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Multiplicity of positive weak solutions to subcritical singular elliptic Dirichlet problems

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Abstract. We study a superlinear subcritical problem at infinity of the form $-\Delta u = a(x)u^{-\alpha} + f(\lambda, x, u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, where Ω is a bounded domain in $\mathbb{R}^n$, $0 \le a \in L^\infty(\Omega)$, and $0 < \alpha < 3$. Under suitable assumptions on f, we prove that there exists $\Lambda > 0$ such that this problem has at least one weak solution in $H_0^1(\Omega)$ if and only if $\lambda \in [0, \Lambda]$; and also that there exists Λ^* such that for any $\lambda \in (0, \Lambda^*)$, at least two solutions exist.

Keywords: singular elliptic problems, positive solutions, sub- and supersolutions, bifurcation problems.

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1 Introduction and statement of the main results

In this work we consider the following singular semilinear elliptic problem with a parameter λ:

$$\begin{cases} -\Delta u = a(x)u^{-\alpha} + f(\lambda, x, u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, $0 < \alpha < 3$, $0 \le \lambda < \infty$ and a, f are functions defined on Ω and $[0,\infty) \times \overline{\Omega} \times [0,\infty)$ respectively.

Singular elliptic problems have been widely studied, they arise in applications to heat conduction in electrical conductors, in chemical catalysts processes, and in non Newtonian flows (see e.g., [7, 11, 16, 20] and the references therein). The existence of solutions to problem (1.1) was proved, for the case $f \equiv 0$, and under a variety of assumptions on a, in [4, 12, 14, 16, 20, 35]. Classical solutions to problem (1.1) were obtained by Shi and Yao in [40], when Ω and a are regular enough, $f(\lambda, x, s) = \lambda s^p$, $0 < \alpha < 1$, and $0 < p < 1$. Free boundary singular elliptic bifurcation problems of the form $-\Delta u = \chi_{\{u>0\}}(-u^{-\alpha} + \lambda g(\cdot, u))$ in Ω, $u = 0$ on $\partial\Omega$, $u \ge 0$ in Ω, $u \not\equiv 0$ (that is: $|\{x \in \Omega : u(x) > 0\}| > 0$) were studied by Dávila and Montenegro in [13]. Problems of the form $-\Delta u = g(x, u) + h(x, \lambda u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, were

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studied by Coclite and Palmieri [10]. They proved that, if $g(x,u) = au^{-\alpha}$, $a \in C^1(\overline{\Omega})$, $a > 0$ in $\overline{\Omega}$, and $h \in C^1(\overline{\Omega} \times [0,\infty))$, then there exists $\lambda^* > 0$ such that, for any $\lambda \in [0,\lambda^*)$, (1.1) has a positive classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and that, if in addition, $\overline{\lim}_{s\to\infty} \frac{h(x,s)}{s} \leq 0$ uniformly on $x \in \overline{\Omega}$, then a positive classical solution exists for any $\lambda \geq 0$.

The singular biparametric bifurcation problem $-\Delta u = g(u) + \lambda |\nabla u|^p + \mu h(\cdot, u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω was studied, by Ghergu and Rădulescu, in [24]. Dupaigne, Ghergu and Rădulescu [19] treated Lane–Emden–Fowler equations with convection term and singular potential. Rădulescu [38] studied blow-up boundary solutions for logistic equations, and for Lane–Emden–Fowler equations, with a singular nonlinearity, and a subquadratic convection term. The existence of positive solutions to the inequality $Lu \geq K(x)u^p$ on the punctured ball $\Omega = B_r(0) \setminus \{0\}$ was investigated by Ghergu, Liskevich and Sobol [22] for a second order linear elliptic operator L. Singular initial value parabolic problems involving the p-Laplacian were treated by Bougherara and Giacomoni [3], and concentration phenomena for singularly perturbed elliptic problems on an annulus were studied by Manna and Srikanth [36].

Gao and Yan [21] proved the existence of positive solutions $u \in C^{2,\beta}(\Omega) \cap C(\overline{\Omega})$ to the problem $-\Delta u + f(u) - u^{-\gamma} = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, in the case when Ω is a bounded domain with $C^{2,\beta}$ boundary, $f \in C([0,\infty))$, $s \to s^{-1}f(s)$ is strictly increasing on $(0,\infty)$, $\gamma > 0$ and $\lambda > \lambda_1$, where λ_1 denotes the principal eigenvalue for $-\Delta$ on Ω, with homogeneous Dirichlet boundary condition. They also proved that, when $0 < \gamma < 1$, such a solution $u = u_\lambda$ is unique, and that if, in addition, f is strictly increasing on $[0,\infty)$, then u_λ is strictly increasing with respect to λ.

Ghergu and Rădulescu [25] proved several existence and nonexistence theorems for the boundary value problem with two parameters $-\Delta u + K(x)g(u) = \lambda f(x,u) + \mu h(x)$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, and λ and μ are positive parameters. The function h is positive in Ω and Hölder continuous on $\overline{\Omega}$, K is Hölder continuous on $\overline{\Omega}$ and may change sign. The function $f : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is Hölder continuous, sublinear at infinity, superlinear at the origin, satisfies some monotonicity assumptions, and is positive on $\overline{\Omega} \times (0,\infty)$. They also assume that $g : (0,\infty) \to \mathbb{R}$ is nonnegative, nonincreasing, Hölder continuous, singular at the origin, and $\sup_{s>0} s^\alpha g(s) < \infty$ for some $\alpha \in (0,1)$.

The problem $-\Delta u = ag(u) + \lambda h(u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω was considered by Cîrstea, Ghergu and Rădulescu [9] in the case when Ω is a regular enough bounded domain in $\mathbb{R}^n$, $0 \leq a \in C^\beta(\overline{\Omega})$, $0 < h \in C^{0,\beta}[0,\infty)$ for some $\beta \in (0,1)$, h is nondecreasing on $[0,\infty)$, $s^{-1}h(s)$ is nonincreasing for $s > 0$, g is nonincreasing on $(0,\infty)$, $\lim_{s\to 0^+} g(s) = +\infty$, and $\sup_{s\in(0,\sigma_0)} s^\alpha g(s) < \infty$ for some $\alpha \in (0,1)$ and $\sigma_0 > 0$.

Godoy and Kaufmann [33] stated sufficient conditions for the existence of positive solutions to problems of the form $-\Delta u = Ku^{-\alpha} - \lambda M u^{-\gamma}$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, K and M are nonnegative functions on Ω, $\alpha > 0$, $\gamma > 0$, and $\lambda > 0$ is a real parameter.

Kaufmann and Medri [34] obtained existence and nonexistence results for positive solutions of one dimensional singular problems of the form $-\left((u')^{p-2}u'\right)' = m(x)u^{-\gamma}$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}$ is a bounded open interval, $p > 1$, $\gamma > 0$, and $m : \Omega \to \mathbb{R}$ is a function that may change sign in Ω.

Orpel [37] gave sufficient conditions for the existence of classical positive solutions to problems of the form $\operatorname{div}(a(|x|)\nabla u(x)) + f(x, u(x)) - u(x)^{-\alpha} |\nabla u(x)|^\beta + \langle x, \nabla u(x)\rangle g(|x|) = 0$ in Ω_R, $\lim_{|x|\to\infty} u(x) = 0$; where $R > 1$, $\Omega_R := \{x \in \mathbb{R}^n : |x| > R\}$, $n > 2$, $0 < 2\alpha \leq \beta \leq 2$ and a, g are sufficiently smooth functions defined on $[1,\infty)$, a is positive, and g is eventually nonnegative. Additionally, the rate of decay of u at infinity is investigated.

The existence of nonnegative and non identically zero weak solutions $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to problems of the form $-\Delta u = au^{-\alpha} - bu^p$ in Ω, $u = 0$ on $\partial\Omega$ was studied in [31] when Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, $0 \le a \in L^\infty(\Omega)$, $a \not\equiv 0$ (that is: $|\{x \in \Omega : a(x) \neq 0\}| > 0$), $0 < \alpha < 1$, $0 < p < \frac{n+2}{n-2}$, and $0 \le b \in L^r(\Omega)$ for suitable values of r. More general problems of the form $-\Delta u = \chi_{\{u>0\}} au^{-\alpha} + h(\cdot, u)$ in Ω, $u = 0$ on $\partial\Omega$, were studied in [32] under the assumptions that Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, $0 < \alpha < 3$, $a \in L^\infty(\Omega)$, $0 \not\equiv a \ge 0$, and $h : \Omega \times [0, \infty) \to \mathbb{R}$ is a suitable Carathéodory function that is sublinear at infinity. There it was also considered the problem with a parameter $-\Delta u = \chi_{\{u>0\}} au^{-\alpha} + \lambda h(\cdot, u)$ in Ω, $u \ge 0$ in Ω, $u = 0$ on $\partial\Omega$.

Giacomoni, Schindler and Takac [26] considered the problem $-\Delta_p u = \lambda u^{-\alpha} + u^q$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, in the case $0 < \alpha < 1$, $1 < p < \infty$, $p - 1 < q \le p^* - 1$. There it was proved that there exists $\Lambda \in (0, \infty)$ such that this problem has a solution if $\lambda \in (0, \Lambda]$, has no solution if $\lambda > \Lambda$, and has at least two solutions if $\lambda \in (0, \Lambda)$.

Aranda and Godoy [2], obtained multiplicity results for positive solutions in $W_{\text{loc}}^{1,p}(\Omega) \cap C(\overline{\Omega})$ to the problem $-\Delta_p u = g(u) + \lambda h(u)$ in Ω, $u = 0$ on $\partial\Omega$, for the case when Ω is a C^2 bounded and strictly convex domain in $\mathbb{R}^n$, $1 < p \le 2$; and g, h are locally Lipschitz functions on $(0, \infty)$ and $[0, \infty)$ respectively, with g nonincreasing, and allowed to be singular at the origin; and h nondecreasing, with subcritical growth at infinity, and satisfying $\inf_{s>0} s^{-p+1} h(s) > 0$.

Recently Saoudi, Agarwal and Mursaleen [39], obtained a multiplicity result for positive solutions of problems of the form $-\operatorname{div}(A(x)\nabla u) = u^{-\alpha} + \lambda u^p$ in Ω, $u = 0$ on $\partial\Omega$, with $0 < \alpha < 1 < p < \frac{n+2}{n-2}$.

Additional references, and a comprehensive treatment of the subject, can be found in [23], [38], see also [15].

For $b \in L^\infty(\Omega)$ such that $b^+ \not\equiv 0$, $\lambda_1(b)$ will denote the positive principal eigenvalue for $-\Delta$ in Ω, with Dirichlet boundary condition, and weight function b (see Remark 2.2 below).

The aim of this work is to prove the following three theorems concerning the existence, and the multiplicity, of weak solutions to problem (1.1).

Theorem 1.1. *Assume that Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, and that the following conditions H1)–H5) hold:*

H1) $0 < \alpha < 3$.

H2) $a \in L^\infty(\Omega)$, $a \ge 0$ *and* $a \not\equiv 0$.

H3) $f \in C([0, \infty) \times \overline{\Omega} \times [0, \infty))$, $f \ge 0$ *on* $[0, \infty) \times \overline{\Omega} \times [0, \infty)$ *and* $f(0, \cdot, \cdot) \equiv 0$ *on* $\overline{\Omega} \times [0, \infty)$.

H4) There exist numbers $\eta_0 > 0$, $q \ge 1$ and a function $b \in L^\infty(\Omega)$, such that $b^+ \not\equiv 0$ and $f(\lambda, \cdot, s) \ge \lambda b s^q$ a.e. in Ω whenever $\lambda \ge \eta_0$ and $s \ge 0$.

H5) There exist $p \in \left(1, \frac{n+2}{n-2}\right)$, and a function $h \in C((0, \infty] \times \overline{\Omega})$ that satisfy $\min_{[\eta,\infty) \times \overline{\Omega}} h > 0$ for any $\eta > 0$, and such that, for every $\sigma > 0$,

$$\lim_{(\lambda,s) \to (\sigma,\infty)} s^{-p} f(\lambda, \cdot, s) = h(\sigma, \cdot) \quad \textit{uniformly on } \overline{\Omega}.$$

Then there exists $\Lambda \in (0, \infty)$ with the following property: (1.1) has a weak solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ if and only if $0 \le \lambda \le \Lambda$. Moreover, for every $\lambda \in [0, \Lambda]$, every weak solution u in $H_0^1(\Omega) \cap L^\infty(\Omega)$ belongs to $C(\overline{\Omega})$, and satisfies $u \ge c d_\Omega^\kappa$ in Ω, for some positive constant c independent of λ

and u*, where* $\kappa := 1$ *if* $0 < \alpha < 1$, $\kappa := \frac{2}{1+\alpha}$ *if* $1 \leq \alpha < 3$*, and* d_Ω *is the distance to the boundary function, defined by*

$$d_\Omega(x) := \operatorname{dist}(x, \partial\Omega). \tag{1.2}$$

From now on (unless otherwise stated), the notion of weak solution that we use is the usual one: Let ρ be a measurable function on Ω such that $\rho\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$. We say that u is a weak solution of the problem $-\Delta u = \rho$ in Ω, $u = 0$ on $\partial\Omega$ if $u \in H_0^1(\Omega)$ and $\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega \rho\varphi$ for any $\varphi \in H_0^1(\Omega)$. Additionally, we will write $-\Delta u \geq \rho$ in Ω (respectively $-\Delta u \leq \rho$ in Ω) to mean that $\int_\Omega \langle \nabla u, \nabla \varphi \rangle \geq \int_\Omega \rho\varphi$ (resp. $\int_\Omega \langle \nabla u, \nabla \varphi \rangle \leq \int_\Omega \rho\varphi$) for any nonnegative $\varphi \in H_0^1(\Omega)$.

Theorem 1.2. *Under the same hypothesis of Theorem 1.1, there exists* $\Lambda^* > 0$ *such that, for every* $\lambda \in (0, \Lambda^*)$, (1.1) *has at least two positive weak solutions in* $H_0^1(\Omega) \cap C(\overline{\Omega})$*. Moreover* $\lambda = 0$ *is a bifurcation point from* ∞ *of* (1.1)*.*

As a consequence of Theorems 1.1 and 1.2, we obtain the following theorem.

Theorem 1.3. *Assume that the conditions H1)–H3) of Theorem 1.1 are fulfilled, and let* $g : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ *satisfy the following conditions H4')–H6'):*

H4') $g \in C(\overline{\Omega} \times [0, \infty))$, $g \geq 0$ *on* $\overline{\Omega} \times [0, \infty)$.

H5') There exist a number $q \geq 1$ *and a function* $b \in L^\infty(\Omega)$*, such that* $b^+ \not\equiv 0$ *and* $g(\cdot, s) \geq bs^q$ *for any* $s \geq 0$.

H6') There exist $h \in C(\overline{\Omega})$ *and* $p \in \left(1, \frac{n+2}{n-2}\right)$ *such that* $\min_{\overline{\Omega}} h > 0$ *and*

$$\lim_{s \to \infty} s^{-p} g(\cdot, s) = h \text{ uniformly on } \overline{\Omega}.$$

Then Theorems 1.1 and 1.2 hold for $f(\lambda, \cdot, s) := \lambda g(\cdot, s)$*. If, in addition,* $g(\cdot, 0) = 0$*, then Theorems 1.1 and 1.2 hold for* $f(\lambda, \cdot, s) := g(\cdot, \lambda s)$*.*

Our approach follows that in [2], however, there are significant differences between the two works. Here we are concerned with weak solutions in $H_0^1(\Omega) \cap C(\overline{\Omega})$; whereas solutions in $W_{\mathrm{loc}}^{1,p}(\Omega) \cap C(\overline{\Omega})$ are considered in [2]. Also, in this paper we do not assume that Ω is convex, and we do not require that $f(\lambda, x, s)$ be a local Lipschitz function.

It is a well known fact that, when a is Hölder continuous on $\overline{\Omega}$, and $\min_{\overline{\Omega}} a > 0$, the classical solution of $-\Delta u = au^{-\alpha}$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, belongs to $H_0^1(\Omega)$ if, and only if, $\alpha < 3$ (see theorem 2 in [35]). It is therefore reasonable, in order to obtain weak solutions in $H_0^1(\Omega)$ to problem (1.1), we restrict ourselves to the case when the singular term of the nonlinearity has the form $au^{-\alpha}$, with a nonnegative and nonidentically zero function in $L^\infty(\Omega)$, and $0 < \alpha < 3$.

In Section 2 we consider, for $\varepsilon \geq 0$, and $0 \leq \zeta \in L^\infty(\Omega)$, the problem $-\Delta u = a(u+\varepsilon)^{-\alpha} + \zeta$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω. We show that, under the assumptions *H1)–H3)*, this problem has a unique weak solution $u_\varepsilon \in H_0^1(\Omega)$, and that its associated solution operator S_ε, defined by $S_\varepsilon(\zeta) := u_\varepsilon$, satisfies $S_\varepsilon(P) \subset P$, where $P := \{\zeta \in C(\overline{\Omega}) : \zeta \geq 0 \text{ in } \Omega\}$ is the positive cone in $C(\overline{\Omega})$. Monotonicity and compactness properties of the map $(\zeta, \varepsilon) \to S(\zeta, \varepsilon) := S_\varepsilon(\zeta)$ are proved.

In Section 3 we obtain an a priori bound for the L^∞ norm of the bounded solutions of $-\Delta u = a(u+\varepsilon)^{-\alpha} + f(\lambda, \cdot, u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω This is achieved by adapting, to our singular setting, the well known Gidas–Spruck blow up technique.

In Section 4, we consider problem (1.1); which we rewrite as $u = S_0(f(\lambda, \cdot, u))$. We use the properties of S_0, and a classical fixed point theorem for nonlinear eigenvalue problems, to prove that, for any λ small enough, (1.1) has at least one positive weak solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$; moreover, the solution set for this problem (i.e., the set of the pairs (λ, u) that solve it) contains an unbounded subcontinuum (i.e., an unbounded connected subset) emanating from $(0, S_0(0))$. Using this subcontinuum, and the a priori estimate obtained in Section 3, we prove that, for every λ positive small enough, there exist at least two positive weak solutions of (1.1). Finally, a number Λ with the properties stated in Theorem 1.1 is obtained by using the sub and supersolution method (as well as the properties of the operator S), applied to the approximating problems $u_\varepsilon = S_\varepsilon(f(\lambda, \cdot, u_\varepsilon))$.

2 Preliminary results

We assume, from now on, that Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, and that α and a satisfy the conditions *H1)–H3)* in the statement of Theorem 1.1. The next two remarks collect some well known facts from the linear theory of elliptic problems.

