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First integrals and phase portraits of planar polynomial differential cubic systems with invariant straight lines of total multiplicity eight

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Abstract. In [C. Bujac, J. Llibre, N. Vulpe, *Qual. Theory Dyn. Syst.* **15**(2016), 327–348] all first integrals and phase portraits were constructed for the family of cubic differential systems with the maximum number of invariant straight lines, i.e. 9 (considered with their multiplicities). Here we continue this investigation for systems with invariant straight lines of total multiplicity eight. For such systems the classification according to the configurations of invariant lines in terms of affine invariant polynomials was done in [C. Bujac, *Bul. Acad. Ştiinţe Repub. Mold. Mat.* **75**(2014), 102–105], [C. Bujac, N. Vulpe, *J. Math. Anal. Appl.* **423**(2015), 1025–1080], [C. Bujac, N. Vulpe, *Qual. Theory Dyn. Syst.* **14**(2015), 109–137], [C. Bujac, N. Vulpe, *Electron. J. Qual. Theory Differ. Equ.* **2015**, No. 74, 1–38], [C. Bujac, N. Vulpe, *Qual. Theory Dyn. Syst.* **16**(2017), 1–30] and all possible 51 configurations were constructed. In this article we prove that all systems in this class are integrable. For each one of the 51 such classes we compute the corresponding first integral and we draw the corresponding phase portrait.

Keywords: quadratic vector fields, infinite and finite singularities, affine invariant polynomials, Poincaré compactification, configuration of singularities, geometric equivalence relation.

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1 Introduction

Polynomial differential systems on the plane are systems of the form

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1.1}$$

where $P, Q \in \mathbb{R}[x,y]$, i.e. P and Q are the polynomials over $\mathbb{R}$. To a system (1.1) we can associate the vector field

$$\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{1.2}$$

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We call *cubic* a differential system (1.1) with degree $n = \max\{\deg P, \deg Q\} = 3$.

There are several open problems on polynomial differential systems, especially on the class of all cubic systems (1.1) (denote by $\mathbb{CS}$ the whole class of such systems). In this paper we are concerned with questions regarding *integrability in the sense of Darboux* and *classification of all phase portraits of* $\mathbb{CS}$. These problems are very hard even in the simplest case of quadratic differential systems.

The method of integration of Darboux uses multiple-valued complex functions of the form:

$$F = e^{G(x,y)} f_1(x,y)^{\lambda_1} \cdots f_s(x,y)^{\lambda_s}, \quad G = G_1/G_2,\ G_i \in \mathbb{C}[x,y], \tag{1.3}$$

and f_i irreducible over $\mathbb{C}$. It is clear that in general the last expression makes sense only for $G_2 \neq 0$ and for points $(x,y) \in \mathbb{C}^2 \setminus \big(\{G_2(x,y)=0\} \cup \{f_1(x,y)=0\} \cup \cdots \cup \{f_s(x,y)=0\}\big)$.

Consider the polynomial system of differential equations (1.1). The equation $f(x,y) = 0$ ($f \in \mathbb{C}[x,y]$, where $\mathbb{C}[x,y]$ denotes the ring of polynomials in two variables x and y with complex coefficients) which describes implicitly some trajectories of systems (1.1), can be seen as an affine representation of an algebraic curve of degree m. Suppose that (1.1) has a solution curve which is not a singular point, contained in an algebraic curve $f(x,y) = 0$. It is clear that the derivative of $f\big(x(t), y(t)\big)$ with respect to t must vanish on the algebraic curve $f(x,y) = 0$, so $\frac{df}{dt}|_{f=0} = \big(\frac{df}{\partial x}P(x,y) + \frac{df}{\partial y}Q(x,y)\big)|_{f=0} = 0$.

In 1878 Darboux introduced the notion of the invariant algebraic curve for differential equations on the complex projective plane. This notion can be adapted for systems (1.1). According to [13] the next definition follows.

Definition 1.1. An algebraic curve $f(x,y) = 0$ in $\mathbb{C}^2$ with $f \in \mathbb{C}[x,y]$ is an ***invariant algebraic curve*** (an algebraic particular integral) of a polynomial system (1.1) if $\mathbb{X}(f) = fK$ for some polynomial $K(x,y) \in \mathbb{C}[x,y]$ called the cofactor of the invariant algebraic curve $f(x,y) = 0$.

In view of Darboux's definition, *an algebraic solution* of a system of equations (1.1) is an invariant algebraic curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ ($\deg f \geq 1$) with f an irreducible polynomial over $\mathbb{C}$. Darboux showed that if a system (1.1) possesses a sufficient number of such invariant algebraic solutions $f_i(x,y) = 0$, $f_i \in \mathbb{C}$, $i = 1,2,\ldots,s$, then the system has a first integral of the form (1.3).

