma^{\eta_i}}{M_i \phi_p(L_1)} \right), \quad i = 1,2$$

provided that $M_i \phi_p(L_1) r_0^{p-1-\xi_i} < m_i \phi_p(L_2) R_0^{p-1-\xi_i} \gamma^{\xi_i+\eta_i}$ $(i = 1,2)$.

Theorem 3.5. *Assume that* $f_i(t,1,1) \neq 0$, $t \in [0,1]$ *and* (H_1)–(H_3) *hold. In addition, assume*

(H_6) $f_1(t,x,y)$ *is nondecreasing in* x *and nonincreasing in* y, *and there exist* $\xi_1, \eta_1 \in [0,1)$ *such that*

$$c^{\xi_1} f_1(t,x,y) \leq f_1(t,cx,y), \quad f_1(t,x,cy) \leq c^{-\eta_1} f_1(t,x,y), \quad \forall\, x,y > 0, c \in (0,1),$$

and $f_2(t,x,y)$ *is nonincreasing in* x *and nondecreasing in* y, *and there exist* $\xi_2, \eta_2 \in [0,1)$ *such that*

$$c^{\xi_2} f_2(t,x,y) \leq f_2(t,x,cy), \quad f_2(t,cx,y) \leq c^{-\eta_2} f_2(t,x,y), \quad \forall\, x,y > 0, c \in (0,1).$$

Then for any $0 < r_0 < 1 < R_0 < \infty$, *the system* (1.1) *has at least one symmetric positive solution for*

$$\lambda_i \in \left(\frac{r_0^{p-1-\xi_i}}{m_i \gamma^{\xi_i} \phi_p(L_2)}, \frac{R_0^{p-1-\xi_i} \gamma^{\eta_i}}{M_i \phi_p(L_1)} \right), \quad i = 1,2$$

provided that $M_i \phi_p(L_1) r_0^{p-1-\xi_i} < m_i \phi_p(L_2) R_0^{p-1-\xi_i} \gamma^{\xi_i+\eta_i}$ $(i = 1,2)$.

Theorem 3.6. *Assume that* $f_i(t,1,1) \neq 0$, $t \in [0,1]$ *and* (H_1)–(H_3) *hold. In addition, assume*

(H_7) $f_1(t,x,y)$ *is nonincreasing in* x *and nondecreasing in* y, *and there exist* $\xi_1, \eta_1 \in [0,1)$ *such that*

$$c^{\xi_1} f_1(t,x,y) \leq f_1(t,x,cy), \quad f_1(t,cx,y) \leq c^{-\eta_1} f_1(t,x,y), \quad \forall x,y > 0, c \in (0,1),$$

and $f_2(t,x,y)$ *is nondecreasing in* x *and nonincreasing in* y, *and there exist* $\xi_2, \eta_2 \in [0,1)$ *such that*

$$c^{\xi_2} f_2(t,x,y) \leq f_2(t,cx,y), \quad f_2(t,x,cy) \leq c^{-\eta_2} f_2(t,x,y), \quad \forall x,y > 0, c \in (0,1).$$

Then for any $0 < r_0 < 1 < R_0 < \infty$, *the system* (1.1) *has at least one symmetric positive solution for*

$$\lambda_i \in \left(\frac{r_0^{p-1-\xi_i}}{m_i \gamma^{\xi_i} \phi_p(L_2)}, \frac{R_0^{p-1-\xi_i} \gamma^{\eta_i}}{M_i \phi_p(L_1)} \right), \quad i = 1,2$$

provided that $M_i \phi_p(L_1) r_0^{p-1-\xi_i} < m_i \phi_p(L_2) R_0^{p-1-\xi_i} \gamma^{\xi_i+\eta_i}$ $(i = 1,2)$.

Theorem 3.7. *Assume that* (H_1)–(H_3) *hold and there exist* $R > r > 0$ *such that*

$$\lambda_i \min_{\substack{t\in[0,1] \\ \gamma r \leq x,y \leq r}} f_i(t,x,y) \geq \phi_p\left(\frac{\gamma r}{L_2}\right), \quad \lambda_i \max_{\substack{t\in[0,1] \\ 0 \leq x,y \leq R}} f_i(t,x,y) \leq \phi_p\left(\frac{R}{L_1}\right), \quad i = 1,2. \tag{3.6}$$

