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Extremal solutions for second-order fully discontinuous problems with nonlinear functional boundary conditions

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Abstract. We provide new results concerning the existence of extremal solutions for a class of second-order problems with nonlinear functional boundary conditions where the nonlinearity considered may be discontinuous with respect to all of its variables. The main result relies on recent fixed point theorems for discontinuous operators and the lower and upper solution method.

Keywords: discontinuous differential equations, upper and lower solutions, fixed point theorems.

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1 Introduction and preliminaries

We study the existence of extremal solutions for the differential equation

$$x''(t) = f(t, x, x'), \qquad t \in I = [a, b], \tag{1.1}$$

where the nonlinear term f may be discontinuous in all the arguments. More specifically, we shall prove existence of extremal solutions to (1.1) coupled with nonlinear functional boundary conditions

$$\begin{aligned} 0 &= L_1(x(a), x(b), x'(a), x'(b), x), \\ 0 &= L_2(x(a), x(b)), \end{aligned} \tag{1.2}$$

where $L_1 \in \mathcal{C}\left(\mathbb{R}^4 \times \mathcal{C}(I), \mathbb{R}\right)$ is nonincreasing in the third and in the fifth variables, and nondecreasing in the fourth one; and $L_2 : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and it is nondecreasing with respect to its first argument.

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In particular, the nonlinear boundary conditions (1.2) contain Dirichlet boundary conditions

$$x(a) = x(b) = 0, \tag{1.3}$$

and periodic conditions

$$x(a) = x(b), \qquad x'(a) = x'(b). \tag{1.4}$$

Since f may be discontinuous in all the arguments, we are forced to use new fixed point theorems (see [6,9]) combined with the lower and upper solutions method [3,5]. Similar fixed point methods were employed in [7] in the analysis of first-order differential problems with initial functional conditions.

Let us start with some preliminary results and definitions. Let K be a nonempty closed convex subset of a normed space $(X, \|\cdot\|)$ and $T : K \longrightarrow K$ an operator, not necessarily continuous.

Definition 1.1. The closed–convex envelope (or Krasovskij envelope [8]) of an operator $T : K \longrightarrow K$ is the multivalued mapping $\mathbb{T} : K \longrightarrow 2^K$ given by

$$\mathbb{T}x = \bigcap_{\varepsilon>0} \overline{\text{co}}\, T\left(\overline{B}_\varepsilon(x) \cap K\right) \quad \text{for every } x \in K, \tag{1.5}$$

where $\overline{B}_\varepsilon(x)$ denotes the closed ball centered at x and radius ε, and $\overline{\text{co}}$ means closed convex hull.

Remark 1.2. Note that $\mathbb{T}$ is an upper semicontinuous multivalued mapping which assumes closed and convex values (see [2,9]) provided that $T\,K$ is a relatively compact subset of X.

Theorem 1.3 ([9, Theorem 3.1])**.** *Let K be a nonempty, convex and compact subset of X.*

Any mapping $T : K \longrightarrow K$ has at least one fixed point provided that for every $x \in K$ we have

$$\{x\} \cap \mathbb{T}x \subset \{Tx\}, \tag{1.6}$$

where $\mathbb{T}$ denotes the closed–convex envelope of T.

Remark 1.4. Condition (1.6) is equivalent to $\text{Fix}(\mathbb{T}) \subset \text{Fix}(T)$, where $\text{Fix}(S)$ denotes the set of fixed points of the operator S.

Theorem 1.5 ([6, Theorem 2.7])**.** *Let K be a nonempty, closed and convex subset of X and $T : K \longrightarrow K$ be a mapping such that $T\,K$ is a relatively compact subset of X and it satisfies condition (1.6). Then T has a fixed point in K.*

2 Existence of solution for discontinuous BVP with nonlinear boundary conditions

We shall work in the Banach space $X = \mathcal{C}^1(I)$ endowed with its usual norm

$$\|x\|_{\mathcal{C}^1} = \|x\|_\infty + \|x'\|_\infty = \max_{t\in I} |x(t)| + \max_{t\in I} |x'(t)|.$$

Following [5] and the review article [3] we shall use lower and upper solutions for obtaining an existence result for problem (1.1)–(1.2). In the proof of the main result we shall consider a modified problem in the line of [4].

Definition 2.1. We say that $\alpha \in \mathcal{C}(I)$ is a lower solution for the differential problem (1.1)–(1.2) if it satisfies the following conditions.

(i) For any $t_0 \in (a,b)$, either $D_-\alpha(t_0) < D^+\alpha(t_0)$,
or there exists an open interval I_0 such that $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$ and

$$\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)) \quad \text{for a.a. } t \in I_0.$$

(ii) $D^+\alpha(a),\ D_-\alpha(b) \in \mathbb{R}$ and $L_1\left(\alpha(a), \alpha(b), D^+\alpha(a), D_-\alpha(b), \alpha\right) \leq 0$.

(iii) $L_2\left(\alpha(a), \alpha(b)\right) = 0$, and $L_2\left(\alpha(a), \cdot\right)$ is injective.

Similarly $\beta \in \mathcal{C}(I)$ is an upper solution for (1.1)–(1.2) if it satisfies the inequalities in the reverse order.

Now we present a Nagumo condition which provides a priori bound on the first derivative of all possible solutions between the lower and upper solutions for the differential problem.

Proposition 2.2. *Let $\bar{\alpha}, \bar{\beta} \in \mathcal{C}(I)$ be such that $\bar{\alpha} \leq \bar{\beta}$ and define*

$$r = \max\left\{\bar{\beta}(b) - \bar{\alpha}(a), \bar{\beta}(a) - \bar{\alpha}(b)\right\} / (b-a).$$

