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Perturbed Li–Yorke homoclinic chaos

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Abstract. It is rigorously proved that a Li–Yorke chaotic perturbation of a system with a homoclinic orbit creates chaos along each periodic trajectory. The structure of the chaos is investigated, and the existence of infinitely many almost periodic orbits out of the scrambled sets is revealed. Ott–Grebogi–Yorke and Pyragas control methods are utilized to stabilize almost periodic motions. A Duffing oscillator is considered to show the effectiveness of our technique, and simulations that support the theoretical results are depicted.

Keywords: homoclinic orbit, Li–Yorke chaos, almost periodic orbits, Duffing oscillator.

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1 Introduction

Traditionally, analysis of nonlinear dynamical systems has been restricted to smooth problems, that is, to smooth differential equations. Besides stability analysis of fixed points and periodic orbits, another fascinating phenomenon is the existence of chaotic orbits. The presence of such orbits has the consequence that the motions of the system depend sensitively on initial conditions, and the behavior of orbits in the future is unpredictable. Such a chaotic behavior of solutions can be explained mathematically by showing the existence of a transverse homoclinic point of the time map with the corresponding invariant Smale horseshoe [21,33,34]. In general, however, it is not easy to demonstrate the existence of a transverse homoclinic point. The perturbation approach, which is now known as the Melnikov method, is a powerful tool for that purpose [17–19]. The starting point is a nonautonomous system, the unperturbed system/equation, with a (necessarily) nontransverse homoclinic orbit. It is known that if we set up a perturbed system by adding a periodic (or almost periodic) perturbation of a

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sufficiently small amplitude to the unperturbed system and a certain Melnikov function has a simple zero at some point, then the perturbed system has a transverse homoclinic point with the corresponding Smale horseshoe [11].

Endogenously generated chaotic behavior of systems are well investigated in the literature. The systems of Lorenz [28], Rössler [36] and Chua [13–15] as well as the Van der Pol [12, 24, 25] and Duffing [29, 31, 41, 42] oscillators can be considered as systems which are capable of generating chaos endogenously. Chaotification of systems with asymptotically stable equilibria through different types of perturbations can be found in the papers [1–3] and the book [5]. Moreover, the study [4] is concerned with the existence of homoclinic and heteroclinic motions in economic models perturbed with exogenous shocks. In the present study, we consider a system with a homoclinic solution under the influence of a chaotic forcing term. The formation of exogenous chaos is theoretically investigated. Our results are based on the Li–Yorke definition of chaos [27] in a modified sense that was introduced in the papers [6, 8]. To emphasize the role of the homoclinic solution in the paper, we call the dynamics Li-*Yorke homoclinic chaos. An example based on Duffing oscillator is presented to show the effectiveness of our results. Moreover, the controllability of the obtained chaos is shown by means of the Ott–Grebogi–Yorke (OGY) [32] and Pyragas [35] control methods.

Our suggested results are of a significant interest due to the theoretical importance and perspectives for applications. This is the first time in the literature that chaos is obtained as a union of infinitely many sets of chaotic motions for a single equation by means of a perturbation. It was observed from experimental data that chaos is a positive factor for brain activities [39] as well as for robotic dynamics [30, 40]. This is the reason why the presence of infinitely many sets of chaotic motions in the dynamics of a single equation may shed light on the capacity of the brain and provide an opportunity for new designs in robotics.

In the next section, we will introduce the systems which are the main objects of our investigation and will give information concerning their properties under some conditions.

2 The model

Let $\mathscr{A}$ be an equicontinuous family of functions defined on $\mathbb{R}$ with range Λ, where Λ is a compact subset of $\mathbb{R}^m$. In order to generate chaos, we perturb the system

$$z' = f(z,t) \tag{2.1}$$

with the elements of the family $\mathscr{A}$ and set up the system

$$u' = f(u,t) + h(x(t)), \tag{2.2}$$

where $x(t) \in \mathscr{A}$, the function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is twice continuously differentiable in u and continuous in t, and the function $h : \Lambda \to \mathbb{R}^n$ is continuous.

In the remaining parts of the paper, we will make use of the usual Euclidean norm for vectors and the norm induced by the Euclidean norm for matrices.

The following conditions are required.

(C1) $f(u,t+1) = f(u,t)$ for all $u \in \mathbb{R}^n$ and $t \in \mathbb{R}$;

(C2) There exist positive numbers L_1 and L_2 such that

$$L_1 \|x_1 - x_2\| \leq \|h(x_1) - h(x_2)\| \leq L_2 \|x_1 - x_2\|$$

for all $x_1, x_2 \in \Lambda$.

We suppose that system (2.1) has a hyperbolic periodic solution $p(t)$ with a homoclinic solution $q(t)$ such that the variational equation $w' = D_u f(q(t), t)w$ has only the zero solution bounded on the real axis. Under this assumption, it is known that $q(t)$ is a transversal homoclinic orbit, i.e., the 1-time map $G : \mathbb{R}^n \to \mathbb{R}^n$ of system (2.1) has a hyperbolic fixed point $p(0)$ with a transversal homoclinic orbit $\{q(j)\}_{j\in\mathbb{Z}}$. Following Sections 3 and 4 of [33], especially Theorem 4.8 of [33], we get a collection of uniformly bounded solutions $\{\nu_\beta(t)\}_{\beta\in S_\sigma}$ of system (2.1) orbitally near $q(t)$, where the index set S_σ, $\sigma \geq 2$, is the set of doubly infinite sequences $d = (\dots, d_{-1}, d_0, d_1, \dots)$ with $d_i \in \{1, \dots, \sigma\}$ for all $i \in \mathbb{Z}$, i.e., $S_\sigma = \{1, 2, \dots, \sigma\}^{\mathbb{Z}}$ such that each linear system

$$w' = D_u f(\nu_\beta(t), t)w \tag{2.3}$$

has an exponential dichotomy on $\mathbb{R}$ with uniform positive constants K and α, and projections Q_β:

$$\begin{aligned} \|W_\beta(t)Q_\beta W_\beta^{-1}(s)\| &\leq Ke^{-\alpha(t-s)} \quad \text{for all } t,s,\ t \geq s, \\ \|W_\beta(t)(I - Q_\beta)W_\beta^{-1}(s)\| &\leq Ke^{\alpha(t-s)} \quad \text{for all } t,s,\ t \leq s, \end{aligned} \tag{2.4}$$

where I is the $n \times n$ identity matrix and W_β is the fundamental matrix of system (2.3) satisfying $W_\beta(0) = I$. It is worth noting that system (2.2) may not possess a homoclinic solution.

By Theorem 4.8 of [33], an iterative G^{k_0}, for some fixed $k_0 \in \mathbb{N}$, is conjugate to the Bernoulli shift on an invariant compact subset $\mathscr{H} \subset \mathbb{R}^n$, $G^{k_0} : \mathscr{H} \to \mathscr{H}$. So G^{k_0} has i-periodic orbits in $\mathscr{H}$ for any natural number i. This gives that the map G has periodic orbits with periods ik_0 starting in $\mathscr{H}$. Since by definition $\nu_\beta(0) = \varsigma_\beta$ for some $\varsigma_\beta \in \mathscr{H}$ and then $G^j(\varsigma_\beta) = \nu_\beta(j)$ for any $j \in \mathbb{Z}$, we see that among these $\nu_\beta(t)$ there are ik_0-periodic solutions of (2.1) for any $i \in \mathbb{N}$. In what follows we will denote by $\mathcal{P}_\sigma \subset S_\sigma$, $\sigma \geq 2$, the index set for which the bounded solutions $\{\nu_\beta(t)\}_{\beta\in\mathcal{P}_\sigma}$ of system (2.1) are periodic.