Then the system (1.1) *has at least one symmetric positive solution* (u^*, v^*) *satisfying*

$$r \leq \|(u^*, v^*)\|_1 \leq R.$$

Proof. Set $K_r = \{(u,v) \in K : \|(u,v)\|_1 < r\}$. For any $(u,v) \in \partial K_r$, by the definition of $\|\cdot\|$, we have

$$\gamma r \le \gamma\|(u,v)\|_1 \le u(t) \le r, \quad \gamma r \le \gamma\|(u,v)\|_1 \le v(t) \le r, \quad t \in [0,1]. \tag{3.7}$$

Thus, for any $(u,v) \in \partial K_r$, by the first inequality of (3.6), we have

$$\lambda_i \min_{\substack{t\in[0,1]\\ \gamma r\le u(t),v(t)\le r}} f_i(t,u(t),v(t)) \ge \phi_p\left(\frac{\gamma r}{L_2}\right), \quad i=1,2. \tag{3.8}$$

So, for any $(u,v) \in \partial K_r$, by (3.8) and Remark 2.6, we have

$$\begin{aligned}
T_i(u,v)(t) \ge{}& \nu \int_0^1 G(s,s)\phi_p^{-1}\Bigg(\lambda_i\tilde{\nu}\int_0^1 G(\tau,\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_j\tilde{\nu}\int_0^1 G(\tau,\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau\Bigg)ds \\
&+\nu \int_0^1 G(s,s)\phi_p^{-1}\Bigg(\lambda_j\tilde{\nu}\int_0^1 G(\tau,\tau)a_j(\tau)f_j(\tau,u(\tau),v(\tau))d\tau \\
&+\lambda_i\tilde{\nu}\int_0^1 G(\tau,\tau)a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau\Bigg)ds \\
\ge{}& \nu\Bigg[\phi_p^{-1}\Bigg(\lambda_i\tilde{\nu}\min_{\substack{\tau\in[0,1]\\ \gamma r\le u(\tau),v(\tau)\le r}} f_i(\tau,u(\tau),v(\tau))\int_0^1 G(\tau,\tau)a_i(\tau)d\tau\Bigg) \\
&+\phi_p^{-1}\Bigg(\lambda_j\tilde{\nu}\min_{\substack{\tau\in[0,1]\\ \gamma r\le u(\tau),v(\tau)\le r}} f_j(\tau,u(\tau),v(\tau))\int_0^1 G(\tau,\tau)a_j(\tau)d\tau\Bigg)\Bigg] \\
\ge{}& \phi_p^{-1}\Bigg(\lambda_i\min_{\substack{\tau\in[0,1]\\ \gamma r\le u(\tau),v(\tau)\le r}} f_i(\tau,u(\tau),v(\tau))\Bigg)\frac{L_2}{2\gamma} \\
&+\phi_p^{-1}\Bigg(\lambda_i\min_{\substack{\tau\in[0,1]\\ \gamma r\le u(\tau),v(\tau)\le r}} f_j(\tau,u(\tau),v(\tau))\Bigg)\frac{L_2}{2\gamma} \\
\ge{}& r, \quad t\in[0,1],\ i=1,2,\ i+j=3.
\end{aligned}$$

Therefore, we have

$$\|T(u,v)\|_1 = \max\{\|T_1(u,v)\|, \|T_2(u,v)\|\} \ge r = \|(u,v)\|_1, \quad (u,v) \in \partial K_r. \tag{3.9}$$

Take $K_R = \{(u,v) \in K : \|(u,v)\|_1 < R\}$. For any $(u,v) \in \partial K_R$, by the definition of $\|\cdot\|$, we have

$$u(t) \le \|u\| \le \|(u,v)\|_1 \le R, \quad v(t) \le \|v\| \le \|(u,v)\|_1 \le R, \quad t \in [0,1]. \tag{3.10}$$

Thus, for any $(u,v) \in \partial K_R$, by the second inequality of (3.6) and (3.10), we have

$$\lambda_i \max_{\substack{t\in[0,1]\\ 0\le u(t),v(t)\le R}} f_i(t,u(t),v(t)) \le \phi_p\left(\frac{R}{L_1}\right). \tag{3.11}$$