We say that a system (1.1) has a generalized Darboux first integral (respectively generalized Darboux integrating factor) if it admits a first integral (respectively integrating factor) of the form $e^{G(x,y)} \prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$, where $G(x,y) \in \mathbb{C}(x,y)$ and $f_i \in \mathbb{C}[x,y]$, $\deg f_i \geq 1$, $i = 1,2,\ldots,s$, f_i irreducible over $\mathbb{C}$ and $\lambda_i \in \mathbb{C}$. If a system (1.1) has an integrating factor (or first integral) of the form $F = \prod_{i=1}^{s} f_i^{\lambda_i}$ then $\forall i \in \{1,\ldots,s\}$, $f_i = 0$ is an algebraic invariant curve of (1.1).

In [13] Darboux proved the following remarkable theorem of integrability using invariant algebraic solutions of systems (1.1).

Theorem 1.2. *Consider a differential system* (1.1) *with* $P, Q \in \mathbb{C}[x,y]$. *Let us assume that* $m = \max(\deg P, \deg Q)$ *and that this system admits s algebraic solutions* $f_i(x,y) = 0$, $i = 1,2,\ldots,s$ ($\deg f_i \geq 1$). *Then we have:*

I. if $s = m(m+1)/2$ *then there exists* $\lambda = (\lambda_1,\ldots,\lambda_s) \in \mathbb{C}^s \setminus \{0\}$ *such that* $R = \prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$ *is an integrating factor of* (1.1);

II. if $s \geq m(m+1)/2 + 1$ *then there exists* $\lambda = (\lambda_1,\ldots,\lambda_s) \in \mathbb{C}^s \setminus \{0\}$ *such that* $F = \prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$ *is a first integral of* (1.1).

In 1979 Jouanolou proved the next theorem which completes part II of Darboux's Theorem.

Theorem 1.3. *Consider a polynomial differential system* (1.1) *over* $\mathbb{C}$ *and assume that it has s algebraic solutions* $f_i(x,y)=0$, $i=1,2,\ldots,s$ $(\deg f_i \geq 1)$. *Suppose that* $s \geq m(m+1)/2+2$. *Then there exists* $(n_1,\ldots,n_s) \in \mathbb{Z}^s \setminus \{0\}$ *such that* $F=\prod_{i=1}^{s} f_i(x,y)^{n_i}$ *is a first integral of* (1.1). *In this case* $F \in \mathbb{C}(x,y)$, *i.e. F is rational function over* $\mathbb{C}$.

The following theorem from [19] improves the Darboux theory of integrability and the above result of Jouanolou taking into account not only the invariant algebraic curves (in particular invariant straight lines) but also their algebraic multiplicities. We mention here this result adapted for two-dimensional vector fields.

Theorem 1.4 ([12, 19])**.** *Assume that the polynomial vector field* $\mathbb{X}$ *in* $\mathbb{C}^2$ *of degree* $d>0$ *has irreducible invariant algebraic curves.*

(i) If some of these irreducible invariant algebraic curves have no defined algebraic multiplicity, then the vector field $\mathbb{X}$ *has a rational first integral.*

(ii) Suppose that all the irreducible invariant algebraic curves $f_i=0$ *have defined algebraic multiplicity* q_i *for* $i=1,\ldots,p$. *If* $\mathbb{X}$ *restricted to each curve* $f_i=0$ *having multiplicity larger than 1 has no rational first integral, then the following statements hold.*

(a) If $\sum_{i=1}^{p} q_i \geq N+1$, *then the vector field* $\mathbb{X}$ *has a Darboux first integral, where* $N=\binom{2+d-1}{2}$

(b) If $\sum_{i=1}^{p} q_i \geq N+2$, *then the vector field* $\mathbb{X}$ *has a rational first integral.*

We note that the notion of "algebraic multiplicity" of an algebraic invariant curve is given in [12] where in particular the authors proved the equivalence of "geometric" and "algebraic" multiplicities of an invariant curve for the polynomial systems (1.1).

If $f(x,y)=ux+vy+w=0$, $(u,v)\neq(0,0)$ and $\mathbb{X}(f)=fK$ where $K(x,y)\in\mathbb{C}[x,y]$, then $f(x,y)=0$ is an invariant line of the family of systems (1.1). We point out that if we have an invariant line $f(x,y)=0$ over $\mathbb{C}$ it could happen that multiplying the equation by a number $\lambda \in \mathbb{C}^*=\mathbb{C}\setminus\{0\}$, the coefficients of the new equation become real, i.e. $(u\lambda,v\lambda,w\lambda)\in\mathbb{R}^3$. In this case, along with the curve $f(x,y)=0$ (sitting in in $\mathbb{C}^2$) we also have an associated real curve (sitting in $\mathbb{R}^2$) defined by $\lambda f(x,y)$.