3 Bounded solutions

Introducing the new variable y through $y = u - \nu_\beta(t)$, $\beta \in S_\sigma$, system (2.2) can be written in the form

$$y' = D_u f(\nu_\beta(t), t)y + F_\beta(y, t) + h(x(t)), \tag{3.1}$$

where the function $F_\beta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is defined as

$$F_\beta(y, t) = f(y + \nu_\beta(t), t) - f(\nu_\beta(t), t) - D_u f(\nu_\beta(t), t)y. \tag{3.2}$$

Using the dichotomy theory [16], one can verify that for a fixed $x(t) \in \mathscr{A}$, a function $y(t)$ which is bounded on the real axis is a solution of system (3.1) if and only if the integral equation

$$y(t) = \int_{-\infty}^{\infty} G_\beta(t, s)[F_\beta(y(s), s) + h(x(s))]ds \tag{3.3}$$

is satisfied, where

$$G_\beta(t, s) = \begin{cases} W_\beta(t)Q_\beta W_\beta^{-1}(s), & t > s, \\ -W_\beta(t)(I - Q_\beta)W_\beta^{-1}(s), & t < s. \end{cases} \tag{3.4}$$

Since f is twice continuously differentiable in u, under the condition $(C1)$, there exist positive numbers N_1 and N_2 such that

$$\sup_{t\in\mathbb{R},\ \beta\in S_\sigma} \|D_u f(\nu_\beta(t), t)\| \leq N_1$$

and

$$\sup_{t\in\mathbb{R},\ \beta\in S_\sigma} \left\|D_{uu}f(\nu_\beta(t),t)\right\| \leq N_2$$

for each bounded solution $\nu_\beta(t)$, $\beta \in S_\sigma$, of (2.1).

The following condition is also required.

(C3) $M_h < \dfrac{\alpha^2}{16K^2N_2}$, where $M_h = \sup\limits_{x\in\Lambda} \|h(x)\|$.

Under the condition $(C3)$, let us denote

$$R_0 = \frac{\alpha - \sqrt{\alpha^2 - 16K^2N_2M_h}}{4KN_2}.$$

The following lemma is concerned with the existence and uniqueness of bounded solutions of system (3.1).

Lemma 3.1. *Suppose that the conditions* $(C1)$–$(C3)$ *hold. For each* $x(t) \in \mathscr{A}$ *system* (3.1) *possesses a unique solution* $\phi^\beta_{x(t)}(t)$, $\beta \in S_\sigma$, *which is bounded on the real axis such that* $\sup_{t\in\mathbb{R}} \left\|\phi^\beta_{x(t)}(t)\right\| \leq R_0$.

Proof. Let $\mathcal{C}_0$ be the set of uniformly bounded, continuous functions $w(t) : \mathbb{R} \to \mathbb{R}^n$ satisfying $\|w\|_\infty \leq R_0$, where the norm $\|\cdot\|_\infty$ is defined by

$$\|w\|_\infty = \sup_{t\in\mathbb{R}} \|w(t)\|. \tag{3.5}$$

One can confirm that $\mathcal{C}_0$ is complete with the metric induced by the norm $\|\cdot\|_\infty$.

Fix an arbitrary function $x(t) \in \mathscr{A}$ and define an operator Π on $\mathcal{C}_0$ through the equation

$$\Pi w(t) = \int_{-\infty}^{\infty} G_\beta(t,s)[F_\beta(w(s),s) + h(x(s))]ds, \tag{3.6}$$

where F_β and G_β are defined by (3.2) and (3.4), respectively. Let $w(t)$ belong to $\mathcal{C}_0$. One can obtain for each $t \in \mathbb{R}$ that

$$\begin{aligned}\|F_\beta(w(t),t)\| &\leq \|w(t)\| \int_0^1 \left\|D_u f(\theta w(t) + \nu_\beta(t),t) - D_u f(\nu_\beta(t),t)\right\| d\theta \\ &\leq \|w(t)\|^2 \int_0^1 \int_0^1 \left\|D_{uu} f(\tau\theta w(t) + \nu_\beta(t),t)\right\| d\tau d\theta \\ &\leq N_2\|w(t)\|^2.\end{aligned} \tag{3.7}$$

The inequality (3.7) yields

$$\|\Pi w\|_\infty \leq \frac{2K(N_2\|w\|^2_\infty + M_h)}{\alpha} \leq R_0,$$

and therefore, $\Pi(\mathcal{C}_0) \subseteq \mathcal{C}_0$.

Now, suppose that $w_1(t)$ and $w_2(t)$ belong to the set $\mathcal{C}_0$. It can be verified that

$$\Pi w_1(t) - \Pi w_2(t) = \int_{-\infty}^{\infty} G_\beta(t,s)\widetilde{F}_\beta(w_1(s),w_2(s),s)ds,$$

where the function $\widetilde{F}_\beta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is defined through the equation

$$\widetilde{F}_\beta(z_1,z_2,t) = f(z_1 + \nu_\beta(t),t) - f(z_2 + \nu_\beta(t),t) - D_u f(\nu_\beta(t),t)(z_1 - z_2). \tag{3.8}$$

For each $t \in \mathbb{R}$, we have that

$$\begin{aligned}
&\|\widetilde{F}_\beta(w_1(t), w_2(t), t)\| \\
&\quad\leq \|w_1(t) - w_2(t)\| \int_0^1 \left\|D_u f(\theta w_1(t) + (1-\theta)w_2(t) + \nu_\beta(t), t) - D_u f(\nu_\beta(t), t)\right\| d\theta \\
&\quad\leq \|w_1(t) - w_2(t)\| \int_0^1 \int_0^1 \left\|D_{uu} f(\tau(\theta w_1(t) + (1-\theta)w_2(t)) + \nu_\beta(t), t)\right\| \\
&\qquad\qquad \times (\theta\|w_1(t)\| + (1-\theta)\|w_2(t)\|) d\tau d\theta \\
&\quad\leq \|w_1(t) - w_2(t)\| \max\{\|w_1(t)\|, \|w_2(t)\|\} \\
&\qquad \times \int_0^1 \int_0^1 \left\|D_{uu} f(\tau(\theta w_1(t) + (1-\theta)w_2(t)) + \nu_\beta(t), t)\right\| d\tau d\theta \\
&\quad\leq N_2 \|w_1(t) - w_2(t)\| \max\{\|w_1(t)\|, \|w_2(t)\|\}.
\end{aligned} \tag{3.9}$$

One can confirm using the last inequality that

$$\|\Pi w_1 - \Pi w_2\|_\infty \leq \frac{2KN_2R_0}{\alpha} \|w_1 - w_2\|_\infty < \frac{1}{2} \|w_1 - w_2\|_\infty .$$

Hence, the operator Π is a contraction. Consequently, for each $x(t) \in \mathscr{A}$, there exists a unique solution $\phi^\beta_{x(t)}(t)$ of system (3.1) which is bounded on the real axis such that $\sup_{t\in\mathbb{R}} \left\|\phi^\beta_{x(t)}(t)\right\| \leq R_0$. □

4 Li–Yorke chaos

In the pioneer paper [27], chaos is considered with infinitely many periodic solutions separated from the elements of a scrambled set. In the present study, we will make use of a modified version of Li–Yorke chaos such that infinitely many almost periodic motions take place in the dynamics instead of periodic ones. Such a modification was first considered in the paper [8]. Since the concept of chaotic set of functions will be used in the theoretical discussions, let us explain the ingredients of Li–Yorke chaos with infinitely many almost periodic motions [6–8].