Remark 2.1. Let ν be the unit outward normal to $\partial\Omega$ and let $d_\Omega : \Omega \to \mathbb{R}$ be defined by (1.2), Then:

i) If $\rho \in L^r(\Omega)$ for some $r > n$ and if $u \in H_0^1(\Omega)$ satisfies $-\Delta u = \rho$ in $D'(\Omega)$ then $u \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$, and so $u \in C^{1,\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$. If in addition, $\rho \geq 0$ and $|\{x \in \Omega : \rho(x) > 0\}| > 0$ then $u > 0$ in Ω, $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, and there exist positive constants c_1 and c_2 such that $c_1 d_\Omega \leq u \leq c_2 d_\Omega$ in Ω.

ii) The following form of the Hopf maximum principle holds (see [6, Lemma 3.2]): suppose that $\rho \geq 0$ belongs to $L^\infty(\Omega)$. Let v be the solution of $-\Delta v = \rho$ in Ω, $v = 0$ on $\partial\Omega$. Then

$$v(x) \geq c d_\Omega(x) \int_\Omega \rho d_\Omega \quad \text{a.e. in } \Omega, \tag{2.1}$$

where c is a positive constant depending only on Ω.

iii) (2.1) holds also, with the same constant c, when $0 \leq \rho \in L^1_{\text{loc}}(\Omega)$ and $v \in H_0^1(\Omega)$ satisfies $-\Delta v \geq \rho$ in the sense of distributions. Indeed, for $\delta > 0$ let $\rho_\delta := \min\{\delta^{-1}, \rho\}$. Then $0 \leq \rho_\delta \in L^\infty(\Omega)$. Let $v_\delta \in H_0^1(\Omega)$ be the solution of $-\Delta v_\delta = \rho_\delta$ in Ω, $v_\delta = 0$ on $\partial\Omega$. Then $-\Delta(v - v_\delta) \geq 0$ in $D'(\Omega)$ and so, since $v - v_\delta \in H_0^1(\Omega)$, we have $-\Delta(v - v_\delta) \geq 0$ in Ω. Thus, by the weak maximum principle, $v \geq v_\delta$ in Ω. Now, by ii), $v \geq v_\delta \geq c d_\Omega \int_\Omega \rho_\delta d_\Omega$ a.e. in Ω, and so, by taking the limit as $\delta \to 0^+$, we obtain (2.1).

We recall that $\lambda \in \mathbb{R}$ is called a principal eigenvalue for $-\Delta$ in Ω, with homogeneous Dirichlet boundary condition and weight function b, if the problem $-\Delta u = \lambda b u$ in Ω, $u = 0$ on $\partial\Omega$ has a solution ϕ (called a principal eigenfunction) such that $\phi > 0$ in Ω.

Remark 2.2. Let us mention some properties of principal eigenvalues and principal eigenfunctions (for a proof of i)–iii), see e.g., [17], also [30]), and [29]). If Ω is a $C^{1,1}$ domain in $\mathbb{R}^n$, $b \in L^\infty(\Omega)$ and $b^+ \not\equiv 0$ then:

i) There exists a unique positive principal eigenvalue for $-\Delta$ in Ω, with homogeneous Dirichlet boundary condition and weight function b, denoted by $\lambda_1(b)$; its associated eigenspace is one dimensional and it is included in $C^1(\overline{\Omega})$. Moreover, $\lambda_1(b)$ has the following variational characterization:

$$\lambda_1(b) = \inf\left\{ \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega b\varphi^2} : \varphi \in H_0^1(\Omega) \text{ and } \int_\Omega b\varphi^2 > 0 \right\}.$$

Furthermore, for each positive eigenfunction ϕ associated to $\lambda_1(b)$, and for δ positive and small enough, there are positive constants c_1, c_2 such that $c_1 d_\Omega \le \phi \le c_2 d_\Omega$ in Ω and $|\nabla\phi| \ge c_1$ in A_δ, where $A_\delta := \{x \in \Omega : d_\Omega(x) \le \delta\}$. In particular, ϕ^γ is integrable if, and only if, $\gamma > -1$.

We recall also that $\lambda_1(kb) = k^{-1}\lambda_1(b)$ for all $k \in (0,\infty)$, and that, if $b^* \in L^\infty(\Omega)$ and $b \le b^*$, then $\lambda_1(b^*) \le \lambda_1(b)$.

ii) If $0 < \lambda < \lambda_1(b)$ and $\rho \in L^\infty(\Omega)$, the problem $-\Delta u = \lambda bu + \rho$ in Ω, $u = 0$ on $\partial\Omega$, has a unique solution $u \in \cap_{1\le p<\infty} W^{2,p}(\Omega)$, and the corresponding solution operator $(-\Delta - \lambda b)^{-1} : L^\infty(\Omega) \to C_0^1(\overline{\Omega})$ is bounded and strongly positive, i.e., if $\rho \in L^\infty(\Omega)$ and $0 \le \rho \not\equiv 0$ then u belongs to the interior of the positive cone of $C_0^1(\overline{\Omega})$ where $C_0^1(\overline{\Omega}) := \{v \in C^1(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\}$. Moreover, if in addition $b \ge 0$ in Ω, the same property holds for all $\lambda \in (-\infty, \lambda_1(b))$.

iii) Let ρ be a nonnegative function in $C(\overline{\Omega})$ such that $\rho \not\equiv 0$ in Ω, and let $\lambda \in [0,\infty)$. If the problem $-\Delta u = \lambda bu + \rho$ in Ω, $u = 0$ on $\partial\Omega$ has a nonnegative weak solution $u \in H_0^1(\Omega)$ then $\lambda < \lambda_1(b)$.

iv) Let ρ be a nonnegative function in $L_{\text{loc}}^\infty(\Omega)$ such that $\rho\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$. If $\rho \not\equiv 0$ in Ω, $\lambda > 0$, and if $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$ satisfies, for some positive constant c, $u \ge cd_\Omega$ in Ω and, in weak sense,

$$-\Delta u = \lambda bu + \rho \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{2.2}$$

then $\lambda \le \lambda_1(b)$. To prove this assertion we can proceed as in the proof of Proposition 2.4 in [29], where a similar result was proved for Neumann problems. Indeed, let $v := -\ln u$ and let $w \in C_c^\infty(\Omega)$. Since $u \ge cd_\Omega$ in Ω and w has compact support we have $u^{-1}w^2 \in H_0^1(\Omega)$. Taking $u^{-1}w^2$ as a test function in (2.2), a computation gives $\lambda \int_\Omega bw^2 = \int_\Omega |\nabla w|^2 - \int_\Omega \rho u^{-1}w^2 - \int_\Omega |\nabla w + w\nabla v|^2$ and so $\lambda \int_\Omega bw^2 \le \int_\Omega |\nabla w|^2$. Now, for $\varphi \in H_0^1(\Omega)$ such that $\int_\Omega b\varphi^2 > 0$, since φ is the limit in $H_0^1(\Omega)$ of some sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\Omega)$, and since $\lambda \int_\Omega b\varphi_j^2 \le \int_\Omega |\nabla\varphi_j|^2$, we get $\lambda \int_\Omega b\varphi^2 \le \int_\Omega |\nabla\varphi|^2$, and so $\lambda \le \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega b\varphi^2}$. Then, by the variational characterization of $\lambda_1(b)$, we obtain $\lambda \le \lambda_1(b)$.

We will need the following comparison principle.

Lemma 2.3. *Let U be a bounded domain in $\mathbb{R}^n$ and $\varepsilon \ge 0$. Let u and v be two positive functions in $H^1(U) \cap C(\overline{U})$, such that $a(u+\varepsilon)^{-\alpha}$ and $a(v+\varepsilon)^{-\alpha}$ belong to $L_{\text{loc}}^1(U)$. If*

$$\begin{cases} -\Delta u - a(u+\varepsilon)^{-\alpha} \le -\Delta v - a(v+\varepsilon)^{-\alpha} & \text{in } D'(U), \\ u - v \le 0 & \text{on } \partial U, \end{cases} \tag{2.3}$$

then $u \le v$ in U.

Proof. We proceed by contradiction. Let $V := \{x \in U : u(x) > v(x)\}$ and suppose that V is nonempty. Thus $u - v \in H^1(V)$ and $u = v$ on ∂V. Then $u - v \in H_0^1(V)$ (see e.g., Theorem 8.17 and also Remark 19 in [5]). Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a sequence in $C_c^\infty(V)$ such that $\{\varphi_j\}_{j\in\mathbb{N}}$ converges to $u - v$ in $H^1(V)$. Thus $\{\varphi_j^+\}_{j\in\mathbb{N}}$ is a sequence of nonnegative functions in $C_c(V) \cap H_0^1(V)$ which converges to $u - v$ in $H^1(V)$. Now, using suitable mollifiers, we obtain a sequence $\{\psi_j\}_{j\in\mathbb{N}}$ of nonnegative functions in $C_c^\infty(V)$ that converges to $u - v$ in $H^1(V)$. From (2.3) we have $\int_V \langle \nabla(u - v), \nabla\psi_j\rangle \leq \int_V a\left((u+\varepsilon)^{-\alpha} - (v+\varepsilon)^{-\alpha}\right)\psi_j \leq 0$ for any $j \in \mathbb{N}$. Thus $\int_V |\nabla(u-v)|^2 \leq 0$, and so $u - v = 0$ on V. □

Remark 2.4. The following forms of the comparison principle hold: if $\varepsilon \geq 0$, and if u, v are two functions in $H^1(\Omega)$ (respectively in $H^1(\Omega) \cap L^\infty(\Omega)$) which are positive a.e. in Ω and satisfy that, for any nonnegative $\varphi \in H_0^1(\Omega)$ (resp. $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$), $a(u+\varepsilon)^{-\alpha}\varphi \in L^1(\Omega)$, $a(v+\varepsilon)^{-\alpha}\varphi \in L^1(\Omega)$ and

$$\int_\Omega \langle \nabla u, \nabla\varphi\rangle - \int_\Omega a(u+\varepsilon)^{-\alpha}\varphi \leq \int_\Omega \langle \nabla v, \nabla\varphi\rangle - \int_\Omega a(v+\varepsilon)^{-\alpha}\varphi,$$

and if, in addition, $u - v \leq 0$ on $\partial\Omega$ (i.e., $(u-v)^+ \in H_0^1(\Omega)$), then $u \leq v$ in Ω. Indeed, by taking $\varphi = (u-v)^+$ as a test function we get $\int_\Omega \left|\nabla\left((u-v)^+\right)\right|^2 \leq 0$, and so $u \leq v$ in Ω.

If a and u are functions defined on Ω, we will write $\chi_{\{u>0\}} a u^{-\alpha}$ to denote the function $w : \Omega \to \mathbb{R}$ defined by $w(x) := a(x)u(x)^{-\alpha}$ if $u(x) \neq 0$, and $w(x) = 0$ otherwise.

Lemma 2.5. *If $\zeta \in L^\infty(\Omega)$, then there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that:*

i) u satisfies

$$\begin{cases} -\Delta u = \chi_{\{u>0\}} a u^{-\alpha} + \zeta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \geq 0 \quad \text{in } \Omega, & u > 0 \quad \text{a.e. in } \{a > 0\} \end{cases} \tag{2.4}$$

in the following sense: for any $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, it holds that $\left(\chi_{\{u>0\}} a u^{-\alpha} + \zeta\right)\varphi \in L^1(\Omega)$ and $\int_U \langle \nabla u, \nabla\varphi\rangle = \int_U \left(\chi_{\{u>0\}} a u^{-\alpha} + \zeta\right)\varphi$;

ii) if, in addition, $\zeta \geq 0$ then u is the unique solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ to the above problem (in the sense stated in i)) and there exists a positive constant c, independent of ζ, such that $u \geq c d_\Omega$ a.e. in Ω.

Proof. i) follows as a particular case of [32, Theorem 1.2]. To see ii), observe that if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution of (2.4) in the sense of i), then $\left(\chi_{\{u>0\}} a u^{-\alpha} + \zeta\right)\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and so $\chi_{\{u>0\}} a u^{-\alpha} \in L^1_{\mathrm{loc}}(\Omega)$. Also $\chi_{\{u>0\}} a u^{-\alpha} \not\equiv 0$ and $-\Delta u \geq \chi_{\{u>0\}} a u^{-\alpha}$ in $D'(\Omega)$. Thus, by Remark 2.1 iii), there exists a positive constant c (in principle, depending perhaps on u) such that $u \geq c d_\Omega \int \chi_{\{u>0\}} a u^{-\alpha} d_\Omega$ in Ω. Then, for some positive constant c', we have $u \geq c' d_\Omega$ in Ω and so $\chi_{\{u>0\}} a u^{-\alpha} + \zeta = a u^{-\alpha} + \zeta$ in Ω. Let w be a solution of (2.4), in the sense of i), corresponding to $\zeta = 0$. By Remark 2.4 we have $u \geq w$ in Ω, and, as above, we have $w \geq c d_\Omega$ in Ω for some constant $c > 0$. Since c is independent of ζ, the last assertion of ii) holds. In particular, u is positive in Ω. Now, the uniqueness assertion follows from Remark 2.4. □

Lemma 2.6. *If ζ is a nonnegative function in $L^{\infty}(\Omega)$, then for each $\varepsilon > 0$ the problem:*

$$\begin{cases} -\Delta u = a(u+\varepsilon)^{-\alpha} + \zeta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{a.e. in } \Omega. \end{cases} \tag{2.5}$$

has a unique weak solution $u \in H_0^1(\Omega)$ to (2.5). Moreover, $u \in L^{\infty}(\Omega)$, and there exists a positive constant c such that $u \geq c d_{\Omega}$ in Ω.

Proof. Let ψ be the solution of $-\Delta\psi = a$ in Ω, $\psi = 0$ on $\partial\Omega$, thus $\psi \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ for any $r \in (1,\infty)$ and there exist positive constants c_1, c_2 such that $c_1 d_{\Omega} \leq \psi \leq c_2 d_{\Omega}$ in Ω. Define $\underline{u} := \eta\psi$, where η is a small enough positive number such that $\eta a \leq a(\eta\psi+\varepsilon)^{-\alpha}$ in Ω. Thus $-\Delta\underline{u} = \eta a \leq a(\eta\psi+\varepsilon)^{-\alpha} \leq a(\underline{u}+\varepsilon)^{-\alpha} + \zeta$ in Ω, also $\underline{u} = 0$ on $\partial\Omega$, and so $\underline{u}$ is a subsolution of (2.5). Let $\overline{u}$ be the solution of $-\Delta\overline{u} = \varepsilon^{-1}a + \zeta$ in Ω, $\zeta = 0$ on $\partial\Omega$. Thus $\overline{u} \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ for any $r \in (1,\infty)$ and there exists a positive constant c_3 such that $\overline{u} \geq c_3 d_{\Omega}$ in Ω. Also, $-\Delta\overline{u} \geq a(\overline{u}+\varepsilon)^{-\alpha} + \zeta$ in Ω, i.e., $\overline{u}$ is a supersolution of (2.5). Taking into account that $\psi \leq c_2 d_{\Omega}$ in Ω and $\overline{u} \geq c_3 d_{\Omega}$ in Ω we can assume, by diminishing η if necessary, that $\underline{u} \leq \overline{u}$ in Ω. Thus [18, Theorem 4.9] gives a weak solution $u \in H_0^1(\Omega)$ to problem (2.5) such that $\underline{u} \leq u \leq \overline{u}$ in Ω. Then $u \geq \eta c_1 d_{\Omega}$ in Ω (with η depending on ε and ζ) and $u \in L^{\infty}(\Omega)$. Finally, if u and v are two weak solutions in $H_0^1(\Omega)$ to problem (2.5), Remark 2.4 gives $u = v$. □

Lemma 2.7. *If $0 \leq \zeta \in L^{\infty}(\Omega)$ and $\varepsilon \in (0,1]$, then the solution u to problem (2.5), given by Lemma 2.6, satisfies $u \geq c d_{\Omega}$ in Ω for some positive constant c independent of ε and ζ.*

Proof. By Lemma 2.6, $u > 0$ a.e. in Ω. Let w be as in the proof of Lemma 2.5. Thus there exists a positive constant c such that $w \geq c d_{\Omega}$ in Ω. As in Lemma 2.5 we have $u \geq w$ in Ω. Thus $u \geq c d_{\Omega}$ in Ω. Since c is independent of ε and ζ, the lemma follows. □

Remark 2.8. Let us recall the Hardy inequality (see e.g., [5], p. 313): There exists a positive constant c such that $\left\|\frac{\varphi}{d_{\Omega}}\right\|_{L^2(\Omega)} \leq c\left\|\nabla\varphi\right\|_{L^2(\Omega)}$ for all $\varphi \in H_0^1(\Omega)$.

Lemma 2.9. *Let $\zeta \in L^{\infty}(\Omega)$ be such that $\zeta \geq 0$, and let $\varepsilon \in (0,1]$ (respectively $\varepsilon = 0$), and let u be the solution to problem*

$$\begin{cases} -\Delta u = a(u+\varepsilon)^{-\alpha} + \zeta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

given by Lemma 2.6 (resp. by Lemma 2.5, in the sense stated there). Then:

i) if $1 < \alpha < 3$ then there exists a positive constant c such that $u \leq c d_{\Omega}^{\frac{2}{1+\alpha}}$ in Ω, whenever $\max\{\|a\|_{\infty}, \|\zeta\|_{\infty}\} \leq M$;

ii) if $0 < \alpha \leq 1$ and $\gamma \in (0,1)$ then there exists a positive constant c such that $u \leq c d_{\Omega}^{\gamma}$ in Ω, whenever $\max\{\|a\|_{\infty}, \|\zeta\|_{\infty}\} \leq M$.

