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Schrödinger–Maxwell systems on compact Riemannian manifolds

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Abstract. In this paper, we are focusing to the following Schrödinger–Maxwell system:

$$\begin{cases} -\Delta_g u + \beta(x)u + eu\phi = \Psi(\lambda, x)f(u) & \text{in } M, \\ -\Delta_g \phi + \phi = qu^2 & \text{in } M, \end{cases} \qquad (\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$$

where (M, g) is a 3-dimensional compact Riemannian manifold without boundary, $e, q > 0$ are positive numbers, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\beta \in C^\infty(M)$ and $\Psi \in C^\infty(\mathbb{R}_+ \times M)$ are positive functions. By various variational approaches, existence of multiple solutions of the problem $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ is established.

Keywords: Schrödinger–Maxwell systems, critical points, compact Riemannian manifolds.

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1 Introduction and statement of the main results

We are concerned with the nonlinear Schrödinger–Maxwell system

$$\begin{cases} -\Delta_g u + \beta(x)u + eu\phi = \Psi(\lambda, x)f(u) & \text{in } M, \\ -\Delta_g \phi + \phi = qu^2 & \text{in } M, \end{cases} \qquad (\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$$

where (M, g) is a 3-dimensional compact Riemannian manifold without boundary, $e, q > 0$ are positive numbers, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\beta \in C^\infty(M)$ and $\Psi \in C^\infty(\mathbb{R}_+ \times M)$ are positive functions.

From physical point of view, the Schrödinger–Maxwell systems

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + \omega u + eu\phi = f(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 4\pi e u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

describe the statical behavior of a charged non-relativistic quantum mechanical particle interacting with the electromagnetic field. More precisely, the unknown terms $u : \mathbb{R}^3 \to \mathbb{R}$ and

$\phi : \mathbb{R}^3 \to \mathbb{R}$ are the fields associated to the particle and the electric potential, respectively, the nonlinear term f models the interaction between the particles and the coupled term ϕu concerns the interaction with the electric field. Note that the quantities m, e, ω and $\hbar$ are the mass, charge, phase, and Planck's constant.

In fact, system (1.1) comes from the evolutionary nonlinear Schrödinger equation by using a Lyapunov–Schmidt reduction.

The Schrödinger–Maxwell system (or its variants) has been the object of various investigations in the last two decades, the existence/non-existence of positive solutions, sign-changing solutions, ground states, radial, non-radial solutions, and semi-classical states has been investigated by several authors. Without sake of completeness, we recall in the sequel some important contributions to the study of system (1.1). Benci and Fortunato [7] considered the case of $f(x,s) = |s|^{p-2}s$ with $p \in (4,6)$ by proving the existence of infinitely many radial solutions for (1.1); their main step relies on the reduction of system (1.1) to the investigation of critical points of a "one-variable" energy functional associated with (1.1).

Based on the idea of Benci and Fortunato, under various growth assumptions on f further existence/multiplicity results can be found in Ambrosetti and Ruiz [4], Azzolini [5], in [6] Azzollini, d'Avenia, and Pomponio were concerned with the existence of a positive radial solution to system (1.1) under the effect of a general nonlinear term, in [11] the existence of a non radially symmetric solution was established when $p \in (4,6)$, by means of a Pohozaev-type identity, d'Aprile and Mugnai [12, 13] proved the non-existence of non-trivial solutions to system (1.1) whenever $f \equiv 0$ or $f(x,s) = |s|^{p-2}s$ and $p \in (0,2] \cup [6,\infty)$, the same authors proved the existence of a non-trivial radial solution to (1.1), for $p \in [4.6)$. Other existence and multiplicity result can be found in the works of Cerami and Vaira [8], Kristály and Repovs [23], Ruiz [27], Sun, Chen, and Nieto [28], Wang and Zhou [31], and references therein.

In the last five years Schrödinger–Maxwell systems has been studied on $n-$dimensional *compact or non-compact Riemannian manifolds* $(2 \le n \le 5)$ by Druet and Hebey [14], Farkas and Kristály [16], Hebey and Wei [19], Ghimenti and Micheletti [17, 18] and Thizy [29, 30]. More precisely, in the aforementioned papers various forms of the system

$$\begin{cases} -\frac{\hbar^2}{m}\Delta_g u + \omega u + eu\phi = f(x,u) & \text{in } M, \\ -\Delta_g \phi + \phi = 4\pi e u^2 & \text{in } M, \end{cases} \tag{1.2}$$

have been considered, where (M,g) is a Riemannian manifold.

The aim of this paper is threefold. First, we consider the system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ with $\Psi(\lambda,x) = \lambda\alpha(x)$, where α is a suitable function and we assume that f is a sublinear nonlinearity (see the assumptions (f_1)–(f_3) below). In this case, we prove that if the parameter λ is small enough the system $(\mathcal{SM}^e_\lambda)$ has only the trivial solution, while if λ is large enough then the system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ has at least two solutions, see Theorem 1.1. It is natural to ask what happens between this two threshold values. In this gap interval we have no information on the number of solutions $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$; in the case when $q \to 0$ these two threshold values may be arbitrary close to each other. Similar bifurcation type result for a perturbed sublinear elliptic problem was obtained by Kristály, see [20].

Second, we consider the system $(\mathcal{SM}^\lambda_{\Psi(\lambda,\cdot)})$ with $\Psi(\lambda,x) = \lambda\alpha(x) + \mu_0\beta(x)$, where α and β are suitable functions. In order to prove a new kind of multiplicity for the system $(\mathcal{SM}^\lambda_{\Psi(\lambda,\cdot)})$ (i.e. $e = \lambda$), we show that certain properties of the nonlinearity, concerning the set of all global minima, can be reflected to the energy functional associated to the problem, see Theorem 1.3.

Third, as a counterpart of Theorem 1.1 we will consider the system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ with $\Psi(\lambda,x) = \lambda$, and f here satisfies the so called Ambrosetti–Rabinowitz condition. This type of result is motivated by the result of Anello [3] and Ricceri [24], where the authors studied the classical Ambrosetti–Rabinowitz problem, without the assumption $\lim_{t\to 0}\frac{f(t)}{t}=0$, i.e. the authors proved that if the nonlinearity f satisfies the so called (AR) condition and a subcritical growth condition, then if λ is small enough the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least two weak solutions in $H_0^1(\Omega)$.

In the sequel we present precisely our results. As we mentioned before, we first consider a continuous function $f : [0,\infty) \to \mathbb{R}$ which verifies the following assumptions:

(f_1) $\frac{f(s)}{s} \to 0$ as $s \to 0^+$;

(f_2) $\frac{f(s)}{s} \to 0$ as $s \to \infty$;

(f_3) $F(s_0) > 0$ for some $s_0 > 0$, where $F(s) = \int_0^s f(t)\mathrm{d}t$, $s \geq 0$.