Assume there exist a continuous function $\bar{N} : [0,\infty) \to (0,\infty)$, $\bar{M} \in L^1(I)$ and $R > r$ such that

$$\int_r^R \frac{1}{\bar{N}(s)} ds > \|\bar{M}\|_{L^1}.$$

Define $E := \left\{(t,x,y) \in I \times \mathbb{R}^2 : \bar{\alpha}(t) \leq x \leq \bar{\beta}(t)\right\}$. Then, for every function $f : E \to \mathbb{R}$ such that for a.e. $t \in I$ and all $(x,y) \in \mathbb{R}^2$ with $(t,x,y) \in E$,

$$|f(t,x,y)| \leq \bar{M}(t)\bar{N}(|y|),$$

and for every solution x of (1.1) such that $\bar{\alpha} \leq x \leq \bar{\beta}$, we have

$$\|x'\|_\infty < R.$$

Proof. Let x be a solution of (1.1) and $t \in I$ such that $x'(t) > R$. Notice that

$$-r \leq \frac{\bar{\alpha}(b) - \bar{\beta}(a)}{b-a} \leq \frac{x(b) - x(a)}{b-a} \leq \frac{\bar{\beta}(b) - \bar{\alpha}(a)}{b-a} \leq r,$$

and then by Lagrange Theorem there exists $\tau \in I$ such that

$$|x'(\tau)| = \left|\frac{x(b) - x(a)}{b-a}\right| \leq r.$$

Thus we can choose $t_0 < t_1$ (or $t_1 < t_0$) such that $x'(t_0) = r$, $x'(t_1) = R$ and $r \leq x'(s) \leq R$ in $[t_0, t_1]$ (or $[t_1, t_0]$).

Therefore we have

$$\int_r^R \frac{1}{\bar{N}(s)} ds = \int_{t_0}^{t_1} \frac{x''(s)}{\bar{N}(x'(s))} ds = \int_{t_0}^{t_1} \frac{f(s, x(s), x'(s))}{\bar{N}(x'(s))} ds \leq \left|\int_{t_0}^{t_1} \bar{M}(s)\, ds\right| \leq \|\bar{M}\|_{L^1},$$

a contradiction, so we deduce that $x'(t) < R$. In the same way we prove that $x'(t) > -R$. □

We consider the differential problem (1.1)–(1.2), under weaker conditions about f than the well-known Carathéodory's conditions, and we look for solutions for this problem, namely functions $x \in W^{2,1}(I)$ satisfying (1.1)–(1.2).

We shall allow f to be discontinuous in the second argument over countably many curves in the conditions of the following definition. They imply a 'transversality' condition whose geometrical idea recalls that of the discontinuity surfaces described in [8].

Definition 2.3. An admissible discontinuity curve for the differential equation (1.1) is a $W^{2,1}$ function $\gamma : [c,d] \subset I \longrightarrow \mathbb{R}$ satisfying one of the following conditions:

either $\gamma''(t) = f(t,\gamma(t),\gamma'(t))$ for a.a. $t \in [c,d]$ (and we then say that γ is viable for the differential equation),

or there exist $\varepsilon > 0$ and $\psi \in L^1(c,d)$, $\psi(t) > 0$ for a.a. $t \in [c,d]$, such that
either

$$\gamma''(t) + \psi(t) < f(t,y,z) \quad \text{for a.a. } t \in [c,d], \text{ all } y \in [\gamma(t)-\varepsilon,\gamma(t)+\varepsilon] \text{ and all } z \in [\gamma'(t)-\varepsilon,\gamma'(t)+\varepsilon], \tag{2.1}$$

or

$$\gamma''(t) - \psi(t) > f(t,y,z) \quad \text{for a.a. } t \in [c,d], \text{ all } y \in [\gamma(t)-\varepsilon,\gamma(t)+\varepsilon] \text{ and all } z \in [\gamma'(t)-\varepsilon,\gamma'(t)+\varepsilon]. \tag{2.2}$$

We say that the admissible discontinuity curve γ is inviable for the differential equation if it satisfies (2.1) or (2.2).

Moreover, we shall allow f to be discontinuous in the third argument over some curves satisfying the conditions of the following definition, slightly different from the previous one. As far as the authors are aware, this is the first time that such discontinuity sets are considered.

Definition 2.4. Given α and β lower and upper solutions for problem (1.1)–(1.2) such that $\alpha \le \beta$ on I, an inviable discontinuity curve for the derivative of the differential equation (1.1) is an absolutely continuous function $\Gamma : [c,d] \subset I \longrightarrow \mathbb{R}$ satisfying that there exist $\varepsilon > 0$ and $\psi \in L^1(c,d)$, $\psi(t) > 0$ for a.a. $t \in [c,d]$, such that
either

$$\Gamma'(t) + \psi(t) < f(t,y,z) \quad \text{for a.a. } t \in [c,d], \text{ all } y \in [\alpha(t),\beta(t)] \text{ and all } z \in [\Gamma(t)-\varepsilon,\Gamma(t)+\varepsilon] \cup \{\alpha'(t),\beta'(t)\}, \tag{2.3}$$

or

$$\Gamma'(t) - \psi(t) > f(t,y,z) \quad \text{for a.a. } t \in [c,d], \text{ all } y \in [\alpha(t),\beta(t)] \text{ and all } z \in [\Gamma(t)-\varepsilon,\Gamma(t)+\varepsilon] \cup \{\alpha'(t),\beta'(t)\}. \tag{2.4}$$

Now we state three technical results that we need in the proof of our main existence result of this section for (1.1)–(1.2). Their proofs can be lookep up in [9].

In the sequel m denotes the Lebesgue measure in $\mathbb{R}$.

Lemma 2.5. *Let $a,b \in \mathbb{R}$, $a < b$, and let $g,h \in L^1(a,b)$, $g \ge 0$ a.e., and $h > 0$ a.e. on (a,b).*

For every measurable set $J \subset (a,b)$ such that $m(J) > 0$ there is a measurable set $J_0 \subset J$ such that $m(J \setminus J_0) = 0$ and for all $\tau_0 \in J_0$ we have

$$\lim_{t\to\tau_0^+} \frac{\int_{[\tau_0,t]\setminus J} g(s)\,ds}{\int_{\tau_0}^{t} h(s)\,ds} = 0 = \lim_{t\to\tau_0^-} \frac{\int_{[t,\tau_0]\setminus J} g(s)\,ds}{\int_{t}^{\tau_0} h(s)\,ds}.$$

Corollary 2.6. *Let $a, b \in \mathbb{R}$, $a < b$, and let $h \in L^1(a,b)$ be such that $h > 0$ a.e. on (a,b).*

For every measurable set $J \subset (a,b)$ such that $m(J) > 0$ there is a measurable set $J_0 \subset J$ such that $m(J \setminus J_0) = 0$ and for all $\tau_0 \in J_0$ we have

$$\lim_{t\to\tau_0^+} \frac{\int_{[\tau_0,t]\cap J} h(s)\,ds}{\int_{\tau_0}^{t} h(s)\,ds} = 1 = \lim_{t\to\tau_0^-} \frac{\int_{[t,\tau_0]\cap J} h(s)\,ds}{\int_{t}^{\tau_0} h(s)\,ds}.$$

