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Moving average network examples for asymptotically stable periodic orbits of monotone maps

To Professor László Hatvani, with respect and affection

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Abstract. For a certain type of discrete-time nonlinear consensus dynamics, asymptotically stable periodic orbits are constructed. Based on a simple ordinal pattern assumption, the Frucht graph, two Petersen septets, hypercubes, a technical class of circulant graphs (containing Paley graphs of prime order), and complete graphs are considered – they are all carrying moving average monotone dynamics admitting asymptotically stable periodic orbits with period 2. Carried by a directed graph with 594 (multiple and multiple loop) edges on 3 vertices, also the existence of asymptotically stable r-periodic orbits, $r = 3, 4, \ldots$ is shown.

Keywords: consensus dynamics, periodic orbits, monotone maps, graph eigenvectors, ordinal patterns

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1 Introduction and the main result

Let G be a (simple, undirected) graph with vertices $V(G) = \{1, 2, \ldots, N\}$ and edges $E(G)$. As usual, A_G denotes the adjacency matrix of G (defined by letting $a_{ij} = 1$ if $(i, j) \in E(G)$ and 0 if $(i, j) \notin E(G)$). Let $D_G = \operatorname{diag}(d_1, d_2, \ldots, d_N)$ denote the degree matrix of G. Assuming $d_i \geq 1$ for $i = 1, 2, \ldots, N$, set $P_G = D_G^{-1} A_G$. Matrix P_G is a row stochastic matrix, the transition matrix of the random walk on G. The diagonal elements of P_G are zeros. Formula $P_G = D_G^{-1/2} (D_G^{-1/2} A_G \, D_G^{-1/2}) D_G^{1/2}$ shows that P_G is conjugate to a symmetric matrix and its eigenvalues $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_N$ are real [22]. The greatest eigenvalue of P_G is $\nu_1 = 1$ and $\mathbf{1}_N = \operatorname{col}(1, 1, \ldots, 1) \in \mathbb{R}^N$ is an eigenvector belonging to ν_1. By the trace theorem, $\sum_{i=1}^{N} \nu_i = 0$.

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The aim of this paper is to study the existence of periodic orbits in iterates of the nonlinear mapping

$$\mathcal{F} : [\omega,\Omega]^N \to [\omega,\Omega]^N, \qquad (\mathcal{F}(\mathbf{x}))_i = f\big((P_G\mathbf{x})_i\big) \qquad (\text{for } i = 1,2,\dots,N). \tag{1.1}$$

Here, once for all, $[\omega,\Omega]$ stands for a finite interval and $f : [\omega,\Omega] \to [\omega,\Omega]$ denotes a C^∞ function with the property that

$$f'(x) > 0 \quad \text{for each } x \in [\omega,\Omega]. \tag{1.2}$$

Note that

$$(P_G\mathbf{x})_i = \frac{1}{d_i} \sum_{\{j \mid (i,j)\in E(G)\}} x_j,$$

the local average of the neighboring x_j's at vertex i, $i = 1,2,\dots,N$.

Inequality (1.2) is a natural requirement on f implying that $\mathcal{F}$ is, in the sense of Hirsch, a monotone mapping. If matrix P_G is primitive, then $\mathcal{F}$ is eventually strongly monotone. Both implications follow directly from formula

$$\mathcal{F}'(\mathbf{x}) = \operatorname{diag}\big(f'((P_G\mathbf{x})_1),\dots,f'((P_G\mathbf{x})_N)\big)P_G \quad \text{for each } \mathbf{x} \in [\omega,\Omega]^N. \tag{1.3}$$

Adapted to the case to be investigated, we recall the definitions from [18] for convenience. By letting $\mathbf{x} \le \mathbf{y}$ if and only if $x_i \le y_i$ for each i, a closed partial order on $[\omega,\Omega]^N$ is introduced. We write $\mathbf{x} \prec \mathbf{y}$ if $x_i < y_i$ for each i. Mapping $\mathcal{F}$ is monotone if $\mathcal{F}(\mathbf{x}) \le \mathcal{F}(\mathbf{y})$ whenever $\mathbf{x} \le \mathbf{y}$. Monotonicity is strong if $\mathcal{F}(\mathbf{x}) \prec \mathcal{F}(\mathbf{y})$ whenever $\mathbf{x} \le \mathbf{y}$ and $\mathbf{x} \ne \mathbf{y}$. If only $\mathcal{F}^k(\mathbf{x}) \prec \mathcal{F}^k(\mathbf{y})$ for some integer $k > 1$, then $\mathcal{F}$ is eventually strongly monotone. Finally, recall that a non–negative square matrix A is primitive if A^k is positive for some integer $k \ge 1$. (Both non–negativity and positivity are understood for all matrix entries.)

In the special case $f = \mathrm{id}_{[\omega,\Omega]}$ formula (1.1) reduces to $\mathcal{F}(\mathbf{x}) = P_G\mathbf{x}$ for $\mathbf{x} \in [\omega,\Omega]^N$, the standard example both for random walks [22] and for consensus dynamics [20]. If matrix A_G is primitive, then $|\nu_i| < 1$ for $i \ne 1$ and, with $\mathbf{e}_L$ and $\mathbf{e}_R = \mathbf{1}_N$ denoting the left and the right Perron–Frobenius eigenvectors (normalized by the scalar product requirement $\langle \mathbf{e}_L, \mathbf{e}_R\rangle = 1$), $\mathcal{F}^k(\mathbf{x}) = P_G^k\mathbf{x} \to \langle \mathbf{e}_L, \mathbf{x}\rangle \mathbf{e}_R$ as $k \to \infty$. In particular, periodic orbits of $\mathcal{F} = P_G$ are homogeneous equilibria and vice versa. Composition with a nonlinear function in (1.1) makes the existence of nontrivial, asymptotically stable periodic orbits possible.

Let us recall here that existence versus nonexistence of nontrivial, asymptotically stable periodic orbits is one of the most striking differences between discrete-time and continuous-time monotone dynamics. This is thoroughly discussed in the long survey paper by Morris W. Hirsch and Hal L. Smith [18]. See also Remark 5.1 in the last Section of the present paper.

Under suitable conditions on matrix P_G, our Theorem 1.1 below is a simple construction for a mapping $\mathcal{F}$ satisfying (1.1)–(1.2) which admits an asymptotically stable periodic orbit of period 2. Our second main result is of a somewhat different character and concerns the 3 by 3 matrix

$$P_{G_*} = \frac{1}{198}\begin{pmatrix} 38 & 20 & 140 \\ 89 & 104 & 5 \\ 71 & 74 & 53 \end{pmatrix}, \tag{1.4}$$

the transition matrix of the random walk on a directed graph G_* on three vertices with multiple and (multiple) loop edges. The total number of edges is 594. Starting from matrix P_{G_*}, two families of nonlinear mappings $\mathcal{F}_r : [\omega,\Omega]^3 \to [\omega,\Omega]^3$ and $f_r : [\omega,\Omega] \to [\omega,\Omega]$ with properties (1.1)–(1.2) will be constructed in such a way that $\mathcal{F}_r$ admits an asymptotically stable periodic orbit of minimal period r, $r = 3, 4, 5, \dots$ Details, with the construction of P_{G_*} included, will be given in Section 4 devoted entirely to Theorem 4.2. After case $r = 3$, the induction step $r \to r + 1$ is well-prepared and easy.

Now we are in a position to state our result on asymptotically stable periodic orbits of period 2.

Theorem 1.1. *Suppose we are given a pair of vectors* $\mathbf{p} \nparallel \pm\mathbf{1}_N$ *and* $\mathbf{u} \neq \mathbf{0}_N$ *in* $\mathbb{R}^N$ *with the properties that*

$$p_i \lesseqgtr p_j \text{ if and only if } (P_G\mathbf{p})_i \lesseqgtr (P_G\mathbf{p})_j \text{ for } i, j = 1, 2, \dots, N\,, \tag{1.5}$$

$$u_i \lesseqgtr 0 \text{ if and only if } -(P_G\mathbf{u})_i \lesseqgtr 0 \text{ for } i = 1, 2, \dots, N \tag{1.6}$$

and requiring also

$$u_i = u_j \text{ and } (P_G\mathbf{u})_i = (P_G\mathbf{u})_j \text{ whenever } p_i = p_j \ (1 \le i, j \le N)\,. \tag{1.7}$$

Then, there exist an interval $[\omega, \Omega] \subset \mathbb{R}$ *and a* C^∞ *function* $f : [\omega, \Omega] \to [\omega, \Omega]$ *satisfying* (1.2) *such that the iteration dynamics of mapping* (1.1) *has an asymptotically stable periodic orbit of period* 2.