Due to the assumptions (f_1)–(f_3), the numbers

$$c_f = \max_{s>0}\frac{f(s)}{s}$$

and

$$c_F = \max_{s>0}\frac{4F(s)}{2s^2+eqs^4}$$

are well-defined and positive. Now, we are in the position to state the first result of the paper. In order to do this, first we recall the definition of the weak solutions of the problem $(\mathcal{SM}_\lambda^e)$: The pair $(u,\phi) \in H_g^1(M) \times H_g^1(M)$ is a *weak solution* to the system $(\mathcal{SM}_\lambda^e)$ if

$$\int_M (\langle \nabla_g u, \nabla_g v\rangle + \beta(x)uv + eu\phi v)\mathrm{d}v_g = \int_M \Psi(\lambda,x)f(u)v\mathrm{d}v_g \text{ for all } v \in H_g^1(M), \tag{1.3}$$

$$\int_M (\langle \nabla_g \phi, \nabla_g \psi\rangle + \phi\psi)\mathrm{d}v_g = q\int_M u^2\psi\mathrm{d}v_g \text{ for all } \psi \in H_g^1(M). \tag{1.4}$$

Our first result reads as follows.

Theorem 1.1. *Let (M,g) be a 3-dimensional compact Riemannian manifold without boundary, and let $\beta \equiv 1$. Assume that $\Psi(\lambda,x) = \lambda\alpha(x)$ and $\alpha \in C^\infty(M)$ is a positive function. If the continuous function $f : [0,\infty) \to \mathbb{R}$ satisfies assumptions (f_1)–(f_3), then*

(a) *if $0 \leq \lambda < c_f^{-1}\|\alpha\|_{L^\infty}^{-1}$, system $(\mathcal{SM}_{\Psi(\lambda,\cdot)}^\lambda)$ has only the trivial solution;*

(b) *for every $\lambda \geq c_F^{-1}\|\alpha\|_{L^1}^{-1}$, system $(\mathcal{SM}_{\Psi(\lambda,\cdot)}^\lambda)$ has at least two distinct non-zero, non-negative weak solutions in $H_g^1(M) \times H_g^1(M)$.*

Remark 1.2.

(a) Due to (f_1), it is clear that $f(0) = 0$, thus we can extend continuously the function $f : [0,\infty) \to \mathbb{R}$ to the whole $\mathbb{R}$ by $f(s) = 0$ for $s \leq 0$; thus, $F(s) = 0$ for $s \leq 0$.

(b) (f_1) and (f_2) mean that f is superlinear at the origin and sublinear at infinity, respectively. Typical functions which fulfill hypotheses (f_1)–(f_3) are

$$f(s) = \min(s^r, s^p), \qquad 0 < r < 1 < p, \ \ s \geq 0$$

or

$$f(s) = \ln(1+s^2), \qquad s \geq 0.$$

(c) By a three critical points result of Ricceri [26], one can prove that the number of solutions of the problem $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ for $\lambda > \tilde{\lambda}$ is stable under small nonlinear perturbations $g : \mathbb{R} \to \mathbb{R}$ of subcritical type, i.e., $g(s) = o(|s|^{2^*-1})$ as $|s| \to \infty$, $2^* = \frac{2N}{N-2}$, $N > 2$.

In order to obtain new kind of multiplicity result for the system $(\mathcal{SM}^\lambda_{\Psi(\lambda,\cdot)})$ (with the choice $e = \lambda$), instead of the assumption (f_1) we require the following one:

(f_4) There exists $\mu_0 > 0$ such that the set of all global minima of the function

$$t \mapsto \Phi_{\mu_0}(t) := \frac{1}{2}t^2 - \mu_0 F(t)$$

has at least $m \geq 2$ connected components.

In this case we can state the following result.

Theorem 1.3. *Let (M, g) be a 3-dimensional compact Riemannian manifold without boundary. Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function which satisfies (f_2) and (f_4), $\beta \in C^\infty(M)$ is a positive function. Assume that $\Psi(\lambda, x) = \lambda\alpha(x) + \mu_0\beta(x)$, where $\alpha \in C^\infty(M)$ is a positive function. Then for every $\tau > \|\beta\|_{L^1(M)} \inf_t \Phi_{\mu_0}(t)$ there exists $\lambda_\tau > 0$ such that for every $\lambda \in (0, \lambda_\tau)$ the problem $(\mathcal{SM}^\lambda_{\Psi(\lambda,\cdot)})$ has at least $m+1$ weak solutions, m of which satisfy the inequality*

$$\frac{1}{2}\int_M (|\nabla_g u|^2 + \beta(x)u^2)\, dv_g - \mu_0 \int_M \beta(x)F(u) dv_g < \tau.$$

Remark 1.4. Taking into account the result of Cordaro [10] and Anello [2] one can prove the following: consider the following system:

$$\begin{cases} -\Delta_g u + \alpha(x)u + \lambda\phi u = \alpha(x)f(u) + \lambda g(x,u), & \text{in } M \\ -\Delta_g \phi + \phi = qu^2, & \text{in } M \end{cases}$$

where $\alpha \in L^\infty(M)$ with $\operatorname{ess\,inf} \alpha > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $g : M \times \mathbb{R} \to \mathbb{R}$, besides being a Carathéodory function, is such that, for some $p > 3\,(= \dim M)$, $\sup_{|s|\leq t} g(\cdot, s) \in L^p(M)$ and $g(\cdot, t) \in L^\infty(M)$ for all $t \in \mathbb{R}$. If the set

$$G_f = \left\{ t \in \mathbb{R} : \frac{1}{2}t^2 - \int_0^t f(s)ds = \inf_{\xi\in\mathbb{R}} \left(\frac{1}{2}\xi^2 - \int_0^\xi f(s)ds \right) \right\}$$

has $m \geq 2$ bounded connected components, then the system has at least $m + \left[\frac{m}{2}\right]$ weak solutions. For the proof, one can use a truncation argument combining with the abstract critical point theory result of Anello [1, Theorem 2.1]. Note that the similar truncation method which was presented in [10] fails, due to the extra term $\int_M \phi_u u^2$. To overcome this difficulty, one can use the same method as in [16, Proposition 3.1 (i) & (ii)] (see also [21]).

Note also that similar multiplicity results was obtained by Kristály and Rădulescu in [22], for Emden–Fowler type equations.

Our abstract tool for proving the Theorem 1.3 is the following abstract theorem that we recall here (see [25]).

Theorem A. Let H be a separable and reflexive real Banach space, and let $\mathcal{N}, \mathcal{G} : H \to \mathbb{R}$ be two sequentially weakly lower semi-continuous and continuously Gateaux differentiable functionals, with $\mathcal{N}$ coercive. Assume that the functional $\mathcal{N} + \lambda\mathcal{G}$ satisfies the Palais–Smale condition for every $\lambda > 0$ small enough and that the set of all global minima of $\mathcal{N}$ has at least m connected components in the weak topology, with $m \geq 2$. Then, for every $\eta > \inf_H \mathcal{N}$, there exists $\overline{\lambda} > 0$ such that for every $\lambda \in (0, \overline{\lambda})$ the functional $\mathcal{N} + \lambda\mathcal{G}$ has at least $m + 1$ critical points, m of which are in $\mathcal{N}^{-1}((-\infty, \eta))$.