Corollary 2.7. *Let $a, b \in \mathbb{R}$, $a < b$, and let $f,\ f_n : [a,b] \to \mathbb{R}$ be absolutely continuous functions on $[a,b]$ ($n \in \mathbb{N}$), such that $f_n \to f$ uniformly on $[a,b]$ and for a measurable set $A \subset [a,b]$ with $m(A) > 0$ we have*

$$\lim_{n\to\infty} f_n'(t) = g(t) \quad \textit{for a.a. } t \in A.$$

If there exists $M \in L^1(a,b)$ such that $|f'(t)| \le M(t)$ a.e. in $[a,b]$ and also $|f_n'(t)| \le M(t)$ a.e. in $[a,b]$ ($n \in \mathbb{N}$), then $f'(t) = g(t)$ for a.a. $t \in A$.

Now we present the main result in this paper.

Theorem 2.8. *Suppose that there exist $\alpha, \beta \in W^{1,\infty}((a,b))$ lower and upper solutions to (1.1)–(1.2), respectively, such that $\alpha \le \beta$ on I. Let*

$$r = \max\{\beta(b) - \alpha(a), \beta(a) - \alpha(b)\}/(b-a).$$

Assume that for $f : I \times \mathbb{R}^2 \to \mathbb{R}$ the following conditions hold:

(C1) *compositions $t \in I \mapsto f(t, x(t), y(t))$ are measurable whenever $x(t)$ and $y(t)$ are measurable;*

(C2) *there exist a continuous function $N : [0,\infty) \to (0,\infty)$ and $M \in L^1(I)$ such that:*

  (a) for a.a. $t \in I$, all $x \in [\alpha(t), \beta(t)]$ and all $y \in \mathbb{R}$, we have $|f(t,x,y)| \le M(t)N(|y|)$;

  (b) there exists $R > r$ such that
$$\int_r^R \frac{1}{N(s)} ds > \|M\|_{L^1};$$

(C3) *there exist admissible discontinuity curves $\gamma_n : I_n = [a_n, b_n] \longrightarrow \mathbb{R}$ ($n \in \mathbb{N}$) such that $\alpha \le \gamma_n \le \beta$ on I_n and their derivatives are uniformly bounded, and for all $y \in \mathbb{R}$ and for a.a. $t \in I$ the function $x \mapsto f(t,x,y)$ is continuous on $[\alpha(t), \beta(t)] \setminus \bigcup_{\{n : t \in I_n\}} \{\gamma_n(t)\}$;*

(C4) *there exist inviable discontinuity curves for the derivative $\Gamma_n : \tilde{I}_n = [c_n, d_n] \longrightarrow \mathbb{R}$ ($n \in \mathbb{N}$) such that they are uniformly bounded and for a.a. $t \in I$ and all $x \in [\alpha(t), \beta(t)]$, the mapping $y \mapsto f(t,x,y)$ is continuous on $[-R, R] \setminus \bigcup_{\{n : t \in \tilde{I}_n\}} \{\Gamma_n(t)\}$.*

Then problem (1.1)–(1.2) has at least a solution $x \in W^{2,1}(I)$ between α and β such that $\|x'\|_\infty < R$.

Proof. Without loss of generality, suppose that $R > \max_{t\in I}\{|\alpha'(t)|, |\beta'(t)|, |\gamma_n'(t)|, |\Gamma_n(t)|\}$ for all $n \in \mathbb{N}$ and define an integrable function

$$\tilde{M}(t) := \max_{s\in[0,R]}\{N(s)\}\, M(t).$$

Let us also define $\delta_R(z) = \max\{\min\{z, R\}, -R\}$ for all $z \in \mathbb{R}$ and

$$f^*(t,x,y) = f(t,x,\delta_R(y)) \quad \text{for all } (t,x,y) \in I \times \mathbb{R}^2. \tag{2.5}$$

Consider the modified problem

$$\begin{cases} x''(t) = f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))') & \text{for a.a. } t \in I, \\ x(a) = L_1^*\left(x(a), x(b), x'(a), x'(b), x\right), \\ x(b) = L_2^*\left(x(a), x(b)\right), \end{cases} \tag{2.6}$$

where

$$\varphi(t, x) = \max\left\{\min\left\{x, \beta(t)\right\}, \alpha(t)\right\} \quad \text{for } (t, x) \in I \times \mathbb{R}, \tag{2.7}$$

and $L_1^*\left(x, y, z, w, \xi\right) = \varphi\left(a, x - L_1\left(x, y, z, w, \xi\right)\right)$ for all $(x, y, z, w, \xi) \in \mathbb{R}^4 \times \mathcal{C}(I)$ and $L_2^*\left(x, y\right) = \varphi\left(b, y + L_2(x, y)\right)$ for all $(x, y) \in \mathbb{R}^2$.

We know from [10, Lemma 2] that if v, $v_n \in \mathcal{C}^1(I)$ are such that $v_n \to v$ in $\mathcal{C}^1(I)$, then

(a) $(\varphi(t, v(t)))'$ exists for a.a. $t \in I$;

(b) $(\varphi(t, v_n(t)))' \to (\varphi(t, v(t)))'$ for a.a. $t \in I$.

Now we consider the compact and convex subset of $X = \mathcal{C}^1(I)$,

$$K = \Big\{ x \in X : \alpha(a) \le x(a) \le \beta(a),\ \alpha(b) \le x(b) \le \beta(b), \\ |x'(t) - x'(s)| \le \int_s^t \tilde{M}(r)\, dr \ (a \le s \le t \le b) \Big\}$$

and for each $x \in K$ define

$$Tx(t) = L_1^*(x) + \frac{t-a}{b-a}\left(L_2^*(x) - L_1^*(x) - \int_a^b \int_a^s f^*(r, \varphi(r, x(r)), (\varphi(r, x(r)))')\, dr\, ds\right) \\ + \int_a^t \int_a^s f^*(r, \varphi(r, x(r)), (\varphi(r, x(r)))')\, dr\, ds,$$

where, for simplicity, we use the notations $L_1^*(x) = L_1^*\left(x(a), x(b), x'(a), x'(b), x\right)$ and $L_2^*(x) = L_2^*\left(x(a), x(b)\right)$. Observe that $y = Tx$ is just the solution of

$$\begin{cases} y''(t) = f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))') & \text{for a.a. } t \in I, \\ y(a) = L_1^*(x), \quad y(b) = L_2^*(x). \end{cases} \tag{2.8}$$

Conditions $(C1)$ and $(C2)$ guarantee that the operator T is well defined. Moreover, T maps K into itself. Indeed, for any $x \in K$ and $y = Tx$ we have, thanks to $(C2)$ (a), that

$$|y''(t)| = |f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))')| \le M(t) N(|\delta_R((\varphi(t, x(t)))')|) \le \tilde{M}(t),$$

which, along with $y(a) = L_1^*(x)$ and $y(b) = L_2^*(x)$, imply that $y \in K$.