In the special case $p_i \neq p_j$ for $i \neq j$, assumption (1.7) is dropped and the remaining assumptions can be reformulated as

the ordinal patterns of $\mathbf{p}$ and of $P_G\mathbf{p}$ are (strict and) the same

and

the sign patterns of $\mathbf{u}$ and of $-P_G\mathbf{u}$ are the same.

For completeness, recall that the *sign pattern of vector* $\mathbf{u} \in \mathbb{R}^N$ is

$$\sigma = \sigma(u) = \operatorname{col}(\operatorname{sgn}(u_1), \operatorname{sgn}(u_2), \dots, \operatorname{sgn}(u_N)) \in \{-1, 0, 1\}^N \subset \mathbb{R}^N\,.$$

A *vector* $\mathbf{p} = \operatorname{col}(p_1, p_2, \dots, p_N) \in \mathbb{R}^N$ *has ordinal pattern*

$$\pi = (\pi_1, \pi_2, \dots, \pi_N) \text{ if } \pi = \pi(p) \text{ is a permutation of the set } \{1, 2, \dots, N\}$$

with the properties that $p_{\pi_1} \ge p_{\pi_2} \ge \dots \ge p_{\pi_N}$ and, given integers $1 \le k \le N - 1$ and $1 \le \ell \le N - k$ arbitrarily, $p_{\pi_k} = p_{\pi_{k+1}} = \dots = p_{\pi_{k+\ell}}$ implies that $\pi_k > \pi_{k+1} > \dots > \pi_{k+\ell}$. For brevity, we say that *the ordinal pattern of vector* $\mathbf{p}$ *is strict* if $p_{\pi_1} > p_{\pi_2} > \dots > p_{\pi_N}$.

In various contexts, ordinal patterns have a long history in statistics and time series analysis [27] (Parsons code for melodic contours), [35] (as far as we know, the first paper with the term *ordinal pattern analysis* – a term coined by Warren Thorngate – in the title). From about 2005 onward [3], ordinal patterns play an increasingly important role in dynamical systems theory, too [2]. For a recent survey, we suggest [19].

The pair of assumptions (1.5) and (1.7) can be replaced by the requirement

$$p_i \neq p_j \text{ and } p_i \lessgtr p_j \text{ if and only if } (P_G\mathbf{p})_i \lessgtr (P_G\mathbf{p})_j \text{ for } i \neq j\,. \tag{1.8}$$

Properties (1.5) and (1.6) are satisfied by eigenvectors associated to positive and negative eigenvalues, respectively. (Actually, assumption (1.6) is always satisfied for eigenvector $\mathbf{u} = \mathbf{v}^N$ associated to the smallest eigenvalue $\nu_N < 0$.) Example 2.1 below shows that property $p_i \neq p_j$ for $i \neq j$ cannot be granted: assumptions (1.5)–(1.7) are satisfied but (1.8) is violated. For a sufficient condition implying property (1.8), we refer to Lemma 2.2 in Section 2.

Sign patterns and nodal domains of (adjacency and signed Laplacian) graph eigenvectors are thoroughly discussed in the monographs [4, 10, 25]. Also repeated entries of graph eigenvectors have been investigated in the literature from various viewpoints [24] (eigenvectors and graph operations), [7, 8] (eigenvector characterization of certain regular graphs and their regular subsets), [29, 30] (eigenspace bases with entries only from the set $\{-1,0,1\}$). However, to the best of our knowledge, there are many more results on graph eigenvalues than on graph eigenvectors.

For a given P_G and a given ordinal pattern, the quest for a vector $\mathbf{p} \in \mathbb{R}^N$ with property (1.5) or property (1.8) reduces to a standard form feasibility problem in linear programming. The real question behind is the characterization of ordinal pattern sequences defined by orbits of linear maps in finite dimension. The same question makes sense in the nonlinear as well as in the time series settings [19, 21], too. The construction of periodic orbits in iterates of $\mathcal{F}$ is essentially a finite problem in constrained combinatorics. The constraint is property (1.2) which makes $\mathcal{F}$ monotone in the sense of Hirsch [18]. The higher the period, the more complicated the construction – if any. For long periodic orbits on the Boolean cube $\{0,1\}^N$, see [12]. Also primitive circulant matrices were investigated in the Boolean setting [6].

The paper is organized as follows. Section 2 begins with two examples and ends with the proof of Theorem 1.1. Section 3 is devoted to direct constructions of period 2 orbits in various graph classes. Section 4 is centered about matrix P_{G_*} defined in (1.4) and contains the construction of general periodic orbits (i.e., of arbitrary periods) within the associated discrete-time strongly monotone maps in $\mathbb{R}^3$. The paper ends with open questions in Section 5.

With the exception of Examples 2.1, 3.1, 3.3, 4.1, and Remark 3.7, only regular graphs are considered. For regular graphs and their spectra, the most recent monograph is the one by Stanić [33]. Our general reference book for algebraic graph theory is the monograph by Godsil and Royle [17].

2 Two examples and the proof of Theorem 1.1

Example 2.1. Consider graph $\mathcal{G}$ with vertices $V(\mathcal{G}) = \{1,2,3,4,5\}$ and transition matrix

$$P_{\mathcal{G}} = \begin{pmatrix} 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \end{pmatrix}$$

For $\mathbf{p} = \mathrm{col}\,(5,5,8,3,3)$ and $\mathbf{u} = \mathrm{col}\,(-1,-1,1,1,1)$, it is readily checked that the assumptions of Theorem 1.1 are satisfied.

Note that, as a trivial consequence of assumption (1.5), $p_1 = p_2$ and $p_4 = p_5$ are a must. We go on presenting a sufficient condition implying property (1.8).

Lemma 2.2. *Let $\nu_1 \geq \ldots \geq \nu_M$ be the positive eigenvalues of matrix P_G (counted with multiplicity) and let $\mathbf{v}^m = col\,(v_1^m, v_2^m, \ldots, v_N^m)$ be an eigenvector associated to eigenvalue ν_m, $m = 1,2,\ldots,M$. Given $j,k \in \{1,\ldots,N\}$, $j \neq k$ arbitrarily, assume there exists an index $i_* = i_*(j,k) \in \{1,\ldots,M\}$ with the property that*

$$v_j^i = v_k^i \quad \textit{whenever} \quad i = 1,\ldots,i_* - 1 \quad \textit{but} \quad v_j^{i_*} \neq v_k^{i_*} \,. \tag{2.1}$$

Then there exists a vector $\mathbf{p} = col\,(p_1, p_2, \ldots, p_N)$ *for which assumption* (1.8) *is satisfied. Actually,* $\mathbf{p}$ *can be chosen from the convex hull of* $\{\mathbf{v}^2, \ldots, \mathbf{v}^M\}$.

Proof. The proof of Lemma 2.2 is trivial. In fact, we can take

$$\mathbf{p} = c_2\mathbf{v}^2 + \cdots + c_M\mathbf{v}^M \quad \text{where} \quad c_2 \gg \cdots \gg c_M > 0\,. \tag{2.2}$$

To put it differently, the coefficients in (2.2) have to be chosen in descending order of different magnitude. (Since $\nu_1 = 1$ and $\mathbf{v}^1 = \mathbf{1}_N \in \mathbb{R}^N$, it is convenient to start the summation in (2.2) at $m = 2$.) It is worth mentioning that variants of Lemma 2.2 can be used in eigenspaces as well as in the linear span of eigenvectors associated to the collection of negative eigenvalues. The latter variant of Lemma 2.2 is particularly useful in looking for a pair of vectors $\mathbf{p} \nparallel \pm\mathbf{1}_N$ and $\mathbf{u} \neq \mathbf{0}_N$ in $\mathbb{R}^N$ satisfying assumption (1.7). □

Returning to Example 2.1, note also that the automorphism group of graph $\mathcal{G}$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ where the first and the second factors correspond to the transpositions of vertices 1, 2 and of vertices 4, 5, respectively. With $\tau = \sqrt{2}$, we see that an alternative choice for $\mathbf{p}$ and $\mathbf{u}$ in Example 2.1 is

$$\mathbf{v}^2 = \text{col}\,(\tau-1, \tau-1, 3, -\tau, -\tau) \quad \text{and} \quad \mathbf{v}^5 = \text{col}\,(\tau+1, \tau+1, -3, -\tau, -\tau)\,,$$

eigenvectors associated to eigenvalues $\nu_2 = \frac{\tau-1}{3} > 0$ and $\nu_5 = -\frac{\tau+1}{3} < 0$, respectively. As already indicated, the largest eigenvalue of matrix $P_{\mathcal{G}}$ is $\nu_1 = 1$ with eigenvector $\mathbf{v}^1 = \mathbf{1}_5$. The remaining two eigenvalues are $\nu_3 = 0$ and $\nu_4 = -\frac{1}{3} < 0$ with eigenvectors $\mathbf{v}^3 = \text{col}\,(1, -1, 0, 0, 0)$ and $\mathbf{v}^4 = \text{col}\,(0, 0, 0, 1, -1)$, respectively. In particular, $\mathbf{p}$ and $\mathbf{u}$ in Example 2.1 cannot be taken for $\mathbf{p} = \mathbf{v}^2$ and $\mathbf{u} = \mathbf{v}^4$.