Let Γ be a set of uniformly bounded functions $\psi : \mathbb{R} \to \mathbb{R}^r$. A couple of functions $\left(\psi(t), \overline{\psi}(t)\right) \in \Gamma \times \Gamma$ is called proximal if for an arbitrary small number $\epsilon > 0$ and an arbitrary large number $E > 0$ there exists an interval $J \subset \mathbb{R}$ with a length no less than E such that $\left\|\psi(t) - \overline{\psi}(t)\right\| < \epsilon$ for all $t \in J$. Besides, a couple of functions $\left(\psi(t), \overline{\psi}(t)\right) \in \Gamma \times \Gamma$ is frequently (ϵ_0, Δ)-separated if there exist numbers $\epsilon_0 > 0$, $\Delta > 0$ and infinitely many disjoint intervals each with a length no less than Δ such that $\left\|\psi(t) - \overline{\psi}(t)\right\| > \epsilon_0$ for each t from these intervals. It is worth noting that the numbers ϵ_0 and Δ may depend on the functions $\psi(t)$ and $\overline{\psi}(t)$.

We say that a couple of functions $\left(\psi(t), \overline{\psi}(t)\right) \in \Gamma \times \Gamma$ is a Li–Yorke pair if they are proximal and frequently (ϵ_0, Δ)-separated for some positive numbers ϵ_0 and Δ.

The description of a Li–Yorke chaotic set with infinitely many almost periodic motions is given in the next definition [6–8].

Definition 4.1. Γ is called a Li–Yorke chaotic set with infinitely many almost periodic motions if:

(i) there exists a countably infinite set $\mathscr{C} \subset \Gamma$ of almost periodic functions;

(ii) there exists an uncountable set $\mathscr{U} \subset \Gamma$, the scrambled set, such that the intersection of $\mathscr{U}$ and $\mathscr{C}$ is empty and each couple of different functions inside $\mathscr{U} \times \mathscr{U}$ is a Li–Yorke pair;

(iii) for any function $\psi(t) \in \mathscr{U}$ and any almost periodic function $\overline{\psi}(t) \in \mathscr{C}$, the couple $\big(\psi(t), \overline{\psi}(t)\big)$ is frequently $(\epsilon_0, \Delta)-$separated for some positive numbers ϵ_0 and Δ.

In order to study the existence of chaos theoretically in the dynamics of system (2.2), let us introduce the sets of functions

$$\mathscr{B}_\beta = \left\{\phi^{\beta}_{x(t)}(t) : \; x(t) \in \mathscr{A}\right\}, \qquad \beta \in S_\sigma. \tag{4.1}$$

Lemma 4.2. *Under the conditions* (C1)–(C3)*, if a couple* $(x(t), \overline{x}(t)) \in \mathscr{A} \times \mathscr{A}$ *is proximal, then the same is true for the couple* $\big(\phi^{\beta}_{x(t)}(t), \phi^{\beta}_{\overline{x}(t)}(t)\big) \in \mathscr{B}_\beta \times \mathscr{B}_\beta$, $\beta \in S_\sigma$.

Proof. Fix an arbitrary small positive number ϵ, and let μ be a number such that

$$\mu \geq 1 + \frac{2KL_2}{\alpha - 2KN_2R_0}.$$

Suppose that E is an arbitrary large positive number satisfying $E > \frac{1}{\delta} \ln\left(\frac{2H_0\mu}{\epsilon}\right)$, where

$$\delta = \sqrt{\alpha^2 - 2KN_2R_0\alpha}$$

and

$$H_0 = \frac{2\left(\alpha - \sqrt{\alpha^2 - 2KN_2R_0\alpha}\right)(M_h + N_2R_0^2)}{\alpha N_2 R_0}.$$

Because the couple $(x(t), \overline{x}(t)) \in \mathscr{A} \times \mathscr{A}$ is proximal, there exist a real number a_0 and a number $E_0 \geq E$ such that the inequality $\|x(t) - \overline{x}(t)\| < \frac{\epsilon}{\mu}$ holds for all $t \in [a_0, a_0 + 3E_0]$.

The bounded solutions $\phi^{\beta}_{x(t)}(t)$ and $\phi^{\beta}_{\overline{x}(t)}(t)$ of system (3.1) satisfy the relation

$$\phi^{\beta}_{x(t)}(t) - \phi^{\beta}_{\overline{x}(t)}(t) = \int_{-\infty}^{\infty} G_\beta(t,s)\left(\widetilde{F}_\beta\left(\phi^{\beta}_{x(t)}(s), \phi^{\beta}_{\overline{x}(t)}(s), s\right) + h(x(s)) - h(\overline{x}(s))\right) ds,$$

where the functions G_β and $\widetilde{F}_\beta$ are defined by (3.4) and (3.8), respectively. According to (3.9) we have for $t \in \mathbb{R}$ that

$$\left\|\widetilde{F}_\beta\left(\phi^{\beta}_{x(t)}(t), \phi^{\beta}_{\overline{x}(t)}(t), t\right)\right\| \leq N_2R_0\left\|\phi^{\beta}_{x(t)}(t) - \phi^{\beta}_{\overline{x}(t)}(t)\right\| \leq 2N_2R_0^2. \tag{4.2}$$

Making use of the inequality (4.2) it can be verified for $t \in [a_0, a_0 + 3E_0]$ that

$$\begin{aligned}\left\|\phi^{\beta}_{x(t)}(t) - \phi^{\beta}_{\overline{x}(t)}(t)\right\| \leq{}& \frac{2KL_2\epsilon}{\mu\alpha} + \frac{2K(M_h + N_2R_0^2)}{\alpha}\left(e^{-\alpha(t-a_0)} + e^{-\alpha(a_0+3E_0-t)}\right) \\ &+ KN_2R_0\int_{a_0}^{a_0+3E_0} e^{-\alpha|t-s|}\left\|\phi^{\beta}_{x(t)}(s) - \phi^{\beta}_{\overline{x}(t)}(s)\right\| ds.\end{aligned}$$

Now, we obtain by applying Theorem A.1 given in the Appendix that

$$\left\|\phi^{\beta}_{x(t)}(t) - \phi^{\beta}_{\overline{x}(t)}(t)\right\| \leq \frac{2KL_2\epsilon}{\mu(\alpha - 2KN_2R_0)} + H_0\left(e^{-\delta(t-a_0)} + e^{-\delta(a_0+3E_0-t)}\right).$$

Since $E > \frac{1}{\delta} \ln \left(\frac{2H_0\mu}{\epsilon}\right)$, the inequality

$$H_0 \left(e^{-\delta(t-a_0)} + e^{-\delta(a_0+3E_0-t)}\right) < \frac{\epsilon}{\mu}$$

is valid for $t \in J$, where $J = [a_0 + E_0, a_0 + 2E_0]$. Thus, if t belongs to the interval J, then we have that

$$\left\| \phi^{\beta}_{x(t)}(t) - \phi^{\beta}_{\overline{x}(t)}(t) \right\| < \left(1 + \frac{2KL_2}{\alpha - 2KN_2R_0}\right) \frac{\epsilon}{\mu} \leq \epsilon.$$

Consequently, the couple $\left(\phi^{\beta}_{x(t)}(t), \phi^{\beta}_{\overline{x}(t)}(t)\right) \in \mathscr{B}_\beta \times \mathscr{B}_\beta$ is proximal. □

Remark 4.3. The interval J mentioned in the proof of Lemma 4.2 is uniform for each $\beta \in S_\sigma$.