Next we prove that the operator T satisfies condition (1.6) for all $x \in K$ and then Theorem 1.3 ensures the existence of a fixed point or, equivalently, a solution to the modified problem (2.6). This part of the proof follows the steps of that in [9, Theorem 4.4], but here some changes are necessary due to the use of lower and upper solutions and the derivative dependence in the ODE.

We fix an arbitrary function $x \in K$ and we consider four different cases.

Case 1: $m(\{t \in I_n : x(t) = \gamma_n(t)\} \cup \{t \in \tilde{I}_n : x'(t) = \Gamma_n(t)\}) = 0$ for all $n \in \mathbb{N}$. Let us prove that then T is continuous at x.

The assumption implies that for a.a. $t \in I$ the mapping $f(t,\cdot,\cdot)$ is continuous at the point $(\varphi(t,x(t)),(\varphi(t,x(t))')$. Hence if $x_k \to x$ in K, then

$$f^*(t,\varphi(t,x_k(t)),(\varphi(t,x_k(t)))') \to f^*(t,\varphi(t,x(t)),(\varphi(t,x(t)))') \quad \text{for a.a. } t \in I,$$

as one can easily check by considering all possible combinations of the cases $x(t) \in [\alpha(t),\beta(t)]$, $x(t) > \beta(t)$ or $x(t) < \alpha(t)$, and $|x'(t)| \leq R$ or $|x'(t)| > R$.

Moreover,

$$\left|f^*(t,\varphi(t,x(t)),(\varphi(t,x(t)))')\right| \leq \tilde{M}(t) \tag{2.9}$$

for a.a. $t \in I$, hence $Tx_k \to Tx$ in $\mathcal{C}^1(I)$.

Case 2: $m(\{t \in I_n : x(t) = \gamma_n(t)\}) > 0$ for some $n \in \mathbb{N}$ such that γ_n is inviable. In this case we can prove that $x \notin \mathbb{T}x$.

First, we fix some notation. Let us assume that for some $n \in \mathbb{N}$ we have $m(\{t \in I_n : x(t) = \gamma_n(t)\}) > 0$ and there exist $\varepsilon > 0$ and $\psi \in L^1(I_n)$, $\psi(t) > 0$ for a.a. $t \in I_n$, such that (2.2) holds with γ replaced by γ_n. (The proof is similar if we assume (2.1) instead of (2.2), so we omit it.)

We denote $J = \{t \in I_n : x(t) = \gamma_n(t)\}$, and we observe that $m(\{t \in J : \gamma_n(t) = \beta(t)\}) = 0$. Indeed, if $m(\{t \in J : \gamma_n(t) = \beta(t)\}) > 0$, then from (2.2) it follows that $\beta''(t) - \psi(t) > f(t,\beta(t),\beta'(t))$ on a set of positive measure, which is a contradiction with the definition of upper solution.

Now we distinguish between two sub-cases.

Case 2.1: $m(\{t \in J : x(t) = \gamma_n(t) = \alpha(t)\}) > 0$. Since $m(\{t \in J : \gamma_n(t) = \beta(t)\}) = 0$, we deduce that $m(\{t \in J : x(t) = \alpha(t) \neq \beta(t)\}) > 0$, so there exists $n_0 \in \mathbb{N}$ such that

$$m\left(\left\{t \in J : x(t) = \alpha(t),\ x(t) < \beta(t) - \frac{1}{n_0}\right\}\right) > 0.$$

We denote $A = \{t \in J : x(t) = \alpha(t),\ x(t) < \beta(t) - 1/n_0\}$ and we deduce from Lemma 2.5 that there is a measurable set $J_0 \subset A$ with $m(J_0) = m(A) > 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t\to\tau_0^+} \frac{2\int_{[\tau_0,t]\setminus A} \tilde{M}(s)\,ds}{(1/4)\int_{\tau_0}^{t} \psi(s)\,ds} = 0 = \lim_{t\to\tau_0^-} \frac{2\int_{[t,\tau_0]\setminus A} \tilde{M}(s)\,ds}{(1/4)\int_{t}^{\tau_0} \psi(s)\,ds}. \tag{2.10}$$

By Corollary 2.6 there exists $J_1 \subset J_0$ with $m(J_0 \setminus J_1) = 0$ such that for all $\tau_0 \in J_1$ we have

$$\lim_{t\to\tau_0^+} \frac{\int_{[\tau_0,t]\cap J_0} \psi(s)\,ds}{\int_{\tau_0}^{t} \psi(s)\,ds} = 1 = \lim_{t\to\tau_0^-} \frac{\int_{[t,\tau_0]\cap J_0} \psi(s)\,ds}{\int_{t}^{\tau_0} \psi(s)\,ds}. \tag{2.11}$$

Let us now fix a point $\tau_0 \in J_1$. From (2.10) and (2.11) we deduce that there exist $t_- < \tau_0$ and $t_+ > \tau_0$, $t_\pm$ sufficiently close to τ_0 so that the following inequalities are satisfied:

$$2\int_{[\tau_0,t_+]\setminus A} \tilde{M}(s)\,ds < \frac{1}{4}\int_{\tau_0}^{t_+} \psi(s)\,ds, \tag{2.12}$$

$$\int_{[\tau_0,t_+]\cap A} \psi(s)\,ds \geq \int_{[\tau_0,t_+]\cap J_0} \psi(s)\,ds > \frac{1}{2}\int_{\tau_0}^{t_+} \psi(s)\,ds, \tag{2.13}$$

$$2\int_{[t_-,\tau_0]\setminus A} \tilde{M}(s)\,ds < \frac{1}{4}\int_{t_-}^{\tau_0} \psi(s)\,ds, \tag{2.14}$$

$$\int_{[t_-,\tau_0]\cap A} \psi(s)\,ds > \frac{1}{2}\int_{t_-}^{\tau_0} \psi(s)\,ds. \tag{2.15}$$