Finally, as a counterpart of the Theorem 1.1 we consider the case when the continuous function $f : [0, +\infty) \to \mathbb{R}$ satisfies the following assumptions:

($\tilde{f}_1$) $|f(s)| \leq C(1 + |s|^{p-1})$, for all $s \in \mathbb{R}$, where $p \in (2, 6)$;

($\tilde{f}_2$) there exists $\eta > 4$ and $\tau_0 > 0$ such that

$$0 < \eta F(s) \leq sf(s), \qquad \forall |s| \geq \tau_0.$$

Theorem 1.5. *Let (M, g) be a 3-dimensional compact Riemannian manifold without boundary, and let $\beta \equiv 1$. Assume that $\Psi(\lambda, x) = \lambda$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, which satisfies hypotheses ($\tilde{f}_1$), ($\tilde{f}_2$). Then there exists λ_0 such that for every $0 < \lambda < \lambda_0$ the problem $(\mathcal{SM}^e_\lambda)$ has at least two weak solutions.*

Our abstract tool for proving the previous theorem is the following abstract theorem that we recall here (see [24]).

Theorem B. Let E be a reflexive real Banach space, and let $\Phi, \Psi : E \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is sequentially weakly lower semi-continuous and coercive. Further, assume that Ψ is sequentially weakly continuous. In addition, assume that for each $\mu > 0$, the functional $J_\mu := \mu\Phi - \Psi$ satisfies the classical compactness Palais–Smale condition. Then for each $\rho > \inf_E \Phi$ and each

$$\mu > \inf_{u \in \Phi^{-1}((-\infty,\rho))} \frac{\sup_{v \in \Phi^{-1}((-\infty,\rho))} \Psi(v) - \Psi(u)}{\rho - \Phi(u)},$$

the following alternative holds: either the functional J_μ has a strict global minimum which lies in $\Phi^{-1}((-\infty, \rho))$, or J_μ has at least two critical points one of which lies in $\Phi^{-1}((-\infty, \rho))$.

2 Proof of the main results

Let $\beta \in C^\infty(M)$ be a positive function. For every $u \in C^\infty(M)$ let us denote by

$$\|u\|^2_\beta = \int_M |\nabla_g u|^2 + \beta(x)u^2 dv_g.$$

The Sobolev space H^1_β is defined as the completion of $C^\infty(M)$ with respect to the norm $\|\cdot\|_\beta$. Clearly, H^1_β is a Hilbert space. Note that, since β is positive, the norm $\|\cdot\|_\beta$ is equivalent to the standard norm, i.e. we have that

$$\min\left\{1, \min_M \sqrt{\beta(x)}\right\} \|u\|_{H^1_g(M)} \le \|u\|_\beta \le \max\left\{1, \sqrt{\|\beta\|_{L^\infty(M)}}\right\} \|u\|_{H^1_g(M)}. \tag{2.1}$$

Note that $H^1_\beta(M)$ is compactly embedded in $L^p(M)$, $p \in [1,6)$; the Sobolev embedding constant will be denoted by κ_p.

We define the energy functional $\mathscr{J}_\lambda : H^1_g(M) \times H^1_g(M) \to \mathbb{R}$ associated with system $(\mathcal{SM}^e_\lambda)$, namely,

$$\mathscr{J}_\lambda(u,\phi) = \frac{1}{2}\|u\|^2_\beta + \frac{e}{2}\int_M \phi u^2 \mathrm{d}v_g - \frac{e}{4q}\int_M |\nabla_g \phi|^2 \mathrm{d}v_g - \frac{e}{4q}\int_M \phi^2 \mathrm{d}v_g - \int_M \Psi(x,\lambda)F(u)\mathrm{d}v_g.$$

It is easy to see that the functional $\mathscr{J}_\lambda$ is well-defined and of class C^1 on $H^1_g(M) \times H^1_g(M)$. Moreover, due to relations (1.3) and (1.4) the pair $(u,\phi) \in H^1_g(M) \times H^1_g(M)$ is a weak solution of $(\mathcal{SM}^e_\lambda)$ if and only if (u,ϕ) is a critical point of $\mathscr{J}_\lambda$.

Using the Lax–Milgram theorem one can see that the equation

$$-\Delta_g \phi + \phi = qu^2, \quad \text{in } M$$

has a unique solution for any fixed u. By exploring an idea of Benci and Fortunato [7], we introduce the map $\phi_u : H^1_g(M) \to H^1_g(M)$ by associating to every $u \in H^1_g(M)$ the unique solution $\phi = \phi_u$ of the Maxwell equation. Thus, one can define the "one-variable" energy functional $\mathcal{E}_\lambda : H^1_g(M) \to \mathbb{R}$ associated with system $(\mathcal{SM}^e_\lambda)$:

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\|u\|^2_\beta + \frac{e}{4}\int_M \phi_u u^2 \mathrm{d}v_g - \mathcal{F}(u), \tag{2.2}$$

where $\mathcal{F} : H^1_g(M) \to \mathbb{R}$ is the functional defined by

$$\mathcal{F}(u) = \int_M \Psi(x,\lambda)F(u)\mathrm{d}v_g.$$

By using standard variational arguments, one has that the pair $(u,\phi) \in H^1_g(M) \times H^1_g(M)$ is a critical point of $\mathscr{J}_\lambda$ if and only if u is a critical point of $\mathcal{E}_\lambda$ and $\phi = \phi_u$, see for instance [16]. Moreover, we have that

$$\mathcal{E}'_\lambda(u)(v) = \int_M (\langle \nabla_g u, \nabla_g v\rangle + \beta(x)uv + e\phi_u uv)\mathrm{d}v_g - \int_M \Psi(x,\lambda)f(u)v\mathrm{d}v_g. \tag{2.3}$$

2.1 Schrödinger–Maxwell systems involving sublinear nonlinearity

In this section, we set $\Psi(x,\lambda) = \lambda\alpha(x) + \mu_0\beta(x)$. Recall that

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\|u\|^2_\beta + \frac{e}{4}\int_M \phi_u u^2 \mathrm{d}v_g - \int_M \Psi(x,\lambda)F(u)\mathrm{d}v_g.$$

In order to apply variational methods, we prove some elementary properties of the functional $\mathcal{E}_\lambda$.