Proof. Let λ_1 be the principal eigenvalue for $-\Delta$ on Ω, with weight function $\mathbf{1}$ and let φ_1 be the corresponding positive principal eigenfunction normalized by $\|\varphi_1\|_{\infty} = 1$. For $\delta > 0$ let $A_{\delta} := \{x \in \Omega : d_{\Omega}(x) \leq \delta\}$ and let $\Omega_{\delta} := \{x \in \Omega : d_{\Omega}(x) > \delta\}$. For δ positive and small enough there exists a positive constant c_{δ} such that $|\nabla\varphi_1| \geq c_{\delta}$ in A_{δ}, and, by diminishing c_{δ} if necessary, we can assume that $\varphi_1 \geq c_{\delta}$ in Ω_{δ}. To see i), we consider first the case when $1 <$

$\alpha < 3$ and $\varepsilon = 0$. Clearly $\varphi_1^{\frac{2}{1+\alpha}} \in L^2(\Omega)$ and, since $\nabla(\varphi_1^{\frac{2}{1+\alpha}}) = \frac{2}{1+\alpha}\varphi_1^{\frac{1-\alpha}{1+\alpha}}\nabla(\varphi_1)$ and $\frac{1-\alpha}{1+\alpha} > -\frac{1}{2}$ we have also $\nabla(\varphi_1^{\frac{2}{1+\alpha}}) \in L^2(\Omega)$. Thus $\varphi_1^{\frac{2}{1+\alpha}} \in H_0^1(\Omega)$. Let $q := \left(\frac{(1+\alpha)M}{2c_\delta^2}\max\left\{\frac{1+\alpha}{\alpha-1}, \frac{1}{\lambda_1}\right\}\right)^{\frac{1}{1+\alpha}}$. A computation gives

$$-\Delta\left(q\varphi_1^{\frac{2}{1+\alpha}}\right) = q\frac{2}{1+\alpha}\lambda_1\varphi_1^{\frac{2}{1+\alpha}} + q\frac{2}{1+\alpha}\frac{\alpha-1}{1+\alpha}\left(\varphi_1^{\frac{2}{1+\alpha}}\right)^{-\alpha}|\nabla\varphi_1|^2, \tag{2.7}$$

and thus

$$\begin{aligned}
-\Delta\left(q\varphi_1^{\frac{2}{1+\alpha}}\right) &\geq q^{1+\alpha}\frac{2}{1+\alpha}\frac{\alpha-1}{1+\alpha}c_\delta^2\left(q\varphi_1^{\frac{2}{1+\alpha}}\right)^{-\alpha} \geq a\left(q\varphi_1^{\frac{2}{1+\alpha}}\right)^{-\alpha} \quad \text{in } A_\delta,\\
-\Delta\left(q\varphi_1^{\frac{2}{1+\alpha}}\right) &\geq \frac{2}{1+\alpha}\lambda_1 q\varphi_1^{\frac{2}{1+\alpha}} \geq a\left(q\varphi_1^{\frac{2}{1+\alpha}}\right)^{-\alpha} \quad \text{in } \Omega_\delta.
\end{aligned}$$

Then

$$-\Delta\left(q\varphi_1^{\frac{2}{1+\alpha}}\right) \geq a\left(q\varphi_1^{\frac{2}{1+\alpha}}\right)^{-\alpha} \quad \text{in } \Omega. \tag{2.8}$$

Let $\theta \in \cap_{1<r<\infty}\left(W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)\right)$ be the solution of $-\Delta\theta = \zeta$ in Ω, $\theta = 0$ on $\partial\Omega$. Thus

$$-\Delta\left(q\varphi_1^{\frac{2}{1+\alpha}} + \theta\right) \geq a\left(q\varphi_1^{\frac{2}{1+\alpha}}\right)^{-\alpha} + \zeta \geq a\left(q\varphi_1^{\frac{2}{1+\alpha}} + \theta\right)^{-\alpha} + \zeta \quad \text{in } \Omega. \tag{2.9}$$

Then

$$\int_\Omega \left\langle \nabla\left(q\varphi_1^{\frac{2}{1+\alpha}} + \theta\right), \nabla\psi \right\rangle \geq \int_\Omega a\left(q\varphi_1^{\frac{2}{1+\alpha}} + \theta\right)^{-\alpha}\psi$$

for any nonnegative $\psi \in H_0^1(\Omega)$; also $q\varphi_1^{\frac{2}{1+\alpha}} + \theta = 0$ on $\partial\Omega$. Since u satisfies (2.4) and $u > 0$ a.e. in Ω, the comparison principle of Remark 2.4 gives $u \leq q\varphi_1^{\frac{2}{1+\alpha}} + \theta$ a.e. in Ω. Finally, since $\|\zeta\|_\infty \leq M$ and $\frac{2}{1+\alpha} < 1$, we have $\theta \leq M(-\Delta)^{-1}(\mathbf{1}) \leq Mc'\varphi_1 \leq Mc'\varphi_1^{\frac{2}{1+\alpha}}$ in Ω, where c' is a positive constant depending only on n and Ω. Also, for some constant $c'' > 0$, $\varphi_1^{\frac{2}{1+\alpha}} \leq c'' d_\Omega^{\frac{2}{1+\alpha}}$ in Ω and so $u \leq c d_\Omega^{\frac{2}{1+\alpha}}$ in Ω, for a positive constant c depending only on M, α, and Ω, therefore i) holds when $\varepsilon = 0$. The proof of i) for the case $\varepsilon \in (0,1]$ reduces to the previous one. Indeed, Remark 2.4 gives $u \leq u_0$ in Ω, where u_0 is the solution (given by Lemma 2.5) to problem (2.6) and corresponding to $\varepsilon = 0$.

The proof of ii) follows similar lines: suppose $0 < \alpha \leq 1$ and $\gamma \in (0,1)$. Define

$$q := \left(\frac{M}{\gamma}\max\left\{\frac{1}{\lambda_1 c_\delta^{\gamma(1+\alpha)}}, \frac{1}{(1-\gamma)c_\delta^2}\right\}\right)^{\frac{1}{1+\alpha}}.$$

Then

$$-\Delta\left(q\varphi_1^\gamma\right) = \gamma q\lambda_1\varphi_1^\gamma + q\gamma(1-\gamma)\varphi_1^{\gamma-2}|\nabla\varphi_1|^2 \quad \text{in } \Omega,$$

and so

$$\begin{aligned}
-\Delta\left(q\varphi_1^\gamma\right) &\geq q\gamma(1-\gamma)\varphi_1^{\gamma-2}|\nabla\varphi_1|^2 \geq a\left(q\varphi_1^\gamma\right)^{-\alpha} \quad \text{in } A_\delta,\\
-\Delta\left(q\varphi_1^\gamma\right) &\geq \gamma q\lambda_1\varphi_1^\gamma \geq a\left(q\varphi_1^\gamma\right)^{-\alpha} \quad \text{in } \Omega_\delta.
\end{aligned}$$

Thus $-\Delta\left(q\varphi_1^\gamma\right) \geq a\left(q\varphi_1^\gamma\right)^{-\alpha}$ in Ω, which is the analogue of (2.8). From this point, the proof of ii) follows exactly as in i), replacing $\varphi_1^{\frac{2}{1+\alpha}}$ and $d_\Omega^{\frac{2}{1+\alpha}}$ by φ_1^γ and d_Ω^γ respectively. □

Lemma 2.10. *Let ζ be a nonnegative function belonging to $L^{\infty}(\Omega)$ and let $M \geq \max\{\|a\|_{\infty}, \|\zeta\|_{\infty}\}$. Let $\varepsilon \in (0,1]$ (respectively $\varepsilon = 0$); and let u be the solution to problem (2.6) given by Lemma 2.6 (resp. by Lemma 2.5, in the sense stated there). Then $u \in C(\overline{\Omega})$.*

Proof. Let Ω' be a subdomain of Ω such that $\overline{\Omega'} \subset \Omega$; and let Ω'' be a subdomain of Ω such that $\overline{\Omega'} \subset \Omega'' \subset \overline{\Omega''} \subset \Omega$. By Lemmas 2.7 and 2.9 there exist positive constants c_1, c_2 and $\gamma > 0$ such that $c_1 d_{\Omega} \leq u \leq c_2 d_{\Omega}^{\gamma}$ in Ω and so $(au^{-\alpha} + \zeta)_{|\Omega''} \in L^{\infty}(\Omega'')$. Also, $u_{|\Omega''} \in L^{\infty}(\Omega'')$. Then, by [28, Theorem 8.24], $u_{|\Omega'} \in C^{\beta}(\overline{\Omega'})$ for some $\beta \in (0,1)$. Since this holds for any domain Ω' such that $\overline{\Omega'} \subset \Omega$, it follows that $u \in C(\Omega)$. Also, $c_1 d_{\Omega} \leq u \leq c_2 d_{\Omega}^{\gamma}$ in Ω, and so u is continuous on $\partial\Omega$. Then $u \in C(\overline{\Omega})$. □

Lemma 2.11. *Assume $1 < \alpha < 3$, and let $\zeta \in L^{\infty}(\Omega)$ be such that $\zeta \geq 0$. Let u be the solution to problem (2.5) given by Lemma 2.5 (in the sense stated there). Then there exists a positive constant c independent of ζ such that $u \geq c d_{\Omega}^{\frac{2}{1+\alpha}}$ in Ω.*

Proof. We consider first the case when $\underline{a} := \inf_{\Omega} a > 0$. Let λ_1 be the principal eigenvalue for $-\Delta$ in Ω with homogeneous Dirichlet boundary condition and weight function a, and let φ_1 be the corresponding positive principal eigenfunction, normalized by $\|\varphi_1\|_{\infty} = 1$. Observe that $\varphi_1^{\frac{2}{1+\alpha}} \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$ and

$$
\begin{aligned}
-\Delta\left(\varphi_1^{\frac{2}{1+\alpha}}\right) &= \frac{2}{1+\alpha}\lambda_1 a \varphi_1^{\frac{2}{1+\alpha}} + \frac{2}{1+\alpha}\frac{\alpha-1}{1+\alpha}\left(\varphi_1^{\frac{2}{1+\alpha}}\right)^{-\alpha}|\nabla\varphi_1|^2 \\
&\leq \beta a\left(\varphi_1^{\frac{2}{1+\alpha}}\right)^{-\alpha} \quad \text{a.e. in } \Omega,
\end{aligned}
$$

where $\beta := \frac{2}{1+\alpha}\lambda_1 + \frac{2}{1+\alpha}\frac{\alpha-1}{1+\alpha}\frac{1}{\underline{a}}\|\nabla\varphi_1\|_{\infty}^2$. Then

$$
-\Delta\left(\beta^{-\frac{1}{1+\alpha}}\varphi_1^{\frac{2}{1+\alpha}}\right) \leq a\left(\beta^{-\frac{1}{1+\alpha}}\varphi_1^{\frac{2}{1+\alpha}}\right)^{-\alpha}
$$

in the weak sense of Lemma 2.5, (i.e., with test functions in $H_0^1(\Omega) \cap L^{\infty}(\Omega)$). We have also, again in the weak sense of Lemma 2.5, $-\Delta u \geq au^{-\alpha}$ in Ω. Then, by Lemma 2.3, $u \geq \beta^{-\frac{1}{1+\alpha}}\varphi_1^{\frac{2}{1+\alpha}}$ in Ω and so $u \geq c d_{\Omega}^{\frac{2}{1+\alpha}}$ in Ω for some positive constant c independent of ζ. Thus the lemma holds when $\inf_{\Omega} a > 0$.

To prove the lemma in the general case, consider the solution θ to the problem $-\Delta\theta = a$ in Ω, $\theta = 0$ on $\partial\Omega$. Thus $\theta \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ for any $r \in [1,\infty)$ and, for some positive constant c_1, $\theta \geq c_1 d_{\Omega}$ in Ω. Let $w \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$ be a solution, in the sense of Lemma 2.5, of problem (2.4) corresponding to $\zeta = 0$. By Lemma 2.3 we have $u \geq w$ in Ω and, by Lemma 2.9, there exists a positive constant c_2 such that $w \leq c_2 d_{\Omega}$ in Ω. Now, for $\varepsilon \in (0,1)$ and $\beta \in (0,1)$, we have, in the weak sense of Lemma 2.5,

$$
\begin{aligned}
-\Delta\left((w+\varepsilon)^{\beta}\right) &= -\alpha(w+\varepsilon)^{\beta-1}\Delta w - \beta(\beta-1)(w+\varepsilon)^{\beta-2}|\nabla w|^2 \qquad (2.10)\\
&\geq \alpha a (w+\varepsilon)^{\beta-1} w^{-\alpha} \geq \alpha a (c_2 d_{\Omega} + \varepsilon)^{\beta-\alpha-1} \\
&\geq -\alpha(c_3\theta + \varepsilon)^{\beta-\alpha-1}\Delta\theta \quad \text{in } \Omega
\end{aligned}
$$

with $c_3 = c_1^{-1} c_2$. Also,

$$\begin{aligned} & -\Delta\left((c_3\theta+\varepsilon)^{\frac{2\alpha}{1+\alpha}}\right) \\ & = -\frac{2\alpha}{1+\alpha}(c_3\theta+\varepsilon)^{\frac{\alpha-1}{\alpha+1}}\Delta\theta - \frac{2\alpha}{1+\alpha}\left(\frac{2\alpha}{1+\alpha}-1\right)(c_3\theta+\varepsilon)^{\frac{2\alpha}{1+\alpha}-2}\left|\nabla(c_3\theta)\right|^2 \\ & \le -\frac{2\alpha}{1+\alpha}(c_3\theta+\varepsilon)^{\frac{\alpha-1}{\alpha+1}}\Delta\theta \le -\frac{2\alpha M_\beta}{1+\alpha}(c_3\theta+\varepsilon)^{\beta-\alpha-1}\Delta\theta, \end{aligned} \tag{2.11}$$

where $M_\beta := (c_3\|\theta\|_\infty + 1)^{\frac{\alpha-1}{\alpha+1}+\alpha+1-\beta}$. Thus, from (2.10) and (2.11), we get

$$-\Delta\left((w+\varepsilon)^\beta\right) \ge -\frac{1+\alpha}{2M_\beta}\Delta\left((c_3\theta+\varepsilon)^{\frac{2\alpha}{1+\alpha}}\right) \quad \text{in } D'(\Omega),$$

also, for ε small enough, $(w+\varepsilon)^\beta = \varepsilon^\beta \ge \frac{1+\alpha}{2M_\beta}\varepsilon^{\frac{2\alpha}{1+\alpha}} = \frac{1+\alpha}{2M_\beta}(c_3\theta+\varepsilon)^{\frac{2\alpha}{1+\alpha}}$ on $\partial\Omega$ and so, by the weak maximum principle, we have, for ε small enough, $(w+\varepsilon)^\beta \ge \frac{1+\alpha}{2M_\beta}(c_3\theta+\varepsilon)^{\frac{2\alpha}{1+\alpha}}$ a.e. in Ω. By taking $\lim_{\varepsilon\to 0^+}$ in this inequality we get, for any $\beta \in (0,1)$,

$$w^\beta \ge \frac{1+\alpha}{2M_\beta}(c_3\theta)^{\frac{2\alpha}{1+\alpha}} \quad \text{a.e. in } \Omega. \tag{2.12}$$

By taking $\lim_{\beta\to 0^+}$ in (2.12), using that $\lim_{\beta\to 0^+} M_\beta = (c_3\|\theta\|_\infty + 1)^{\frac{2\alpha}{\alpha+1}}$, recalling that $u \ge w$ in Ω and that $\theta \ge c_1 d_\Omega$ in Ω, we get

$$u \ge \frac{1}{2}(1+\alpha)\left(\frac{c_1c_3}{c_3\|\theta\|_\infty+1}\right)^{\frac{2\alpha}{1+\alpha}} d_\Omega^{\frac{2\alpha}{1+\alpha}} \quad \text{a.e. in } \Omega,$$

which ends the proof of the lemma. □

Lemma 2.12. *Let ζ be a nonnegative function in $L^\infty(\Omega)$, and let $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$ be the solution (in the sense of Lemma 2.5) to (2.4). Then, for some positive constant c, $u \ge cd_\Omega$ in Ω if $0<\alpha\le 1$, and $u \ge cd_\Omega^{\frac{2}{1+\alpha}}$ in Ω if $1\le\alpha<3$. Moreover, u is the unique weak solution, in the usual $H_0^1(\Omega)$ sense, to the problem*

$$\begin{cases} -\Delta u = au^{-\alpha}+\zeta \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega. \end{cases} \tag{2.13}$$

Proof. To see that u is a weak solution of (2.4), i.e., that, for any $\psi \in H_0^1(\Omega)$, $au^{-\alpha}\psi \in L^1(\Omega)$ and

$$\int_\Omega \langle\nabla u, \nabla\psi\rangle = \int_\Omega (au^{-\alpha}+\zeta)\psi, \tag{2.14}$$

we consider first the case $0<\alpha\le 1$. Let $\psi\in H_0^1(\Omega)$ and, for $j\in\mathbb{N}$ and $x\in\Omega$, let $\psi_j(x) := \psi(x)$ if $|\psi(x)|\le j$, $\psi_j(x) := j$ if $\psi(x) > j$, and $\psi_j(x) := -j$ if $\psi(x) < -j$. Then $\psi_j \in H_0^1(\Omega)\cap L^\infty(\Omega)$ and so, for all j,

$$\int_\Omega \langle\nabla u, \nabla\psi_j\rangle = \int_\Omega (au^{-\alpha}+\zeta)\psi_j \tag{2.15}$$

Also, by Lemma 2.7, there exists a positive constant c such that $u \ge cd_\Omega$ in Ω, and so, for any $j\in\mathbb{N}$,

$$\left|(au^{-\alpha}+\zeta)\psi_j\right| \le \|a\|_\infty c^{-\alpha} d_\Omega^{1-\alpha}\left|\frac{\psi_j}{d_\Omega}\right| + \zeta|\psi_j| \le c' d_\Omega^{1-\alpha}\left|\frac{\psi}{d_\Omega}\right| + \zeta|\psi| \tag{2.16}$$

with $c' = \|a\|_\infty c^{-\alpha}$; applying Hardy's inequality we get

$$\left\| c' d_\Omega^{1-\alpha} \left| \frac{\psi}{d_\Omega} \right| + \zeta \left| \psi \right| \right\|_1 \leq c \left\| d_\Omega^{1-\alpha} \right\|_2 \|\nabla \psi\|_2 + \|\zeta\|_\infty |\Omega|^{\frac{1}{2}} \|\psi\|_2 < \infty. \tag{2.17}$$

for some positive constant c. Since $\{\psi_j\}_{j\in\mathbb{N}}$ converges to ψ in $H_0^1(\Omega)$ and a.e. in Ω, Lebesgue's dominated convergence theorem gives $(au^{-\alpha} + \zeta)\psi \in L^1(\Omega)$ and (2.14). Thus u is a weak solution (in the usual $H_0^1(\Omega)$ sense) of (2.4), it satisfies $u \geq cd_\Omega$ in Ω and, by Lemma 2.5, u is the unique weak solution to (2.4). Thus i) and ii) holds when $0 < \alpha \leq 1$.

Consider now the case $1 < \alpha < 3$. To see that u is a weak solution of (2.13) we proceed as in the case $0 < \alpha \leq 1$, except that instead of (2.16) we use now that, by Lemma 2.11, there exists a positive constant c such that $u \geq cd_\Omega^{\frac{2}{1+\alpha}}$. Thus,

$$\left|\left(au^{-\alpha} + \zeta\right)\psi_j\right| \leq \|a\|_\infty c^{-\frac{2\alpha}{1+\alpha}} d_\Omega^{1-\frac{2\alpha}{1+\alpha}} \left|\frac{\psi_j}{d_\Omega}\right| + \zeta\left|\psi_j\right| \leq c' d_\Omega^{-\frac{\alpha-1}{\alpha+1}} \left|\frac{\psi}{d_\Omega}\right| + \zeta\left|\psi\right| \tag{2.18}$$

with c' a constant independent of j. Since $\alpha < 3$ we have $\|d_\Omega^{-\frac{\alpha-1}{\alpha+1}}\|_2 < \infty$ and so, by Hardy's inequality, $\|c' d_\Omega^{-\frac{\alpha-1}{\alpha+1}} |\frac{\psi}{d_\Omega}| + \zeta|\psi|\|_1 < \infty$. Then, as in the case $0 < \alpha \leq 1$, Lebesgue's dominated convergence theorem gives $(au^{-\alpha} + \zeta)\psi \in L^1(\Omega)$ and (2.14). Thus u is a weak solution (in the usual $H_0^1(\Omega)$ sense) of (2.13), it satisfies $u \geq cd_\Omega^{\frac{2}{1+\alpha}}$ a.e. in Ω and, by the comparison principle in Remark 2.4, u is the unique weak solution (in the usual $H_0^1(\Omega)$ sense) to problem (2.13). □

Let $P_\infty := \{\zeta \in L^\infty(\Omega) : \zeta \geq 0 \text{ a.e. in } \Omega\}$, and, for $\varepsilon \geq 0$, let $S_\varepsilon : P_\infty \to H_0^1(\Omega) \cap L^\infty(\Omega)$ be defined by $S_\varepsilon(\zeta) := u$, where u is the unique weak solution (provided by Lemma 2.6 when $\varepsilon > 0$, and by Lemma 2.12 when $\varepsilon = 0$) to problem (2.5). Consider $S : P_\infty \times [0,\infty) \to H_0^1(\Omega) \cap L^\infty(\Omega)$ defined by $S(\zeta, \varepsilon) := S_\varepsilon(\zeta)$. Except explicit mention on the contrary, we will consider P_∞ endowed with the topology of the L^∞ norm.