Finally, we define a positive number

$$\rho = \min\left\{\frac{1}{4}\int_{t_-}^{\tau_0}\psi(s)\,ds, \frac{1}{4}\int_{\tau_0}^{t_+}\psi(s)\,ds\right\}, \tag{2.16}$$

and we are now in a position to prove that $x \notin \mathbb{T}x$. It is sufficient to prove the following claim: *Claim – Let $\tilde{\varepsilon} > 0$ be defined as $\tilde{\varepsilon} = \min\{\varepsilon, 1/n_0\}$, where ε is given by our assumptions over γ_n and $1/n_0$ by the definition of the set A, and let ρ be as in (2.16). For every finite family $x_i \in \overline{B}_{\tilde{\varepsilon}}(x) \cap K$ and $\lambda_i \in [0,1]$ $(i = 1, 2, \dots, m)$, with $\sum \lambda_i = 1$, we have $\|x - \sum \lambda_i T x_i\|_{\mathcal{C}^1} \geq \rho$.*

Let x_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum \lambda_i T x_i$. For a.a. $t \in J = \{t \in I_n : x(t) = \gamma_n(t)\}$ we have

$$y''(t) = \sum_{i=1}^{m} \lambda_i (Tx_i)''(t) = \sum_{i=1}^{m} \lambda_i f^*(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))'). \tag{2.17}$$

On the other hand, for every $i \in \{1, 2, \dots, m\}$ and for a.a. $t \in J$ we have

$$|x_i(t) - \gamma_n(t)| + |x_i'(t) - \gamma_n'(t)| = |x_i(t) - x(t)| + |x_i'(t) - x'(t)| < \varepsilon, \tag{2.18}$$

but by continuity $x'(t) = \gamma_n'(t)$ for all $t \in J$, so (2.18) holds for every $t \in J$. Since $\gamma_n(t) \in [\alpha(t), \beta(t)]$, for a.a. $t \in A$ we have $|\varphi(t, x_i(t)) - \gamma_n(t)| \leq |x_i(t) - \gamma_n(t)|$, and

$$\left|(\varphi(t, x_i(t)))' - \gamma_n'(t)\right| \leq \left|x_i'(t) - \gamma_n'(t)\right|$$

because if $x_i(t) < \alpha(t)$, then $(\varphi(t, x_i(t)))' = \alpha'(t) = \gamma_n'(t)$.

Hence, from (2.2) it follows that

$$\gamma_n''(t) - \psi(t) > f(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))')$$

for a.a. $t \in A$ and for all $x_i(t)$ satisfying (2.18).

Moreover, since for a.a. $t \in A$ we have $|\gamma_n'(t)| < R$ and $|x_i'(t) - \gamma_n'(t)| < \varepsilon$, without loss of generality we can suppose $|(\varphi(t, x_i(t)))'| \leq R$ and thus

$$\gamma_n''(t) - \psi(t) > f^*(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))') \tag{2.19}$$

for a.a. $t \in A$.

Therefore the assumptions on γ_n ensure that for a.a. $t \in A$ we have

$$y''(t) = \sum_{i=1}^{m} \lambda_i f^*(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))') < \sum_{i=1}^{m} \lambda_i (\gamma_n''(t) - \psi(t)) = x''(t) - \psi(t). \tag{2.20}$$

Now we compute

$$\begin{aligned}
y'(\tau_0) - y'(t_-) &= \int_{t_-}^{\tau_0} y''(s)\,ds = \int_{[t_-,\tau_0]\cap A} y''(s)\,ds + \int_{[t_-,\tau_0]\setminus A} y''(s)\,ds \\
&< \int_{[t_-,\tau_0]\cap A} x''(s)\,ds - \int_{[t_-,\tau_0]\cap A} \psi(s)\,ds \\
&\quad + \int_{[t_-,\tau_0]\setminus A} \tilde{M}(s)\,ds \qquad \text{(by (2.20), (2.17) and (2.9))} \\
&= x'(\tau_0) - x'(t_-) - \int_{[t_-,\tau_0]\setminus A} x''(s)\,ds - \int_{[t_-,\tau_0]\cap A} \psi(s)\,ds \\
&\quad + \int_{[t_-,\tau_0]\setminus A} \tilde{M}(s)\,ds \\
&\leq x'(\tau_0) - x'(t_-) - \int_{[t_-,\tau_0]\cap A} \psi(s)\,ds + 2\int_{[t_-,\tau_0]\setminus A} \tilde{M}(s)\,ds \\
&< x'(\tau_0) - x'(t_-) - \frac{1}{4}\int_{t_-}^{\tau_0} \psi(s)\,ds \quad \text{(by (2.14) and (2.15))},
\end{aligned}$$

hence $\|x-y\|_{\mathcal{C}^1} \geq y'(t_-) - x'(t_-) \geq \rho$ provided that $y'(\tau_0) \geq x'(\tau_0)$.

Similar computations with t_+ instead of t_- show that if $y'(\tau_0) \leq x'(\tau_0)$ then we also have $\|x-y\|_{\mathcal{C}^1} \geq \rho$. The claim is proven.

Case 2.2: $m(\{t \in J : \gamma_n(t) \in (\alpha(t), \beta(t))\}) > 0$.

The set $\{t \in J : \gamma_n(t) \in (\alpha(t), \beta(t))\}$ can be written as the following countable union

$$\bigcup_{n \in \mathbb{N}} \left\{ t \in J : \alpha(t) + \frac{1}{n} < x(t) < \beta(t) - \frac{1}{n} \right\},$$

so there exists some $n_0 \in \mathbb{N}$ such that $m(\{t \in J : \alpha(t) + 1/n_0 < x(t) < \beta(t) - 1/n_0\}) > 0$. Now we denote $A = \{t \in J : \alpha(t) + 1/n_0 < x(t) < \beta(t) - 1/n_0\}$. Since A is a set of positive measure we can argue as in Case 2.1 for obtaining inequalities (2.12)–(2.15) and we are in position to prove the Claim again.