The considerations above may suggest there is a close relationship between the automorphism group of a graph and its eigenvalue–eigenvector structure. The next example – found by an anonymous participant in a discussion on MathOverflow under the title *Eigenvectors of asymmetric graphs* and checked via symbolic computation with Wolfram's Mathematica by Douglas Zare in the same discussion – shows that such a relationship cannot be too close: For F being the Frucht graph, all eigenvalues of $P_F = \frac{1}{3}\,A_F$ are simple and all eigenvectors have repeated entries. Of course the result is independent of the numbering of the vertices of F. Linear combinations of $\mathbf{v}^2$ and $\mathbf{v}^5$ have repeated entries, too. Note that $\nu_5 > \nu_6 = 0 > \nu_7$. Recall that a cubic or 3–regular graph is a graph in which all vertices have degree 3. Asymmetric graphs are defined by possessing only a single graph automorphism, the identity.

Example 2.3. Frucht graph: Let F be the famous asymmetric cubic graph on 12 vertices constructed by Robert Frucht [13] in 1939. Symbolic computation shows that the conditions of Theorem 1.1 are satisfied for $\mathbf{p} = \mathbf{v}^2$ and $\mathbf{u} = \mathbf{v}^{12}$. Replacing $\mathbf{v}^2$ by $c\,\mathbf{v}^2 + (1-c)\mathbf{v}^3$ with $0 < c < 1$ suitably chosen, also property $p_i \neq p_j$ for $i, j = 1, 2, \ldots, 11, 12$ $(i \neq j)$ holds true.

Commented ball–and–stick models of all the 85 connected simple cubic graphs on 12 vertices can be found in Wikipedia under the title '*Table of simple cubic graphs*'. Only 5 of them are asymmetric. The 1949 paper of Frucht [14] presents two planar asymmetric cubic graphs on 12 vertices. The remaining three asymmetric cubic graphs on 12 vertices are non-planar and were discovered by computer search – please see the historical remarks in [5]. Forgetting about the asymmetric non-planar cubic graph having one cycle of length 3 and three cycles of length 4, all the respective $48 = 4 \times 12$ graph eigenvalues are simple and all eigenvectors have repeated entries. In the exceptional case, $1 = \nu_1 > \cdots > \nu_6 = 0 = \nu_7 > \cdots >$

$\nu_{10} = -\frac{2}{3} > \cdots$ All eigenvectors belonging to 1, 0, and $-\frac{2}{3}$ have repeated entries whereas the remaining eigenvectors have no repeated entries. (We note that the exceptional case in the aforementioned '*Table of simple cubic graphs*' is the only asymmetric graph on 12 vertices which has five different Lederberg–Coxeter–Frucht (LCF) descriptions.) For basic results on symmetry and graph eigenvectors, we refer to [9].

It is routine to check that the conditions of Theorem 1.1 are satisfied for the quintuplets containing the Frucht graph above.

Proof. With $\mathbf{f}(\mathbf{x}) = \mathrm{col}\,(f(x_1), f(x_2), \ldots, f(x_N)) \in [\omega, \Omega]^N$ for $\mathbf{x} \in [\omega, \Omega]^N$, the periodic orbit will be constructed according to the scheme

$$\mathbf{p} \xrightarrow{P_G} \mathbf{a} = P_G\mathbf{p} \xrightarrow{\mathbf{f}} \mathbf{q} \xrightarrow{P_G} \mathbf{b} = P_G\mathbf{q} \xrightarrow{\mathbf{f}} \mathbf{p}\,. \tag{2.3}$$

Set $\mathbf{a} = P_G\mathbf{p}$, fix $\varepsilon > 0$ in such a way that

$$2\varepsilon \max_i |u_i| < \min_{p_i \neq p_j} |p_i - p_j| \quad \text{and} \quad 2\varepsilon \max_i |(P_G\mathbf{u})_i| < \min_{a_i \neq a_j} |a_i - a_j| \tag{2.4}$$

and take $\mathbf{q} = \mathbf{p} + \varepsilon\mathbf{u}$ and $\mathbf{b} = P_G\mathbf{q}$.

Consider a pair of indices i, $j \neq i$. There is no loss of generality in assuming that $p_i < p_j$ or $p_i = p_j$. If $p_i < p_j$, then $a_i < a_j$ by (1.5) and $p_i < q_j$, $q_i < q_j$ by the first part of (2.4). In view of the second part of (2.4), we conclude that $b_i < a_j$ and $b_i < b_j$. If $p_i = p_j$, then $a_i = a_j$ by (1.5) and $q_i = q_j$, $b_i = b_j$ by (1.7). In particular, $a_i = b_j$ if and only if $a_i = b_i$ and $q_i = p_j$ if and only if $q_i = p_i$. By using the $u_i = (P_G\mathbf{u})_i = 0$ case of assumption (1.6), $a_i = b_i$ and $q_i = p_i$ are equivalent. Hence $a_i = b_j$ if and only if $q_i = p_j$ (still under the conditions that $p_i = p_j$, $j \neq i$).

Exploiting the full power of assumption (1.6), we obtain that $a_i \leq b_i$ if and only if $q_i \leq p_i$ and $a_i \geq b_i$ if and only if $q_i \geq p_i$, $i = 1, 2, \ldots, N$. For indices i_* with $u_{i_*} \neq 0$, we have $q_{i_*} \neq p_{i_*}$ (and also $a_{i_*} \neq b_{i_*}$). In particular, $\mathbf{q} \neq \mathbf{p}$.

Now we are in a position to let $f(a_i) = q_i$ and $f(b_i) = p_i$ for $i = 1, 2, \ldots, N$. By a step by step reconsideration of the separate cases, f is well-defined and extends to a strictly increasing C^∞ real function on some interval $[\omega, \Omega]$. With a little more care, also properties $\omega \leq f(\omega)$, $f(\Omega) \leq \Omega$ and (1.2) can be taken for granted. By the construction, $\mathcal{F}(\mathbf{p}) = \mathbf{q}$ and $\mathcal{F}(\mathbf{q}) = \mathbf{p}$.

Asymptotic stability is ensured by choosing f in such a way that the norms of the Jacobians $\mathcal{F}'(\mathbf{p})$ and $\mathcal{F}'(\mathbf{q})$ are < 1. In view of formula (1.3), this is possible by making $f'(a_1), \ldots, f'(a_N) > 0$ and $f'(b_1), \ldots, f'(b_N) > 0$ sufficiently small. □

3 Examples for Theorem 1.1 and beyond

The analysis of the quintuplets containing the Frucht graph in Section 2 is followed by investigating two famous septets containing the Petersen graph P. First we consider the Petersen family [28], the family of graphs

$$K_6, K_{3,3,1}, G_7, K_{4,4}\backslash\{e\}, G_8, G_9, P \tag{3.1}$$

listed in a nondecreasing order of the number of vertices. Then we pass to the collection of all symmetric graphs among the class of generalized Petersen graphs $GP(n,k)$, i.e., generalized Petersen graphs with parameters [15]

$$(n,k) = (4,1),\ (5,2),\ (8,3),\ (10,2),\ (10,3),\ (12,5),\ (24,5) \tag{3.2}$$

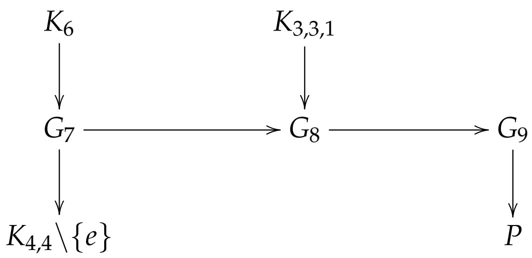

Figure 3.1: The Petersen family. Each arrow represents a Δ–Y transform [28].

known also under the names Cube, Petersen, Möbius–Kantor, dodecahedral, Desargues, Nauru, and Foster F048A graphs, respectively. For all the 14 graphs above, the desired asymptotically stable periodic orbits of period 2 can be constructed along the general scheme (2.3).