Lemma 2.1. *The energy functional $\mathcal{E}_\lambda$ is coercive, for every $\lambda \geq 0$.*

Proof. Indeed, due to (f_2), we have that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|F(s)| \leq \varepsilon|s|^2$, for every $|s| > \delta$. Thus, since $\Psi(x,\lambda) \in L^\infty(M)$ we have that

$$\begin{aligned}\mathcal{F}(u) &= \int_{\{u>\delta\}} \Psi(x,\lambda)F(u)\mathrm{d}v_g + \int_{\{u\leq\delta\}} \Psi(x,\lambda)F(u)\mathrm{d}v_g \\ &\leq \varepsilon \|\Psi(\cdot,\lambda)\|_{L^\infty(M)}\, \kappa_2^2\|u\|_\beta^2 + \|\Psi(\cdot,\lambda)\|_{L^\infty(M)}\mathrm{Vol}_g M \max_{|s|\leq\delta}|F(s)|.\end{aligned}$$

Therefore,

$$\mathcal{E}_\lambda(u) \geq \left(\frac{1}{2} - \varepsilon\kappa_2^2\, \|\Psi(\cdot,\lambda)\|_{L^\infty(M)}\right) \|u\|_\beta^2 - \mathrm{Vol_g} M \cdot \|\Psi(\cdot,\lambda)\|_{L^\infty(M)} \max_{|s|\leq\delta}|F(s)|.$$

In particular, if $0 < \varepsilon < (2\kappa_2^2\|\Psi(\cdot,\lambda)\|_{L^\infty(M)})^{-1}$, then $\mathcal{E}_\lambda(u) \to \infty$ as $\|u\|_\beta \to \infty$. □

Lemma 2.2. *The energy functional $\mathcal{E}_\lambda$ satisfies the Palais–Smale condition for every $\lambda \geq 0$.*

Proof. Let $\{u_j\}_j \subset H_g^1(M)$ be a Palais–Smale sequence, i.e. $\{\mathcal{E}_\lambda(u_j)\}_j$ is bounded and

$$\|(\mathcal{E}_\lambda)'(u_j)\|_{H_g^1(M)^*} \to 0$$

as $j \to \infty$. Since $\mathcal{E}_\lambda$ is coercive (see Lemma 2.1), the sequence $\{u_j\}_j$ is bounded in $H_g^1(M)$. Therefore, up to a subsequence, then $\{u_j\}_j$ converges weakly in $H_g^1(M)$ and strongly in $L^p(M)$, $p \in (2, 2^*)$, to an element $u \in H_g^1(M)$.

First we claim that for all $u, v \in H_g^1(M)$ we have that

$$\int_M (u\phi_u - v\phi_v)\,(u - v)\mathrm{d}v_g \geq 0. \tag{2.4}$$

This inequality is equivalent with the following one:

$$\int_M \phi_u u^2\mathrm{d}v_g + \int_M \phi_v v^2\mathrm{d}v_g \geq \int_M (\phi_u uv + \phi_v uv)\mathrm{d}v_g.$$

On the other hand, using the Cauchy–Schwarz inequality, we have, that

$$\begin{aligned}\int_M (\phi_u uv + \phi_v uv)\mathrm{d}v_g &\leq \left(\int_M \phi_u u^2\mathrm{d}v_g\right)^{1/2} \left(\int_M \phi_u v^2\mathrm{d}v_g\right)^{1/2} \\ &\quad + \left(\int_M \phi_v u^2\mathrm{d}v_g\right)^{1/2} \left(\int_M \phi_v v^2\mathrm{d}v_g\right)^{1/2} \\ &= \frac{1}{q}\left(\int_M (\nabla_g\phi_u \nabla_g\phi_v + \phi_u\phi_v)\mathrm{d}v_g\right)^{1/2} (\|\phi_u\|_{H_g^1(M)} + \|\phi_v\|_{H_g^1(M)}) \\ &\leq \frac{1}{q}\|\phi_u\|_{H_g^1(M)}^{1/2}\|\phi_v\|_{H_g^1(M)}^{1/2} \left(\|\phi_u\|_{H_g^1(M)} + \|\phi_v\|_{H_g^1(M)}\right).\end{aligned}$$

Taking into account the following algebraic inequality $(xy)^{1/2}(x+y) \leq (x^2+y^2)$, $(\forall) x, y \geq 0$, we have that

$$\|\phi_u\|_{H_g^1(M)}^{1/2}\|\phi_v\|_{H_g^1(M)}^{1/2} \left(\|\phi_u\|_{H_g^1(M)} + \|\phi_v\|_{H_g^1(M)}\right) \leq \|\phi_u\|_{H_g^1(M)}^2 + \|\phi_v\|_{H_g^1(M)}^2.$$

Therefore,

$$\int_M (\phi_u uv + \phi_v uv) \mathrm{d}v_g \le \frac{1}{q}\left(\|\phi_u\|^2_{H^1_g(M)} + \|\phi_v\|^2_{H^1_g(M)}\right) = \int_M \phi_u u^2 \mathrm{d}v_g + \int_M \phi_v v^2 \mathrm{d}v_g,$$

which proves the claim.

Now, using inequality (2.4) one has

$$\int_M |\nabla_g u_j - \nabla_g u|^2 \mathrm{d}v_g + \int_M \beta(x)\left(u_j - u\right)^2 \mathrm{d}v_g$$
$$\le (\mathcal{E}_\lambda)'(u_j)(u_j - u) + (\mathcal{E}_\lambda)'(u)(u - u_j) + \int_M \Psi(x,\lambda)[f(u_j(x)) - f(u(x))](u_j - u)\mathrm{d}v_g.$$

Since $\|(\mathcal{E}_\lambda)'(u_j)\|_{H^1_g(M)^*} \to 0$, and $u_j \rightharpoonup u$ in $H^1_g(M)$, the first two terms at the right hand side tend to 0. Let $p \in (2, 2^*)$.

By the assumptions on f, for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|f(s)| \le \varepsilon|s| + C_\varepsilon |s|^{p-1},$$

for every $s \in \mathbb{R}$. The latter relation, Hölder inequality and the fact that $u_j \to u$ in $L^p(M)$ imply that

$$\left|\int_M \Psi(x,\lambda)[f(u_j) - f(u)](u_j - u)\mathrm{d}v_g\right| \to 0,$$

as $j \to \infty$. Therefore, $\|u_j - u\|^2_{H^1_g(M)} \to 0$ as $j \to \infty$, which proves our claim. □

Before we prove Theorem 1.1, we prove the following lemma.