Let x_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum \lambda_i T x_i$. Then for every $i \in \{1, 2, \ldots, m\}$ and all $t \in A$ we have $x_i(t) \in (\alpha(t), \beta(t))$, so $\varphi(t, x_i(t)) = x_i(t)$ and $(\varphi(t, x_i(t)))' = x_i'(t)$ and thus

$$|\varphi(t, x_i(t)) - \gamma_n(t)| + \left|(\varphi(t, x_i(t)))' - \gamma_n'(t)\right| = |x_i(t) - x(t)| + \left|x_i'(t) - x'(t)\right| < \varepsilon, \qquad (2.21)$$

for all $t \in A$.

Hence, from (2.2) it follows that

$$\gamma_n''(t) - \psi(t) > f(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))')$$

for a.a. $t \in A$ and all $x_i \in \overline{B}_{\tilde{\varepsilon}}(x)$.

Now the proof of the Claim follows exactly as in Case 2.1.

Case 3: $m(\{t \in \tilde{I}_n : x'(t) = \Gamma_n(t)\}) > 0$ *for some* $n \in \mathbb{N}$ *such that* Γ_n *is an inviable discontinuity curve for the derivative.* In this case, we can prove again that $x \notin \mathbb{T}x$.

As before, let us assume that for some $n \in \mathbb{N}$ we have $m(\{t \in \tilde{I}_n : x'(t) = \Gamma_n(t)\}) > 0$ and there exist $\varepsilon > 0$ and $\psi \in L^1(\tilde{I}_n)$, $\psi(t) > 0$ for a.a. $t \in \tilde{I}_n$, such that (2.4) holds with Γ replaced by Γ_n. Similarly, we can define ρ as in (2.16) and we shall prove the Claim.

Let x_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum \lambda_i T x_i$. For a.a. $t \in J = \{t \in \tilde{I}_n : x'(t) = \Gamma_n(t)\}$ we have (2.17). On the other hand, for every $i \in \{1, 2, \ldots, m\}$ and for every $t \in J$ we have

$$|x_i'(t) - \Gamma_n(t)| = |x_i'(t) - x'(t)| < \varepsilon.$$

Moreover, from (2.4) it follows that

$$\Gamma_n'(t) - \psi(t) > f^*(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))')$$

for a.a. $t \in I_n$ and for all $x_i(t)$ since $\varphi(t, x_i(t)) \in [\alpha(t), \beta(t)]$ and $(\varphi(t, x_i(t)))' \in \{x_i'(t), \alpha'(t), \beta'(t)\}$.

Therefore the assumptions on Γ_n ensure that for a.a. $t \in J$ we have

$$y''(t) = \sum_{i=1}^{m} \lambda_i f^*(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))') < \sum_{i=1}^{m} \lambda_i \left(\Gamma_n'(t) - \psi(t)\right) = x''(t) - \psi(t),$$

and the proof of Case 3 follows as in Case 2.1, but now the set J plays the role of the set A there.

Case 4 – $m(\{t \in I_n : x(t) = \gamma_n(t)\}) > 0$ only for some of those $n \in \mathbb{N}$ such that γ_n is viable and $m(\{t \in \tilde{I}_n : x'(t) = \Gamma_n(t)\}) = 0$ for all $n \in \mathbb{N}$. Let us prove that in this case the relation $x \in \mathbb{T}x$ implies $x = Tx$.

Note first that $x \in \mathbb{T}x$ implies that x satisfies the boundary conditions in (2.6), because every element in $\mathbb{T}x$ is, roughly speaking, a limit of convex combinations of functions y satisfying (2.8).

Now it only remains to show that $x \in \mathbb{T}x$ implies that x satisfies the ODE in (2.6).

Let us consider the subsequence of all viable admissible discontinuity curves in the conditions of Case 4, which we denote again by $\{\gamma_n\}_{n\in\mathbb{N}}$ to avoid overloading notation. We have $m(J_n) > 0$ for all $n \in \mathbb{N}$, where

$$J_n = \{t \in I_n : x(t) = \gamma_n(t)\}.$$

For each $n \in \mathbb{N}$ and for a.a. $t \in J_n$ we have $\gamma_n''(t) = f(t, \gamma_n(t), \gamma_n'(t))$ and from $\alpha \le \gamma_n \le \beta$ and $|\gamma_n'(t)| < R$ it follows that $\gamma_n''(t) = f^*(t, \varphi(t, \gamma_n(t)), (\varphi(t, \gamma_n(t)))')$, so γ_n is viable for (2.6). Then for a.a. $t \in J_n$ we have

$$x''(t) = \gamma_n''(t) = f^*(t, \varphi(t, \gamma_n(t)), (\varphi(t, \gamma_n(t)))') = f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))'),$$

and therefore

$$x''(t) = f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))') \quad \text{a.e. in } J = \textstyle\bigcup_{n\in\mathbb{N}} J_n. \tag{2.22}$$

Now we assume that $x \in \mathbb{T}x$ and we prove that it implies that

$$x''(t) = f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))') \quad \text{a.e. in } I \setminus J,$$

thus showing that $x = Tx$.

Since $x \in \mathbb{T}x$ then for each $k \in \mathbb{N}$ we can choose $\varepsilon = \rho = 1/k$ to guarantee that we can find functions $x_{k,i} \in B_{1/k}(x) \cap K$ and coefficients $\lambda_{k,i} \in [0,1]$ ($i = 1, 2, \dots, m(k)$) such that $\sum \lambda_{k,i} = 1$ and

$$\left\| x - \sum_{i=1}^{m(k)} \lambda_{k,i} T x_{k,i} \right\|_{\mathcal{C}^1} < \frac{1}{k}.$$

Let us denote $y_k = \sum_{i=1}^{m(k)} \lambda_{k,i} T x_{k,i}$, and notice that $y_k' \to x'$ uniformly in I and $\|x_{k,i} - x\|_{\mathcal{C}^1} \le 1/k$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2, \dots, m(k)\}$. Note also that

$$y_k''(t) = \sum_{i=1}^{m(k)} \lambda_{k,i} f^*(t, \varphi(t, x_{k,i}(t)), (\varphi(t, x_{k,i}(t)))') \quad \text{for a.a. } t \in I. \tag{2.23}$$