Lemma 2.13. *Let $\{\zeta_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $L^\infty(\Omega)$ such that $\zeta_j \geq 0$ for all j, and let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence in $[0,\infty)$. Then $\{S_{\varepsilon_j}(\zeta_j)\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$.*

Proof. For $j \in \mathbb{N}$, let $u_j := S_{\varepsilon_j}(\zeta_j)$. Since u_j is a weak solution of $-\Delta u_j = a(u_j + \varepsilon_j)^{-\alpha} + \zeta_j$ in Ω, $u_j = 0$ on $\partial\Omega$, and using u_j as a test function, we get

$$\int_\Omega |\nabla u_j|^2 = \int_\Omega a(u_j + \varepsilon_j)^{-\alpha} u_j + \int_\Omega u_j \zeta_j. \tag{2.19}$$

If $0 < \alpha \leq 1$, since $\{\zeta_j\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, (2.19), the Poincaré inequality gives

$$\|\nabla u_j\|_2^2 \leq c\left(\|\nabla u_j\|_2^{1-\alpha} + \|\nabla u_j\|_2\right)$$

with c independent of j, which proves the lemma when $0 < \alpha \leq 1$. Let us consider now the case $1 < \alpha < 3$. The function $z := S_0(0)$ is a weak solution of $-\Delta z = az^{-\alpha}$ in Ω, $z = 0$ on $\partial\Omega$ and, by Lemma 2.10, $z \in C(\overline{\Omega})$. Also $-\Delta u_j \geq au_j^{-\alpha}$ in Ω, $u_j = 0$ on $\partial\Omega$, and then, by Lemma 2.3, $u_j \geq z$ in Ω. By Lemma 2.9, $u_j \leq cd_\Omega^{\frac{2}{1+\alpha}}$ for any j, with c a positive constant independent of j. Thus, from (2.19), we have

$$\int_\Omega |\nabla u_j|^2 \leq c \int_\Omega \left(\|a\|_\infty z_j^{-\alpha} + \|\zeta_j\|_\infty\right) d_\Omega^{\frac{2}{1+\alpha}}. \tag{2.20}$$

Also, by Lemma 2.11, $z \geq c' d_\Omega^{\frac{2}{1+\alpha}}$ in Ω, for some positive constant c'. Then, from (2.20),

$$\int_\Omega |\nabla u_j|^2 \leq c \int_\Omega \left(\|a\|_\infty (c')^{-\alpha} d_\Omega^{-\frac{2\alpha}{1+\alpha}} + \|\zeta_j\|_\infty \right) d_\Omega^{\frac{2}{1+\alpha}}. \tag{2.21}$$

Since $1 < \alpha < 3$, we have $\frac{2(1-\alpha)}{1+\alpha} > -1$, therefore $\int_\Omega d_\Omega^{-\frac{2\alpha}{1+\alpha}} d_\Omega^{\frac{2}{1+\alpha}} < \infty$, and then, since $\{\zeta_j\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, the lemma follows. □

Lemma 2.14.

i) $\zeta \to S_\varepsilon(\zeta)$ *is nondecreasing on* P_∞ *for any* $\varepsilon \geq 0$.

ii) $\varepsilon \to S_\varepsilon(\zeta)$ *is nonincreasing on* $[0,\infty)$ *for any* $\zeta \in P_\infty$.

iii) $S : P_\infty \times [0,\infty) \to C(\overline{\Omega})$ *is continuous.*

iv) $S : P_\infty \times [0,\infty) \to C(\overline{\Omega})$ *is a compact map.*

Proof. i) and ii) follow directly from Lemma 2.3. To prove iii) it is enough to show that if $(\zeta,\varepsilon) \in P_\infty \times [0,\infty)$, and $\{(\zeta_j,\varepsilon_j)\}_{j\in\mathbb{N}} \subset P_\infty \times [0,\infty)$ converges to (ζ,ε) in $P_\infty \times [0,\infty)$, then there exists a subsequence $\{(\zeta_{j_k},\varepsilon_{j_k})\}_{k\in\mathbb{N}}$ such that $\{S(\zeta_{j_k},\varepsilon_{j_k})\}_{k\in\mathbb{N}}$ converges to $S(\zeta,\varepsilon)$ in $C(\overline{\Omega})$.

Let $(\zeta,\varepsilon) \in P_\infty \times [0,\infty)$, and let $\{(\zeta_j,\varepsilon_j)\}_{j\in\mathbb{N}} \subset P_\infty \times [0,\infty)$ be a sequence that converges to (ζ,ε) in $P_\infty \times [0,\infty)$. For $j \in \mathbb{N}$, let $u_j := S(\zeta_j,\varepsilon_j)$. By Lemma 2.13 $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$, therefore there exist $u \in H_0^1(\Omega)$, and a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$, such that $\{u_{j_k}\}_{k\in\mathbb{N}}$ converges strongly in $L^2(\Omega)$ to u, and $\{\nabla u_{j_k}\}_{k\in\mathbb{N}}$ converges weakly in $L^2(\Omega,\mathbb{R}^n)$ to ∇u. Taking a further subsequence if necessary, we can assume that $\{u_{j_k}\}_{k\in\mathbb{N}}$ converges to u, a.e. in Ω.

Let us see that $u = S(\zeta,\varepsilon)$, i.e., that $a(u+\varepsilon)^{-\alpha}\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$, and

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega \left(a(u+\varepsilon)^{-\alpha} + \zeta \right) \varphi. \tag{2.22}$$

Let $\varphi \in H_0^1(\Omega)$. For $k \in \mathbb{N}$, $\int_\Omega \langle \nabla u_{j_k}, \nabla\varphi \rangle = \int_\Omega \left(a(u_{j_k} + \varepsilon_{j_k})^{-\alpha} + \zeta_{j_k} \right)\varphi$. Now, as $\{\nabla u_{j_k}\}_{k\in\mathbb{N}}$ converges weakly to ∇u in $L^2(\Omega,\mathbb{R}^n)$, we have that $\lim_{k\to\infty} \int_\Omega \langle \nabla u_{j_k}, \nabla\varphi \rangle = \int_\Omega \langle \nabla u, \nabla \varphi \rangle$. Also, $\zeta\varphi \in L^1(\Omega)$ and, applying the Lebesgue dominated convergence theorem, we get $\lim_{k\to\infty} \int_\Omega \zeta_{j_k}\varphi = \int_\Omega \zeta\varphi$. Therefore, to prove (2.22), it is enough to show that $a(u+\varepsilon)^{-\alpha}\varphi \in L^1(\Omega)$ and

$$\lim_{k\to\infty} \int_\Omega a\left(u_{j_k} + \varepsilon_{j_k}\right)^{-\alpha} \varphi = \int_\Omega a(u+\varepsilon)^{-\alpha}\varphi. \tag{2.23}$$

In order to prove this, we consider two cases; suppose first that $0 < \alpha \leq 1$. Lemma 2.7 gives a constant $c_1 > 0$ such that $u_{j_k} \geq c_1 d_\Omega$ in Ω for every k. Then, for any k, $\left|a\left(u_{j_k}+\varepsilon_{j_k}\right)^{-\alpha}\varphi\right| \leq c_1^{-\alpha}\left\|d_\Omega^{1-\alpha} a\right\|_\infty \frac{|\varphi|}{d_\Omega}$ and, by the Hardy inequality, $\frac{\varphi}{d_\Omega} \in L^2(\Omega) \subset L^1(\Omega)$. Taking into account that $\lim_{k\to\infty} a\left(u_{j_k}+\varepsilon_{j_k}\right)^{-\alpha}\varphi = a(u+\varepsilon)^{-\alpha}\varphi$ a.e. in Ω, Lebesgue's dominated convergence theorem gives $a(u+\varepsilon)^{-\alpha}\varphi \in L^1(\Omega)$; and (2.23).

Let us now consider the case $1 < \alpha < 3$. Define $z := S_0(0)$. Observe that $u_{j_k} + \varepsilon_{j_k} \in H^1(\Omega) \cap C(\overline{\Omega})$, $-\Delta\left(u_{j_k}+\varepsilon_{j_k}\right) = a\left(u_{j_k}+\varepsilon_{j_k}\right)^{-\alpha} + \zeta_{j_k} \geq a\left(u_{j_k}+\varepsilon_{j_k}\right)^{-\alpha}$ in $D'(\Omega)$, $z \in H^1(\Omega) \cap$

$C\left(\overline{\Omega}\right)$, $-\Delta z = az^{-\alpha}$ in $D'(\Omega)$, and $u_{j_k} + \varepsilon_{j_k} \geq z$ on $\partial\Omega$. Thus, by Lemma 2.3, $u_{j_k} + \varepsilon_{j_k} \geq z$ in Ω. By Lemma 2.7, there exists a positive constant c such that $z \geq c d_\Omega^{\frac{2}{1+\alpha}}$ in Ω. Then

$$\left| a \left(u_{j_k} + \varepsilon_{j_k}\right)^{-\alpha} \varphi \right| \leq a z^{-\alpha} |\varphi| \leq c^{-\alpha} \|a\|_\infty d_\Omega^{1-\frac{2\alpha}{1+\alpha}} \frac{|\varphi|}{d_\Omega} = c^{-\alpha} \|a\|_\infty d_\Omega^{-\frac{1-\alpha}{1+\alpha}} \frac{|\varphi|}{d_\Omega} \quad \text{in } \Omega.$$

Since $1 < \alpha < 3$, we have $-2\frac{\alpha-1}{\alpha+1} > -1$, and so, by the Hardy inequality, $\left\| d_\Omega^{-\frac{\alpha-1}{\alpha+1}} \frac{|\varphi|}{d_\Omega} \right\|_1 < \infty$. Since $\lim_{k\to\infty} a\left(u_{j_k} + \varepsilon_{j_k}\right)^{-\alpha} \varphi = a(u+\varepsilon)^{-\alpha}\varphi$ a.e. in Ω, Lebesgue's dominated convergence theorem applies to get $a(u+\varepsilon)^{-\alpha}\varphi \in L^1(\Omega)$ and (2.23). Thus $u = S_\varepsilon(\zeta)$.

To complete the proof of iii), it only remains to prove that $\{u_{j_k}\}_{k\in\mathbb{N}}$ (or some subsequence of it) converges to u in $C\left(\overline{\Omega}\right)$. Let $B > 0$ be such that $\left\|\zeta_{j_k}\right\|_\infty \leq B$ for all $k \in \mathbb{N}$. Since $0 \leq \zeta_{j_k} \leq B$ we have $0 \leq \zeta \leq B$. Now, $0 \leq u_{j_k} = S_{\varepsilon_{j_k}}\left(\zeta_{j_k}\right) \leq S_0(B)$. Also $0 \leq u = S_\varepsilon(\zeta) \leq S_0(B)$. Now, $S_0(B) \in C\left(\overline{\Omega}\right)$ and, by Lemmas 2.6 and 2.7, there exist positive constants c_1', c_2' and τ' such that $c_1' d_\Omega \leq S_0(B) \leq c_2' d_\Omega^{\tau'}$ in Ω. Then $S_0(B) = 0$ on $\partial\Omega$ pointwise, and so, for any $\mu > 0$ there exists $\eta > 0$ such that $0 \leq S_0(B) \leq \mu$ in $A_\eta := \{x \in \Omega : d_\Omega(x) \leq \eta\}$. Thus $0 \leq S_{\varepsilon_{j_k}}\left(\zeta_{j_k}\right) \leq \mu$ in A_η for all $k \in \mathbb{N}$. Also $0 \leq S_\varepsilon(\zeta) \leq S_0(B) \leq \mu$ in A_η. Then

$$\left\| S_{\varepsilon_{j_k}}\left(\zeta_{j_k}\right) - S_\varepsilon(\zeta) \right\|_{L^\infty(A_\eta)} \leq 2\mu \quad \text{for all } k \in \mathbb{N}. \tag{2.24}$$

Let $\Omega' := \Omega \setminus A_\eta$, and let Ω'' be a subdomain of Ω such that $\overline{\Omega'} \subset \Omega'' \subset \overline{\Omega''} \subset \Omega$. By Lemmas 2.6 and 2.7 there exist positive constants c_1, c_2 and τ such that for all k, $c_1 d_\Omega \leq u_{j_k} \leq c_2 d_\Omega^\tau$ in Ω. Thus there exists $B' > 0$ such that, for all k, $\|u_{j_k|\Omega''}\|_{L^\infty(\Omega'')} \leq B'$ and

$$\left\| \left(a\left(u_{j_k} + \varepsilon_{j_k}\right)^{-\alpha} + \zeta_{j_k} \right)_{|\Omega''} \right\|_{L^\infty(\Omega'')} \leq B'.$$

Then, by the inner elliptic estimates in [28, Theorem 8.24], there exist $B'' > 0$ and $\gamma \in (0,1)$ such that, for all k, $\|u_{j_k|\Omega'}\|_{C^\gamma\left(\overline{\Omega'}\right)} \leq B''$. Thus, the Ascoli–Arzelà theorem applies to give a subsequence, still denoted $\{u_{j_k}\}_{k\in\mathbb{N}}$, that converges uniformly to some function v in Ω'. Since $\{u_{j_k}\}_{k\in\mathbb{N}}$ converges to u a.e. in Ω, we have $u = v$ in Ω'. Then there exists $k_0 > 0$ such that $\|\left(u_{j_k} - u\right)_{|\Omega''}\|_{L^\infty(\Omega')} \leq \mu$ for $k \geq k_0$, i.e.,

$$\left\| S_{\varepsilon_{j_k}}\left(\zeta_{j_k}\right) - S_\varepsilon(\zeta) \right\|_{L^\infty(\Omega\setminus A_\eta)} \leq \mu \qquad \text{for all } k \geq k_0, \tag{2.25}$$

and so, by (2.24) and (2.25), $\lim_{k\to\infty} S_{\varepsilon_{j_k}}\left(\zeta_{j_k}\right) = S_\varepsilon(\zeta)$, with convergence in $C\left(\overline{\Omega}\right)$. Thus S is continuous.

To prove iv), consider a bounded sequence $\{(\zeta_j, \varepsilon_j)\}_{j\in\mathbb{N}} \subset P_\infty \times [0,\infty)$. Taking a subsequence if necessary, we can assume that $\{\varepsilon_j\}_{j\in\mathbb{N}}$ converges to some $\varepsilon \in [0,\infty)$. Let $\{\Omega_r\}_{r\in\mathbb{N}}$ be a sequence of subdomains of Ω such that $\overline{\Omega_r} \subset \Omega_{r+1}$ for all r, and $\Omega = \cup_{r=1}^\infty \Omega_r$. Let $u_j = S_{\varepsilon_j}\left(\zeta_j\right)$. Let $B > 0$ be such that $\left\|\zeta_j\right\|_\infty \leq B$ for all j. Since $0 \leq \zeta_j \leq B$ we have $0 \leq u_j = S_{\varepsilon_j}\left(\zeta_j\right) \leq S_0(B)$. By Lemmas 2.6 and 2.7, there exist positive constants c_1', c_2' and τ' such that, for all j, $c_1' d_\Omega \leq u_j \leq c_2' d_\Omega^{\tau'}$ in Ω. Thus, for each r there exists a positive constant $B_r > 0$ such that, for all j, $\|u_{j|\Omega_{r+1}}\|_{L^\infty(\Omega_{r+1})} \leq B_r$ and $\|\left(a\left(u_j + \varepsilon_j\right)^{-\alpha} + \zeta_j\right)_{|\Omega_{r+1}}\|_{L^\infty(\Omega_{r+1})} \leq B_r$. Then, by [28, Theorem 8.24], for each r there exist constants $B_r' > 0$ and $\gamma_r \in (0,1)$ such that, for all j, $\left\|u_{j|\Omega_r}\right\|_{C^\gamma\left(\overline{\Omega_r}\right)} \leq B_r'$. Then, for each r, the Ascoli–Arzelà theorem gives a subsequence, still denoted by $\{u_j\}_{j\in\mathbb{N}}$ which converges uniformly in $\overline{\Omega_r}$. Now, a Cantor diagonal process gives a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ which converges uniformly on each $\overline{\Omega_r}$ to a function u independent of r (therefore $\{u_{j_k}\}_{k\in\mathbb{N}}$ converges uniformly to u on each compact subset of Ω). Let us show that $\{u_{j_k}\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $C\left(\overline{\Omega}\right)$: Since $0 \leq \zeta_{j_k} \leq B$ we have $0 \leq S_{\varepsilon_{j_k}}\left(\zeta_{j_k}\right) \leq S_0\left(B\right)$. Also $S_0\left(B\right) \in C\left(\overline{\Omega}\right)$, and $S_0\left(B\right) = 0$ on $\partial\Omega$ pointwise. Since $S_0\left(B\right) \in C\left(\overline{\Omega}\right)$ we have that for any $\mu > 0$ there exists $\eta > 0$ such that $0 \leq S_0\left(B\right) \leq \mu$ in A_η. Thus $0 \leq S_{\varepsilon_{j_k}}\left(\zeta_{j_k}\right) \leq \mu$ in A_η , for all $k \in \mathbb{N}$. Then $\left\|S\left(\zeta_{j_l}\right) - S\left(\zeta_{j_s}\right)\right\|_{L^\infty\left(A_\eta\right)} \leq 2\mu$ for all $l, s \in \mathbb{N}$. Let $\Omega^\eta := \Omega \setminus A_\eta$. Since $\{u_{j_k}\}_{l\in\mathbb{N}}$ is uniformly convergent in $\overline{\Omega^\eta}$ then there exists $l_0 \in \mathbb{N}$ such that $\left\|S_{\varepsilon_{j_l}}\left(\zeta_{j_l}\right) - S\left(\zeta_{j_s}\right)\right\|_{L^\infty\left(\overline{\Omega^\eta}\right)} \leq 2\mu$ for $l \geq l_0$ and $s \geq l_0$. Then $\{u_{j_k}\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $C\left(\overline{\Omega}\right)$, and so $\{u_{j_k}\}_{l\in\mathbb{N}}$ converges in $C\left(\overline{\Omega}\right)$. □

3 A priori estimates

We assume for the whole section that *H1)–H5)* of Theorem 1.1 are satisfied.

Remark 3.1. If $v \in C^1\left(\mathbb{R}^n\right)$ satisfies $-\Delta v \geq 0$ in $D'\left(\mathbb{R}^n\right)$, $v \geq 0$ in $\mathbb{R}^n$ and $v\left(x_0\right) > 0$ for some $x_0 \in \mathbb{R}^n$, then $v\left(x\right) > 0$ for all $x \in \mathbb{R}^n$. Indeed, let $\varphi \in C_c^\infty\left(\mathbb{R}^n\right)$ be a nonnegative radial function with support in the unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$, and such that $\int_B \varphi = 1$. For $\varepsilon > 0$ let $\varphi_\varepsilon\left(x\right) := \varepsilon^{-n}\varphi\left(\varepsilon^{-1}x\right)$ and let $\left(\varphi_\varepsilon\right)^\vee\left(x\right) := \varphi_\varepsilon\left(-x\right)$. Then $v * \varphi_\varepsilon \in C^\infty\left(\mathbb{R}^n\right)$. A computation gives

$$\int_{\mathbb{R}^n} \psi\left(-\Delta\left(v * \varphi_\varepsilon\right)\right) = \left\langle -\Delta v, \left(\left(\varphi_\varepsilon\right)^\vee * \psi\right)\right\rangle \geq 0$$

for all nonnegative $\psi \in C_c^\infty\left(\mathbb{R}^n\right)$. Then $-\Delta\left(v * \varphi_\varepsilon\right) \geq 0$ in $\mathbb{R}^n$, and so $u * \varphi_\varepsilon$ is a C^∞ superharmonic function on $\mathbb{R}^n$. Thus

$$\left(v * \varphi_\varepsilon\right)\left(x\right) \geq \frac{1}{\alpha\left(n\right) r^n} \int_{B_r(x)} \left(v * \varphi_\varepsilon\right)\left(y\right) dy$$

for all $x \in \mathbb{R}^n$, $\varepsilon > 0$ and $r > 0$. We have also (see e.g., [5], Theorem 4.22) $\lim_{\varepsilon\to 0^+}\left(v * \varphi_\varepsilon\right)\left(x\right) = v\left(x\right)$, and since $0 \leq \left(v * \varphi_\varepsilon\right)\left(y\right) \leq \|v\|_{L^\infty(B_r(x))}$ for any $y \in B_r\left(x\right)$, $r > 0$ and $\varepsilon > 0$, Lebesgue's dominated convergence theorem gives that $v\left(x\right) \geq \frac{1}{\alpha(n)r^n}\int_{B_r(x)} v\left(y\right) dy$ for $x \in \mathbb{R}^n$, Now we take r such that $x_0 \in B_r\left(x\right)$ to obtain $v\left(x\right) > 0$ for any $x \in \mathbb{R}^n$.