Lemma 2.3. *Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous function satisfying the assumptions (f_1)–(f_3) . Then*

$$c_f := \max_{s>0} \frac{f(s)}{s} > c_F := \max_{s>0} \frac{4F(s)}{2s^2 + eqs^4}.$$

Proof. Let $s_0 > 0$ be a maximum point for the function $s \mapsto \dfrac{4F(s)}{2s^2 + eqs^4}$, therefore

$$c_F = \frac{4F(s_0)}{2s_0^2 + eqs_0^4} = \frac{f(s_0)}{s_0 + eqs_0^3} \le \frac{f(s_0)}{s_0} \le c_f.$$

Now we assume that $c_f = c_F := \theta$. Let

$$\widetilde{s}_0 := \inf\left\{s > 0 : \theta = \frac{4F(s)}{2s^2 + eqs^4}\right\}.$$

Note that $\widetilde{s}_0 > 0$. Fix $t_0 \in (0, \widetilde{s}_0)$, in particular $4F(t_0) < \theta(2t_0^3 + eqt_0^4)$. On the other hand, from the definition of c_f, one has $f(t) \le \theta(s + eqs^3)$. Therefore

$$0 = 4F(\widetilde{s}_0) - \theta(2\widetilde{s}_0 + eq\widetilde{s}_0^4) = \left(4F(t_0) - \theta(2t_0^2 + eqt_0^4)\right) + 4\int_{t_0}^{\widetilde{s}_0} \left(f(t) - \theta(s + eqs^3)\right) ds < 0,$$

which is a contradiction, thus $c_f > c_F$. □

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. First recall that, in this case, $\beta(x) \equiv 1$ and $\Psi(\lambda, x) = \lambda\alpha(x)$, and $\alpha \in C^\infty(M)$ is a positive function.
(a) Let $\lambda \geq 0$. If we choose $v = u$ in (1.3) we obtain that

$$\int_M \left(|\nabla_g u|^2 + u^2 + e\phi_u u^2\right) \mathrm{d}v_g = \lambda \int_M \alpha(x) f(u) u \mathrm{d}v_g.$$

As we already mentioned, due to the assumptions (f_1)–(f_3), the number $c_f = \max_{s>0} \frac{f(s)}{s}$ is well-defined and positive. Thus, since $\|\phi_u\|^2_{H^1_g(M)} = q\int_M \phi_u u^2 \mathrm{d}v_g \geq 0$, we have that

$$\|u\|^2_{H^1_g(M)} \leq \|u\|^2_{H^1_g(M)} + e\int_M \phi_u u^2 \mathrm{d}v_g \leq \lambda c_f \|\alpha\|_{L^\infty(M)} \int_M u^2 \mathrm{d}v_g \leq \lambda c_f \|\alpha\|_{L^\infty(M)} \|u\|^2_{H^1_g(M)}.$$

Therefore, if $\lambda < c_f^{-1}\|\alpha\|^{-1}_{L^\infty(M)}$, then the last inequality gives $u = 0$. By the Maxwell's equation we also have that $\phi = 0$, which concludes the proof of (a).
(b) By using assumptions (f_1) and (f_2), one has

$$\lim_{\mathscr{H}(u)\to 0} \frac{\mathcal{F}(u)}{\mathscr{H}(u)} = \lim_{\mathscr{H}(u)\to\infty} \frac{\mathcal{F}(u)}{\mathscr{H}(u)} = 0,$$

where $\mathscr{H}(u) = \frac{1}{2}\|u\|^2_\beta + \frac{e}{4}\int_M \phi_u u^2 \mathrm{d}v_g$. Since $\alpha \in C^\infty(M)_+ \setminus \{0\}$, on account of ($f_3$), one can guarantee the existence of a suitable truncation function $u_T \in H^1_g(M) \setminus \{0\}$ such that $\mathcal{F}(u_T) > 0$. Therefore, we may define

$$\lambda_0 = \inf_{\substack{u\in H^1_g(M)\setminus\{0\} \\ \mathcal{F}(u)>0}} \frac{\mathscr{H}(u)}{\mathcal{F}(u)}.$$

The above limits imply that $0 < \lambda_0 < \infty$. Since $H^1_g(M)$ contains the positive constant functions on M, we have

$$\lambda_0 = \inf_{\substack{u\in H^1_g(M)\setminus\{0\} \\ \mathcal{F}(u)>0}} \frac{\mathscr{H}(u)}{\mathcal{F}(u)} \leq \max_{s>0} \frac{2s^2 + eqs^4}{4F(s)\|\alpha\|_{L^1(M)}} = c_F^{-1}\|\alpha\|^{-1}_{L^1(M)}.$$

For every $\lambda > \lambda_0$, the functional $\mathcal{E}_\lambda$ is bounded from below, coercive and satisfies the Palais–Smale condition (see Lemma 2.1, Lemma 2.2). If we fix $\lambda > \lambda_0$ one can choose a function $w \in H^1_g(M)$ such that $\mathcal{F}(w) > 0$ and

$$\lambda > \frac{\mathscr{H}(w)}{\mathcal{F}(w)} \geq \lambda_0.$$

In particular,

$$c_1 := \inf_{H^1_g(M)} \mathcal{E}_\lambda \leq \mathcal{E}_\lambda(w) = \mathscr{H}(w) - \lambda\mathcal{F}(w) < 0.$$

The latter inequality proves that the global minimum $u^1_\lambda \in H^1_g(M)$ of $\mathcal{E}_\lambda$ on $H^1_g(M)$ has negative energy level.

In particular, $(u^1_\lambda, \phi_{u^1_\lambda}) \in H^1_g(M) \times H^1_g(M)$ is a nontrivial weak solution to $(\mathcal{SM}^e_\lambda)$.

Let $\nu \in (2, 6)$ be fixed. By assumptions, for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$0 \leq |f(s)| \leq \frac{\varepsilon}{\|\alpha\|_{L^\infty(M)}}|s| + C_\varepsilon |s|^{\nu-1} \quad \text{for all } s \in \mathbb{R}.$$

Thus

$$\begin{aligned}0 \leq |\mathcal{F}(u)| &\leq \int_M \alpha(x)|F(u(x))|\mathrm{d}v_g \\ &\leq \int_M \alpha(x)\left(\frac{\varepsilon}{2\|\alpha\|_{L^\infty(M)}}u^2(x) + \frac{C_\varepsilon}{\nu}|u(x)|^\nu\right)\mathrm{d}v_g \\ &\leq \frac{\varepsilon}{2}\|u\|^2_{H^1_g(M)} + \frac{C_\varepsilon}{\nu}\|\alpha\|_{L^\infty(M)}\widetilde{\kappa}_\nu^\nu\|u\|^\nu_{H^1_g(M)},\end{aligned}$$

where $\widetilde{\kappa}_\nu$ is the embedding constant in the compact embedding $H^1_g(M) \hookrightarrow L^\nu(M)$, $\nu \in [1,6)$.