By (3.11) and Remark 2.6, we have

$$\begin{aligned}
T_i(u,v)(t) \le \mu \int_0^1 G(s,s)\phi_p^{-1}\Bigg(&\lambda_i\tilde{\mu} \max_{\substack{\tau\in[0,1]\\ 0\le u(\tau),v(\tau)\le R}} f_i(\tau,u(\tau),v(\tau)) \int_0^1 G(\tau,\tau)a_i(\tau)d\tau \\
&+\lambda_j\tilde{\mu} \max_{\substack{\tau\in[0,1]\\ 0\le u(\tau),v(\tau)\le R}} f_j(\tau,u(\tau),v(\tau)) \int_0^1 G(\tau,\tau)a_j(\tau)d\tau\Bigg) ds \\
&+ \mu \int_0^1 G(s,s)\phi_p^{-1}\Bigg(\lambda_j\tilde{\mu} \max_{\substack{\tau\in[0,1]\\ 0\le u(\tau),v(\tau)\le R}} f_j(\tau,u(\tau),v(\tau)) \int_0^1 G(\tau,\tau)a_j(\tau)d\tau \\
&+\lambda_i\tilde{\mu} \max_{\substack{\tau\in[0,1]\\ 0\le u(\tau),v(\tau)\le R}} f_i(\tau,u(\tau),v(\tau)) \int_0^1 G(\tau,\tau)a_i(\tau)d\tau\Bigg) ds \\
&\le \frac{\mu R}{L_1}\phi_p^{-1}\left(\tilde{\mu}\int_0^1 G(\tau,\tau)a_i(\tau)d\tau\right) + \frac{\mu R}{L_1}\phi_p^{-1}\left(\tilde{\mu}\int_0^1 G(\tau,\tau)a_j(\tau)d\tau\right) \\
&\le \frac{R}{L_1}\times\frac{L_1}{2} + \frac{R}{L_1}\times\frac{L_1}{2} = R, \quad t\in[0,1],\ i=1,2,\ i+j=3.
\end{aligned}$$

Therefore, we have

$$\|T(u,v)\|_1 = \max\{\|T_1(u,v)\|, \|T_2(u,v)\|\} \le R = \|(u,v)\|_1, \quad \forall\ (u,v)\in\partial K_R. \tag{3.12}$$

It follows from (3.9) and (3.12), T has a fixed point $(u_0,v_0)\in K_R\setminus\bar{K}_r$ with $r\le\|(u_0,v_0)\|_1\le R$. Moreover, (u_0,v_0) is positive. In fact, from $\|(u_0,v_0)\|_1\ge r>0$, by construction of the cone K, we have

$$\min_{t\in[0,1]} u_0(t) \ge \gamma\|(u_0,v_0)\|_1 > 0,$$

which implies that $u_0(t)>0$ for all $t\in[0,1]$. Similarly, we also have $v_0(t)>0$ for all $t\in[0,1]$. Hence, (u_0,v_0) is a positive solution of the system (2.1). Let

$$u^*(t) = \begin{cases} u_0(-t), & -1\le t<0,\\ u_0(t), & 0\le t\le 1,\end{cases} \qquad v^*(t) = \begin{cases} v_0(-t), & -1\le t<0,\\ v_0(t), & 0\le t\le 1.\end{cases}$$

By Remark 2.1 we know that (u^*,v^*) is the desired symmetric positive solution for the system (1.1). □

Corollary 3.8. *From the proof of Theorem 3.7, then for*

$$\lambda_1 \min_{\substack{t\in[0,1]\\ \gamma r\le x,y\le r}} f_1(t,x,y) \ge \phi_p\left(\frac{\gamma r}{L_{21}}\right), \quad \lambda_i \max_{\substack{t\in[0,1]\\ 0\le x,y\le R}} f_i(t,x,y) \le \phi_p\left(\frac{R}{L_1}\right), \quad i=1,2,$$

or

$$\lambda_2 \min_{\substack{t\in[0,1]\\ \gamma r\le x,y\le r}} f_2(t,x,y) \ge \phi_p\left(\frac{\gamma r}{L_{22}}\right), \quad \lambda_i \max_{\substack{t\in[0,1]\\ 0\le x,y\le R}} f_i(t,x,y) \le \phi_p\left(\frac{R}{L_1}\right), \quad i=1,2,$$

the conclusion of Theorem 3.5 is valid, where

$$L_{21} = \gamma\nu\phi_p^{-1}\left(\tilde{\nu}\int_0^1 G(\tau,\tau)a_1(\tau)d\tau\right), \quad L_{22} = \gamma\nu\phi_p^{-1}\left(\tilde{\nu}\int_0^1 G(\tau,\tau)a_2(\tau)d\tau\right).$$