The following lemma is an adaptation, suitable for our purpose here, of the blow up method developed in [27], to obtain a priori estimates for the L^∞ norm of solutions to subcritical superlinear elliptic problems. For the convenience of the reader, and as our statement is somewhat different to that in Theorem 1.1 of [27], we provide a detailed proof of it.

For $r > 0$, and $x \in \mathbb{R}^n$, we will write $B_r\left(x\right)$ (respectively $\overline{B}_r\left(x\right)$) to denote the open (resp. closed) ball in $\mathbb{R}^n$ of radius r and centered at x.

Lemma 3.2. *Let Θ be an equibounded family of nonnegative measurable functions in $L^\infty\left(\Omega\right)$, and let $\mathcal{G}$ be a family of nonnegative functions in $C\left(\overline{\Omega} \times [0,\infty)\right)$. Assume that there exist $p \in \left(1, \frac{n+2}{n-2}\right)$, and $h \in C\left(\overline{\Omega}\right)$, such that $\min_{\overline{\Omega}} h > 0$ and $\lim_{s\to\infty} \frac{g(x,s)}{s^p} = h\left(x\right)$ uniformly on $g \in \mathcal{G}$ and $x \in \overline{\Omega}$. Then there exists a constant C such that $\|u\|_\infty < C$ whenever $u \in H_0^1\left(\Omega\right) \cap L^\infty\left(\Omega\right)$ is a weak solution, for* *some $\theta \in \Theta$ and $g \in \mathcal{G}$, to the problem*

$$\begin{cases} -\Delta u = \theta + g(\cdot, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases}$$

Proof. To prove the lemma we proceed by contradiction. Suppose that for any $k \in \mathbb{N}$ there exist $\theta_k \in \Theta$, $g_k \in \mathcal{G}$, and a weak solution $u_k \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to the problem

$$\begin{cases} -\Delta u_k = \theta_k + g_k(\cdot, u_k) & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \\ u_k > 0 & \text{in } \Omega \end{cases} \tag{3.1}$$

such that $\lim_{k\to\infty} \|u_k\|_\infty = \infty$. Let $f_k : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ be defined by $f_k(x,s) := \theta_k(x) + g_k(x,s)$. Since $u_k \in L^\infty(\Omega)$ we have $f_k(\cdot, u_k) \in L^\infty(\Omega)$, and so $u_k \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ for any $r \in (1,\infty)$. Thus $u_k \in C(\overline{\Omega})$, and u_k is a strong solution of (3.1). Let $P_k \in \Omega$ be such that $\|u_k\|_\infty = u_k(P_k)$. Taking a subsequence if necessary, we can assume that $\lim_{k\to\infty} P_k = P$ for some $P \in \overline{\Omega}$.

Case a): $P \in \Omega$: Let $d := \frac{1}{4} d_\Omega(P)$, $M_k := u_k(P_k)$ and $\sigma_k := M_k^{-\frac{p-1}{2}}$. Then $\sigma_k^{\frac{2}{p-1}} M_k = 1$ and $\lim_{k\to\infty} \sigma_k = 0$. For k large enough we have $\|P_k - P\| < d$ and $\sigma_k < 1$. Thus $\|\sigma_k y + P_k - P\| \le \|\sigma_k y\| + \|P_k - P\| < 2d$. Then, taking a further subsequence if necessary, we can assume that $\sigma_k y + P_k \in B_{2d}(P) \subset \Omega$ for any $y \in B_{\sigma_k^{-1} d}$ and $k \in \mathbb{N}$. Let $v_k : \overline{B}_{\sigma_k^{-1} d}(0) \to \mathbb{R}$ be defined by

$$v_k(y) := \sigma_k^{\frac{2}{p-1}} u_k(\sigma_k y + P_k).$$

Then $v_k \in W^{2,r}\left(B_{\sigma_k^{-1} d}(0)\right)$ for any $r \in (1,\infty)$, and so $v_k \in C\left(\overline{B_{\sigma_k^{-1} d}(0)}\right)$. Also, $v_k \le \sigma_k^{\frac{2}{p-1}} M_k = 1$ in $\overline{B_{\sigma_k^{-1} d}(0)}$, and $v_k(0) = \sigma_k^{\frac{2}{p-1}} M_k = 1$. Therefore

$$\|v_k\|_{L^\infty\left(B_{\sigma_k^{-1} d}(0)\right)} = 1.$$

From (3.1), a computation shows that $-\Delta v_k = F_k$ in $B_{\sigma_k^{-1} d}(0)$, with F_k defined by $F_k(y) := \sigma_k^{\frac{2p}{p-1}} f_k(\sigma_k y + P_k, u_k(\sigma_k y + P_k))$. Also, $\lim_{k\to\infty} \sigma_k = 0$, and so, for $R > 0$, there exists $k(R) \in \mathbb{N}$ such that $B_{2R}(0) \subset B_{\sigma_k^{-1} d}(0)$ for $k \ge k(R)$. Our assumptions on Θ and $\mathcal{G}$ imply that there exists a positive constant c such that $f_k(x,s) \le c(s^p + 1)$ for any $(x,s) \in \overline{\Omega} \times [0,\infty)$ and $k \in \mathbb{N}$. Then, for $y \in B_{2R}(0)$ and $k \ge k(R)$, we have, for some positive constant c' independent of y and k,

$$0 \le F_k(y) = \sigma_k^{\frac{2p}{p-1}} f_k(\sigma_k y + P_k, u_k(\sigma_k y + P_k)) \le c \sigma_k^{\frac{2p}{p-1}} \left(1 + M_k^p\right) \le c'. \tag{3.2}$$

Thus $\|F_k\|_{L^\infty(B_{2R}(0))} \le c'$ for $k \ge k(R)$. Also $\|v_k\|_{L^\infty(B_{2R}(0))} = 1$. Thus, since $-\Delta v_k = F_k$ in $B_{2R}(0)$, the standard inner elliptic estimates (as stated, e.g., in [8, Proposition 4.1.2]), imply that $\|v_k\|_{W^{2,r}(B_R(0))} \le c_r''$ for any $r > n$ and $k \ge k(R)$, with c_r'' a positive constant independent of k. Therefore there exists a subsequence, still denoted by $\{v_k\}_{k\in\mathbb{N}}$, that converges in $C^{1,\gamma}\left(\overline{B_R(0)}\right)$ for some $\gamma \in (0,1)$. Let $\{R_l\}_{l\in\mathbb{N}}$ be an increasing sequence such that $\lim_{l\to\infty} R_l = \infty$. A Cantor diagonal process gives a further subsequence, still denoted by

$\{v_k\}_{k\in\mathbb{N}}$, and a function $v \in C^1(\mathbb{R}^n)$ such that $\{v_k\}_{k\in\mathbb{N}}$ converges to v, in the C^1 norm, on each compact subset of $\mathbb{R}^n$. Moreover, $v \geq 0$ in $\mathbb{R}^n$, $v(0) = 1$ and $\|v\|_{L^\infty(\mathbb{R}^n)} = 1$. Note that, for each l, and for k large enough, $-\Delta v_k = F_k \geq 0$ in $B_{R_l}(0)$, then $-\Delta v \geq 0$ in $D'(\mathbb{R}^n)$. Since $v \geq 0$ in $\mathbb{R}^n$ and $v(0) = 1$, we have, by Remark 3.1, $v(x) > 0$ for any $x \in \mathbb{R}^n$.

From our assumptions on the family $\mathcal{G}$ we have $f_k(x,s) = \theta_k(x) + s^p h(x) + s^p \psi_k(x,s)$, with $\lim_{s\to\infty} \psi_k(x,s) = 0$ uniformly on $x \in \overline{\Omega}$ and $k \in \mathbb{N}$. Then, for $R > 0$, $y \in B_R(0)$ and $k \geq k(R)$,

$$\begin{aligned} 0 \leq F_k(y) &= \sigma_k^{\frac{2p}{p-1}} f_k(\sigma_k y + P_k, u_k(\sigma_k y + P_k)) \\ &= \sigma_k^{\frac{2p}{p-1}} \theta_k(\sigma_k y + P_k) + (v_k(y))^p h(\sigma_k y + P_k) \\ &\quad + v_k^p(y)\, \psi_k\left(\sigma_k y + P_k, \sigma_k^{-\frac{2}{p-1}} v_k(y)\right). \end{aligned}$$

Now, $\lim_{k\to\infty} v_k(y) = v(y) > 0$ for any $y \in \mathbb{R}^n$, then $\lim_{k\to\infty} \sigma_k^{-\frac{2}{p-1}} v_k(y) = \infty$, and so, taking into account that $\lim_{s\to\infty} \psi_k(x,s) = 0$ uniformly on $x \in \overline{\Omega}$ and $k \in \mathbb{N}$, we get, for $y \in B_R(0)$,

$$\lim_{k\to\infty} \psi_k(\sigma_k y + P_k, u_k(\sigma_k y + P_k)) = \lim_{k\to\infty} \psi_k\left(\sigma_k y + P_k, \sigma_k^{-\frac{2}{p-1}} v_k(y)\right) = 0.$$

Also, $\lim_{k\to\infty} \sigma_k^{\frac{2p}{p-1}} \theta_k(\sigma_k y + P_k) = 0$. Then $\lim_{k\to\infty} F_k(y) = h(P)\, v^p(y)$ for $y \in B_R(0)$ and, from (3.2), $\sup_k \|F_k\|_{L^\infty(B_R(0))} < \infty$ for $k \geq k(R)$. Thus $\{F_k\}_{k\in\mathbb{N}}$ converges to $h(P)\, v^p$ in $D'(\mathbb{R}^n)$ and so v satisfies $-\Delta v = h(P)\, v^p$ in $D'(\mathbb{R}^n)$. Also, $v > 0$ in $\mathbb{R}^n$, and $\|v\|_{L^\infty(\mathbb{R}^n)} = 1$. Then, by elliptic regularity theory (see e.g., [8, Proposition 4.1.2]), $v \in W^{2,r}_{\text{loc}}(\mathbb{R}^n)$ for any $r \in (1,\infty)$, and v satisfies, in strong sense, $-\Delta v = h(P)\, v^p$ in $\mathbb{R}^n$. Let $\eta := (h(P))^{\frac{1}{1-p}}$, and let $w := \eta v$. Thus $\eta > 0$, $w \in W^{2,r}_{\text{loc}}(\mathbb{R}^n)$ for any $r \in (1,\infty)$, and w is a bounded positive strong solution to the problem $-\Delta w = w^p$ in $\mathbb{R}^n$. Moreover, for each open ball $B \subset \mathbb{R}^n$ we have $w^p \in C^\gamma(U)$ for some $\gamma \in (0,1)$. Then, by [28, Theorem 9.19], $w \in C^2(\mathbb{R}^n)$. But Theorem 1.2 in [27] says that such a solution w does not exist. Contradiction.

Case b): $P \in \partial\Omega$: Since Ω is a C^2 domain, there exists an open ball $B = B_r(P)$ with radius $r > 0$, centered at P; and a one to one mapping $\Phi = \Phi(x) = (\Phi_1(x), \ldots, \Phi_n(x))$ from B onto a bounded open set $D \subset \mathbb{R}^n$ such that i) $\Phi(B \cap \Omega) \subset \mathbb{R}^n_+$, ii) $\Phi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$ iii) $\Phi \in C^2(B)$, $\Phi^{-1} \in C^2(D)$; where $\mathbb{R}^n_+$ denotes the open upper halfspace $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ (see e.g., [28, p. 94]). After compositions with a suitable translation, and with a linear endomorphism, we can assume $\Phi(P) = 0$, and that $\Phi'(P)$ (the Jacobian matrix of Φ at P) is an orthogonal matrix. Diminishing B and D if necessary, we can also assume that $\Phi \in C^2(\overline{B})$ and $\Phi^{-1} \in C^2(\overline{D})$.

For $k \in \mathbb{N}$ and $y \in \Phi(B \cap \Omega)$, let $z_k(y) := u_k(\Phi^{-1}(y))$. Since $u_k \in W^{2,r}(\Omega)$ for any $r \in (1,\infty)$, we have $z_k \in W^{2,r}(\Phi(B \cap \Omega))$ for $1 < r < \infty$. From (3.1), a computation using the chain rule shows that z_k satisfies

$$\begin{aligned} -\sum_{1\leq i,m\leq n} a_{i,m}(y) \frac{\partial^2 z_k}{\partial y_i \partial y_m}(y) + \sum_{1\leq m\leq n} b_m(y) \frac{\partial z_k}{\partial y_m}(y) \\ = f_k\left(\Phi^{-1}(y), z_k(y)\right) \quad \text{for } y \in \Phi(B \cap \Omega), \end{aligned} \tag{3.3}$$

where each $a_{i,m} \in C^1\left(\overline{\Phi(B\cap\Omega)}\right)$, $A := (a_{i,m})$ is uniformly elliptic on $\overline{\Phi(B\cap\Omega)}$; and $b = (b_1, \dots, b_n) \in C(\overline{\Phi(B\cap\Omega)}, \mathbb{R}^n)$. Moreover, a computation gives

$$a_{i,m}(y) = \sum_{\gamma=1}^{n} \frac{\partial \Phi_i}{\partial x_\gamma}\left(\Phi^{-1}(y)\right) \frac{\partial \Phi_m}{\partial x_\gamma}\left(\Phi^{-1}(y)\right)$$

and so, in particular, $A(P) = I$.

For k large enough $P_k \in B\cap\Omega$. For such k, let

$$\delta_k := \text{dist}\left(\Phi(P_k), \partial\Phi(B\cap\Omega)\right).$$

Note that for k sufficiently large, $\delta_k = \langle \Phi(P_k), e_n\rangle$ where $e_n = (0, \dots, 0, 1)$. Then, taking a subsequence if necessary, we can assume that $P_k \in B\cap\Omega$ and $\delta_k = \langle\Phi(P_k), e_n\rangle$ for all $k \in \mathbb{N}$. Define M_k and σ_k as in case a). Then $z_k(\Phi(P_k)) = u_k(P_k) = M_k$, $\sigma_k^{\frac{2}{p-1}} M_k = 1$, and $\lim_{k\to\infty} \sigma_k = 0$.

For $\delta > 0$ such that $[-4\delta, 4\delta]^n \cap \mathbb{R}^n_+ \subset \Phi\left(B\cap\overline{\Omega}\right)$ and for $k \in \mathbb{N}$, let $Q_k := \left(-\sigma_k^{-1}\delta, \sigma_k^{-1}\delta\right)^{n-1} \times \left(-\sigma_k^{-1}\delta_k, \sigma_k^{-1}\delta\right)$. For k large enough $\sigma_k y + \Phi(P_k) \in \Phi\left(\overline{\Omega}\cap B\right)$ for any $y \in \overline{Q}_k$ and so, taking a further subsequence, we can assume that $\sigma_k y + \Phi(P_k) \in \Phi\left(\overline{\Omega}\cap B\right)$ for any $y \in \overline{Q}_k$ and $k \in \mathbb{N}$. For $k \in \mathbb{N}$, let $v_k : \overline{Q}_k \to \mathbb{R}$ be defined by

$$v_k(y) := \sigma_k^{\frac{2}{p-1}} z_k\left(\sigma_k y + \Phi(P_k)\right).$$

Then $v_k \in C\left(\overline{Q}_k\right)$, $v_k = 0$ on $\left[-\sigma_k^{-1}\delta, \sigma_k^{-1}\delta\right]^{n-1} \times \left\{-\sigma_k^{-1}\delta_k\right\}$, $v_k \le \sigma_k^{\frac{2}{p-1}} M_k = 1$ in $\overline{Q}_k$, and $v_k(0) = \sigma_k^{\frac{2}{p-1}} M_k = 1$. Thus $\|v_k\|_{L^\infty(Q_k)} = 1$. Also, $v_k \in W^{2,r}(Q_k)$ for $1 < r < \infty$. From (3.3), a computation shows that, for $y \in Q_k$ and $k \in \mathbb{N}$,

$$-\sum_{1\le m,i\le n} \alpha_{i,m,k}(y) \frac{\partial^2 v_k}{\partial y_q \partial y_m}(y) + \sum_{1\le m\le n} \beta_{m,k}(y) \frac{\partial v_k}{\partial y_m}(y) = \sigma_k^{\frac{2p}{p-1}} f_k\left(\Phi^{-1}\left(\sigma_k y + \Phi(P_k)\right), \sigma_k^{-\frac{2}{p-1}} v_k(y)\right) \quad (3.4)$$

where $\alpha_{i,m.k}(y) := a_{i,m}\left(\sigma_k y + \Phi(P_k)\right)$, and $\beta_{m.k}(y) := \sigma_k b_m\left(\sigma_k y + \Phi(P_k)\right)$.

Note that $\left\{\sigma_k^{-1}\delta_k\right\}$ is bounded from above. Indeed, if $\sup_{k\in\mathbb{N}} \sigma_k^{-1}\delta_k = \infty$ then there exists a subsequence $\left\{\sigma_{k_q}^{-1}\delta_{k_q}\right\}_{q\in\mathbb{N}}$ such that $\lim_{q\to\infty} \sigma_{k_q}^{-1}\delta_{k_q} = \infty$. Since v_{k_q} is well defined on $B_{\sigma_{k_q}^{-1}\delta_{k_q}}(0)$ and $v_{k_q}(0) = 1$, the same arguments of the case a) apply to obtain a positive and bounded solution $v \in C^2(\mathbb{R}^n)$ of

$$-\sum_{1\le m,i\le n} \alpha_{i,m}(\Phi(P)) \frac{\partial^2 v}{\partial y_q \partial y_m}(y) = h(P)\, v^p(y) \quad \text{in } \mathbb{R}^n.$$

Now, $A(P)$ is a symmetric and positive matrix, and then there exists an invertible matrix B such that $BA(P)B^t = I$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Ty = yB^t$. Thus $w := (h(P))^{\frac{1}{1-p}} v \circ T$ satisfies $-\Delta w = w^p$ in $\mathbb{R}^n$ which contradicts Theorem 1.2 in [27].