Therefore,

$$\mathcal{E}_\lambda(u) \geq \frac{1}{2}(1-\lambda\varepsilon)\|u\|^2_{H^1_g(M)} - \frac{\lambda C_\varepsilon}{\nu}\|\alpha\|_{L^\infty(M)}\widetilde{\kappa}_\nu^\nu\|u\|^\nu_{H^1_g(M)}.$$

Bearing in mind that $\nu > 2$, for enough small $\rho > 0$ and $\varepsilon < \lambda^{-1}$ we have that

$$\inf_{\|u\|_{H^1_g(M)}=\rho} \mathcal{E}_\lambda(u) \geq \frac{1}{2}\left(1-\varepsilon\lambda\right)\rho - \frac{\lambda C_\varepsilon}{\nu}\|\alpha\|_{L^\infty(M)}\widetilde{\kappa}_\nu^\nu\rho^{\frac{\nu}{2}} > 0.$$

A standard mountain pass argument (see, for instance, Willem [32]) implies the existence of a critical point $u^2_\lambda \in H^1_g(M)$ for $\mathcal{E}_\lambda$ with positive energy level. Thus $(u^2_\lambda, \phi_{u^2_\lambda}) \in H^1_g(M) \times H^1_g(M)$ is also a nontrivial weak solution to $(\mathcal{SM}^e_\lambda)$. Clearly, $u^1_\lambda \neq u^2_\lambda$. □

It is also clear that the function $q \mapsto \max_{s>0} \frac{4F(s)}{2s^2+eqs^4}$ is non-increasing. Let $a > 1$ be a real number. Now, consider the following function

$$f(s) = \begin{cases} 0, & 0 \leq s < 1, \\ s + g(s), & 1 \leq s < a, \\ a + g(a), & s \geq a, \end{cases}$$

where $g : [1, +\infty) \to \mathbb{R}$ is a continuous function with the following properties

(g_1) $g(1) = -1$;

(g_2) the function $s \mapsto \frac{g(s)}{s}$ is non-decreasing on $[1, +\infty)$;

(g_3) $\lim_{s\to\infty} g(s) < \infty$.

In this case the

$$F(s) = \begin{cases} 0, & 0 \leq s < 1, \\ \frac{s^2}{2} + G(s) - \frac{1}{2}, & 1 \leq s < a, \\ (a+g(a))s - \frac{a^2}{2} + G(a) - ag(a) - \frac{1}{2}, & s \geq a, \end{cases}$$

where $G(s) = \int_1^s g(t)dt$. It is also clear that f satisfies the assumptions (f_1)–(f_3).

Thus, a simple calculation shows that

$$c_f = \frac{a + g(a)}{a}.$$

We also claim that

$$\widehat{c}_F = \lim_{q\to 0} c_F = \frac{(a+g(a))^2}{a^2+2ag(a)-2G(a)+1}.$$

Indeed,

$$\widehat{c}_F = \max_{s>0} \frac{2F(s)}{s^2}.$$

It is clear that it is enough to show that the maximum of the function $\frac{2F(s)}{s^2}$ is achieved on the interval $s \geq a$, i.e.,

$$sg(s) - 2G(s) > -1, \qquad s > 1.$$

Now, using a result of [9, page 42, equation (4.3)] (see also [15, Theorem 1.3]), we have that the function $\frac{G(s)}{\frac{s^2-1}{2}}$ is increasing, thus

$$sg(s) - 2G(s) \geq \frac{g(s)}{s} \geq -1, \qquad s \geq -1,$$

which proves our claim.

One can see from the assumptions on g, that the values c_f and $\widehat{c}_F$ may be arbitrary close to each other. Indeed, when

$$\lim_{a\to\infty} c_f = \lim_{a\to\infty} \widehat{c}_F = 1.$$

Therefore, if $\alpha \equiv 1$ then the threshold values are c_f^{-1} and c_F^{-1} (which are constructed independently), i.e., if $\lambda \in (0, c_f^{-1})$ we have just the trivial solution, while if $\lambda \in (c_F^{-1}, +\infty)$ we have at least two solutions. λ lying in the gap-interval $[c_f^{-1}, c_F^{-1}]$ we have no information on the number of solutions for $(\mathcal{SM}_\lambda^e)$.

Taking into account the above example we see that if the "impact" of the Maxwell equation is small ($q \to 0$), then the values c_f and c_F may be arbitrary close to each other.

Remark 2.4. Typical examples for function g can be:

(a) $g(s) = -1$. In this case $c_f = \frac{a-1}{a}$ and $\widehat{c}_F = \frac{a-1}{a+1}$.

(b) $g(s) = \frac{1}{s} - 2$. In this case $c_f = \frac{(a-1)^2}{a^2}$ and $\widehat{c}_F = \frac{(a-1)^4}{a^2(a^2-2\ln a-1)}$.

Proof of Theorem 1.3. We follow the idea presented in [22]. First, we claim that the set of all global minima of the functional $\mathcal{N} : H_g^1(M) \to \mathbb{R}$,

$$\mathcal{N}(u) = \frac{1}{2}\|u\|_\beta^2 - \mu_0 \int_M \beta(x)F(u)\mathrm{d}v_g$$

has at least m connected components in the weak topology on $H_g^1(M)$. Indeed, for every $u \in H_\beta^1(M)$ one has

$$\begin{aligned}\mathcal{N}(u) &= \frac{1}{2}\|u\|_\beta^2 - \mu_0 \int_M \beta(x)F(u)\mathrm{d}v_g \\ &= \frac{1}{2}\int_M |\nabla_g u|^2 \mathrm{d}v_g + \int_M \beta(x)\Phi_{\mu_0}(u)\mathrm{d}v_g \\ &\geq \|\beta\|_{L^1(M)} \inf_t \Phi_{\mu_0}(t).\end{aligned}$$

Moreover, if we consider $u = \tilde{t}$ for a.e. $x \in M$, where $\tilde{t} \in \mathbb{R}$ is the minimum point of the function $t \mapsto \Phi_{\mu_0}(t)$, then we have equality in the previous estimate. Thus,

$$\inf_{u \in H^1_\beta(M)} \mathcal{N}(u) = \|\beta\|_{L^1(M)} \inf_t \Phi_{\mu_0}(t).$$

On the other hand, if $u \in H^1_g(M)$ is not a constant function, then $|\nabla_g u|^2 > 0$ on a positive measure set in M, i.e.

$$\mathcal{N}(u) > \|\beta\|_{L^1(M)} \inf_t \Phi_{\mu_0}(t).$$

Consequently, there is a one-to-one correspondence between the sets

$$\mathrm{Min}(\mathcal{N}) = \left\{ u \in H^1_g(M) : \mathcal{N}(u) = \inf_{u \in H^1_g(M)} \mathcal{N}(u) \right\}$$

and

$$\mathrm{Min}\left(\Phi_{\mu_0}\right) = \left\{ t \in \mathbb{R} : \Phi_{\mu_0}(t) = \inf_{t \in \mathbb{R}} \Phi_{\mu_0}(t) \right\}.$$

Let ξ be the function that associates to every $t \in \mathbb{R}$ the equivalence class of those functions which are a.e. equal to t on the whole M. Then $\xi : \mathrm{Min}(\mathcal{N}) \to \mathrm{Min}\left(\Phi_{\mu_0}\right)$ is actually a homeomorphism, where $\mathrm{Min}(\mathcal{N})$ is considered with the relativization of the weak topology on $H^1_g(M)$. On account of (f_4), the set $\mathrm{Min}\left(\Phi_{\mu_0}\right)$ has at least $m \geq 2$ connected components. Therefore, the same is true for the set $\mathrm{Min}(\mathcal{N})$, which proves the claim.