Now we start discussing the Petersen family (3.1) in a nutshell. Please see the accompanying Figure 1 and recall that the Δ–Y transform is a graph operation (invented originally by electrical engineers) in which a cycle of length 3 is replaced by a vertex of degree 3. The Petersen family plays a somewhat similar role in $\mathbb{R}^3$ as the complete graph K_5 and the complete bipartite graph $K_{3,3}$ in Wagner's graph minor theorem (we refer to [23]) on the imbeddability of graphs into $\mathbb{R}^2$. Members of the Petersen family constitute the set of forbidden minors for linkless imbeddability of graphs into $\mathbb{R}^3$. An imbedding of graph G in $\mathbb{R}^3$ is linkless if every cycle of the imbedded copy is the boundary of a topological disc whose relative interior is disjoint from the imbedded copy itself.

Example 3.1. PETERSEN FAMILY: For K_6 and $K_{3,3,1}$, we have $\nu_2 \leq 0$ and thus the easy way, the way based on eigenvectors (we applied successfully in Section 2) is blocked. The proof of Therem 1.1 still applies but the constructions of a period 2 point $\mathbf{p}$ and of the associated nonlinear function $\mathbf{f}$ in (2.3) need ad hoc methods. The complete tripartite graph $K_{3,3,1}$ will be settled in Example 3.3 below. As for K_6, we refer to Example 3.6. For the remaining five graphs in (3.1), (we have $\nu_2 > 0$ and) the argument we used in Lemma 2.2 shows that assumptions (1.5)–(1.7) are satisfied.

For convenience, graphs $K_{4,4} \backslash \{e\}$ and G_8 will be discussed in some details. Case $K_{4,4} \backslash \{e\}$ is particularly easy, it can be handled by hands. Letting $e = (1,8) \in E(K_{4,4})$ and $a = \frac{1}{\sqrt{38}}$, the eigenvalues of matrix $P_{K_{4,4} \backslash \{e\}}$ are

$$\nu_1 = 1 > \nu_2 = 1/4 > \nu_3 = \cdots = \nu_6 = 0 > \nu_7 = -1/4 > \nu_8 = -1$$

and

$$\mathbf{v}^2 = \mathrm{col}\,(-4a, a, a, a, -a, -a, -a, 4a) \quad \text{and} \quad \mathbf{v}^7 = \mathrm{col}\,(-4a, a, a, a, a, a, a, -4a)$$

are unit eigenvectors associated to ν_2 and ν_7, respectively. Just on the line, for $\mathbf{p} = \mathbf{v}^2$ and $\mathbf{u} = \mathbf{v}^7$, assumptions (1.5)–(1.7) are satisfied.

Now we turn our attention to the transition matrix P_{G_8} of the random walk on G_8. The output of Wolfram's Mathematica contains

$$\mathrm{Root}\big[-1 - 14\,\sharp 1 + 45\,\sharp 1^2 + 90\,\sharp 1^3\ \&, i\,\big], \qquad i = 1,2,3. \tag{3.3}$$

This refers to the i–th root of the cubic polynomial $-1 - 14\lambda + 45\lambda^2 + 90\lambda^3$, a factor of the characteristic polynomial of P_{G_8}. The same step of symbolic computation provides all eigenvalues of matrix P_{G_8}. In addition to the three eigenvalues in (3.3), the remaining five eigenvalues are

$$1,\ \frac{1}{12}\big(-3+\sqrt{33}\big),\ 0,0,\ \frac{1}{12}\big(-3-\sqrt{33}\big).$$

Observe that the three formulas in (3.3) appear in certain coordinates of three eigenvectors of P_{G_8}. Comparing symbolic expressions, it is readily checked that both $\mathbf{p}=\mathbf{v^2}$ and $\mathbf{u}=\mathbf{v^7}$ have only 4 different coordinate values and

$$\begin{aligned}\mathbf{p} &= \mathrm{col}\,\big(p_1,p_2,p_3,p_4,p_3,p_3,p_4,p_3\big)\in\mathbb{R}^8\\ \mathbf{u} &= \mathrm{col}\,\big(u_1,u_2,u_3,u_4,u_3,u_3,u_4,u_3\big)\in\mathbb{R}^8\end{aligned}$$

where $\nu_2>0$ and $\nu_7<0$. Here again, just on the line, assumptions (1.5)–(1.7) are satisfied.

Now we pass to the other famous septet containing the Petersen graph P. The generalized Petersen graph $G(n,k)$ is a graph with vertex set

$$V\big(G(n,k)\big)=\{U_0,U_1,\dots,U_{n-1}\}\cup\{V_0,V_1,\dots,V_{n-1}\}$$

and edge set

$$E\big(G(n,k)\big)=\{(U_i,U_{i+1}),(V_i,V_{i+k}),(U_i,V_i)\mid i=0,\dots,n-1\}$$

where subscripts are to be read modulo n and $1\le k<n/2$. The standard geometrical representation of $G(n,k)$ is the union of a regular n-gon (the subgraph spanned by the U-vertices lying on a circle of radius $r>0$) and of a regular $\{n/k\}$-star polygon (the subgraph spanned by the V-vertices, a figure formed by connecting with straight line segments every k-th point out of n regularly spaced points lying on a circle of radius $0<\rho<r$) plus n individual straight line segments between U_i and V_i, $i=0,1,\dots,n-1$. The two circles are concentric and the connections between U_i and V_i are radial.

A graph G is termed symmetric if any edge can be mapped to any other edge by a pair of elements of its automorphism group. More precisely, for any given pair $(i,j),(k,\ell)\in E(G)$ of edges, there exists a graph automorphism mapping vertex i and j to vertex k and ℓ as well as a second graph automorphism mapping vertex i and j to vertex ℓ and k, respectively. Now we consider the seven symmetric generalized Petersen graphs with parameters [15] listed in (3.2) above.

Example 3.2. Symmetric generalized Petersen graphs: Recall that they are known under the names Cube, Petersen, Möbius–Kantor, dodecahedral, Desargues, Nauru, and Foster F048A graphs. The transition matrices of the corresponding random walks are of order $8,10,16,20,20,24,48$, respectively. The numbers of different eigenvalues are $4,3,6,6,6,7,11$, respectively. Applying Lemma 2.2 to an eigenspace associated to a suitable positive eigenvector, we conclude that – in all the seven cases – assumptions (1.5)–(1.7) are satisfied. Actually, assumption (1.8) is satisfied in each case.

We restrict ourselves to the four-dimensional eigenspace $\mathcal{L}$ (of the transition matrix) of the Möbius–Kantor graph associated to the positive eigenvalue $\nu_2=\frac{1}{\sqrt{3}}$. Both ν_2 and a basis of the eigenspace $\mathcal{L}$ are provided by Wolfram's Mathematica software:

$$\begin{matrix}(b & 0 & a & B & a & 0 & b & A & 0 & 0 & 0 & b & 0 & 0 & 0 & a)\\ (0 & a & B & a & 0 & b & A & b & 0 & 0 & b & 0 & 0 & 0 & a & 0)\\ (a & B & a & 0 & b & A & b & 0 & 0 & b & 0 & 0 & 0 & a & 0 & 0)\\ (B & a & 0 & b & A & b & 0 & a & b & 0 & 0 & 0 & a & 0 & 0 & 0)\end{matrix}$$

where $a = 1$, $b = -1$, $A = \sqrt{3}$, $B = -\sqrt{3}$. Please observe that the basis "chosen" by Wolfram's Mathematica has an easily recognizable structure which seems to be the result of a heuristic inner optimization. It is immediate that Lemma 2.2 applies and leads to the the fulfilment of assumption (1.8). However, Lemma 2.2 does not apply to the three-dimensional eigenspace belonging to eigenvalue $\nu_6 = \frac{1}{3}$.

Example 3.3. Set $G = K_{3,3,1}$. Then the transition matrix P_G of the random walk on G is defined as

$$(P_G)_{i,j} = \begin{cases} 0 & \text{if } 1 \le i,j \le 3 \text{ or } 4 \le i,j \le 6 \text{ or } i = j = 7 \\ 1/6 & \text{if } i = 7 \text{ and } j = 1,2,\ldots,6 \\ 1/4 & \text{otherwise} \end{cases}$$

for $i,j = 1,2,\ldots,7$.