Observe also that $\inf_{k\in\mathbb{N}} \sigma_k^{-1}\delta_k > 0$. For, if not, taking a subsequence, we can assume $\lim_{k\to\infty} \sigma_k^{-1}\delta_k = 0$ and, taking a further subsequence if necessary, we can also assume that $\sigma_k^{-1}\delta_k < 1$, $\sigma_k < 1$, and $\sigma_k^{-1}\delta > 4$, for any $k \in \mathbb{N}$. Let $E_k := (-\delta, \delta)^{n-1} \times \left(0, \sigma_k^{-1}\delta + \sigma_k^{-1}\delta_k\right)$ and, for $y \in (-\delta, \delta)^{n-1} \times [0, \sigma_k^{-1}\delta + \sigma_k^{-1}\delta_{j_{k_q}})$, let $\widetilde{v}_k(y) = v_k(y - y_k)$ with $y_k := (0', \sigma_k^{-1}\delta_k)$ where, as before, $0'$ denotes the origin of $\mathbb{R}^{n-1}$.

From (3.4) we have, for $y \in E_k$,

$$-\sum_{1\leq i,m\leq n} \widetilde{a}_{i,m,k}(y)\frac{\partial^2 \widetilde{v}_k}{\partial y_q \partial y_m}(y) + \sum_{1\leq m\leq n} \widetilde{b}_{m,k}(y)\frac{\partial \widetilde{v}_k}{\partial y_m}(y) = \varphi_k(y),$$

where $\widetilde{a}_{i,m,k}(y) := \alpha_{i,m,k}(y - y_k)$, $\widetilde{b}_{m,k}(y) := \beta_{m,k}(y - y_k)$, and

$$\varphi_k(y) := \sigma_k^{\frac{2p}{p-1}} f_k\left(\Phi^{-1}(\sigma_k(y - y_k) + \Phi(P_k)), \sigma_k^{-\frac{2}{p-1}} v_k(y - y_k)\right).$$

Let $Q := (-\delta, \delta)^{n-1} \times (0, 4)$, $\Gamma := (-\delta, \delta)^{n-1} \times \{0\}$, and $Q' := \left(-\frac{\delta}{2}, \frac{\delta}{2}\right)^{n-1} \times [0, 2)$. Then $Q' \subset\subset Q \cup \Gamma$, and $Q \subset E_k$. Let $\widetilde{A}_k(y)$ be the $n \times n$ matrix whose (i, m) entry is $\widetilde{a}_{i,m,k}(y)$. Then $\widetilde{A}_k$ is uniformly elliptic on $\overline{Q}$, $\|\widetilde{a}_{i,m,k}\|_{L^\infty(Q)} \leq c$ for $1 \leq i, m \leq n$, with c a positive constant independent of k. Also, the ellipticity constants of $\widetilde{A}_k$ and a modulus of continuity of its coefficients can be chosen independent of k. In addition, $\|\widetilde{b}_{m,k}\|_{L^\infty(Q)} \leq c'$, $\|\widetilde{v}_k\|_{L^\infty(Q)} \leq c'$ for some constant c' independent of k and, as in case a), there exists a positive constant c'' independent of k such that $\|\varphi_k\|_{L^\infty(Q)} \leq c'$. Let $r > n$. By elliptic regularity up to the boundary (see e.g., [28, Theorem 9.13]), there exists a positive constant c_r such that $\|\widetilde{v}_k\|_{W^{2,r}(Q)} \leq c_r$ for any k. Then there exists a positive constant γ such that $\left|v_k(0) - v_k(0', -\sigma_k^{-1}\delta_k)\right| = \left|\widetilde{v}_k(0, \sigma_k^{-1}\delta_k) - \widetilde{v}_k(0', 0)\right| \leq \gamma\sigma_k^{-1}\delta_k$ for any k, i.e., $1 \leq \gamma\sigma_k^{-1}\delta_k$, which contradicts $\lim_{k\to\infty} \sigma_k^{-1}\delta_k = 0$. Then $\inf_{k\in\mathbb{N}} \sigma_k^{-1}\delta_k > 0$.

Thus $\{\sigma_k^{-1}\delta_k\}$ is bounded from above and from below by positive constants, and so, taking a subsequence if necessary, we can assume $\lim_{k\to\infty} \sigma_k^{-1}\delta_k = \tau$, for some $\tau > 0$.

For $k \in \mathbb{N}$, let $w_k : \left[-\sigma_k^{-1}\delta, \sigma_k^{-1}\delta\right]^{n-1} \times [0, \sigma_k^{-1}\delta + \sigma_k^{-1}\delta_k) \to \mathbb{R}$ be given by $w_k(y) := v_k(y - y_k)$, with $y_k := (0', \sigma_k^{-1}\delta_k)$, where $0'$ denotes the origin in $\mathbb{R}^{n-1}$. Thus, w_k satisfies, for $y \in (-\sigma_k\delta, -\sigma_k\delta)^{n-1} \times (0, \sigma_k\delta + \sigma_k^{-1}\delta_k)$,

$$-\sum_{1\leq m,l\leq n} \widetilde{\alpha}_{l,m,k}(y)\frac{\partial^2 w_k}{\partial y_l \partial y_m}(y) + \sum_{1\leq m\leq n} \widetilde{\beta}_{m,k}(y)\frac{\partial w_k}{\partial y_m}(y) = \widetilde{\varphi}_k(y),$$

where $\widetilde{\alpha}_{l,m,k}(y) := \alpha_{l,m,k}(y - y_k)$, $\widetilde{\beta}_{m,k}(y) := \beta_{m,k}(y - y_k)$ and

$$\widetilde{\varphi}_k(y) := \sigma_k^{\frac{2p}{p-1}} f_k\left(\Phi^{-1}(\sigma_k(y - y_k) + \Phi(P_k)), \sigma_k^{-\frac{2}{p-1}} v_k(y - y_k)\right).$$

By repeating compactness arguments used in the case a), and taking into account that, for $y \in \mathbb{R}^n_+$, $\lim_{k\to\infty} \widetilde{\alpha}_{l,m,k}(y) = a_{l,m}(P)$, $\lim_{l\to\infty} \widetilde{\beta}_{m,l}(y) = 0$, and $A(P) = I$ we obtain a subsequence, still denoted $\{w_k\}_{k\in\mathbb{N}}$, that converges in $\mathbb{R}^n_+$ to a function $w \in C^2(\mathbb{R}^n_+)$ such that $w > 0$ in $\mathbb{R}^n_+$, $w(0', \tau) = 1$, and

$$-\Delta w(y) = h(P)\, w^p(y) \quad \text{in } \mathbb{R}^n_+. \tag{3.5}$$

For $R > 0$, let $U^R := B_R^{n-1}(0) \times (0, R)$, where $B_R^{n-1}(0)$ denotes the open ball in $\mathbb{R}^{n-1}$ of radius R and centered at the origin. Let $r > n$. As above, by elliptic regularity up to the boundary [28, Theorem 9.13], we have $\|w_k\|_{W^{2,r}(U^R)} \leq c_r$ for some positive constant c_r independent of k. Thus, taking a further subsequence, still denoted $\{w_k\}_{k\in\mathbb{N}}$, we have that $\{w_k\}_{k\in\mathbb{N}}$ converges uniformly on $\overline{U}$. Now, by considering an increasing sequence of radius $\{R_j\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} R_j = \infty$, a Cantor diagonal process gives a further subsequence, still denoted by $\{w_k\}_{k\in\mathbb{N}}$, which converges uniformly on K, for each compact subset $K \subset \mathbb{R}^n_+$. Then w belongs to $C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ and, since $w_k = 0$ on $\partial\mathbb{R}^n_+$ for each k, we also have also $w = 0$ on $\partial\mathbb{R}^n_+$. Therefore $\widetilde{w} := h(P)^{\frac{1}{1-p}} w$ belongs to $C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ and satisfies $-\Delta\widetilde{w} = \widetilde{w}^p$ in $\mathbb{R}^n_+$, $\widetilde{w} = 0$ on $\partial\mathbb{R}^n_+$ and $\widetilde{w} > 0$ in $\mathbb{R}^n_+$, which contradicts Theorem 1.3 in [27]. □

Next we use Lemma 3.2 to obtain a priori estimates for the L^∞ norm of solutions to subcritical superlinear elliptic problems (in particular of solutions to the singular problem that arises when $\varepsilon = 0$).

Lemma 3.3. *Assume the hypothesis of Theorem 1.1. Then there exists $\lambda^* > 0$ such that $\lambda \le \lambda^*$ whenever the problem*

$$\begin{cases} -\Delta u = a(u+\varepsilon)^{-\alpha} + f(\lambda,\cdot,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{3.6}$$

has a weak solution $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ for some $\varepsilon \in [0,1]$.

Proof. By Lemmas 2.7 and 2.12 we have $u \ge c d_\Omega$ in Ω, with c independent of λ, ε, and of the particular solution u. Let η_0, q, and b, be as in *H4)*. Note that $\lambda \le \max\{\eta_0, \lambda_1(b(cd_\Omega)^{q-1})\}$. Indeed, if $\lambda \ge \eta_0$, by *H4)*, $f(\lambda,\cdot,u) \ge \lambda b u^q \ge \lambda b (cd_\Omega)^{q-1} u$ in Ω, and so, in weak sense, $-\Delta u = \lambda b(cd_\Omega)^{q-1} u + \rho$ in Ω, with $\rho := a(u+\varepsilon)^{-\alpha} + f(\lambda,\cdot,u) - \lambda b(cd_\Omega)^{q-1} u$. Observe that $0 \le \rho \in L^\infty_{\text{loc}}(\Omega)$, and that $\rho \not\equiv 0$ (because $a(u+\varepsilon)^{-\alpha} \not\equiv 0$). Then, by Remark 2.2 iv), $\lambda \le \lambda_1(b(cd_\Omega)^{q-1})$. Thus $\lambda \le \lambda^* := \max\{\eta_0, \lambda_1(b(cd_\Omega)^{q-1})\}$. □

Lemma 3.4. *Assume the hypothesis of Theorem 1.1. Then, for any $\lambda_0 > 0$ there exists $c_{\lambda_0} > 0$ such that $\|u\|_\infty < c_{\lambda_0}$ whenever $\lambda \ge \lambda_0$ and $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ is a weak solution, for some $\varepsilon \in [0,1]$, of problem (3.6).*

Proof. To prove the lemma we proceed by contradiction. Suppose that there exist sequences $\{\lambda_j\}_{j\in\mathbb{N}} \subset [\lambda_0,\infty)$, $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset [0,1]$, $\{u_j\}_{j\in\mathbb{N}} \subset H^1_0(\Omega) \cap L^\infty(\Omega)$ such that, for all $j \in \mathbb{N}$,

$$\begin{cases} -\Delta u_j = a(u_j+\varepsilon_j)^{-\alpha} + f(\lambda_j,\cdot,u_j) & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega, \\ u_j > 0 & \text{in } \Omega, \end{cases} \tag{3.7}$$

and $\lim_{j\to\infty}\|u_j\|_\infty = \infty$. Let $\lambda^* > 0$ be as given by Lemma 3.3. Thus $\lambda_j \le \lambda^*$ for all j. Then $\{\lambda_j\}_{j\in\mathbb{N}}$ is bounded and so, taking a subsequence if necessary, we can assume that $\lim_{j\to\infty}\lambda_j = \lambda$ for some $\lambda \in [\lambda_0,\lambda^*]$. Since $u_j \in L^\infty(\Omega)$ we have $f(\lambda_j,\cdot,u_j) \in L^\infty(\Omega)$, and, since $u_j = S_{\varepsilon_j}(f(\lambda_j,\cdot,u_j))$, we have $u_j \in C(\overline{\Omega})$.

Let ψ_1 and ψ_2 be nonnegative functions in $C^\infty_c(\mathbb{R})$ such that $\psi_1 \equiv 1$ on $[-\infty,\frac{1}{2}]$, $\operatorname{supp}(\psi_1) \subset (-\infty,2)$, $\psi_2 \equiv 1$ on $[2,\infty)$, $\operatorname{supp}(\psi_2) \subset (\frac{1}{2},\infty)$, and $\psi_1 + \psi_2 \equiv 1$ on $\mathbb{R}$. Let $w_j \in H^1_0(\Omega) \cap L^\infty(\Omega)$ be the solution, provided by Lemma 2.6 when $\varepsilon_j > 0$, and by Lemma 2.12 when $\varepsilon_j = 0$ (applied with $\varepsilon = \varepsilon_j$ and with a replaced by $a(\psi_1 \circ u_j)$) to the problem

$$\begin{cases} -\Delta w_j = a(\psi_1\circ u_j)(w_j+\varepsilon_j)^{-\alpha} & \text{in } \Omega, \\ w_j = 0 & \text{on } \partial\Omega, \\ w_j > 0 & \text{in } \Omega. \end{cases} \tag{3.8}$$

From Lemma 2.10, applied with $\varepsilon = \varepsilon_j$, and with a replaced by $a\left(\psi_1 \circ u_j\right)$, we have that $w_j \in C\left(\overline{\Omega}\right)$. Notice that, by Lemma 2.14 i), $w_j \leq \widetilde{w}$ in Ω, where $\widetilde{w} \in H_0^1\left(\Omega\right)$ is the weak solution, given by Lemma 2.12 (applied with a replaced by $a\left(\psi_1 \circ u_j\right)$), to the problem

$$\begin{cases} -\Delta \widetilde{w} = a\left(\psi_1 \circ u_j\right) \widetilde{w}^{-\alpha} & \text{in } \Omega, \\ \widetilde{w} = 0 & \text{on } \partial\Omega, \\ \widetilde{w} > 0 & \text{in } \Omega. \end{cases} \tag{3.9}$$

By Lemma 2.10, $\widetilde{w} \in C\left(\overline{\Omega}\right)$, then $\left\{w_j\right\}_{j \in \mathbb{N}}$ is bounded in $L^\infty\left(\Omega\right)$. Also, in weak sense, $-\Delta u_j = a\left(u_j + \varepsilon_j\right)^{-\alpha} + f\left(\lambda_j, \cdot, u_j\right) \geq a\left(\psi_1 \circ u_j\right)\left(u_j + \varepsilon_j\right)^{-\alpha}$ in Ω, $u_j = 0$ on $\partial\Omega$, and so, in weak sense,

$$\begin{aligned} -\Delta\left(u_j - w_j\right) &\geq a\left(\psi_1 \circ u_j\right)\left(\left(u_j + \varepsilon_j\right)^{-\alpha} - \left(w_j + \varepsilon_j\right)^{-\alpha}\right) \quad \text{in } \Omega, \\ u_j - w_j &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.10}$$

Using $\left(u_j - w_j\right)^-$ as test function in (3.10) we get $-\int_\Omega \left|\nabla\left(\left(u_j - w_j\right)^-\right)\right| \geq 0$, and then $u_j \geq w_j$ in Ω.

We claim that $a\left(\psi_1 \circ u_j\right)\left(u_j + \varepsilon_j\right)^{-\alpha} \varphi \in L^1\left(\Omega\right)$ for any $\varphi \in H_0^1\left(\Omega\right)$, and that there exists a nonnegative weak solution $z_j \in H_0^1\left(\Omega\right) \cap C\left(\overline{\Omega}\right)$ to the problem

$$\begin{cases} -\Delta z_j = a\left(\psi_1 \circ u_j\right)\left(u_j + \varepsilon_j\right)^{-\alpha} & \text{in } \Omega, \\ z_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

To prove this, first observe that, for all $j \in \mathbb{N}$,

$$\left(u_j + \varepsilon_j\right)^{-\alpha} d_\Omega \in L^2\left(\Omega\right). \tag{3.12}$$

Indeed, (3.12) clearly holds when $\varepsilon_j > 0$. If $\varepsilon_j = 0$ and $0 < \alpha \leq 1$ then, by Lemma 2.7 there exists a positive constant c, independent of j, such that $\left(u_j + \varepsilon_j\right)^{-\alpha} d_\Omega = u_j^{-\alpha} d_\Omega \leq c d_\Omega^B 1 - \alpha \in L^2\left(\Omega\right)$. If $\varepsilon_j = 0$ and $1 < \alpha < 3$ then, by Lemma 2.11, we have $\left(u_j + \varepsilon_j\right)^{-\alpha} d_\Omega = u_j^{-\alpha} d_\Omega \leq c d_\Omega^{1 - \frac{2\alpha}{1+\alpha}}$ for some positive constant c independent of j, and, since $1 < \alpha < 3$, we have $1 - \frac{2\alpha}{1+\alpha} > -1$, and so $d_\Omega^{1 - \frac{2\alpha}{1+\alpha}} \in L^2\left(\Omega\right)$. Therefore (3.12) holds for all j. We next prove that, for $\varphi \in H_0^1\left(\Omega\right)$ and for all j, $a\left(\psi_1 \circ u_j\right)\left(u_j + \varepsilon_j\right)^{-\alpha} \varphi \in L^1\left(\Omega\right)$ and that the map $\varphi \to \int_\Omega a\left(\psi_1 \circ u_j\right)\left(u_j + \varepsilon_j\right)^{-\alpha} \varphi$ is continuous on $H_0^1\left(\Omega\right)$. Indeed, from (3.12) and the Hardy inequality, we have, for some positive constant c,

$$\begin{aligned} \int_\Omega \left|a\left(\psi_1 \circ u_j\right)\left(u_j + \varepsilon_j\right)^{-\alpha} \varphi\right| &\leq \|a\|_\infty \int_\Omega \left(u_j + \varepsilon_j\right)^{-\alpha} d_\Omega \left|\frac{\varphi}{d_\Omega}\right| \\ &\leq c \left\|\left(u_j + \varepsilon_j\right)^{-\alpha} d_\Omega\right\|_2 \|\nabla\varphi\|_2. \end{aligned}$$

Then, by the Riesz theorem, there exists a weak solution $z_j \in H_0^1\left(\Omega\right)$ to (3.11), and, by the weak maximum principle, $z_j \geq 0$ a.e. in Ω. Since $u_j \geq w_j$ in Ω, from (3.8) and (3.11) we have

$$\begin{cases} -\Delta\left(z_j - w_j\right) \leq 0 & \text{in } \Omega \\ z_j - w_j = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

and so $z_j \le w_j$ in Ω. Also, $z_j \ge 0$, and $w_j \le \widetilde{w}$ in Ω. Thus $\sup_{j\in\mathbb{N}} \|z_j\|_\infty \le \|\widetilde{w}\|_\infty < \infty$. Now, $u_j \ge c d_\Omega$ in Ω for some positive constant c independent of j, and then, for any domain Ω' such that $\overline{\Omega'} \subset \Omega$, $a\left(\psi_1 \circ u_j\right)\left(u_j + \varepsilon_j\right)^{-\alpha} \in L^\infty(\Omega')$. Using that $z_j \in L^\infty(\Omega')$, and the inner elliptic estimates, we conclude that $z_j \in C(\Omega)$. Since $0 \le z_j \le w_j$ and $w_j \in C(\overline{\Omega})$, then z_j is continuous on $\partial\Omega$ and so $z_j \in C(\overline{\Omega})$. Now,

$$\begin{cases} -\Delta\left(u_j - z_j\right) = \theta_j + f\left(\lambda_j, \cdot, u_j - z_j + z_j\right)\left(\psi_2 \circ u_j\right) & \text{in } \Omega, \\ u_j - z_j = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

with $\theta_j \in L^\infty(\Omega)$ defined by

$$\theta_j(x) := a(x)\,\psi_2\left(u_j(x)\right)\left(u_j(x) + \varepsilon_j\right)^{-\alpha} + f\left(\lambda_j, x, u_j(x)\right)\psi_1\left(u_j(x)\right).$$

Let $\widetilde{u}_j := u_j - z_j$. Since $u_j \ge w_j \ge z_j$ in Ω, we have $\widetilde{u}_j \ge 0$ in Ω. For $j \in \mathbb{N}$ let $g_j : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ be defined by $g_j(x,s) := f\left(\lambda_j, x, s + z_j(x)\right)\psi_2\left(s + z_j(x)\right)$. Thus $\widetilde{u}_j$ is a weak solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of the problem

$$\begin{cases} -\Delta\widetilde{u}_j = \theta_j + g_j\left(\cdot, \widetilde{u}_j\right) & \text{in } \Omega, \\ \widetilde{u}_j = 0 \quad \text{on } \partial\Omega, \\ \widetilde{u}_j \ge 0 \quad \text{in } \Omega. \end{cases} \tag{3.15}$$

Note that $\theta_j + g_j\left(\cdot, \widetilde{u}_j\right)$ is nonnegative and belongs to $L^1(\Omega)$. We claim that, for j large enough,

$$\theta_j + g_j\left(\cdot, \widetilde{u}_j\right) \not\equiv 0 \quad \text{in } \Omega. \tag{3.16}$$

To prove our claim we proceed by contradiction. Taking a subsequence if necessary, we can assume that $\theta_j + g_j\left(\cdot, \widetilde{u}_j\right) = 0$ in Ω for all j. Then, for all j, (3.15) gives $\widetilde{u}_j = 0$ in Ω, and so $u_j = z_j$. Also,

$$\begin{aligned} \theta_j + g_j\left(\cdot, \widetilde{u}_j\right) &= a\left(\psi_2 \circ u_j\right)\left(u_j + \varepsilon_j\right)^{-\alpha} + f\left(\lambda_j, \cdot, u_j\right)\left(\psi_1 \circ u_j\right) + f\left(\lambda_j, \cdot, u_j\right)\left(\psi_2 \circ u_j\right) \\ &= a\left(\psi_2 \circ u_j\right)\left(u_j + \varepsilon_j\right)^{-\alpha} + f\left(\lambda_j, \cdot, u_j\right), \end{aligned}$$

therefore $\theta_j + g_j\left(\cdot, \widetilde{u}_j\right) = 0$ implies $f\left(\lambda_j, \cdot, u_j\right) = 0$ in Ω. Let $P_j \in \Omega$ be such that $u_j\left(P_j\right) = \left\|u_j\right\|_\infty$. Then $f\left(\lambda_j, P_j, u_j\left(P_j\right)\right) = 0$ for any j. Taking a further subsequence we can assume that $\lim_{j\to\infty} P_j = P$ for some $P \in \overline{\Omega}$. Also $\lim_{\to\infty} \lambda_j = \lambda \ge \lambda_0$, and $\lim_{\to\infty} u_j\left(P_j\right) = \infty$. Then, from the uniform convergence in *H5)*, we get $\lim_{j\to\infty} u_j\left(P_j\right)^{-p} f\left(\lambda_j, P_j, u_j\left(P_j\right)\right) = h(\lambda, P) > 0$, which contradicts that $f\left(\lambda_j, \cdot, u_j\right) = 0$ for all j. Thus (3.16) holds.