Note that, since a system (1.1) is real, if its associated complex system has a complex invariant straight line $ux+vy+w=0$, then it also has its conjugate complex invariant straight line $\bar{u}x+\bar{v}y+\bar{w}=0$.

To a line $f(x,y)=ux+vy+w=0$, $(u,v)\neq(0,0)$ we associate its projective completion $F(X,Y,Z)=uX+vY+wZ=0$ under the embedding $\mathbb{C}^2 \hookrightarrow \mathbf{P}_2(\mathbb{C})$, $(x,y)\mapsto[x:y:1]$. The line $Z=0$ in $\mathbf{P}_2(\mathbb{C})$ is called the line at infinity of the affine plane $\mathbb{C}^2$. It follows from the work of Darboux (see, for instance, [13]) that each system of differential equations of the form (1.1) over $\mathbb{C}$ yields a differential equation on the complex projective plane $\mathbf{P}_2(\mathbb{C})$ which is the compactification of the differential equation $Q\,dx-P\,dy=0$ in $\mathbb{C}^2$. The line $Z=0$ is an invariant manifold of this complex differential equation.

For an invariant line $f(x,y)=ux+vy+w=0$ we denote $\hat{a}=(u,v,w)\in\mathbb{C}^3$. We note that the equation $\lambda f(x,y)=0$ where $\lambda\in\mathbb{C}^*$ and $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ yields the same locus of complex points in the plane as the locus induced by $f(x,y)=0$. So that a straight line defined by $\hat{a}$ can be identified with a point $[\hat{a}]=[u:v:w]$ in $\mathbf{P}_2(\mathbb{C})$. We say that a sequence of straight lines $f_i(x,y)=0$ converges to a straight line $f(x,y)=0$ if and only if the sequence of points $[a_i]$ converges to $[\hat{a}]=[u:v:w]$ in the topology of $\mathbf{P}_2(\mathbb{C})$.

Definition 1.5 ([27])**.** We say that an invariant affine straight line $f(x,y) = ux + vy + w = 0$ (respectively the line at infinity $Z = 0$) for a real cubic vector field $\mathbb{X}$ has multiplicity m if there exists a sequence of real cubic vector fields $\mathbb{X}_k$ converging to $\mathbb{X}$, such that each $\mathbb{X}_k$ has m (respectively $m-1$) distinct (complex) invariant affine straight lines $f_i^j = u_i^j x + v_i^j y + w_i^j = 0$, $(u_i^j, v_i^j) \neq (0,0)$, $(u_i^j, v_i^j, w_i^j) \in \mathbb{C}^3$, converging to $f = 0$ as $k \to \infty$, in the topology of $\mathbf{P}_2(\mathbb{C})$, and this does not occur for $m+1$ (respectively m).

We first remark that in the above definition we made an abuse of notation. Indeed, to talk about a complex invariant curve we need to have a complex system. However we said that the real systems $\mathbb{X}_k$ meaning of course the complex systems associtated to the real ones $\mathbb{X}_k$.

We remark that the above definition is a particular case of the definition of geometric multiplicity given in paper [12], and namely the notion of "strong geometric multiplicity" with the restriction, that the corresponding perturbations are cubic systems.

The set $\mathbb{CS}$ of cubic differential systems depends on 20 parameters and for this reason people began by studying particular subclasses of $\mathbb{CS}$. Here we deal with $\mathbb{CS}$ possessing invariant straight lines. We mention some papers devoted to polynomial differential systems possessing invariant straight lines. For quadratic systems see [14, 23, 24, 27–31] and [32]; for cubic systems see [4–10, 17, 18, 20, 21, 25, 35, 36] and [26]; for quartic systems see [34] and [38].

The existence of sufficiently many invariant straight lines of planar polynomial systems could be used for integrability of such systems. During the past 15 years several articles were published on this theme. Investigations concerning polynomial differential systems possessing invariant straight lines were done by Popa, Sibirski, Llibre, Gasull, Kooij, Sokulski, Zhang Xi Kang, Schlomiuk, Vulpe, Dai Guo Ren, Artes as well as Dolov and Kruglov.

According to [1] the maximum number of invariant straight lines taking into account their multiplicities for a polynomial differential system of degree m is $3m$ when we also consider the straight line at infinity. This bound is always reached if we consider the real and the complex invariant straight lines, see [12].