The periodic orbit of period 2 is constructed by letting $f(4) = 0$, $f(6) = 4$, $f(7) = 8$, $f(8) = 16$. Thus the general scheme (2.3) is specified as

$$\mathbf{p} = \begin{pmatrix} 16 \\ \vdots \\ 16 \\ 4 \end{pmatrix} \xrightarrow{P_G} \mathbf{a} = \begin{pmatrix} 4 \\ \vdots \\ 4 \\ 7 \end{pmatrix} \xrightarrow{\mathbf{f}} \mathbf{q} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 8 \end{pmatrix} \xrightarrow{P_G} \mathbf{b} = \begin{pmatrix} 8 \\ \vdots \\ 8 \\ 6 \end{pmatrix} \xrightarrow{\mathbf{f}} \mathbf{p} = \begin{pmatrix} 16 \\ \vdots \\ 16 \\ 4 \end{pmatrix}.$$

Since function $f : \{4,6,7,8\} \to \{0,4,8,16\}$ is strictly increasing, the argument we applied in the last two paragraphs of Section 2 can be repeated.

The next two examples discuss hypercube and circulant graphs in connection with Theorem 1.1. In either case, vectors $\mathbf{p}$ and $\mathbf{u}$ are chosen for eigenvectors with properties (1.8) and (1.6), respectively. Thus, for hypercubes in dimension $N \ge 3$ and for circulant graphs subject to conditions formulated in Example 3.5 below, Theorem 1.1 applies.

Example 3.4. Hypercube Q_N, $N \ge 3$: The standard representation of the adjacency matrix is obtained by the recursion

$$A_{Q_0} = \mathbf{0}_1 \quad \text{and} \quad A_{Q_{M+1}} = \begin{pmatrix} A_{Q_M} & I_{2^M} \\ I_{2^M} & A_{Q_M} \end{pmatrix} \quad \text{for } M = 0,1,\ldots,N-1$$

where I_{2^M} is the $2^M \times 2^M$ identity matrix. It is readily checked that

$$\mathbf{p} = \operatorname{col}\left(2^N - 1, 2^N - 3, 2^N - 5, 2^N - 7, \ldots, -2^N + 1\right) \in \mathbb{R}^{2^N}$$

is an eigenvector of the transition matrix $P_{Q_N} = \frac{1}{N} A_{Q_N}$ associated to the second largest eigenvalue $\nu_2 = 1 - \frac{2}{N} > 0$.

The simplest way of defining a circulant graph is to designate its adjacency matrix. We set

$$A_{\mathcal{C}_N} = \operatorname{circ}\left(c_0, c_{N-1}, c_{N-2}, \ldots, c_2, c_1\right)$$

where $c_0 = 0$ and $c_k = c_{N-k} \in \{0,1\}$ for $k = 1,2,\ldots,N-1$ are parameters with $|\mathbf{c}| = \sum_{k=1}^{N-1} c_k > 0$. Note that the eigenvectors of $A_{\mathcal{C}_N}$ do not depend on the particular choice of the parameters $\{c_k\}_{k=1}^{N-1}$. Due to the symmetry property of the parameters required, the eigenvalues are real (though they are defined via complex roots of unity) and the corresponding eigenspaces – with the exception of at most two separate cases of simple eigenvalues – are two-dimensional subspaces in $\mathbb{R}^N$.

Example 3.5. A technical class of circulant graphs: Let $N \geq 5$. Pick an integer $0 < k_* < N$ and set

$$\lambda_{k_*} = c_0 + c_{N-1}\,\omega_{k_*} + c_{N-2}\,\omega_{k_*}^2 + \cdots + c_2\,\omega_{k_*}^{N-2} + c_1\,\omega_{k_*}^{N-1}$$

where $\omega_{k_*} = \exp\left(2\sqrt{-1}\,\pi\,k_*\,\frac{1}{N}\right)$. Assume that k_* and N are relatively prime and that $\lambda_{k_*} > 0$. Then $\frac{1}{|\mathbf{c}|}\lambda_{k_*} > 0$ is an eigenvalue of the transition matrix $P_{\mathcal{C}_N} = \frac{1}{|\mathbf{c}|}A_{\mathcal{C}_N}$ and, for each $\delta \in \mathbb{R}$,

$$p_n = \cos\left(2\pi(n-1)\,k_*\,\frac{1}{N} + \delta\right), \qquad n = 1, 2, \ldots, N$$

defines an eigenvector $\mathbf{p} = \mathrm{col}\,(p_1, p_2, \ldots, p_N) \in \mathbb{R}^N$ associated to $\frac{1}{|\mathbf{c}|}\lambda_{k_*}$. Now choose $\delta = \delta(k^*, N)$ in such a way that $p_i \neq p_j$ for all $i \neq j$. (This is possible since the exceptional set is finite in every interval.)

The assumptions in Example 3.5 are satisfied for cycle/circular graphs C_N of order $N \geq 5$ and for Paley graphs of prime order (which are Hamiltonian) but not for complete graphs. Actually, for complete graphs of order $N \geq 3$, (1.5) implies that $p_i = p_j$ for all i, j. Hence a direct construction is needed.

Example 3.6. Complete graph $G = K_N$, $N \geq 2$: Observe that $P_{K_N} = \frac{1}{N-1}\,A_{K_N}$ where $\left(A_{K_N}\right)_{i,j} = 0$ if $i = j$ and 1 if $i \neq j$. For $N = 2$ and $N = 3$, the general scheme (2.3) can be specified as

$$\mathbf{p} = \begin{pmatrix}2\\1\end{pmatrix} \xrightarrow{P_G} \mathbf{a} = \begin{pmatrix}1\\2\end{pmatrix} \xrightarrow{\mathbf{f}} \mathbf{q} = \begin{pmatrix}1\\2\end{pmatrix} \xrightarrow{P_G} \mathbf{b} = \begin{pmatrix}2\\1\end{pmatrix} \xrightarrow{\mathbf{f}} \mathbf{p} = \begin{pmatrix}2\\1\end{pmatrix}$$

and

$$\mathbf{p} = \begin{pmatrix}1\\7\\11\end{pmatrix} \xrightarrow{P_G} \mathbf{a} = \begin{pmatrix}9\\6\\4\end{pmatrix} \xrightarrow{\mathbf{f}} \mathbf{q} = \begin{pmatrix}12\\4\\2\end{pmatrix} \xrightarrow{P_G} \mathbf{b} = \begin{pmatrix}3\\7\\8\end{pmatrix} \xrightarrow{\mathbf{f}} \mathbf{p} = \begin{pmatrix}1\\7\\11\end{pmatrix},$$

respectively. The vector diagrams above show that the underlying real functions (defined on the sets $\{1,2\}$ and $\{3,4,6,7,8,9\}$, respectively) are strictly increasing. Thus the argument we applied in the last two paragraphs of Section 2 can be repeated.

Finally, case $N \geq 4$ is settled by letting

$$\begin{pmatrix}1\\\vdots\\1\\N\end{pmatrix} \xrightarrow{P_G} \begin{pmatrix}2\\\vdots\\2\\1\end{pmatrix} \xrightarrow{\mathbf{f}} \begin{pmatrix}2+\frac{1}{N-2}\\\vdots\\2+\frac{1}{N-2}\\0\end{pmatrix} \xrightarrow{P_G} \begin{pmatrix}2-\frac{1}{N-1}\\\vdots\\2-\frac{1}{N-1}\\2+\frac{1}{N-2}\end{pmatrix} \xrightarrow{\mathbf{f}} \begin{pmatrix}1\\\vdots\\1\\N\end{pmatrix}.$$

On complete bipartite graphs, the search for asymptotically stable periodic orbits of period 2 reduces to the one on K_2. However, the case of complete multipartite graphs seems to be considerably more difficult. For convenience, we note that $Q_2 = C_4 = K_{2,2}$ and $Q_1 = K_2 = K_{1,1}$.

Remark 3.7. Complete bipartite graph $G = K_{M,N}$, $N, M \geq 1$: Looking for a period 2 orbit on $K_{M,N}$, it is clear that vertices on the same side of the bipartition can be contracted and thus the problem reduces to the special case $M, N = 1$. The argument works in the reverse direction as well. A period 2 orbit on $K_{1,1}$ gives rise to a uniquely defined period 2 orbit on $K_{M,N}$ assigning the same values (inherited from the period 2 orbit on $K_{1,1}$) to all vertices on the same side of the bipartition. (In general, complete multipartite graphs can be contracted to complete graphs with weighted edges.)