From (3.16), (3.15), and the Hopf maximum principle in Remark 2.1 ii), we conclude that $\widetilde{u}_j > 0$ in Ω.

Finally, observe that, since $\operatorname{supp}(\psi_1) \subset (-\infty, 2)$ and $0 \le \psi_1 \le 1$, then the support of $\psi_1 \circ u_j$ is included in $\left\{x \in \Omega : u_j(x) \le 2\right\}$. Thus, for all j, $\left\|\theta_j\right\|_\infty \le \|a\|_\infty + \sup_{[0,\lambda^*]\times\overline{\Omega}\times[0,2]} f$. Also, noting that

$$g_j(x,s) = f\left(\lambda_j, \cdot, s + z_j(x)\right)\psi_2\left(s + z_j(x)\right),$$

that $z_j \in C(\overline{\Omega})$, and that f and ψ_2 are continuos, we conclude that $g \in C\left(\overline{\Omega} \times [0,\infty)\right)$. Since $\psi_2 \equiv 1$ on $[2,\infty)$, taking into account that $\lambda_j \le \lambda^*$, and the uniform convergence in *H5)*, we get $\lim_{s\to\infty} s^{-p} g_j(x,s) = h(\lambda, x)$ uniformly on j and $x \in \overline{\Omega}$. Then the families $\Theta := \left\{\theta_j\right\}_{j\in\mathbb{N}}$, and $\mathcal{G} := \left\{g_j\right\}_{j\in\mathbb{N}}$ satisfy the assumptions of Lemma 3.2, and so $\left\{\widetilde{u}_j\right\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. Since $\sup_{j\in\mathbb{N}} \left\|z_j\right\|_\infty < \infty$, and $u_j = \widetilde{u}_j + z_j$, we conclude that $\sup_{j\in\mathbb{N}} \left\|u_j\right\|_\infty < \infty$, contradiction. $\square$

4 Proofs of the main results

We assume for the whole section that *H1)–H5)* of Theorem 1.1 hold. Let

$$P := \left\{ \zeta \in C\left(\overline{\Omega}\right) : \zeta \geq 0 \text{ in } \Omega \right\}, \tag{4.1}$$

and for $\varepsilon \geq 0$, let $T_\varepsilon : [0, \infty) \times P \to C\left(\overline{\Omega}\right)$ be defined by

$$T_\varepsilon(\lambda, \zeta) := S_\varepsilon\left(f\left(\lambda, \cdot, \zeta + S_\varepsilon(0)\right)\right) - S_\varepsilon(0). \tag{4.2}$$

Since $0 \leq f \in C\left([0,\infty) \times \overline{\Omega} \times [0,\infty)\right)$ and, since by Lemma 2.10, $S_\varepsilon(0) \in C\left(\overline{\Omega}\right)$ then, for $\zeta \in P$ we have $0 \leq f\left(\lambda, \cdot, \zeta + S_\varepsilon(0)\right) \in L^\infty(\Omega)$, therefore the definition of $T_\varepsilon(\lambda, \zeta)$ makes sense. Moreover, since $f \geq 0$, Lemma 2.14 i) gives $T_\varepsilon(\lambda, \zeta) \in P$.

Remark 4.1. Observe that, for $\varepsilon \geq 0$, $\zeta \in P$, and $\lambda \geq 0$, we have $T_\varepsilon(\lambda, \zeta) = \zeta$ if and only if $S_\varepsilon\left(f\left(\lambda, \cdot, \zeta + S_\varepsilon(0)\right)\right) = \zeta + S_\varepsilon(0)$; i.e., if and only if $w := \zeta + S_\varepsilon(0)$ is a weak solution of (3.6).

Lemma 4.2. *For any $\varepsilon \geq 0$, $T_\varepsilon : [0,\infty) \times P \to C\left(\overline{\Omega}\right)$ is a continuous and compact map.*

Proof. The map $(\lambda, \zeta) \to f\left(\lambda, \cdot, \zeta + S_\varepsilon(0)\right)$ is continuous from $[0,\infty) \times C\left(\overline{\Omega}\right)$ into $C\left(\overline{\Omega}\right)$, therefore the continuity of T_ε follows from Lemma 2.14. The compactness of T_ε is also given by Lemma 2.14, by observing that if $\left\{\left(\lambda_j, \zeta_j\right)\right\}_{j \in \mathbb{N}}$ is a bounded sequence in $[0,\infty) \times C\left(\overline{\Omega}\right)$, then $\left\{f\left(\lambda_j, \cdot, \zeta_j + S(0)\right)\right\}_{j \in \mathbb{N}}$ is bounded in $C\left(\overline{\Omega}\right)$. □

Lemma 4.3. *For any $\varepsilon \geq 0$ the following statements hold:*

i) $T_\varepsilon(0,0) = 0$.

ii) *If $0 \leq \zeta \in C\left(\overline{\Omega}\right)$ and if $T_\varepsilon(0,\zeta) = \zeta$, then $\zeta = 0$.*

iii) *There exists $\rho > 0$ such that, if $u \in \left\{\zeta \in C\left(\overline{\Omega}\right) : \zeta \geq 0 \text{ and } \|\zeta\|_\infty = \rho\right\}$ and $\sigma \in (1,\infty)$, then $T_\varepsilon(0,u) \neq \sigma u$.*

Proof. i) is immediate from the definition of T_ε. If $0 \leq \zeta \in C\left(\overline{\Omega}\right)$ and $T_\varepsilon(0,\zeta) = \zeta$ then, as observed in Remark 4.1, $w := \zeta + S_\varepsilon(0)$ is a solution of $-\Delta w = a(w+\varepsilon)^{-\alpha}$ in Ω, $w = 0$ on $\partial\Omega$; and, by Lemmas 2.6 and 2.12, the unique solution to this problem is $S_\varepsilon(0)$. Thus $\zeta = 0$, and so ii) holds. Finally, iii) follows from the fact that $T_\varepsilon(0,u) = 0$ for all nonnegative $u \in C\left(\overline{\Omega}\right)$.

Let us recall the following result from [1]. □

Remark 4.4 ([1, Theorem 1.17])**.** Let E be an ordered Banach space, let $P := \{\zeta \in E : \zeta \geq 0\}$) be its positive cone, and let $T : [0,\infty) \times P \to P$ be a continuous and compact map. Suppose that $T(0,0) = 0$, and that zero is the only fixed point of $T(0,\cdot)$. Suppose, in addition, that there exists a positive number ρ such that $T(0,\zeta) \neq \sigma\zeta$ for all $\zeta \in S_\rho^+ := \{\zeta \in P : \|\zeta\|_E = \rho\}$ and all $\sigma \geq 1$. Then the set $\Sigma := \{(\lambda,\zeta) \in [0,\infty) \times P : T(\lambda,\zeta) = \zeta\}$ includes an unbounded subcontinuum subset (i.e. an unbounded closed and connected subset) that contains $(0,0)$.

For $\varepsilon \geq 0$, let Σ_ε be defined by

$$\Sigma_\varepsilon := \{(\lambda,\zeta) \in [0,\infty) \times P : T_\varepsilon(\lambda,\zeta) = \zeta\}. \tag{4.3}$$

Lemma 4.5. *For any $\varepsilon \geq 0$, Σ_ε includes an unbounded closed connected subset C_{Σ_ε} that contains* $(0,0)$.

Proof. Follows from Remark 4.4 and Lemmas 4.2 and 4.3. □

Lemma 4.6. *Let $\varepsilon \geq 0$, and let C_{Σ_ε} be the set given by Lemma 4.5. Then:*

i) there exists $\Lambda^{\#} \in (0,\infty)$ such that $\pi_1(\Sigma_\varepsilon) \subset [0,\Lambda^{\#}]$;

ii) there exists $\Lambda_\varepsilon \in (0,\infty)$ such that $[0,\Lambda_\varepsilon) \subset \pi_1(C_{\Sigma_\varepsilon}) \subset [0,\Lambda_\varepsilon]$;

iii) $[0,\Lambda_0) \subset \pi_1(\Sigma_\varepsilon)$ if $\varepsilon > 0$.

Proof. Let $\varepsilon \geq 0$ and $(\lambda,\zeta) \in \Sigma_\varepsilon$. Let λ^* be as given by Lemma 3.3. Then $\lambda \leq \lambda^*$, and so i) holds with $\Lambda^{\#} = \lambda^*$.

To prove ii) observe that, from i), $\pi_1(C_{\Sigma_\varepsilon})$ is a bounded and connected subset of $\mathbb{R}$. If $(0,\zeta) \in C_{\Sigma_\varepsilon}$ then, by Lemma 4.3 ii), $\zeta = 0$. Since, by Corollary 3.5, C_{Σ_ε} is unbounded, we get that $\pi_1(C_{\Sigma_\varepsilon}) \neq \{0\}$. Also, $(0,0) \in C_{\Sigma_\varepsilon}$, and so $0 \in \pi_1(C_{\Sigma_\varepsilon})$. Since $\pi_1(C_{\Sigma_\varepsilon}) \neq \{0\}$ and $\pi_1(C_{\Sigma_\varepsilon})$ is a bounded and connected subset of $\mathbb{R}$, ii) follows.

To see iii), consider $\lambda \in [0,\Lambda_0)$. By ii) there exists a weak solution $\overline{u} \in H_0^1(\Omega) \cap C(\overline{\Omega})$ to the problem $-\Delta\overline{u} = a\overline{u}^{-\alpha} + f(\lambda,\cdot,\overline{u})$ in Ω, $\overline{u} = 0$ on $\partial\Omega$, $\overline{u} > 0$ in Ω. Then $-\Delta\overline{u} \geq a(\overline{u}+\varepsilon)^{-\alpha} + f(\lambda\cdot,\overline{u})$ in Ω and, since $\overline{u} = S_0(f(\lambda,\cdot,\overline{u}))$, by Lemma 2.6 there exists a positive constant c such that $\overline{u} \geq cd_\Omega$ in Ω. For $\delta \in (0,1)$ to be determined later, let z be the weak solution (given e.g., by Lemma 2.6 applied with a replaced by δa) to the problem $-\Delta z = \delta a(z+1)^{-\alpha}$ in Ω, $z = 0$ on $\partial\Omega$, $z > 0$ in Ω. Thus $-\Delta z \leq \delta a$ in Ω, and so there exists a positive constant c', independent of δ, such that $z \leq c'\delta d_\Omega$ in Ω. Then $-\Delta z = \delta a(z+1)^{-\alpha} \geq \delta a(c'\delta d_\Omega + 1)^{-\alpha} \geq \delta a(c' \operatorname{diam}(\Omega)+1)^{-\alpha}$ in Ω, and so $z \geq \delta(c' \operatorname{diam}(\Omega)+1)^{-\alpha}(-\Delta)^{-1}(a)$. Then there exists a positive constant c'' independent of δ such $z \geq c''\delta d_\Omega$ in Ω. Also, $-\Delta z = \delta a(z+1)^{-\alpha} \leq a(z+\varepsilon)^{-\alpha} \leq a(z+\varepsilon)^{-\alpha} + f(\lambda,\cdot,\delta z)$ in Ω. Now we take small enough such $c'\delta \leq c$. Then $z \leq c'\delta d_\Omega \leq cd_\Omega \leq \overline{u}$. Thus [18, Theorem 4.9] gives a weak solution u to the problem $-\Delta u = a(u+\varepsilon)^{-\alpha} + f(\lambda,\cdot,u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, which satisfies $z \leq u \leq \overline{u}$. Then $u = S_\varepsilon(f(\lambda,\cdot,u))$, and so, by Lemma 2.8, $u \in C(\overline{\Omega})$. Since $-\Delta u \geq a(u+\varepsilon)^{-\alpha}$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, and since $v := S_\varepsilon(0)$ satisfies $-\Delta v = a(v+\varepsilon)^{-\alpha}$ in Ω, $v = 0$ on $\partial\Omega$, $v > 0$ in Ω, the comparison principle in Remark 2.4 gives $u \geq v$ in Ω. Then $u \geq S_\varepsilon(0)$ in Ω. Thus $\zeta := u - S_\varepsilon(0) \in P$ and, by Remark 4.1, $(\lambda,\zeta) \in \Sigma_\varepsilon$. □

Lemma 4.7. *Let Λ_0 be as given by Lemma 4.6 ii). Then, for any $\sigma > \|S_0(0)\|_\infty$, there exists $\lambda_\sigma \in (0,\Lambda_0)$ such that $\|\zeta\|_\infty \neq \sigma$ whenever $0 \leq \lambda \leq \lambda_\sigma$, $\varepsilon \in [0,1]$, and $(\lambda,\zeta) \in \Sigma_\varepsilon$.*

Proof. To prove the lemma we proceed by contradiction. Assume that such a λ_σ does not exist. Then, for j large enough, there exist $\lambda_j \in [0,1]$, $\varepsilon_j \in [0,1]$, and a function $\zeta_j \in P$, such that $\lim_{j\to\infty}\lambda_j = 0$, $(\lambda_j,\zeta_j) \in \Sigma_{\varepsilon_j}$ and $\|\zeta_j\|_\infty = \sigma$. Taking a subsequence if necessary, we can assume that $\lim_{j\to\infty}\varepsilon_j = \overline{\varepsilon}$ for some $\overline{\varepsilon} \in [0,1]$. Let $w_j := \zeta_j + S_{\varepsilon_j}(0)$. Then $w_j \in C(\overline{\Omega})$, and $\|w_j\|_\infty \leq \|\zeta_j\|_\infty + \|S_{\varepsilon_j}(0)\|_\infty \leq \sigma + \|S_0(0)\|_\infty := M$, and so $\|f(\lambda_j,\cdot,w_j)\|_\infty \leq \max_{[0,1]\times\overline{\Omega}\times[0,M]} f$. Thus $\{f(\lambda_j,\cdot,w_j)\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. Also, by Remark 4.1, w_j is a weak solution of $-\Delta w_j = a(w_j+\varepsilon_j)^{-\alpha} + f(\lambda_j,\cdot,w_j)$ in Ω, $w_j = 0$ on $\partial\Omega$, and so $w_j = S_{\varepsilon_j}(f(\lambda_j,\cdot,w_j))$. Thus, by Lemma 2.13, $\{w_j\}_{j\in N}$ is bounded in $H_0^1(\Omega)$. Then there exists $w \in H_0^1(\Omega)$, and a subsequence $\{w_{j_k}\}_{k\in N}$, such that $\{w_{j_k}\}_{k\in N}$ converges to w strongly in $L^2(\Omega)$, and $\{\nabla w_{j_k}\}_{k\in N}$ converges weakly to ∇w in $L^2(\Omega,\mathbb{R}^n)$. Taking a subsequence if necessary, we can assume that $\{w_{j_k}\}_{k\in N}$ converges to w a.e. in Ω. Thus $\{f(\lambda_{j_k},\cdot,w_{j_k})\}_{j\in\mathbb{N}}$ is a bounded sequence in $L^\infty(\Omega)$ and converges pointwise to $f(0,\cdot,w)$ in Ω. Then, by Lemma 2.14 iii), $\{S_{\varepsilon_{j_k}}(f(\lambda_{j_k},\cdot,w_{j_k}))\}_{k\in\mathbb{N}}$ converges to $S_{\overline{\varepsilon}}(f(0,\cdot,w))$ in $C(\overline{\Omega})$, i.e., $\{w_{j_k}\}_{k\in\mathbb{N}}$ converges to

$S_{\overline{\varepsilon}}(0)$ in $C(\overline{\Omega})$. Then $w = S_{\overline{\varepsilon}}(0)$, and so $\|w\|_\infty = \lim_{k\to\infty} \|w_{j_k}\|_\infty \geq \lim_{k\to\infty} \|\zeta_{j_k}\|_\infty = \sigma > \|S_0(0)\|_\infty$; which contradicts that $w = S_{\overline{\varepsilon}}(0) \leq S_0(0)$. □

Lemma 4.8. *Let $\lambda_0 > 0$, let $\{\lambda_j\}_{j\in\mathbb{N}}$ be a sequence in $[\lambda_0, \infty)$, and let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence in $[0,1]$. For $j \in \mathbb{N}$, let $w_j \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a weak solution of*

$$\begin{cases} -\Delta w_j = a\left(w_j + \varepsilon_j\right)^{-\alpha} + f\left(\lambda_j, \cdot, w_j\right) & \text{in } \Omega, \\ w_j = 0 \quad \text{on } \partial\Omega, \\ w_j > 0 \quad \text{in } \Omega. \end{cases}$$

Then:

i) $\{w_j\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$;

ii) if $\{w_{j_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{w_j\}_{j\in\mathbb{N}}$ that converges weakly in $H_0^1(\Omega)$ to some $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and if, in addition, $\lim_{k\to\infty} \lambda_{j_k} = \lambda$ and $\lim_{k\to\infty} \varepsilon_{j_k} = \varepsilon$ for some $\varepsilon \in [0,1]$ and $\lambda \in [\lambda_0, \infty)$, then w is a weak solution to (3.6) and there exists a positive constant c such that $w \geq c d_\Omega$ in Ω.