4 An example

Example 4.1. Consider the differential system

$$\begin{cases} (\phi_p(u''(t)))'' = \lambda_1 a_1(t)\dfrac{2u(t)^{1/2}}{(v(t)+7)^{1/3}}, \ \ -1<t<1, \\ (\phi_p(v''(t)))'' = \lambda_2 a_2(t)\dfrac{(1+t^2)u(t)^{1/3}}{(3+v(t))^{1/2}}, \ \ -1<t<1, \\ u(-1)=u(1)=\displaystyle\int_{-1}^{1} v(t)dA_1(t), \ \ \phi_p(u''(-1))=\phi_p(u''(1))=\int_{-1}^{1}\phi_p(v''(t))dB_1(t), \\ v(-1)=v(1)=\displaystyle\int_{-1}^{1} u(t)dA_2(t), \ \ \phi_p(v''(-1))=\phi_p(v''(1))=\int_{-1}^{1}\phi_p(u''(t))dB_2(t), \end{cases} \tag{4.1}$$

where $\lambda_i > 0$ $(i=1,2)$ are some parameters,

$$f_1(t,x,y) = \frac{2x^{1/2}}{(y+7)^{1/3}}, \quad f_2(t,x,y) = \frac{(1+t^2)x^{1/3}}{(3+y)^{1/2}}, \quad (t,x,y) \in [-1,1]\times[0,\infty)\times[0,\infty).$$

Obviously, $f_i(\cdot,x,y)$ are symmetric on $[-1,1]$ for all $x,y\in[0,\infty)$ $(i=1,2)$, so the condition (H_3) holds. Let

$$a_1(t)=t^2, \quad a_2(t)=|t|, \quad A_1(t)=\frac{3}{4}t^5, \quad A_2(t)=\frac{1}{3}t^3, \quad B_1(t)=t, \quad B_2(t)=\frac{1}{2}t^3, \ \ t\in[-1,1],$$

and $p=\frac{5}{2}$, then $q=\frac{5}{3}$, $\phi_p^{-1}(z)=\phi_q(z)=z^{2/3}$ as $z>0$. By Remark 2.1, we only need to consider the following boundary value problem on the interval $[0,1]$,

$$\begin{cases} (\phi_p(u''(t)))'' = \lambda_1\dfrac{2t^2u(t)^{1/2}}{(v(t)+7)^{1/3}}, \quad 0<t<1, \\ (\phi_p(v''(t)))'' = \lambda_2\dfrac{t(1+t^2)u(t)^{1/3}}{(3+v(t))^{1/2}}, \quad 0<t<1, \\ u'(0)=0, \ u(1)=\displaystyle\int_0^1 v(s)dA_1(s), \ \phi_p(u'''(0))=0, \ \phi_p(u''(1))=\int_0^1\phi_p(v''(s))dB_1(s), \\ v'(0)=0, \ v(1)=\displaystyle\int_0^1 u(s)dA_2(s), \ \phi_p(v'''(0))=0, \ \phi_p(v''(1))=\int_0^1\phi_p(u''(s))dB_2(s). \end{cases}$$

By a simple computation, we have

$$\kappa_1=\frac{3}{4}, \quad \kappa_2=\frac{1}{3}, \quad \tilde{\kappa}_1=1, \quad \tilde{\kappa}_2=\frac{1}{2},$$

$$\Delta_1 = 1-\kappa_1\kappa_2=\frac{3}{4}>0, \quad \Delta_2=1-\tilde{\kappa}_1\tilde{\kappa}_2=\frac{1}{2}>0,$$

$$\int_0^1 G(s,s)a_1(s)ds=\frac{1}{12}, \quad \int_0^1 G(s,s)a_2(s)ds=\frac{1}{6}.$$

Clearly, the conditions (H_1) and (H_2) hold. In addition, $f_i(t,x,y)$ $(i=1,2)$ is nondecreasing in x and nonincreasing in y on $[0,1]\times[0,\infty)\times[0,\infty)$. Take $\xi_1=\frac{2}{3}, \xi_2=\frac{1}{2}$, $\eta_1=\frac{1}{3}, \eta_2=\frac{1}{2}$, we can get