For $G = K_2$, a full analysis of the iteration dynamics

$$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{\mathcal{F}} \begin{pmatrix} f(b) \\ f(a) \end{pmatrix} \xrightarrow{\mathcal{F}} \begin{pmatrix} f^2(a) \\ f^2(b) \end{pmatrix} \xrightarrow{\mathcal{F}} \begin{pmatrix} f^3(b) \\ f^3(a) \end{pmatrix} \xrightarrow{\mathcal{F}} \begin{pmatrix} f^4(a) \\ f^4(b) \end{pmatrix} \xrightarrow{\mathcal{F}} \dots$$

of mapping (1.1) can easily be given. Recall that, in view of condition (1.2), the real function f is strictly increasing. As for period 2 orbits on $\mathcal{F}$, there are only two possibilities. Depending on the three cases $a < f^2(a)$, $a > f^2(a)$ and $a = f^2(a)$, the sequence $\{f^{2k}(a)\}_{k=0}^{\infty}$ is strictly increasing, strictly decreasing and constant, respectively. It follows immediately that $\mathbf{p} = \binom{a}{b} \in \mathbb{R}^2$ is periodic if and only if $\mathbf{p} = \mathcal{F}(\mathbf{p})$ $\big($implying $a = b$ and $a = f(a)\big)$ or $\mathbf{p} = \mathcal{F}^2(\mathbf{p})$ $\big($if 2 is the minimal period, then $a = f(a)$, $b = f(b)$ and $a \neq b\big)$.

For $G = K_3$, we conjecture that the minimal period of all periodic orbits (induced by an arbitrary mapping $\mathcal{F}$ that satisfies (1.1)–(1.2)) is ≤ 2. We have only a preliminary result into this direction.

Lemma 3.8. *There is no periodic orbit of minimal period* 4.

Proof. The proof is elementary but not entirely trivial. What is trivial is that – forgetting about fixed points – the minimal period is even. (In fact, inequality $a \geq b \geq c$ implies that $F\big(\frac{b+c}{2}\big) \leq F\big(\frac{c+a}{2}\big) \leq F\big(\frac{a+b}{2}\big)$.)

Suppose we are given a strictly increasing real function F (used in defining $\mathbf{F}(\mathbf{x}) = \operatorname{col}\big(F(x_1), F(x_2), F(x_3)\big)$) and a periodic orbit

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \xrightarrow{P_G} \begin{pmatrix} \frac{b+c}{2} \\ \frac{c+a}{2} \\ \frac{a+b}{2} \end{pmatrix} \xrightarrow{\mathbf{F}} \begin{pmatrix} d \\ e \\ f \end{pmatrix} \xrightarrow{P_G} \begin{pmatrix} \frac{e+f}{2} \\ \frac{f+d}{2} \\ \frac{d+e}{2} \end{pmatrix} \xrightarrow{\mathbf{F}} \begin{pmatrix} g \\ h \\ i \end{pmatrix} \xrightarrow{P_G} \begin{pmatrix} \frac{h+i}{2} \\ \frac{i+g}{2} \\ \frac{g+h}{2} \end{pmatrix} \xrightarrow{\mathbf{F}} \begin{pmatrix} j \\ k \\ \ell \end{pmatrix} \xrightarrow{P_G} \begin{pmatrix} \frac{k+\ell}{2} \\ \frac{\ell+j}{2} \\ \frac{j+k}{2} \end{pmatrix} \xrightarrow{\mathbf{F}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

of minimal period 4. By toroidal symmetry of the diagram above, we see there is no loss of generality in assuming that $h \leq b$ and $i \leq c$. Thus

$$\left.\begin{matrix} h \leq b \\ i \leq c \end{matrix}\right\} \Rightarrow \frac{h+i}{2} \leq \frac{b+c}{2} \Rightarrow j \leq d \tag{3.4}$$

and, in view of inequality $j \leq d$ in (3.4),

$$\left.\begin{matrix} h \leq b \Rightarrow \frac{f+d}{2} \leq \frac{\ell+j}{2} \Rightarrow f \leq \ell \\ i \leq c \Rightarrow \frac{d+e}{2} \leq \frac{j+k}{2} \Rightarrow e \leq k \end{matrix}\right\} \Rightarrow \frac{e+f}{2} \leq \frac{k+\ell}{2} \Rightarrow g \leq a. \tag{3.5}$$

Case 1. If $h = b$ and $i = c$, then $j = d$ by (3.4) and $g = a$ by (3.5). Hence the minimal period is ≤ 2, a contradiction.

Case 2. If $h \leq b$ and $i \leq c$ and at least one of these inequalities is strict, then $j < d$ by (3.4) and $f < \ell$, $e < k$ by (3.5) implying $g < a$ as well. Now we start from the strict inequalities $e < k$, $f < \ell$ and repeat the entire argumentation from the very beginning. As an analogue of inequality $g < a$, we arrive at $d < j$ what is impossible by (3.4). □

4 Asymptotically stable long period orbits

In order to motivate the construction of matrix P_{G_*} in (1.4), it is instrumental to reconsider the proof of Theorem 1.1 in the technically simplest special case where $N = 3$ and

$$p_i \neq p_j \quad \text{for } i \neq j, \quad i, j = 1, 2, 3 \quad \text{and} \quad u_i \neq 0 \quad \text{for } i = 1, 2, 3,$$

and both $\mathbf{p} \in \mathbb{R}^3$ and $\mathbf{u} \in \mathbb{R}^3$ are eigenvectors of the 3 by 3 matrix P_G with eigenvalues $\lambda > 0$ and $\mu < 0$, respectively.

The starting point is scheme (2.3) we recall in its original form

$$\mathbf{p} \xrightarrow{P_G} \mathbf{a} = P_G\mathbf{p} \xrightarrow{\mathbf{f}} \mathbf{q} \xrightarrow{P_G} \mathbf{b} = P_G\mathbf{q} \xrightarrow{\mathbf{f}} \mathbf{p}\,,$$

together with the simplified notation $\mathbf{q} = \mathbf{p} + \varepsilon\,\mathbf{u}$, $\varepsilon > 0$ we used in Section 2. For convenience, recall $\mathbf{f}(\mathbf{x}) = \mathrm{col}\,\big(f(x_1), f(x_2), f(x_3)\big)$, too. Now the proof of Theorem 1.1 reduces to point out that

$$\begin{cases} (P_G\mathbf{p})_i < (P_G\mathbf{p})_j & \text{if and only if } q_i < q_j \\ (P_G\mathbf{q})_i < (P_G\mathbf{q})_j & \text{if and only if } p_i < p_j \\ (P_G\mathbf{p})_i < (P_G\mathbf{q})_j & \text{if and only if } q_i < p_j \end{cases}$$

for each $i, j = 1, 2, 3$.

With $\boldsymbol{\alpha} \in \mathbb{R}^3$ defined by letting $\alpha_i = \varepsilon\, u_i \neq 0$ for each i, this boils down to $\mathbf{q} = \mathbf{p} + \boldsymbol{\alpha}$ and

$$\begin{cases} \lambda p_i < \lambda p_j & \text{if and only if } p_i + \alpha_i < p_j + \alpha_j \\ \lambda p_i + \mu\alpha_i < \lambda p_j + \mu\alpha_j & \text{if and only if } p_i < p_j \\ \lambda p_i < \lambda p_j + \mu\alpha_j & \text{if and only if } p_i + \alpha_i < p_j \end{cases} \tag{4.1}$$

for each $i, j = 1, 2, 3$. By taking the norm of $\boldsymbol{\alpha} \in \mathbb{R}^3$ sufficiently small, $\lambda > 0$ implies (4.1) for each $i \neq j$. If $i = j$, then the first two rows of (4.1) are irrelevant and the last row of (4.1) follows from the equivalence of $0 < \mu\alpha_i$ and $\alpha_i < 0$ which is nothing else but inequality $(P_G\boldsymbol{\alpha})_i\,\boldsymbol{\alpha}_i < 0$ for $i = 1, 2, 3$.

All in all, in the period 2 case where $\boldsymbol{\alpha} + \boldsymbol{\beta} = \mathbf{0}$, we had only to guarantee that $(P_G\boldsymbol{\alpha})_i\,\boldsymbol{\alpha}_i < 0$ which is equivalent to $(P_G\boldsymbol{\beta})_i\,\boldsymbol{\beta}_i < 0$ for $i = 1, 2, 3$.