The next assertion is concerned with the second ingredient, the frequent separation feature, of Li–Yorke chaos.

Lemma 4.4. *Assume that the conditions (C1)–(C3) hold. If a couple $(x(t), \overline{x}(t)) \in \mathscr{A} \times \mathscr{A}$ is frequently (ϵ_0, Δ)-separated for some positive numbers ϵ_0 and Δ, then the couple $\left(\phi^{\beta}_{x(t)}(t), \phi^{\beta}_{\overline{x}(t)}(t)\right) \in \mathscr{B}_\beta \times \mathscr{B}_\beta$ is frequently $(\epsilon_1, \overline{\Delta})$-separated for some positive numbers ϵ_1 and $\overline{\Delta}$ uniform for each $\beta \in S_\sigma$.*

Proof. Because the couple $(x(t), \overline{x}(t)) \in \mathscr{A} \times \mathscr{A}$ is frequently (ϵ_0, Δ)-separated, there exist infinitely many disjoint intervals J_k, $k \in \mathbb{N}$, each with a length no less than Δ such that $\|x(t) - \overline{x}(t)\| > \epsilon_0$ for each t from these intervals.

Suppose that $h(x) = (h_1(x), h_2(x), \ldots, h_n(x))$, where each h_j, $j = 1, 2, \ldots, n$, is a real valued function. The function $\widetilde{h} : \Lambda \times \Lambda \to \mathbb{R}^n$ defined by $\widetilde{h}(z_1, z_2) = h(z_1) - h(z_2)$ is uniformly continuous on $\Lambda \times \Lambda$. Since $\mathscr{A}$ is an equicontinuous family on $\mathbb{R}$, the set of functions whose elements are of the form $h_j(x(t)) - h_j(\overline{x}(t))$, $j = 1, 2, \ldots, n$, where $x(t), \overline{x}(t) \in \mathscr{A}$, is also an equicontinuous family on $\mathbb{R}$. Therefore, there exists a positive number $\tau < \Delta$, which does not depend on the functions $x(t)$ and $\overline{x}(t)$, such that for every $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \tau$, the inequality

$$\left| \left(h_j\left(x(t_1)\right) - h_j\left(\overline{x}(t_1)\right)\right) - \left(h_j\left(x(t_2)\right) - h_j\left(\overline{x}(t_2)\right)\right) \right| < \frac{L_1\epsilon_0}{2\sqrt{n}} \tag{4.3}$$

holds for all $j = 1, 2, \ldots, n$.

Fix a natural number k. Let us denote $\xi_k = \eta_k - \tau/2$, where η_k is the midpoint of the interval J_k. There exists an integer j_k, $1 \leq j_k \leq n$, such that

$$\left| h_{j_k}(x(\eta_k)) - h_{j_k}(\overline{x}(\eta_k)) \right| \geq \frac{L_1}{\sqrt{n}} \left\| x(\eta_k) - \overline{x}(\eta_k) \right\| > \frac{L_1\epsilon_0}{\sqrt{n}}. \tag{4.4}$$

For $t \in [\xi_k, \xi_k + \tau]$, one can confirm by means of (4.3) that

$$\left| h_{j_k}\left(x(\eta_k)\right) - h_{j_k}\left(\overline{x}(\eta_k)\right) \right| - \left| h_{j_k}\left(x(t)\right) - h_{j_k}\left(\overline{x}(t)\right) \right| < \frac{L_1\epsilon_0}{2\sqrt{n}}.$$

Accordingly, the inequality (4.4) yields

$$\left| h_{j_k}\left(x(t)\right) - h_{j_k}\left(\overline{x}(t)\right) \right| > \frac{L_1\epsilon_0}{2\sqrt{n}}, \; t \in [\xi_k, \xi_k + \tau].$$

Since there exist numbers $c_1, c_2, \ldots, c_n \in [\xi_k, \xi_k + \tau]$ such that the equation

$$\left\| \int_{\xi_k}^{\xi_k+\tau} (h(x(s)) - h(\overline{x}(s)))\, ds \right\| = \tau \left(\sum_{j=1}^{n} \left(h_j(x(c_j)) - h_j(\overline{x}(c_j))\right)^2 \right)^{1/2},$$

is valid, we have

$$\left\| \int_{\xi_k}^{\xi_k+\tau} (h(x(s)) - h(\overline{x}(s)))\, ds \right\| \geq \tau \left| h_{j_k}(x(c_{j_k})) - h_{j_k}(\overline{x}(c_{j_k})) \right| > \frac{\tau L_1 \epsilon_0}{2\sqrt{n}}. \tag{4.5}$$

The bounded solutions $\phi^{\beta}_{x(t)}(t)$ and $\phi^{\beta}_{\overline{x}(t)}(t)$ satisfy the equation

$$\begin{aligned}\phi^{\beta}_{x(t)}(t) - \phi^{\beta}_{\overline{x}(t)}(t) = {} & \phi^{\beta}_{x(t)}(\xi_k) - \phi^{\beta}_{\overline{x}(t)}(\xi_k) \\ & + \int_{\xi_k}^{t} \left[\widetilde{F}_{\beta}\left(\phi^{\beta}_{x(t)}(s), \phi^{\beta}_{\overline{x}(t)}(s), s\right) + D_u f(\nu_{\beta}(s), s) \left(\phi^{\beta}_{x(t)}(s) - \phi^{\beta}_{\overline{x}(t)}(s)\right)\right] ds \\ & + \int_{\xi_k}^{t} (h(x(s)) - h(\overline{x}(s)))\, ds,\end{aligned}$$

where the function $\widetilde{F}_{\beta}$ is defined by (3.8). Therefore, making use of the inequalities (4.2) and (4.5) we obtain that

$$\left\|\phi^{\beta}_{x(t)}(\xi_k+\tau) - \phi^{\beta}_{\overline{x}(t)}(\xi_k+\tau)\right\| > \frac{\tau L_1 \epsilon_0}{2\sqrt{n}} - [1 + (N_1 + N_2 R_0)\tau] \max_{t \in [\xi_k, \xi_k+\tau]} \left\|\phi^{\beta}_{x(t)}(t) - \phi^{\beta}_{\overline{x}(t)}(t)\right\|.$$

Hence, the inequality

$$\max_{t \in [\xi_k, \xi_k+\tau]} \left\|\phi^{\beta}_{x(t)}(t) - \phi^{\beta}_{\overline{x}(t)}(t)\right\| > \frac{\tau L_1 \epsilon_0}{2\left[2 + (N_1 + N_2 R_0)\tau\right]\sqrt{n}}$$

holds.

Now, suppose that

$$\max_{t \in [\xi_k, \xi_k+\tau]} \left\|\phi^{\beta}_{x(t)}(t) - \phi^{\beta}_{\overline{x}(t)}(t)\right\| = \left\|\phi^{\beta}_{x(t)}(\rho_k) - \phi^{\beta}_{\overline{x}(t)}(\rho_k)\right\|$$

for some $\rho_k \in [\xi_k, \xi_k + \tau]$.