$$
\begin{aligned}
f_1(t,cx,y) &= \frac{2(cx)^{1/2}}{(y+7)^{1/3}} = \frac{2c^{1/2}x^{1/2}}{(y+7)^{1/3}} > c^{\frac{2}{3}} f_1(t,x,y), \quad \forall\, x,y>0, c\in(0,1),\\
f_1(t,x,cy) &= \frac{2x^{1/2}}{(cy+7)^{1/3}} < \frac{2x^{1/2}}{(cy+7c)^{1/3}} = c^{-\frac{1}{3}} f_1(t,x,y), \quad \forall\, x,y>0, c\in(0,1),\\
f_2(t,cx,y) &= \frac{(1+t^2)(cx)^{1/3}}{(3+y)^{1/2}} = \frac{(1+t^2)c^{1/3}x^{1/3}}{(3+y)^{1/2}} > c^{\frac{1}{2}} f_2(t,x,y), \quad \forall\, x,y>0, c\in(0,1),\\
f_2(t,x,cy) &= \frac{(1+t^2)x^{1/3}}{(3+cy)^{1/2}} < \frac{(1+t^2)x^{1/3}}{(3c+cy)^{1/2}} = c^{-\frac{1}{2}} f_2(t,x,y), \quad \forall\, x,y>0, c\in(0,1).
\end{aligned}
$$

So, the condition (H_4) is also valid. Furthermore,

$$
\begin{aligned}
\varrho_1 &= \int_0^1 G(t,t)dA_1(t) = \frac{1}{8}, \qquad \varrho_2 = \int_0^1 G(t,t)dA_2(t) = \frac{1}{12},\\
\tilde{\varrho}_1 &= \int_0^1 G(t,t)dB_1(t) = \frac{1}{2}, \qquad \tilde{\varrho}_2 = \int_0^1 G(t,t)dB_2(t) = \frac{1}{8},\\
\mu &= \frac{\max\{1,\kappa_1,\kappa_2\}}{\Delta_1} = \frac{4}{3}, \quad \nu = \frac{\min\{\kappa_1\varrho_2,\kappa_2\varrho_1,\varrho_1,\varrho_2\}}{\Delta_1} = \frac{1}{18},\\
\tilde{\mu} &= \frac{\max\{1,\tilde{\kappa}_1,\tilde{\kappa}_2\}}{\Delta_2} = 2, \quad \tilde{\nu} = \frac{\min\{\tilde{\kappa}_1\tilde{\varrho}_2,\tilde{\kappa}_2\tilde{\varrho}_1,\tilde{\varrho}_1,\tilde{\varrho}_2\}}{\Delta_2} = \frac{1}{4},\\
\gamma &= \frac{\nu\phi_p^{-1}(\tilde{\nu})}{\mu\phi_p^{-1}(\tilde{\mu})} = \frac{\nu\cdot\tilde{\nu}^{2/3}}{\mu\cdot\tilde{\mu}^{2/3}} = \frac{1}{96},
\end{aligned}
$$

$$
\begin{aligned}
L_1 &= \max\left\{2\mu\phi_p^{-1}\left(\tilde{\mu}\int_0^1 G(\tau,\tau)a_1(\tau)d\tau\right), 2\mu\phi_p^{-1}\left(\tilde{\mu}\int_0^1 G(\tau,\tau)a_2(\tau)d\tau\right)\right\} = \frac{8}{3^{5/3}},\\
L_2 &= \min\left\{2\nu\phi_p^{-1}\left(\tilde{\nu}\int_0^1 G(\tau,\tau)a_1(\tau)d\tau\right), 2\nu\phi_p^{-1}\left(\tilde{\nu}\int_0^1 G(\tau,\tau)a_2(\tau)d\tau\right)\right\} = \frac{1}{36^{4/3}},
\end{aligned}
$$

$$
\begin{aligned}
M_1 &= \max_{t\in[0,1]} f_1(t,1,1) = 1, \qquad m_1 = \min_{t\in[0,1]} f_1(t,1,1) = 1,\\
M_2 &= \max_{t\in[0,1]} f_2(t,1,1) = 1, \qquad m_2 = \min_{t\in[0,1]} f_2(t,1,1) = \frac{1}{2}.
\end{aligned}
$$

Take $r_0 = \frac{1}{64}, R_0 = 6^6 = 46656$. By Theorem 3.3, for $\lambda_1 \in (849.1203, 1169.9500)$ and $\lambda_2 \in (396.8174, 3280.5000)$, the system (4.1) has at least one symmetric positive solution.

Remark 4.2. Example 4.1 implies that there is a large number of functions that satisfy the conditions of Theorem 3.3. In addition, the conditions of Theorem 3.3 are also easy to check.

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