Now, we are in the position to apply Theorem **A** with $H = H^1_g(M)$, $\mathcal{N}$ and

$$\mathcal{G} = \frac{1}{4} \int_M \phi_u u^2 \mathrm{d}v_g - \int_M \alpha(x) F(u) \mathrm{d}v_g.$$

Now, we prove that the functional $\mathcal{G}$ is sequentially weakly lower semicontinuous. To see this, it is enough to prove that the map

$$H^1_\beta(M) \ni u \mapsto \int_M \phi_u u^2 \mathrm{d}v_g$$

is convex. To prove this, let us fix $u, v \in H^1_\beta(M)$ and $t, s \geq 0$ such that $t + s = 1$. Then we have that

$$\begin{aligned} \mathscr{A}(\phi_{tu+sv}) := -\Delta_g \phi_{tu+sv} + \phi_{tu+sv} &= q(tu+sv)^2 \\ &\leq q(tu^2 + sv^2) \\ &= t(qu^2) + s(qv^2) \\ &= t(-\Delta_g \phi_u + \phi_u) + s(-\Delta_g \phi_v + \phi_v) \\ &= \mathscr{A}(t\phi_u + s\phi_v). \end{aligned}$$

Then, using a comparison principle it follows that

$$\phi_{tu+sv} \leq t\phi_u + s\phi_v.$$

Then, multiplying the equations $-\Delta_g \phi_u + \phi_u = qu^2$ by ϕ_v and $-\Delta_g \phi_v + \phi_v = qv^2$ by ϕ_u, after integration, we obtain that

$$\int_M (\nabla_g \phi_u \nabla_g \phi_v + \phi_u \phi_v) \mathrm{d}v_g = q \int_M u^2 \phi_v \mathrm{d}v_g = q \int_M v^2 \phi_u \mathrm{d}v_g. \tag{2.5}$$

Thus, combining the above outcomes we have

$$\begin{aligned}
\int_M \phi_{tu+sv}(tu+sv)^2 dv_g &\le \int_M (t\phi_u + s\phi_v)\left(tu^2+sv^2\right) dv_g \\
&= t^2\int_M \phi_u u^2 dv_g + ts\int_M \left(\phi_u v^2 + \phi_v u^2\right) dv_g + s^2\int_M \phi_v v^2 dv_g \\
&\overset{(2.5)}{=} \frac{t^2}{q}\left(\int_M |\nabla_g \phi_u|^2 dv_g + \int_M \phi_u^2 dv_g\right) + \frac{2ts}{q}\int_M (\nabla_g\phi_u \nabla_g \phi_v + \phi_u\phi_v) dv_g \\
&\quad + \frac{s^2}{q}\left(\int_M |\nabla_g \phi_v|^2 dv_g + \int_M \phi_v^2 dv_g\right) \\
&= \frac{1}{q}\int_M (t\nabla_g\phi_u + s\nabla_g\phi_v)^2 dv_g + \frac{1}{q}\int_M (t\phi_u + s\phi_v)^2 dv_g \\
&\le t\int_M \phi_u u^2 dv_g + s\int_M \phi_v v^2 dv_g,
\end{aligned}$$

which gives the required inequality, therefore it follows the required convexity. Almost the same way as in Lemma 2.2 we can prove that $\mathcal{N} + \lambda\mathcal{G}$ satisfies the Palais–Smale condition for every $\lambda > 0$ small enough. Therefore, the functionals $\mathcal{N}$ and $\mathcal{G}$ satisfies all the hypotheses of Theorem **A**. Therefore for every $\tau > \max\left\{0, \|\beta\|_1 \inf_t \Phi_{\mu_0}(t)\right\}$ there exists $\lambda_\tau > 0$ such that for every $\lambda \in (0, \lambda_\tau)$ the problem $(\mathcal{SM}_\lambda^\lambda)$ has at least $m+1$ solutions. We know in addition that m elements among the solutions belong to the set $\mathcal{N}_{\nu_0}^{-1}((-\infty, \tau))$, which proves that m solutions satisfy the inequality

$$\frac{1}{2}\int_M \left(|\nabla_g u|^2 + \beta(x)u^2\right) \mathrm{d}v_g - \mu_0 \int_M \beta(x)F(u)\mathrm{d}v_g < \tau. \qquad \square$$

Remark 2.5.

(a) Note that (f_4) implies that the function $t \mapsto \Phi_{\mu_0}(t)$ has at least $m-1$ local maxima. Thus, the function $t \mapsto \mu_0 f(t)$ has at least $2m-1$ fixed points. In particular, if for some $\lambda > 0$

$$\Psi(x,\lambda) = \mu_0\beta(x), \quad \text{for every } x \in M,$$

then the problem $(\mathcal{SM}_\lambda^\lambda)$ has at least $2m-1 \ge 3$ constant solutions.

(b) Using the abstract Theorem **A**, one can guarantee that $\tau > \max\left\{0, \|\beta\|_{L^1(M)} \inf_t \Phi_{\mu_0}(t)\right\}$ It is clear that the assumption (f_2) holds if there exist $\nu \in (0,1)$ and $c > 0$ such that

$$|f(t)| \le c|t|^\nu, \quad \text{for every } t \in \mathbb{R}.$$

In this case, m weak solutions of the problem satisfy the inequality

$$\frac{1}{2}\int_M \left(|\nabla_g u|^2 + \beta(x)u^2\right) \mathrm{d}v_g - \mu_0 \int_M \beta(x)F(u)\mathrm{d}v_g < \tau.$$

Now, it is clear that

$$|F(t)| \le \frac{c}{\nu+1}|t|^{\nu+1}, \quad \text{for every } t \in \mathbb{R}.$$

Using a Hölder inequality

$$\int_M \beta(x)|u|^{\nu+1}\mathrm{d}v_g \le \|\beta\|_{L^1(M)}^{\frac{1-\nu}{2}} \|u\|_{H^1_\beta(M)}^{\nu+1}.$$

One can observe, that since $\tau > 0$ the equation

$$\frac{1}{2}t^2 - \frac{\mu_0 c \|\beta\|_{L^1 M}^{\frac{1-\nu}{2}}}{\nu+1} t^{\nu+1} - \tau = 0,$$

always has a positive solution.

Summing up, the number $\|u\|_{H^1_\beta(M)}$ is less than the greatest solution of the previous algebraic equation. Combining this with (2.1), we have that

$$\|u\|_{H^1_g(M)} \leq \frac{t_*}{\min\{1, \min_M \sqrt{\beta}\}},$$

where t_* the greatest solution of the previous algebraic equation. A similar study for Emden–Fowler equation was done by Kristály and Rădulescu, see [22, Theorem 1.3].