Let us define the numbers

$$\epsilon_1 = \frac{\tau L_1 \epsilon_0}{4\left[2 + (N_1 + N_2 R_0)\tau\right]\sqrt{n}}$$

and

$$\overline{\Delta} = \min\left\{ \frac{\tau}{2}, \frac{\tau L_1 \epsilon_0}{8\left[2 + (N_1 + N_2 R_0)\tau\right]\left[M_h + (N_1 + N_2 R_0) R_0\right]\sqrt{n}} \right\}.$$

It is worth noting that ϵ_1 and $\overline{\Delta}$ do not depend on $\beta \in S_{\sigma}$. Moreover, we denote $\theta_k = \rho_k$ if $\xi_k \leq \rho_k \leq \xi_k + \tau/2$ and $\theta_k = \rho_k - \overline{\Delta}$ if $\xi_k + \tau/2 < \rho_k \leq \xi_k + \tau$.

Using the inequality

$$\begin{aligned}\left\|\phi^{\beta}_{x(t)}(t) - \phi^{\beta}_{\overline{x}(t)}(t)\right\| \geq {} & \left\|\phi^{\beta}_{x(t)}(\rho_k) - \phi^{\beta}_{\overline{x}(t)}(\rho_k)\right\| \\ & - \left| \int_{\rho_k}^{t} \left\| \widetilde{F}_{\beta}\left(\phi^{\beta}_{x(t)}(s), \phi^{\beta}_{\overline{x}(t)}(s), s\right) + D_u f(\nu_{\beta}(s), s)\left(\phi^{\beta}_{x(t)}(s) - \phi^{\beta}_{\overline{x}(t)}(s)\right)\right\| ds \right| \\ & - \left| \int_{\rho_k}^{t} \| h(x(s)) - h(\overline{x}(s)) \|\, ds \right|\end{aligned}$$

together with (4.2), it can be verified for $t \in \left[\theta_k, \theta_k + \overline{\Delta}\right]$ that

$$\left\|\phi^{\beta}_{x(t)}(t) - \phi^{\beta}_{\overline{x}(t)}(t)\right\| > \frac{\tau L_1 \epsilon_0}{2\left[2 + (N_1 + N_2 R_0)\tau\right]\sqrt{n}} - 2[M_h + (N_1 + N_2 R_0) R_0]\overline{\Delta} \geq \epsilon_1.$$

One can confirm that the intervals $[\theta_k, \theta_k + \overline{\Delta}]$, $k \in \mathbb{N}$, are disjoint. Consequently, the couple $\left(\phi^{\beta}_{x(t)}(t), \phi^{\beta}_{\overline{x}(t)}(t)\right) \in \mathscr{B}_\beta \times \mathscr{B}_\beta$ is frequently $(\epsilon_1, \overline{\Delta})$-separated uniform for each $\beta \in S_\sigma$. □

Next, we deal with the almost periodic solutions of system (3.1). A continuous function $\vartheta : \mathbb{R} \to \mathbb{R}^r$ is said to be almost periodic, if for any $\epsilon > 0$ there exists $l > 0$ such that for any interval with length l there exists a number T in this interval satisfying $\|\vartheta(t+T) - \vartheta(t)\| < \epsilon$ for all $t \in \mathbb{R}$ [22,26,37].

In the proof of the following assertion, we will make use of the operator Π defined by (3.6).

Lemma 4.5. *Suppose that the conditions* (C1)–(C3) *are satisfied. If $x(t) \in \mathscr{A}$ is an almost periodic function, then the bounded solution $\phi^{\beta}_{x(t)}(t)$, $\beta \in \mathcal{P}_\sigma$, of system* (3.1) *is also almost periodic.*

Proof. Let us denote by $\mathcal{C}_1$ the set of continuous almost periodic functions $w(t) : \mathbb{R} \to \mathbb{R}^n$ satisfying $\|w\|_\infty \leq R_0$, where the norm $\|\cdot\|_\infty$ is defined by (3.5). If $w(t) \in \mathcal{C}_1$, then one can confirm using the results of [22] that the functions $A_\beta(t) = D_u f(\nu_\beta(t), t)$ and $f_\beta(t) = F_\beta(w(t), t) + h(x(t))$ are almost periodic for $\beta \in \mathcal{P}_\sigma$. On the other hand, according to [16, p. 72], $\Pi w(t)$ is also almost periodic, where Π is the operator defined by (3.6). Therefore, $\Pi(\mathcal{C}_1) \subseteq \mathcal{C}_1$. Since the operator Π is contractive as shown in the proof of Lemma 3.1, its fixed point $\phi^{\beta}_{x(t)}(t)$ is almost periodic for each $\beta \in \mathcal{P}_\sigma$. □

Now, we state and prove our main theorem.

Theorem 4.6. *Suppose that the conditions* (C1)–(C3) *are valid. If the collection $\mathscr{A}$ is Li–Yorke chaotic with infinitely many almost periodic motions, then the same is true for each of the collections $\mathscr{B}_\beta$, $\beta \in \mathcal{P}_\sigma$.*

Proof. Let $\mathscr{C} \subset \mathscr{A}$ be a countably infinite set of almost periodic functions, and for each $\beta \in \mathcal{P}_\sigma$ define the set

$$\mathscr{C}_\beta = \left\{\phi^{\beta}_{x(t)}(t) : \ x(t) \in \mathscr{C}\right\}.$$

Condition (C2) implies that there is a one-to-one correspondence between the sets $\mathscr{C}$ and $\mathscr{C}_\beta$, $\beta \in \mathcal{P}_\sigma$. Therefore, $\mathscr{C}_\beta \subset \mathscr{B}_\beta$ is also countably infinite for each $\beta \in \mathcal{P}_\sigma$. Furthermore, Lemma 4.5 implies that $\mathscr{C}_\beta$, $\beta \in \mathcal{P}_\sigma$, consists of almost periodic functions.

Next, we denote by $\mathscr{U} \subset \mathscr{A}$ an uncountable scrambled set. Let us introduce the sets

$$\mathscr{U}_\beta = \left\{\phi^{\beta}_{x(t)}(t) : \ x(t) \in \mathscr{U}\right\},$$

where $\beta \in \mathcal{P}_\sigma$. It can be verified using condition (C2) one more time that the sets $\mathscr{U}_\beta$, $\beta \in \mathcal{P}_\sigma$, are all uncountable, and no almost periodic functions take place in these sets, i.e., the intersection of $\mathscr{U}_\beta$ and $\mathscr{C}_\beta$ is empty.

Because each couple of functions inside $\mathscr{U} \times \mathscr{U}$ is proximal, Lemma 4.2 implies that the same is true for each couple inside $\mathscr{U}_\beta \times \mathscr{U}_\beta$, $\beta \in \mathcal{P}_\sigma$. On the other hand, according to Lemma 4.4, there exist positive numbers ϵ_1 and $\overline{\Delta}$ such that each couple of functions inside $\mathscr{U}_\beta \times \mathscr{U}_\beta$ is frequently $(\epsilon_1, \overline{\Delta})$-separated. Lemma 4.4 also implies the presence of the frequent separation feature for each couple inside $\left(\mathscr{U}_\beta \times \mathscr{C}_\beta\right)$, $\beta \in \mathcal{P}_\sigma$. Consequently, each of the collections $\mathscr{B}_\beta$, $\beta \in \mathcal{P}_\sigma$, is Li–Yorke chaotic. □

Remark 4.7. System (2.1) may possess bounded solutions other than $\{\nu_\beta(t)\}_{\beta\in S_\sigma}$. Therefore, there may exist a chaotic set corresponding to each of such solutions, but its verification is a difficult task in general, which would require additional assumptions on the system.