Proof. Let c_{λ_0} be as given by Lemma 3.4. Then $\|w_j\|_\infty \leq c_{\lambda_0}$, which implies $\|f(\lambda_j, \cdot, w_j)\|_\infty \leq \sup_{[0,\Lambda^\#]\times\Omega\times[0,c_{\lambda_0}]} f$, with $\Lambda^\#$ given by Lemma 4.6 i). Since $w_j = S_{\varepsilon_j}(f(\lambda_j, \cdot, w_j))$, Lemma 2.13 gives that $\{w_j\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$.

Now suppose that $\{w_{j_k}\}_{k\in\mathbb{N}}$ is a subsequence such that, for some $w \in H_0^1(\Omega)$, $\{w_{j_k}\}_{k\in\mathbb{N}}$ converges to w strongly in $L^2(\Omega)$ and $\{\nabla w_{j_k}\}_{k\in\mathbb{N}}$ converges to ∇w weakly in $L^2(\Omega, \mathbb{R}^n)$. Suppose also that $\lim_{k\to\infty} \lambda_{j_k} = \lambda$ and $\lim_{k\to\infty} \varepsilon_{j_k} = \varepsilon$. Taking a further subsequence if necessary, we can assume that $\{w_{j_k}\}_{k\in\mathbb{N}}$ converges to w a.e. in Ω. Now, $w_{j_k} = S_{\varepsilon_{j_k}}(f(\lambda_{j_k}, \cdot, w_{j_k}))$ and then, by Lemma 2.7, $w_{j_k} \geq c d_\Omega$ in Ω for some positive constant independent of k. Thus $w \geq c d_\Omega$ in Ω. Note that $\{f(\lambda_{j_k}, \cdot, w_{j_k})\}_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty(\Omega)$ that converges pointwise to $f(\lambda, \cdot, w)$. Then, by Lemma 2.14 iv), $\{S_{\varepsilon_{j_k}}(f(\lambda_{j_k}, \cdot, w_{j_k}))\}_{k\in\mathbb{N}}$ converges to $S_\varepsilon(f(\lambda, \cdot, w))$ in $C(\overline{\Omega})$, i.e., $\{w_{j_k}\}_{k\in\mathbb{N}}$ converges to $S_\varepsilon(f(\lambda, \cdot, w))$ in $C(\overline{\Omega})$. Thus $w = S_\varepsilon(f(\lambda, \cdot, w))$, i.e., w solves (3.6). Finally, Lemma 2.7 says that, for some positive constant c, $w \geq c d_\Omega$ in Ω. □

Proof of Theorem 1.1. Let $\Lambda := \sup\{\lambda \geq 0 : (\lambda, \zeta) \in \Sigma_0 \text{ for some } \zeta \in P\}$, and let Λ_0 be given by Lemma 4.6 ii). Thus $\Lambda_0 > 0$ and, for any $\lambda \in [0, \Lambda_0)$, there exists $\zeta \in P$ such that $(\lambda, \zeta) \in \Sigma_0$; and the function $w_\zeta := \zeta + S_0(0)$ is a positive weak solution of (1.1) that belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. Moreover, $w_\zeta \in C(\overline{\Omega})$ and, for some positive constant c, $w_\zeta \geq c d_\Omega$ in Ω. Also, $\Lambda \geq \Lambda_0 > 0$ and clearly, if (1.1) has a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then $\lambda \leq \Lambda$. Consider a sequence $\{(\lambda_j, \zeta_j)\}_{j\in\mathbb{N}} \subset \Sigma_0$ such that $\lim_{j\to\infty} \lambda_j = \Lambda$, and a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,1]$ such that $\lim_{j\to\infty} \varepsilon_j = 0$. Let $w_j := \zeta_j + S_0(0)$. Then $w_j \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and w_j is a solution of (1.1) for $\lambda = \lambda_j$. Thus w_j is a supersolution to the following nonsingular problem

$$\begin{cases} -\Delta z = a\left(z + \varepsilon_j\right)^{-\alpha} + f\left(\lambda_j, \cdot, z\right) & \text{in } \Omega, \\ z = 0 \quad \text{on } \partial\Omega, \\ z > 0 \quad \text{in } \Omega. \end{cases} \tag{4.4}$$

Clearly $v_j := S_{\varepsilon_j}(0)$ is a subsolution of (4.4) and, by Lemma 2.3, $v_{\varepsilon_j} \leq w_j$ in Ω. Thus, by [18, Theorem 4.9], there exists a weak solution $u_j \in H_0^1(\Omega)$ of (4.4) such that $v_j \leq u_j \leq w_j$

in Ω. Since $0 \leq u_j \leq w_j$ we have also $u_j \in L^\infty(\Omega)$. Then, by Lemma 4.8 i), $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Thus there exists a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ and $w \in H_0^1(\Omega)$, such that $\{u_{j_k}\}_{k\in\mathbb{N}}$ converges weakly in $H_0^1(\Omega)$ and a.e. in Ω to $w \in H_0^1(\Omega)$. Also, $\lambda_j \geq \frac{1}{2}\Lambda$ for j large enough, and so, for such j, Lemma 3.4 gives $\|u_j\|_\infty \leq \widetilde{c}_{\frac{1}{2}\Lambda}$. Then $\|w\|_\infty \leq \widetilde{c}_{\frac{1}{2}\Lambda}$ and so, $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Now, by Lemma 4.8 ii), w is a weak solution of $-\Delta w = aw^{-\alpha} + f(\Lambda,\cdot,w)$ in Ω, $w = 0$ on $\partial\Omega$, $w > 0$ in Ω, and, for some positive constant c, it satisfies $w \geq cd_\Omega$ in Ω.

Let $\{v_j\}_{j\in\mathbb{N}}$ and $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be as above, and let $\lambda \in [0,\Lambda)$; then w is a supersolution of (4.4). Clearly v_j is a subsolution to (4.4). Also, by Lemma 2.14 i), $w = S_0(f(\Lambda,\cdot,w)) \geq S_{\varepsilon_j}(f(\Lambda,\cdot,w)) \geq S_{\varepsilon_j}(0) = v_j$ and so, by [18, Theorem 4.9], there exists a solution $\widetilde{u}_j \in H_0^1(\Omega)$ to problem (4.4) such that $v_j \leq \widetilde{u}_j \leq w$. Thus $\widetilde{u}_j \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and by Lemma 3.4, $\|\widetilde{u}_j\|_\infty \leq \widetilde{c}_\lambda$ for all j. By Lemma 4.8 i), $\{\widetilde{u}_j\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Thus there exists a subsequence $\{\widetilde{u}_{j_k}\}_{k\in\mathbb{N}}$, and $\widetilde{w} \in H_0^1(\Omega)$, such that $\{\widetilde{u}_{j_k}\}_{k\in\mathbb{N}}$ converges weakly in $H_0^1(\Omega)$, and a.e. in Ω, to $\widetilde{w} \in H_0^1(\Omega)$, which satisfies $\|\widetilde{w}\|_\infty \leq \widetilde{c}_\lambda$. Thus, by Lemma 4.8 ii), $\widetilde{w}$ is a weak solution to the problem $-\Delta\widetilde{w} = a\widetilde{w}^{-\alpha} + f(\lambda,\cdot,\widetilde{w})$ in Ω, $\widetilde{w} = 0$ on $\partial\Omega$, $\widetilde{w} > 0$ in Ω. Finally, if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (1.1) then, by Lemmas 2.7 and 2.11, $u \geq cd_\Omega$ in Ω if $0 < \alpha < 1$, and $u \geq cd_\Omega^{\frac{2}{1+\alpha}}$ in Ω if $1 \leq \alpha < 3$; in both cases with c a positive constant independent of λ and u. □

Lemma 4.9. *Let $\sigma > \|S_0(0)\|_\infty$ and let λ_σ be as in Lemma 4.7. Then $\Lambda_\varepsilon > \lambda_\sigma$ for any $\varepsilon \in [0,1]$.*

Proof. By way of contradiction, suppose that $\Lambda_\varepsilon \leq \lambda_\sigma$ for some $\varepsilon \in [0,1]$. Then for each $(\lambda,\zeta) \in C_{\Sigma_\varepsilon}$ we have $\lambda \leq \lambda_\sigma$ and so, by Lemma 4.7, $\|\zeta\|_\infty \neq \sigma$. Also, $\lambda < \Lambda^\# + 1$, with $\Lambda^\#$ given by Lemma 4.6 i). Let

$$U_1 := \left\{(\lambda,\zeta) \in \left[0,\Lambda^\# + 1\right) \times P : \|\zeta\|_\infty < \sigma\right\},$$
$$U_2 := \left\{(\lambda,\zeta) \in \left[0,\Lambda^\# + 1\right) \times P : \|\zeta\|_\infty > \sigma\right\}.$$

Note that U_1 and U_2 are disjoint open subsets of $[0,\infty) \times P$, and that $C_{\Sigma_\varepsilon} \subset U_1 \cup U_2$. As C_{Σ_ε} is unbounded, $C_{\Sigma_\varepsilon} \cap U_2$ is nonempty. Also $(0,0) \in C_{\Sigma_\varepsilon} \cap U_1$, therefore $C_{\Sigma_\varepsilon} \cap U_1$ is nonempty. Contradiction, since C_{Σ_ε} is connected. □

Lemma 4.10. *For each $\lambda_0 > 0$ there exists $c_{\lambda_0} > 0$ such that $\|\zeta\|_\infty < c_{\lambda_0}$ whenever $\lambda \geq \lambda_0$ and $(\lambda,\zeta) \in \Sigma_0$.*

Proof. $(\lambda,\zeta) \in \Sigma_0$ if and only if $u := \zeta + S_0(0)$ is a solution of (1.1). Since $\|\zeta\|_\infty \leq \|u\|_\infty + \|S_0(0)\|_\infty$, the lemma follows from Lemma 3.4. □

Lemma 4.11. *Let $\sigma > \|S_0(0)\|_\infty$, and let λ_σ be given by Lemma 4.7. Then, for each $\overline{\lambda} \in (0,\lambda_\sigma)$, the set $\left\{(\lambda,\zeta) \in \Sigma_0 : 0 < \lambda < \overline{\lambda}\right\}$ is an unbounded subset of $[0,\infty) \times P$.*

Proof. Suppose for the sake of contradiction, that for some $\overline{\lambda} \in (0,\lambda_\sigma)$ there exists $M > 0$ such that $\|\zeta\|_\infty \leq M$ whenever $(\lambda,\zeta) \in \Sigma_0$ and $0 < \lambda < \overline{\lambda}$. Let $c_{\overline{\lambda}}$ be as given by Lemma 4.11. Then $C_{\Sigma_0} \subset \left[0,\Lambda^\#\right] \times \left\{\zeta \in P : \|\zeta\|_\infty \leq M + c_{\overline{\lambda}}\right\}$, which contradicts the fact that C_{Σ_0} is unbounded. □

Proof of Theorem 1.2. Let $\sigma > \|S_0(0)\|_\infty$, and let λ_σ be as given by Lemma 4.7. Let $\Lambda^* := \min\{\Lambda_0,\lambda_\sigma\}$ and, for each $\lambda \in (0,\Lambda_0)$, let c_λ be as given by Lemma 4.11. We claim that, for $0 < \lambda < \Lambda^*$, problem (1.1) has at least two weak solutions in P. To see this we proceed by contradiction. Suppose, by way of contradiction, that for some $\lambda^\# \in (0,\Lambda^*)$ problem

(1.1) has a unique solution $u_{\lambda^\#}$ in P (at least one solution exists because $\lambda^\# < \Lambda_0$). Now, $u_{\lambda^\#} = S_0\left(f\left(\lambda^\#, \cdot, u_{\lambda^\#}\right)\right)$ and so, by Lemma 2.14 i), $u_{\lambda^\#} \geq S_0(0)$. Let $\zeta_{\lambda^\#} := u_{\lambda^\#} - S_0(0)$; clearly $\zeta_{\lambda^\#} \in P$ and, by Remark 4.1, $\left(\lambda^\#, \zeta_{\lambda^\#}\right) \in \Sigma_0$. Also, if $\left(\lambda^\#, \zeta\right) \in \Sigma_0$ for some $\zeta \in P$, then $\zeta = \zeta_{\lambda^\#}$. Now, by Lemma 4.7, $\|\zeta_{\lambda^\#}\|_\infty \neq \sigma$; Then either $\|\zeta_{\lambda^\#}\|_\infty < \sigma$ or $\|\zeta_{\lambda^\#}\|_\infty > \sigma$.

If $\|\zeta_{\lambda^\#}\|_\infty < \sigma$, consider the disjoint open sets V_1 and V_2 in $\mathbb{R}^2$ defined by

$$V_1 := \left\{(\lambda, t) \in \mathbb{R}^2 : \lambda < \lambda^\# \text{ and } t > \sigma\right\}$$

and $V_2 := V_{21} \cup V_{22} \cup V_{23}$ where

$$\begin{aligned}
V_{21} &:= \left\{(\lambda, t) \in \mathbb{R}^2 : \lambda > \lambda^\# \text{ and } t < c_{\lambda^\#} + \sigma\right\}, \\
V_{22} &:= \left\{\left(\lambda^\#, t\right) \in \mathbb{R}^2 : t < \sigma\right\}, \\
V_{23} &:= \left\{(\lambda, t) \in \mathbb{R}^2 : \lambda < \lambda^\# \text{ and } t < \sigma\right\},
\end{aligned}$$

and let U_1 and U_2 be the two disjoint open sets in $[0, \infty) \times P$ defined, for $i = 1, 2$, by $U_i := \{(\lambda, \zeta) \in [0, \infty) \times P : (\lambda, \|\zeta\|_\infty) \in V_i\}$. Let $(\lambda, \zeta) \in C_{\Sigma_0}$. If $\lambda > \lambda^\#$ then, by Lemma 4.11, $\|\zeta\|_\infty \leq c_{\lambda^\#}$, and so $(\lambda, \|\zeta\|_\infty) \in V_{21}$. Then $(\lambda, \zeta) \in U_2$. If $\lambda = \lambda^\#$ then $\zeta = \zeta_{\lambda^\#}$, and so $\|\zeta\|_\infty = \|\zeta_{\lambda^\#}\|_\infty < \sigma$. Thus $(\lambda, \|\zeta\|_\infty) \in V_{22}$, which implies $(\lambda, \zeta) \in U_2$. If $\lambda < \lambda^\#$ then, by Lemma 4.7, $\|\zeta\|_\infty \neq \sigma$. If $\|\zeta\|_\infty < \sigma$ then $(\lambda, \|\zeta\|_\infty) \in V_{22}$, and so $(\lambda, \zeta) \in U_2$. If $\|\zeta\|_\infty > \sigma$ then $(\lambda, \|\zeta\|_\infty) \in V_1$, which gives $(\lambda, \zeta) \in U_1$. Then $C_{\Sigma_0} \subset U_1 \cup U_2$. Also, $(0, 0) \in C_{\Sigma_0} \cap U_2$, and so $C_{\Sigma_0} \cap U_2$ is nonempty. On the other hand, $C_{\Sigma_0} \cap U_2 \subset \left\{(\lambda, \zeta) : \lambda \in \left[0, \Lambda^\#\right] \text{ and } \|\zeta\| < c_{\lambda^\#} + \sigma\right\}$, which is bounded in $[0, \infty) \times P$. Since C_{Σ_0} is unbounded and $C_{\Sigma_0} \subset U_1 \cup U_2$, we conclude that also $C_{\Sigma_0} \cap U_1$ is nonempty, contradicting that C_{Σ_0} is a connected set.

When $\|\zeta_{\lambda^\#}\|_\infty > \sigma$ we consider the disjoint open sets $\widetilde{V}_1$ and $\widetilde{V}_2$ in $\mathbb{R}^2$ defined by $\widetilde{V}_1 := V_{23}$ and $\widetilde{V}_2 := V_{21} \cup \widetilde{V}_{22} \cup \widetilde{V}_{23}$, with V_{21} and V_{23} defined as above, and

$$\begin{aligned}
\widetilde{V}_{22} &:= \left\{\left(\lambda^\#, t\right) \in \mathbb{R}^2 : \sigma < t < c_{\lambda^\#} + \sigma\right\}, \\
\widetilde{V}_{23} &:= \left\{(\lambda, t) \in \mathbb{R}^2 : \lambda < \lambda^\# \text{ and } t > \sigma\right\}.
\end{aligned}$$

For $i = 1, 2$, let $\widetilde{U}_i := \left\{(\lambda, \zeta) \in [0, \infty) \times P : (\lambda, \|\zeta\|_\infty) \in \widetilde{V}_i\right\}$. Thus $\widetilde{U}_1$ and $\widetilde{U}_2$ are open and disjoint sets in $[0, \infty) \times P$. Let $(\lambda, \zeta) \in C_{\Sigma_0}$. If $\lambda > \lambda^\#$ then, as before, $(\lambda, \|\zeta\|_\infty) \in V_{21}$, and so $(\lambda, \zeta) \in \widetilde{U}_2$. If $\lambda = \lambda^\#$ then $\zeta = \zeta_{\lambda^\#}$, and so $\|\zeta\|_\infty > \sigma$, also $\|\zeta\|_\infty \leq c_{\lambda^\#}$, and thus $(\lambda, \|\zeta\|_\infty) \in \widetilde{V}_{22}$, which implies $(\lambda, \zeta) \in \widetilde{U}_2$. If $\lambda < \lambda^\#$ then, either $\|\zeta\|_\infty < \sigma$, or $\|\zeta\|_\infty > \sigma$. If $\|\zeta\|_\infty < \sigma$ then $(\lambda, \|\zeta\|_\infty) \in V_{23}$, and so $(\lambda, \zeta) \in \widetilde{U}_1$. If $\|\zeta\|_\infty > \sigma$ then $(\lambda, \|\zeta\|_\infty) \in \widetilde{V}_{23}$ which gives $(\lambda, \zeta) \in \widetilde{U}_2$. Then $C_{\Sigma_0} \subset \widetilde{U}_1 \cup \widetilde{U}_2$. Also, $(0, 0) \in C_{\Sigma_0} \cap \widetilde{U}_1$, and so $C_{\Sigma_0} \cap \widetilde{U}_1$ is nonempty. On the other hand, $C_{\Sigma_0} \cap \widetilde{U}_1$ is bounded in $[0, \infty) \times P$, and then, since C_{Σ_0} is unbounded, $C_{\Sigma_0} \cap U_2$ is nonempty, contradicting that C_{Σ_0} is connected.

The assertion that $\lambda = 0$ is a bifurcation point from ∞ for (1.1), follows from the fact that, by Lemma 4.11, for any $j \in \mathbb{N}$, there exists $\left(\lambda_j, \zeta_j\right) \in \Sigma_0$ such that $\lambda_j < \frac{1}{j}$ and $\left\|\zeta_j\right\|_\infty \geq j$. □

Proof of Theorem 1.3. A direct inspection shows that, in each case, the corresponding function $f(\lambda, \cdot, s)$ satisfies the hypothesis of Theorems 1.1 and 1.2. □

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