2.2 Schrödinger–Maxwell systems involving superlinear nonlinearity

In the sequel, we prove Theorem 1.5. Recall that $\Psi(\lambda, x) = \lambda$ and $\beta \equiv 1$. The energy functional associated with the problem $(\mathcal{SM}^e_\lambda)$ is defined by

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\|u\|^2_{H^1_g(M)} + \frac{e}{4}\int_M \phi_u u^2 \mathrm{d}v_g - \lambda \int_M F(u)\mathrm{d}v_g.$$

Lemma 2.6. *Every (PS) sequence for the functional $\mathcal{E}_\lambda$ is bounded in $H^1_g(M)$.*

Proof. We consider a Palais–Smale sequence $(u_j)_j \subset H^1_g(M)$ for $\mathcal{E}_\lambda$, i.e. $\{\mathcal{E}_\lambda(u_j)\}$ is bounded and

$$\|(\mathcal{E}_\lambda)'(u_j)\|_{H^1_g(M)^*} \to 0 \quad \text{as } j \to \infty.$$

We claim that $(u_j)_j$ is bounded in $H^1_g(M)$. We argue by contradiction, so suppose the contrary. Passing to a subsequence if necessary, we may assume that

$$\|u_j\|_{H^1_g(M)} \to \infty, \quad \text{as } j \to \infty.$$

It follows that there exists $j_0 \in \mathbb{N}$ such that for every $j \geq j_0$ we have that

$$\mathcal{E}_\lambda(u_j) - \frac{\langle \mathcal{E}'_\lambda(u_j), u_j\rangle}{\eta} = \frac{1}{2}\left(\frac{\eta-2}{\eta}\right)\|u_j\|^2_{H^1_g(M)} + \frac{e}{4}\left(\frac{\eta-4}{\eta}\right)\int_M \phi_{u_j} u_j^2 \mathrm{d}v_g + \lambda \int_M \left(\frac{f(u_j)u_j}{\eta} - F(u_j)\right)\mathrm{d}v_g.$$

Thus, bearing in mind that $\int_M \phi_u u^2 \mathrm{d}v_g \geq 0$ and $(\tilde{f}_2)$ one has that

$$\frac{1}{2}\left(\frac{\eta-2}{\eta}\right)\|u_j\|^2_{H^1_g(M)} \leq \mathcal{E}_\lambda(u_j) - \frac{\langle \mathcal{E}'_\lambda(u_j), u_j\rangle}{\eta} + \chi \mathrm{Vol_g}(M),$$

where

$$\chi = \sup\left\{\left|\frac{tf(t)}{\eta} - F(t)\right| : t \leq \tau_0\right\}.$$

Therefore, for every $j \geq j_0$ we have that

$$\frac{1}{2}\left(\frac{\eta-2}{\eta}\right)\|u_j\|^2_{H^1_g(M)} \leq \mathcal{E}_\lambda(u_j) + \frac{1}{\eta}\|(\mathcal{E}_\lambda)'(u_j)\|_{H^1_g *}\|u_j\|_{H^1_g(M)} + \chi \mathrm{Vol_g}(M).$$

Dividing by $\|u_j\|_{H^1_g(M)}$ and letting $j \to \infty$ we get a contradiction, which implies the boundedness of the sequence $\{u_j\}_j$ in $H^1_g(M)$. □

Proof of the Theorem 1.5. Let us consider as before the following functionals:

$$\mathscr{H}(u) = \frac{1}{2}\|u\|^2_{H^1_g(M)} + \frac{e}{4}\int_M \phi_u u^2 \mathrm{d}v_g \quad \text{and} \quad \mathcal{F}(u) = \int_M F(u)\mathrm{d}v_g.$$

Form the positivity and the convexity of functional $u \mapsto \int_M \phi_u u^2$ it follows that the functional $\mathscr{H}$ is sequentially weakly semicontinuous and coercive functional. It is also clear that $\mathcal{F}$ is sequentially weakly continuous. Then, for $\mu = \frac{1}{2\lambda}$, we define the functional $J_\mu(u) = \mu\mathscr{H}(u) - \mathcal{F}(u)$. Integrating, we get from ($\tilde{f}_2$) that

$$F(ts) \geq t^\eta F(s), \qquad t \geq 1 \text{ and } |s| \geq \tau_0.$$

Now, let us consider a fixed function $u_0 \in H^1_g(M)$ such that

$$\mathrm{Vol_g}\left(\{x \in M : |u_0(x)| \geq \tau_0\}\right) > 0,$$

and using the previous inequality and the fact that $\phi_{tu} = t^2\phi_u$, we have that:

$$\begin{aligned} J_\mu(tu_0) &= \mu\mathscr{H}(tu_0) - \mathcal{F}(tu_0) \\ &= \mu\frac{t^2}{2}||u_0||^2_{H^1_g(M)} + \mu\frac{e}{4}t^4\int_M \phi_{u_0}u_0^2 - \int_M F(tu_0) \\ &\leq \mu t^2||u_0||^2_{H^1_g(M)} + \mu\frac{e}{2}t^4\int_M \phi_{u_0}u_0^2 - t^\eta\int_{\{x\in M::|u_0|\geq\tau_0\}} F(u_0) + \chi_2\mathrm{Vol}_g(M) \overset{\eta>4}{\to} -\infty, \end{aligned}$$

as $t \to \infty$, where

$$\chi_2 = \sup\left\{|F(t)| : |t| \leq \tau_0\right\}.$$

Thus, the functional J_μ is unbounded from below. A similar argument as before shows that (taking eventually a subsequence), the functional J_μ satisfies the (PS) condition.

Let us denote by $K_\tau = \left\{x \in M : \|u\|^2_{H^1_g(M)} < \tau\right\}$ and by

$$h(\tau) = \inf_{u\in K_\tau} \frac{\sup\limits_{v\in K_\tau} \mathcal{F}(v) - \mathcal{F}(u)}{\tau - \mathscr{H}(u)}$$

Since $0 \in K_\tau$, we have that

$$h(\tau) \leq \frac{\sup_{v\in K_\tau} \mathcal{F}(v)}{\tau}.$$

On the other hand, bearing in mind the assumption ($\tilde{f}_1$), we have that

$$\mathcal{F}(v) \leq C\|v\|_{H^1_g(M)} + \frac{C}{p}\kappa_p^p\|v\|^p_{H^1_g(M)}.$$

Therefore

$$h(\tau) \leq \frac{C}{2}\tau^{\frac{1}{2}} + \frac{C\kappa_p^p}{p}\tau^{\frac{p-2}{2}}.$$

Thus, if

$$\lambda < \lambda_0 := \frac{p\tau^{\frac{1}{2}}}{2pC + 2C\kappa_p^p\tau^{\frac{p-1}{2}}}$$

one has $\mu = \frac{1}{2\lambda} > h(\tau)$. Therefore, we are in the position to apply Ricceri's result, i.e., Theorem **B**, which concludes our proof. □

Remark 2.7. From the proof of Theorem 1.5, one can see, that

$$\lambda_0 \leq \frac{p}{2C} \max_{\tau>0} \frac{\tau^{\frac{1}{2}}}{p + \kappa_p^p \tau^{\frac{p-1}{2}}}.$$

Since $p > 2$, one can see, that

$$\max_{\tau>0} \frac{p\tau^{\frac{1}{2}}}{2pC + 2C\kappa_p^p \tau^{\frac{p-1}{2}}} < \infty.$$

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