It is worth noting that the criterion in Definition 4.1 for the existence of a countably infinite subset of almost periodic functions in a Li–Yorke chaotic can be replaced with the existence of a countably infinite subset of quasi-periodic functions [7]. In the following section, we will exemplify Li–Yorke homoclinic chaos with infinitely many quasi-periodic motions.

5 An example

This part of the paper is devoted to an illustrative example. First of all, we will take into account a forced Duffing equation, which is Li–Yorke chaotic with infinitely many quasi-periodic motions, as the source of chaotic perturbations. A relay function will be used in this equation as the forcing term to ensure the presence of chaos. Detailed theoretical as well as numerical results concerning relay systems can be found in the studies [1,2,5]. Next, to provide the Li–Yorke homoclinic chaos, we will perturb another Duffing equation, which admits a homoclinic orbit, by the solutions of the former one.

Another issue that we will focus on is the stabilization of unstable quasi-periodic motions. In the literature, control of chaos is understood as the stabilization of unstable periodic orbits embedded in a chaotic attractor [20,38]. However, we will demonstrate the stabilization of quasi-periodic motions instead of periodic ones. The presence of chaos with infinitely many unstable quasi-periodic motions will be revealed by means of an appropriate chaos control technique based on the Ott–Grebogi–Yorke (OGY) [32] and Pyragas [35] control methods.

Let us consider the following forced Duffing equation,

$$x'' + 0.82x' + 1.4x + 0.01x^3 = 0.25\sin(3t) + v(t,\zeta,\lambda), \tag{5.1}$$

where the relay function $v(t,\zeta,\lambda)$ is defined as

$$v(t,\zeta,\lambda) = \begin{cases} 0.3, & \text{if } \zeta_{2j} < t \le \zeta_{2j+1},\ j\in\mathbb{Z}, \\ 1.9, & \text{if } \zeta_{2j-1} < t \le \zeta_{2j},\ j\in\mathbb{Z}. \end{cases} \tag{5.2}$$

In (5.2), the sequence $\zeta = \{\zeta_j\}_{j\in\mathbb{Z}}$, $\zeta_0 \in [0,1]$, of switching moments is defined through the equation $\zeta_j = j + \kappa_j$, $j\in\mathbb{Z}$, and the sequence $\{\kappa_j\}_{j\in\mathbb{Z}}$ is a solution of the logistic map

$$\kappa_{j+1} = \lambda\kappa_j(1-\kappa_j), \tag{5.3}$$

where λ is a real parameter. The interval $[0,1]$ is invariant under the iterations of (5.3) for the values of λ between 1 and 4 [23], and the map possesses Li-Yorke chaos for $\lambda = 3.9$ [27].

Making use of the new variables $x_1 = x$ and $x_2 = x'$, one can reduce (5.1) to the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -1.4x_1 - 0.82x_2 - 0.01x_1^3 + 0.25\sin(3t) + v(t,\zeta,\lambda). \end{aligned} \tag{5.4}$$

For each $\zeta_0 \in [0,1]$, system (5.4) with $\lambda = 3.9$ possesses a solution which is bounded on the whole real axis, and the collection $\mathscr{A}$ consisting of all such bounded solutions is Li–Yorke chaotic with infinitely many quasi-periodic motions [2,6,7]. Moreover, for each natural number ρ, system (5.4) admits unstable periodic solutions with periods 2ρ.

Figure 5.1 shows the solution of system (5.4) with $\lambda = 3.9$ and $\zeta_0 = 0.41$ corresponding to the initial data $x_1(0.41) = 0.8$, $x_2(0.41) = 0.7$. The simulation results seen in Figure 5.1 confirm the presence of chaos in (5.4).

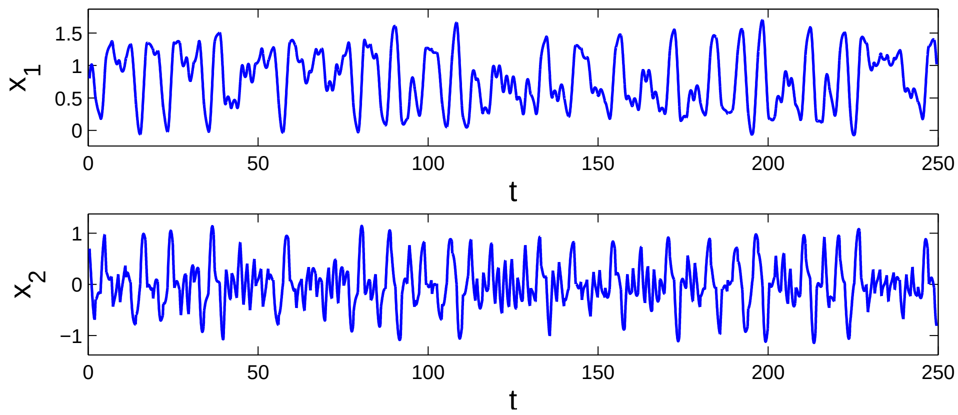

Figure 5.1: The chaotic behavior of system (5.4).

Next, let us consider the following Duffing equation [9],

$$z'' + 0.15z' - 0.5z(1 - z^2) = 0.2\sin(0.9t). \tag{5.5}$$

It was mentioned in paper [9] that the equation (5.5) is chaotic, and it admits a homoclinic orbit.

By means of the variables $z_1 = z$ and $z_2 = z'$, equation (5.5) can be written as a system in the form,

$$\begin{aligned} z_1' &= z_2 \\ z_2' &= -0.15z_2 + 0.5z_1(1 - z_1^2) + 0.2\sin(0.9t). \end{aligned} \tag{5.6}$$

We perturb system (5.6) with the solutions of (5.4) and set up the system

$$\begin{aligned} u_1' &= u_2 + 1.9(x_1(t) + 0.4\sin(x_1(t))) \\ u_2' &= -0.15u_2 + 0.5u_1(1 - u_1^2) + 1.3x_2(t) + 0.2\sin(0.9t). \end{aligned} \tag{5.7}$$

System (5.7) is in the form of (2.2) with

$$f(u_1, u_2, t) = \begin{pmatrix} u_2 \\ -0.15u_2 + 0.5u_1(1 - u_1^2) + 0.2\sin(0.9t) \end{pmatrix}$$

and

$$h(x_1, x_2) = \begin{pmatrix} 1.9(x_1 + 0.4\sin(x_1)) \\ 1.3x_2 \end{pmatrix}.$$

According to Theorem 4.6, system (5.7) possesses Li–Yorke chaos with infinitely many quasi-periodic motions.

In order to simulate the chaotic behavior, we use the solution $(x_1(t), x_2(t))$ of (5.4) which is represented in Figure 5.1 as the perturbation in (5.7), and depict in Figure 5.2 the solution of (5.7) with the initial data $u_1(0.41) = 0.12$, $u_2(0.41) = 0.013$. Figure 5.2 supports the result of Theorem 4.6 such that the perturbed system (5.7) exhibits chaos.