For a.a. $t \in I \setminus J$ we have that either $x(t) \in [\alpha(t), \beta(t)]$, and then $f^*(t, \varphi(t, \cdot), (\varphi(t, \cdot))')$ is continuous at $x(t)$, so for any $\varepsilon > 0$ there is some $k_0 = k_0(t) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \ge k_0$, we have

$$|f^*(t, \varphi(t, x_{k,i}(t)), (\varphi(t, x_{k,i}(t)))') - f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))')| < \varepsilon \quad \text{for all } i \in \{1, 2, \dots, m(k)\},$$

or $x(t) < \alpha(t)$ (analogously if $x(t) > \beta(t)$), so there is some $k_0 = k_0(t) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \ge k_0$ we have $x_{k,i}(t) < \alpha(t)$ for all $i \in \{1, 2, \dots, m(k)\}$ and then $\varphi(t, x(t)) = \alpha(t) = \varphi(t, x_{k,i}(t))$, which implies

$$|f^*(t, \varphi(t, x_{k,i}(t)), (\varphi(t, x_{k,i}(t)))') - f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))')| = 0 \quad \text{for all } i \in \{1, 2, \dots, m(k)\}.$$

Now we deduce from (2.23) that $y_k''(t) \to f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))')$ for a.a. $t \in I \setminus J$, and then Corollary 2.7 guarantees that $x''(t) = f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))')$ for a.a. $t \in I \setminus J$. Combining this result with (2.22), we see that x solves (2.6), which implies that x is a fixed point of T.

So far, we have proven that the operator T satisfies condition (1.6) for all $x \in K$ and then Theorem 1.3 ensures the existence of a fixed point or, equivalently, a solution to the modified problem (2.6). It remains to prove that every solution of (2.6) is also a solution of the former problem (1.1)–(1.2).

First we will see that if x is a solution for (2.6), then $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in I$. Suppose that there exists $t_0 \in I$ such that

$$x(t_0) - \alpha(t_0) = \min_{t \in I} (x(t) - \alpha(t)) < 0.$$

By the boundary conditions we have $\alpha(a) \leq x(a) \leq \beta(a)$ and $\alpha(b) \leq x(b) \leq \beta(b)$, so t_0 has to belong to the open interval (a, b). Suppose that $x(t_0) - \alpha(t_0) < x(t) - \alpha(t)$ for all $t \in (t_0, b]$. Then we have

$$x'(t_0) - D_-\alpha(t_0) \leq x'(t_0) - D^+\alpha(t_0)$$

so, by the definition of lower solution, there exists an open interval I_0 such that $t_0 \in I_0$ and

$$\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)) \quad \text{for a.a. } t \in I_0.$$

Further $x'(t_0) = \alpha'(t_0)$ and

$$\forall r > 0 \ \exists\, t_r \in (t_0, t_0 + r) \quad \text{such that } \alpha'(t_r) < x'(t_r). \tag{2.24}$$

On the other hand, by the continuity of $x - \alpha$ there exists $\varepsilon > 0$ such that for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ we have $x(t) - \alpha(t) < 0$. Then by definition of solution for (2.6), we obtain that

$$x''(t) = f(t, \alpha(t), \alpha'(t)) \qquad \text{for a.e. } t \in [t_0, t_0 + \varepsilon],$$

and for $t \in [t_0, t_0 + \varepsilon]$,

$$x'(t) - \alpha'(t) = \int_{t_0}^{t} \big(x''(s) - \alpha''(s)\big)\, ds = \int_{t_0}^{t} \big(f(s, \alpha(s), \alpha'(s)) - \alpha''(s)\big)\, ds \leq 0,$$

a contradiction with (2.24). In a similar way we can see that $x \leq \beta$, so $\varphi(t, x(t)) = x(t)$.

In addition, by the Nagumo condition given in Proposition 2.2 it is immediate that $\|x'\|_\infty < R$.

To finish we will see that if x is a solution of (2.6) then x satisfies (1.2). We follow the steps of [4, Lemma 3.5].

If $x(b) + L_2(x(a), x(b)) < \alpha(b)$ the definition of L_2^* gives us that $x(b) = \alpha(b)$. Since L_2 is nondecreasing with respect to its first variable we get a contradiction:

$$\alpha(b) > x(b) + L_2(x(a), x(b)) \geq \alpha(b) + L_2(\alpha(a), \alpha(b)) = \alpha(b).$$

Similarly if $x(b) + L_2(x(a), x(b)) > \beta(b)$ we have $x(b) = \beta(b)$ and we get a contradiction as above. Then $\alpha(b) \leq x(b) + L_2(x(a), x(b)) \leq \beta(b)$, so $L_2^*(x(a), x(b)) = x(b) + L_2(x(a), x(b))$ and $L_2^*(x(a), x(b)) = x(b)$ imply $L_2(x(a), x(b)) = 0$.

In a similar way, to prove that $L_1(x(a), x(b), x'(a), x'(b), x) = 0$ it is enough to show that

$$\alpha(a) \leq x(a) - L_1(x(a), x(b), x'(a), x'(b), x) \leq \beta(a).$$

If $x(a) - L_1(x(a), x(b), x'(a), x'(b), x) < \alpha(a)$ then $x(a) = \alpha(a)$ and thus $0 = L_2(x(a), x(b)) = L_2(\alpha(a), x(b))$. Now, since $L_2(\alpha(a), \cdot)$ is injective and $L_2(\alpha(a), \alpha(b)) = 0$, we have that $x(b) = \alpha(b)$. Previously, we saw that $x - \alpha$ is nonnegative in I and thus it attains its minimum at a and b, so $x'(a) \geq D^+\alpha(a)$ and $x'(b) \leq D_-\alpha(b)$. Using the definition of lower solution and the properties of L_1 we obtain a contradiction:

$$\alpha(a) > x(a) - L_1(x(a), x(b), x'(a), x'(b), x) \geq \alpha(a) - L_1(\alpha(a), \alpha(b), D^+\alpha(a), D_-\alpha(b), \alpha) \geq \alpha(a).$$

Analogously it is possible to prove that $x(a) - L_1(x(a), x(b), x'(a), x'(b), x) \leq \beta(a)$.

Hence every solution for the modified problem (2.6) is a solution for (1.1)–(1.2). □

3 Existence of extremal solutions and an example

Now sufficient conditions for the existence of extremal solutions for problem (1.1)–(1.2) are given.