Remaining in $\mathbb{R}^3$, we hope that the previous considerations based on (2.3) and (4.1) can be repeated for the period 3 scheme

$$\mathbf{p} \xrightarrow{\mathbf{A}} \mathbf{a} = \mathbf{A}\mathbf{p} \xrightarrow{\mathbf{f}} \mathbf{q} \xrightarrow{\mathbf{A}} \mathbf{b} = \mathbf{A}\mathbf{q} \xrightarrow{\mathbf{f}} \mathbf{r} \xrightarrow{\mathbf{A}} \mathbf{c} = \mathbf{A}\mathbf{r} \xrightarrow{\mathbf{f}} \mathbf{p} \tag{4.2}$$

where $\mathbf{A}$ is a 3 by 3 row stochastic positive matrix (all entries are positive and the sum of the entries in each row equals 1) with rational entries. Now $\mathbf{q} = \mathbf{p} + \boldsymbol{\alpha}$, $\mathbf{r} = \mathbf{q} + \boldsymbol{\beta}$, $\mathbf{p} = \mathbf{r} + \boldsymbol{\gamma}$. Clearly $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} = \mathbf{0}$. We assume that

$$p_i \neq p_j,\ q_i \neq q_j,\ r_i \neq r_j \quad \text{for } i \neq j \quad \text{and} \quad \alpha_i,\ \beta_i,\ \gamma_i \neq 0 \quad \text{for } i, j = 1, 2, 3,$$

and both $\boldsymbol{\alpha} = \mathbf{p}$ $\big($this is the trick!$\big)$ and $\boldsymbol{\beta}$ are eigenvectors of matrix $\mathbf{A}$ with eigenvalues $\lambda > 0$ and $\mu < 0$, respectively.

In order to make the real function f (defined for a while only on the nine coordinate values of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and to be extended to an interval $[\omega, \Omega]$ only at a later moment) strictly increasing, we end up with the requirements

$$\left.\begin{matrix} (\mathbf{A}\boldsymbol{\alpha})_i\,\beta_i > 0 \\ (\mathbf{A}\boldsymbol{\beta})_i\,\gamma_i > 0 \\ (\mathbf{A}\boldsymbol{\gamma})_i\,\alpha_i > 0 \end{matrix}\right\} \quad \Leftrightarrow \quad \begin{cases} \lambda\,\alpha_i\,\beta_i > 0 \\ \mu\,\beta_i\big(-\alpha_i - \beta_i\big) > 0 \\ \big(-\lambda\,\alpha_i - \mu\,\beta_i\big)\alpha_i > 0. \end{cases} \tag{4.3}$$

Now we look for a 3 by 3 matrix $\mathbf{A}$ such that the assumptions in (4.3) and in the paragraph centered about (4.2) are all satisfied. Actually, matrix $\mathbf{A}$ will be constructed via a dyadic decomposition of the form

$$\mathbf{A} = \zeta\big(\mathbf{1}\,\mathbf{1}^T\big) \;+\; \eta\big(\boldsymbol{\alpha}\,\mathbf{a}^T\big) \;+\; \vartheta\big(\boldsymbol{\beta}\,\mathbf{b}^T\big)\,.$$

Here $\mathbf{1} = \mathbf{1}_3 = \operatorname{col}(1,1,1) \in \mathbb{R}^3$ is the normal vector of the two-dimensional linear subspace spanned by vectors $\mathbf{a}$ and $\mathbf{b}$. In addition, $\boldsymbol{\alpha} = \mathbf{1}\times\mathbf{b}$ and $\boldsymbol{\beta} = \mathbf{1}\times\mathbf{a}$. Since $\mathbf{1}^T\boldsymbol{\alpha} = 0 \in \mathbb{R}$ and $\mathbf{b}^T\boldsymbol{\alpha} = 0 \in \mathbb{R}$, the associativity property of matrix products implies that

$$\mathbf{A}\boldsymbol{\alpha} = \lambda\,\boldsymbol{\alpha} \quad \text{with } \lambda = \eta\big(\mathbf{a}^T\boldsymbol{\alpha}\big) \quad \text{and similarly,} \quad \mathbf{A}\boldsymbol{\beta} = \mu\,\boldsymbol{\beta} \quad \text{with } \mu = \vartheta\big(\mathbf{b}^T\boldsymbol{\beta}\big).$$

Property $A\mathbf{1} = \mathbf{1}$ is obvious for $\zeta = \frac{1}{3}$.

We are left to choose vectors $\mathbf{a}^T$, $\mathbf{b}^T$ and scalars η, ϑ in such a way that the nine conditions in (4.3) are all satisfied. We follow an intuitive argument and check retrospectively if it is successful or not.

Assume for the moment that $\lambda > 0$ and $\mu < 0$. Then we choose vectors $\mathbf{a},\mathbf{b} \in \mathbb{R}^3$ in such a way that the first six conditions

$$\alpha_i\,\beta_i > 0 \quad \text{and} \quad \beta_i(\alpha_i+\beta_i) > 0 \quad \text{for } i = 1,2,3$$

in (4.3) are all satisfied. The 'more' the vectors $\boldsymbol{\alpha} \in \mathbb{R}^3$ and $\boldsymbol{\beta} \in \mathbb{R}^3$ are 'parallel', the better. Since $\boldsymbol{\alpha} = \mathbf{1}\times\mathbf{b}$ and $\boldsymbol{\beta} = \mathbf{1}\times\mathbf{a}$, the angle between $\mathbf{b}$ and $\mathbf{a}$ is exactly the same as the angle between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Taking $\mathbf{a} = \operatorname{col}(1,2,-3)$ and $\mathbf{b} = \operatorname{col}(2,3,-5)$, the cosine of the angle between them is $23\,\frac{1}{\sqrt{14\cdot 38}} \approx 0.9971$. Thus $\boldsymbol{\alpha} = \operatorname{col}(-8,7,1)$ and $\boldsymbol{\beta} = \operatorname{col}(-5,4,1)$ are 'almost parallel'.

Since $\lambda = \eta\big(\mathbf{a}^T\boldsymbol{\alpha}\big)$ and $\mu = \vartheta\big(\mathbf{b}^T\boldsymbol{\beta}\big)$, we obtain readily that $\lambda = 3\,\eta$ and $\mu = -3\,\vartheta$. Now we take $\vartheta = 2\eta$. Anticipating $\eta > 0$, we have $\lambda > 0$, $\mu < 0$ and see that the last three conditions

$$\begin{aligned}
\big(-\lambda\,\alpha_1 - \mu\,\beta_1\big)\,\alpha_1 > 0 \quad &\Leftrightarrow \quad \big(8\eta - 10\eta\big)\cdot\big(-8\big) > 0\\
\big(-\lambda\,\alpha_2 - \mu\,\beta_2\big)\,\alpha_2 > 0 \quad &\Leftrightarrow \quad \big(-7\eta + 8\eta\big)\cdot 7 > 0\\
\big(-\lambda\,\alpha_3 - \mu\,\beta_3\big)\,\alpha_3 > 0 \quad &\Leftrightarrow \quad \big(-\eta + 2\eta\big)\cdot 1 > 0
\end{aligned}$$

in (4.3) are satisfied. It remains to check that $\mathbf{A} > 0$ for some $\eta > 0$. In fact,

$$\begin{aligned}
\mathbf{A} &= \zeta\big(\mathbf{1}\,\mathbf{1}^T\big) + \eta\big(\boldsymbol{\alpha}\,\mathbf{a}^T\big) + \vartheta\big(\boldsymbol{\beta}\,\mathbf{b}^T\big) = \frac{1}{3}\big(\mathbf{1}\,\mathbf{1}^T\big) + \eta\big(\boldsymbol{\alpha}\,\mathbf{a}^T + 2\,\boldsymbol{\beta}\,\mathbf{b}^T\big)\\
&= \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} + \eta\begin{pmatrix}-28&-46&74\\23&38&-61\\5&8&-13\end{pmatrix} > 0 \quad \text{whenever } \frac{1}{3} - 61\,\eta > 0\,.
\end{aligned}$$

The choice $\eta = \frac{1}{198}$ makes matrix A equal to matrix P_{G^*} introduced in (1.4) and we are almost done.

By the considerations above, we have shown that $\boldsymbol{\alpha} = \mathbf{p}$ is an eigenvector of matrix $\mathbf{A}$ and the corresponding eigenvalue is $\lambda = 3\eta > 0$. In particular,

$$P_{G_*}\mathbf{p} = \frac{1}{\kappa}\,\mathbf{p}, \quad \text{where } \kappa = 66 \text{ and } \mathbf{p} = \operatorname{col}(8,-7,-1) \in \mathbb{R}^3.$$

The remaining two eigenvalues are 1 and $\mu = -6\eta = -\frac{2}{\kappa} = -\frac{1}{33}$, with the respective eigenvectors $\mathbf{1} = \mathbf{1}_3$ and $\boldsymbol{\beta} = \operatorname{col}(-5,4,1)$. In view of formulas (1.1) and (1.3), it remains to construct a C^∞ function $f : [\omega,\Omega] \to [\omega,\Omega]$ satisfying condition (1.2) on a suitable interval $[\omega,\Omega]$. This is the content of our next example. The existence of such an f is immediate from (4.3) already proven. However, it is the particular form of f what plays a pivotal role in proving the forthcoming Theorem 4.2.