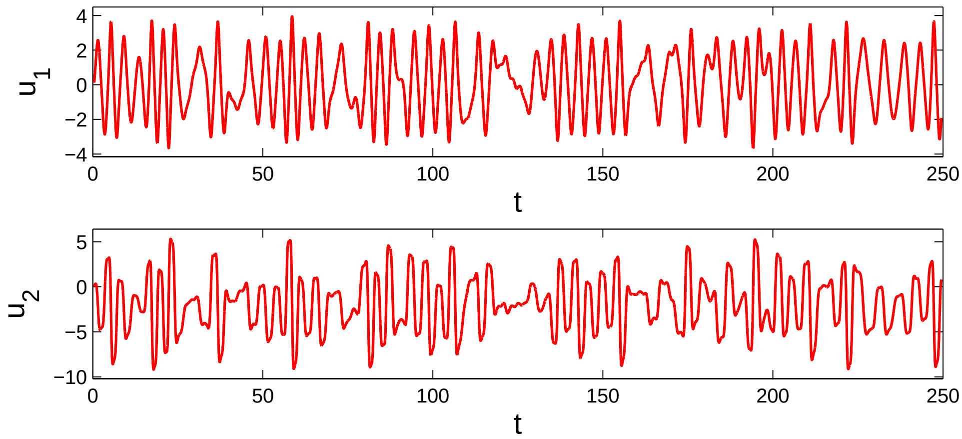

Figure 5.2: The chaotic solution of system (5.7) with $u_1(0.41) = 0.12$ and $u_2(0.41) = 0.013$. The solution $(x_1(t), x_2(t))$ of (5.4), which is represented in Figure 5.1, is used as the perturbation in (5.7).

Next, we depict in Figure 5.3 the trajectories of (5.6) and (5.7) corresponding to the initial data $z_1(0.41) = 0.12$, $z_2(0.41) = 0.013$ and $u_1(0.41) = 0.12$, $u_2(0.41) = 0.013$, respectively. Here, the trajectory of (5.6) is depicted in blue and the trajectory of (5.7) is shown in red. It is seen in Figure 5.3 that even if the same initial data is used, systems (5.6) and (5.7) generate completely different chaotic trajectories.

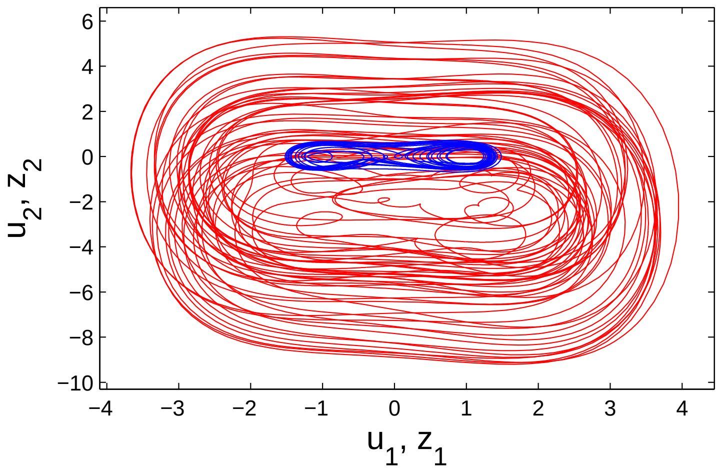

Figure 5.3: Chaotic trajectories of systems (5.6) and (5.7). The trajectory of (5.6) with $z_1(0.41) = 0.12$, $z_2(0.41) = 0.013$ is represented in blue color, while the trajectory of (5.7) corresponding to $u_1(0.41) = 0.12$, $u_2(0.41) = 0.013$ is shown in red color. One can observe that the unperturbed system (5.6) and the perturbed system (5.7) possess different chaotic motions.

Now, we will confirm the presence of chaos with infinitely many quasi-periodic motions in the dynamics of (5.7) by stabilizing one of them through a control technique based on the OGY [32] and Pyragas [35] methods. The idea of the control procedure depends on the usage of both the OGY control for the discrete-time dynamics of the logistic map (5.3), which the source of chaotic motions in the forced Duffing equation (5.4), and the Pyragas control for the continuous-time dynamics of (5.7). The simultaneous usage of both methods will give rise to the stabilization of a quasi-periodic solution of (5.7) since (5.4) and (5.6) admit unstable periodic motions with incommensurate periods.

Let us explain briefly the OGY control method for the map (5.3) [38]. Suppose that the parameter λ in the map (5.3) is allowed to vary in the range $[3.9-\varepsilon, 3.9+\varepsilon]$, where ε is a given small positive number. Consider an arbitrary solution $\{\kappa_j\}$, $\kappa_0 \in [0,1]$, of the map and denote by $\kappa^{(i)}$, $i = 1,2,\ldots,p$, the target p-periodic orbit to be stabilized. In the OGY control method [38], at each iteration step j after the control mechanism is switched on, we consider the logistic map with the parameter value $\lambda = \overline{\lambda}_j$, where

$$\overline{\lambda}_j = 3.9\left(1+\frac{(2\kappa^{(i)}-1)(\kappa_j-\kappa^{(i)})}{\kappa^{(i)}(1-\kappa^{(i)})}\right), \tag{5.8}$$

provided that the number on the right hand side of the formula (5.8) belongs to the interval $[3.9-\varepsilon, 3.9+\varepsilon]$. In other words, formula (5.8) is valid if the trajectory $\{\kappa_j\}$ is sufficiently close to the target periodic orbit. Otherwise, we take $\overline{\lambda}_j = 3.9$ so that the system evolves at its original parameter value and wait until the trajectory $\{\kappa_j\}$ enters in a sufficiently small neighborhood of the periodic orbit $\kappa^{(i)}$, $i = 1,2,\ldots,p$, such that the inequality $-\varepsilon \le 3.9\frac{(2\kappa^{(i)}-1)(\kappa_j-\kappa^{(i)})}{\kappa^{(i)}(1-\kappa^{(i)})} \le \varepsilon$ holds. If this is the case, the control of chaos is not achieved immediately after switching on the control mechanism. Instead, there is a transition time before the desired periodic orbit is stabilized. The transition time increases if the number ε decreases [20].

On the other hand, according to the Pyragas control method [20, 35], an unstable periodic solution with period τ_0 can be stabilized by using an external perturbation of the form $C[s(t-\tau_0)-s(t)]$, where C is the strength of the perturbation, $s(t)$ is a scalar signal which is given by some function of the state of the system and $s(t-\tau_0)$ is the signal measured with a time delay equal to τ_0.

To stabilize an unstable quasi-periodic solution of (5.7), we set up the system

$$\begin{aligned}
w_1' &= w_2 \\
w_2' &= -1.4w_1 - 0.82w_2 - 0.01w_1^3 + 0.25\sin(w_5) + v(t,\zeta,\overline{\lambda}_j) \\
w_3' &= w_4 + 1.9(w_1 + 0.4\sin(w_1)) \\
w_4' &= -0.15w_4 + 0.5w_3(1-w_3^2) + 1.3w_2 + 0.2\sin(0.9w_5) + C[w_4(t-2\pi/0.9) - w_4(t)] \\
w_5' &= 1,
\end{aligned} \tag{5.9}$$

which we call the control system corresponding to the coupled system (5.4)+(5.7).