Theorem 3.1. *Assume hypothesis of Theorem 2.8 hold and $L_2(x, \cdot)$ is injective for all $x \in [\alpha(a), \beta(a)]$, then problem (1.1)–(1.2) has extremal solutions between α and β.*

Proof. Let $S = \{x \in [\alpha, \beta] : x \text{ is a solution for (1.1)–(1.2)}\}$. Notice that

$$S = \left\{x \in \mathcal{C}^1(I) : x \text{ is a solution for (2.6)}\right\} = \{x \in K : x = Tx\}$$

and since condition $\{x\} \cap \mathbb{T}x \subset \{Tx\}$ is satisfied for every $x \in K$ we have that

$$S = \{x \in K : x \in \mathbb{T}x\} = (I - \mathbb{T})^{-1}(\{0\})$$

which is a closed set because $\mathbb{T}$ is an upper semicontinuous mapping and $\{0\}$ is a closed subset of the Banach space (see [1, Lemma 17.4]). Now the fact that $S \subset K$ implies that S is compact.

Define $x_{\min}(t) = \inf\{x(t) : x \in S\}$ for $t \in I$. By the compactness of S in $\mathcal{C}^1(I)$ there exists, for each $t_0 \in I$, a function $x_0 \in S$ such that $x_0(t_0) = x_{\min}(t_0)$ and $x_{\min}$ is continuous in I. Following the steps of [4, Theorem 4.1] is possible to show that $x_{\min}$ is the least solution. □

Finally, we illustrate the previous results with an example.

Example 3.2. Consider the problem (1.1) along with the following functional boundary conditions

$$\begin{aligned} 0 &= L_1(x(0), x(1), x'(0), x'(1), x) = -\max_{t\in[0,1]} x(t), \\ 0 &= L_2(x(0), x(1)) = x(1), \end{aligned}$$

and

$$f(t, x, y) = t^2\lfloor 1/(t^2 + |x|)\rfloor \cos(y) + \frac{(x-1)^2\,|y|}{54} \sin^2(y) \left[1 + H\left(\sin\left(\frac{1}{y + at}\right)\right) H(y)\right]$$

if $y \neq 0$, and $f(t, x, 0) = t^2\lfloor 1/(t^2 + |x|)\rfloor$ for all $x \in \mathbb{R}$, $t \in [0, 1]$, $t > 0$ and where $\lfloor x\rfloor$ denotes the integer part of x, H is the Heaviside step function given by

$$H(y) = \begin{cases} 1 & \text{if } y \geq 0, \\ 0 & \text{if } y < 0, \end{cases}$$

and $a \in (1, \pi/2)$.

Observe that f is unbounded and discontinuous in the second and third arguments.

We take the functions $\alpha(t) = \pi t - \pi$ and $\beta(t) = 0$ for $t \in [0,1]$ which are lower and upper solutions for our problem, respectively. Indeed,

$$f(t, \alpha(t), \alpha'(t)) = -t^2 \lfloor 1/(t^2 + \pi(1-t)) \rfloor \le 0 = \alpha''(t),$$

and

$$f(t, \beta(t), \beta'(t)) = t^2 \lfloor 1/t^2 \rfloor \ge 0 = \beta''(t).$$

For a.a. $t \in [0,1]$ and all $y \in \mathbb{R}$, the function $f(t, \cdot, y)$ is continuous on

$$[\alpha(t), \beta(t)] \setminus \bigcup_{\{n: t \in I_n^i,\ i=1,2\}} \left\{ \gamma_n^i(t) \right\}$$

where for each $n \in \mathbb{N}$,

$$\gamma_n^1(t) = t^2 - n^{-1} \quad \text{for all } t \in I_n^1 = [0, n^{-1/2}],$$

and

$$\gamma_n^2(t) = -t^2 + n^{-1} \quad \text{for all } t \in I_n^2 = [n^{-1/2}, 1].$$

These curves are inviable for the differential equation. Indeed, we can take $\varepsilon_n^1 = \frac{1}{2n(n+1)}$ and $\psi_n^1 \equiv \frac{1}{4}$ and then for all $u \in [\gamma_n^1(t) - \varepsilon_n^1, \gamma_n^1(t) + \varepsilon_n^1]$ and all $v \in [{\gamma_n^1}'(t) - \varepsilon_n^1, {\gamma_n^1}'(t) + \varepsilon_n^1]$ we have

$$f(t,u,v) \le 1 + \frac{1}{9} \max \left\{ t^2 - \frac{1}{n} + \varepsilon_n^1 - 1, t^2 - \frac{1}{n} - \varepsilon_n^1 - 1 \right\}^2 \le 1 + \frac{1}{9} \left(\frac{9}{4} \right)^2 = 1 + \frac{9}{16},$$

so (2.2) holds (analogously for γ_n^2 (2.1) holds) and condition $(C3)$ in Theorem 2.8 is satisfied.

On the other hand, for a.a. $t \in [0,1]$ and all $x \in [\alpha(t), \beta(t)]$, the function $f(t, x, \cdot)$ is continuous on $\mathbb{R} \setminus \bigcup_{\{n: t \in \tilde{I}_n\}} \{\Gamma_n(t)\}$ where for each $n \in \mathbb{N}$,

$$\Gamma_n(t) = -at + \frac{1}{n\pi} \quad \text{for all } t \in \tilde{I}_n = [0, (an\pi)^{-1}].$$

Moreover, for all $t \in [0,1]$, $x \in [\alpha(t), \beta(t)]$ and $y \in [-a + (n\pi)^{-1} - \varepsilon, (n\pi)^{-1} + \varepsilon]$, for $\varepsilon > 0$ small enough, we have $|y| \le \pi/2$ what implies that $f(t,x,y) \ge 0$; and for all $t \in [0,1]$, $x \in [\alpha(t), \beta(t)]$ and $y = \alpha'(t) = \pi$ we have

$$f(t, x, -\pi) = -t^2 \lfloor 1/(t^2 + |x|) \rfloor \ge -1.$$

Therefore the curves Γ_n satisfy (2.3), so they are inviable discontinuity curves for the derivative.

Hence the mentioned theorem guarantees the existence of at least a solution in $W^{2,1}(0,1)$ between α and β. In addition there exist extremal solutions for this problem between the lower and the upper solutions as a consequence of Theorem 3.1.

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