Example 4.1. With P_{G_*}, κ and $\mathbf{p}$ as above, it is readily checked that

$$\kappa\mathbf{p} \xrightarrow{P_{G_*}} \mathbf{p} \xrightarrow{\mathbf{f}} 2\kappa\mathbf{p} \xrightarrow{P_{G_*}} 2\mathbf{p} \xrightarrow{\mathbf{f}} \begin{pmatrix} 21\kappa \\ -18\kappa \\ -3\kappa \end{pmatrix} \xrightarrow{P_{G_*}} \begin{pmatrix} 6 \\ -6 \\ 0 \end{pmatrix} \xrightarrow{\mathbf{f}} \kappa\mathbf{p} \tag{4.4}$$

defines a periodic orbit of period 3. The crucial fact is of course that function

$$f : \{-14,-7,-6,-2,-1,0,6,8,16\} \to \mathbb{R}$$

given by $f(-14) = -18\kappa$, $f(-7) = -14\kappa$, $f(-6) = -7\kappa$, $f(-2) = -3\kappa$, $f(-1) = -2\kappa$, $f(0) = -\kappa$, $f(6) = 8\kappa$, $f(8) = 16\kappa$, and $f(16) = 21\kappa$ is strictly increasing. Thus a slightly modified version of the argument we applied in the last two paragraphs of Section 2 can be repeated.

Now we are in a position to state and prove the second main result of the present paper.

Theorem 4.2. *In order to obtain a periodic orbit of period $4+r$ ($r = 0,1,2\ldots$), the previous period 3 example is modified. The idea is to replace the second map in (4.4) by a chain of $2r+3$ maps obtained via interpolating f on the intervals $[-14,-7]$, $[-2,-1]$, and $[8,16]$ (and redefining it on the "entry set" $\{-7,-1,8\}$). When doing this, we remain in the linear span of vector $\mathbf{p}$ in $\mathbb{R}^3$. This homogeneity of the interpolation is a key factor to ensure that the modified f (still on a finite subset of $\mathbb{R}$) is strictly increasing. We end up with a monotone – in the sense of Hirsch – mapping $\mathcal{F}_r : [-20,20]^3 \to [-20,20]^3$ having an asymptotically stable periodic orbit with minimal period r.*

Proof. For $k = 0,1,\ldots$, set $a_k = 2 - 2^{-k-1}$.

For $r = 0,1,\ldots$ fixed, the second map $\mathbf{p} \xrightarrow{\mathbf{f}} 2\kappa\mathbf{p}$ in (4.4) is replaced by

$$\mathbf{p} \xrightarrow{\mathbf{f}_r} a_0\kappa\mathbf{p} \xrightarrow{P_{G_*}} a_0\mathbf{p} \xrightarrow{\mathbf{f}_r} a_1\kappa\mathbf{p} \xrightarrow{P_{G_*}} a_1\mathbf{p} \xrightarrow{\mathbf{f}_r} a_2\kappa\mathbf{p} \xrightarrow{P_{G_*}} a_2\mathbf{p} \xrightarrow{\mathbf{f}_r} \cdots$$
$$\cdots \xrightarrow{\mathbf{f}_r} a_{r-1}\kappa\mathbf{p} \xrightarrow{P_{G_*}} a_{r-1}\mathbf{p} \xrightarrow{\mathbf{f}_r} a_r\kappa\mathbf{p} \xrightarrow{P_{G_*}} a_r\mathbf{p} \xrightarrow{\mathbf{f}_r} 2\kappa\mathbf{p}\,,$$

a chain of $(2r+3)$ maps. (The fourth and the sixth maps in (4.4) obtain subscript r, too.) Starting with

$$\left.\begin{aligned} f_r(8) &= 16a_0\kappa \\ f_r(-7) &= -14a_0\kappa \\ f_r(-1) &= -2a_0\kappa \end{aligned}\right\}, \quad \text{we set} \quad \left.\begin{aligned} f_r(8a_k) &= 8a_{k+1}\kappa \\ f_r(-7a_k) &= -7a_{k+1}\kappa \\ f_r(-a_k) &= -a_{k+1}\kappa \end{aligned}\right\} \quad \text{for } k = 0,\ldots,r-1$$

$$(\text{there is no such } k \text{ if } r=0) \quad \text{and} \quad \left.\begin{aligned} f_r(8a_r) &= 16\kappa \\ f_r(-7a_r) &= -14\kappa \\ f_r(-a_r) &= -2\kappa \end{aligned}\right\}.$$

We keep f on the finite set $\{-14,-6,-2,0,6,8,16\}$ unaltered. Taking $f_r(-14) = -18\kappa$, $f_r(-6) = -7\kappa$, $f_r(-2) = -3\kappa$, $f_r(0) = -\kappa$, $f_r(6) = 8\kappa$ and $f_r(16) = 21\kappa$, also the modified map f_r (defined on 9 old and $3(r+1)$ new points for $r = 0,1,\ldots$ on the real line) is strictly increasing. The final step is to repeat the argument we applied in the last two paragraphs of Section 2. The domain of f_r can be chosen for $[\omega,\Omega] = [-20,20]$. □

5 Remarks and open questions

Remark 5.1. The nonexistence of asymptotically stable nontrivial periodic orbits is one of the oldest results in the theory of continuous-time strongly monotone dynamical systems in $\mathbb{R}^n$. Actually, this is a consequence of combining Propositions 1.2 and Theorem 1.7 of [18]. Moreover, by Theorem 2.6 of [18], nontrivial periodic orbits of continuous-time eventually strongly monotone semi-dynamical systems are unstable in a well-defined technical sense. The construction of monotone maps with periodic orbits reduces to an extension problem within the class of monotone maps – please see Subsection 5.2 "Definitions and Basic Results" of the survey by Hirsch and Hal Smith [18] and several references therein. In principle one can start from any unordered set $\Gamma = \{\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^{K-1}\} \subset [\omega, \Omega]^N$ and from an $\mathcal{F}$-cycle

$$\mathbf{x}^0 \xrightarrow{\mathcal{F}} \mathbf{x}^1 \xrightarrow{\mathcal{F}} \mathbf{x}^2 \xrightarrow{\mathcal{F}} \cdots \xrightarrow{\mathcal{F}} \mathbf{x}^{K-2} \xrightarrow{\mathcal{F}} \mathbf{x}^{K-1} \xrightarrow{\mathcal{F}} \mathbf{x}^0 \tag{5.1}$$

on Γ. However, within the subclass of monotone maps singled out by the combination of (1.1) and (1.2), the difficulty is of combinatorial nature and lies in choosing (5.1). It is not clear to us if standard extension theorems for real-valued continuous functions (e.g. Nachbin's order-preserving version of Tietze's extension theorem) are of any use in this context.

The major questions on the relation between the structure of graph G and the properties of the nonlinear consensus dynamics $\mathcal{F}$ remain open. Of course everything depends on the properties of function f, too. In an earlier paper of ours [16], we gave a sufficient condition for f implying global consensus in all networks with the property that the transition matrix of the associated random walk is primitive. Synchronization results in [1, 26, 31] are out and away much more interesting and much better.

Given a connected graph G and an integer $K > 1$, construct an f with property (1.2) so that the consensus dynamics in (1.1) has a periodic orbit of minimal period K. The question makes sense for fixed points with the greatest possible number of different coordinates. We think that the solution of related subproblems might be of independent interest, too. For global consensus in control problems, we refer to [11].

Does Theorem 4.2 remain valid for undirected, unweighted graphs containing no multiple or loop edges? Is the conjecture preceding Lemma 3.8 true or false? What about the maximum of minimal periods of asymptotically stable periodic orbits for a given triplet $G, f, \mathcal{F}$ subject to conditions (1.1) and (1.2)? (Due to Theorem 5.25 in [18], there exists a maximum value of the minimal periods.) What about properties, if any, leading to upper and lower bounds for the maximum value? What is the status of property (2.1) among strongly regular graphs (that are known to have a unique nontrivial positive eigenvalue of high multiplicity [32, 33]). It would be nice to have asymptotic and probabilistic results (like those in [34]) for property (2.1) among strongly regular graphs. How can Theorem 1.1 be generalized for directed, weighted graphs containing loop edges?

Finally, it would be nice to find any connections to diffusion models on lattices and graphs or to models in mathematical ecology. Unfortunately, both in Theorem 1.1 and in Theorem 4.2, the second derivative of function f is wildly oscillating. This is an indication that real-world interpretations in the directions above seem to be difficult to find. In contrast to this pessimism, the possibility of replacing arithmetic means by certain Kolmogorov–Nagumo averages is more likely.

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