Let us use the OGY control method around the fixed point 2.9/3.9 of the logistic map (5.3) so that $\overline{\lambda}_j$ in (5.9) is given by the formula (5.8) with $\kappa^{(i)} \equiv 2.9/3.9$. The control mechanism is switched on at $t = \zeta_{70}$ using the values $\varepsilon = 0.085$ and $C = 2.6$. The OGY control is switched off at $t = \zeta_{350}$ and the Pyragas control is switched off at $t = \zeta_{400}$. More precisely, we take $\overline{\lambda}_j \equiv 3.9$, $C = 2.6$ for $\zeta_{350} \le t < \zeta_{400}$, and we take $\overline{\lambda}_j \equiv 3.9$, $C = 0$ for $\zeta_{400} \le t \le \zeta_{550}$. Figure 5.4 shows the simulation results for the w_3 and w_4 coordinates of the control system (5.9) corresponding to the initial data $w_1(0.41) = 0.8$, $w_2(0.41) = 0.7$, $w_3(0.41) = 0.12$, $w_4(0.41) = 0.013$, and $w_5(0.41) = 0.41$. It is seen in Figure 5.4 that one of the quasi-periodic solutions of (5.7) is stabilized such that the control becomes dominant approximately at $t = 144$ and its effect lasts until $t = 400$, after which the instability becomes dominant and irregular behavior develops again. To present a better visuality, the stabilized quasi-periodic solution of (5.7) is shown in Figure 5.5 for $200 \le t \le 300$. Both of the Figures 5.4 and 5.5 confirm the presence of infinitely many quasi-periodic motions in the dynamics of (5.7).

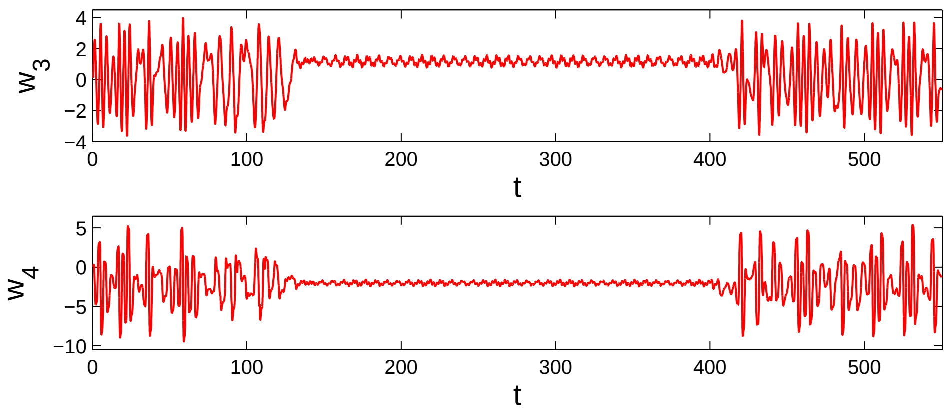

Figure 5.4: Chaos control of system (5.7). We make use of the OGY control method around the fixed point 2.9/3.9 of the logistic map (5.3). The values $\varepsilon = 0.085$ and $C = 2.6$ are used in the simulation.

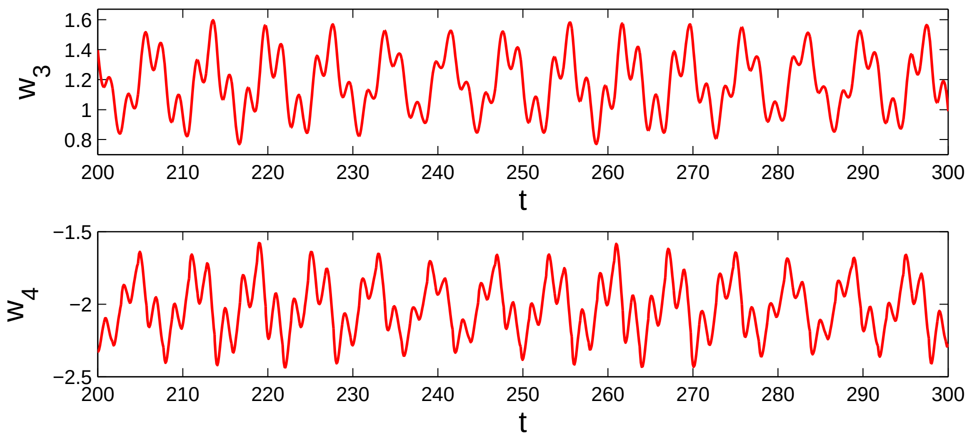

Figure 5.5: The stabilized quasi-periodic solution of system (5.7).

Appendix

For the convenience of the reader, we present and prove a Gronwall–Coppel type inequality (see [10]) result used in this paper.

Theorem A.1. *Let a, b, c, and γ be constants such that $a \ge 0$, $b \ge 0$, $c > 0$, $\gamma > 0$, and suppose that $\varphi(t) \in C([r_1, r_2], \mathbb{R})$ is a nonnegative function satisfying*

$$\varphi(t) \le a + b\left(e^{-\gamma(t-r_1)} + e^{-\gamma(r_2-t)}\right) + c\int_{r_1}^{r_2} e^{-\gamma|t-s|}\varphi(s)ds,\ t \in [r_1, r_2].$$

If $2c < \gamma$*, then*

$$\varphi(t) \le \frac{a\gamma}{\gamma - 2c} + \frac{b}{c}(\gamma - \delta)\left(e^{-\delta(t-r_1)} + e^{-\delta(r_2-t)}\right)$$

for any $t \in [r_1, r_2]$*, where* $\delta = \sqrt{\gamma^2 - 2c\gamma}$*.*

Proof. According to Theorems 2.3, 2.4 [10], the functions

$$\begin{aligned}
\varphi_1(t) &= \frac{b}{c}(\gamma - \delta)e^{-\delta(t-r_1)}, \quad t \ge r_1,\\
\varphi_2(t) &= \frac{b}{c}(\gamma - \delta)e^{-\delta(r_2-t)}, \quad t \le r_2,\\
\varphi_3(t) &= \frac{a\gamma}{\gamma - 2c}
\end{aligned}$$

satisfy the relations

$$\begin{aligned}
\varphi_1(t) &= be^{-\gamma(t-r_1)} + c\int_{r_1}^{\infty} e^{-\gamma|t-s|}\varphi_1(s)ds,\\
\varphi_2(t) &= be^{-\gamma(r_2-t)} + c\int_{-\infty}^{r_2} e^{-\gamma|t-s|}\varphi_2(s)ds,\\
\varphi_3(t) &= a + c\int_{-\infty}^{\infty} e^{-\gamma|t-s|}\varphi_3(s)ds,
\end{aligned}$$

respectively. Since $\varphi_1(t)$, $\varphi_2(t)$, and $\varphi_3(t)$ are all nonnegative, the inequality

$$\varphi_4(t) \ge a + b\left(e^{-\gamma(t-r_1)} + e^{-\gamma(r_2-t)}\right) + c\int_{r_1}^{r_2} e^{-\gamma|t-s|}\varphi_4(s)ds,$$

is valid for $t \in [r_1, r_2]$, where the function $\varphi_4(t)$ is defined by

$$\varphi_4(t) = \varphi_1(t) + \varphi_2(t) + \varphi_3(t).$$

Next, let us consider the operator $\Omega : C([r_1, r_2], \mathbb{R}) \to C([r_1, r_2], \mathbb{R})$ given by

$$\Omega\varphi(t) = a + b\left(e^{-\gamma(t-r_1)} + e^{-\gamma(r_2-t)}\right) + c\int_{r_1}^{r_2} e^{-\gamma|t-s|}\varphi(s)ds.$$

Then it is nondecreasing, and according to [10, p. 14] it is contractive. So it has a unique fixed point $\varphi_*(t) \in C([r_1, r_2], \mathbb{R})$, i.e., $\varphi_* = \Omega\varphi_*$. Since $\varphi(t) \le \Omega\varphi(t)$ and $\varphi_4(t) \ge \Omega\varphi_4(t)$, by standard arguments (see [10, Theorem 2.2]), we get $\varphi(t) \le \varphi_*(t) \le \varphi_4(t)$. The theorem is proved. □

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