So the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems with finite number of infinite singularities is 9. A classification of all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities has been made in [18]. The authors used the notion of *configuration of invariant lines* for cubic systems (as introduced in [27], but without indicating the multiplicities of real singularities) and detected 23 such configurations. Moreover in this paper using invariant polynomials with respect to the action of *the group* $Aff(2,\mathbb{R})$ *of affine transformations and time rescaling* (i.e. $Aff(2,\mathbb{R}) \times \mathbb{R}^*$)), the necessary and sufficient conditions for the realization of each one of 23 configurations were detected. A new class of cubic systems omitted in [18] was constructed in [4].

Definition 1.6 ([31])**.** Consider a real planar cubic system (1.1). We call *configuration of invariant straight lines* of this system, the set of (complex) invariant straight lines (which may have real coefficients) of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

The configurations of invariant straight lines which were detected for various families of systems (1.1) using Poincaré compactification, could serve as a base to complete the whole Poincaré disc with the trajectories of the solutions of corresponding systems, i.e. to give a full topological classification of such systems. For example, in papers [28, 30] for quadratic systems with invariant lines greater than or equal to 4, it was proved that we have a total of 57 distinct configurations of invariant lines which leads to the existence of 135 topologically distinct phase portraits. In [25, 26, 35, 36] the existence of 113 topologically distinct phase portraits was proved for cubic systems with invariant lines of total parallel multiplicity six or seven, taking in consideration the configurations of invariant lines of these systems. The notion of "parallel multiplicity" could be found in [36].

In this paper we consider the analogous problems for a specific class of cubic systems which we denote by $\mathbb{CSL}_8$. We say that a cubic system belongs to the family $\mathbb{CSL}_8$ if it possesses invariant straight lines of total multiplicity 8, including the line at infinity and considering their multiplicities.

The goal of this article is to complete the study we began in [5–9]. More precisely in this work we

- prove that all systems in the class $\mathbb{CSL}_8$ are integrable. We show this by using the geometric method of integration of Darboux. We construct explicit Darboux integrating factors and we give the list of first integrals for each system in this class;
- construct all possible phase portraits of the systems in this class and prove that only 30 of them are topologically distinct;
- give invariant (under the action of the group $Aff(2,\mathbb{R}) \times \mathbb{R}^*$)) necessary and sufficient conditions, in terms of the twenty coefficients of the systems, for the realization of each specific phase portrait.

This article is organized as follows.

In Section 2 we give the list of affine invariant polynomials and some notion and results needed in this article.

In Section 3 we present some preliminary results. More exactly, in Theorem 3.1 we describe all the 51 possible configurations of invariant lines which could possess the cubic systems in the class $\mathbb{CSL}_8$. Moreover we give necessary and sufficient conditions for the realization of each of these configurations. These results (obtained in [5–9]) serve as a base for the construction of the phase portraits as well as for determining of the corresponding first integrals and integrating factors.

Section 4 contains the main results of this article formulated in the Main Theorem. In Table 4.1 we give the canonical forms of systems in $\mathbb{CSL}_8$ as well as the corresponding first integrals and integrating factors. We prove that each one of the 51 configurations given by Theorem 3.1 leads to a single phase portrait, except the configuration *Config. 8.6*, which generates two topologically distinct phase portraits. In Table 4.1 we also present the necessary and sufficient affine invariant conditions for the realization of each one of the phase portraits obtained. Defining some geometric invariants, we prove (see Diagram 4.1) that among the obtained 52 phase portraits only 30 of them are topologically distinct.

2 Invariant polynomials associated with cubic systems possessing invariant lines

As it was mentioned earlier our work here is based on the result of the papers [4,6–9] where the classification theorems according to the configurations of invariant straight lines for different subfamilies (i.e. systems with either 4 or 3 or 2 or 1 infinite distinct singularities) of systems in

$\mathbb{CSL}_8$ were proved (see further below). In what follows we recall some results in [18] which will be needed to state the mentioned theorems.

Consider real cubic systems, i.e. systems of the form:

$$\begin{aligned}\dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv P(x,y),\\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv Q(x,y)\end{aligned} \tag{2.1}$$

with real coefficients and variables x and y. The polynomials p_i and q_i $(i = 0,1,2,3)$ are the following homogeneous polynomials in x and y:

$$\begin{aligned} p_0 &= a_{00}, & p_3(x,y) &= a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3,\\ p_1(x,y) &= a_{10}x + a_{01}y, & p_2(x,y) &= a_{20}x^2 + 2a_{11}xy + a_{02}y^2,\\ q_0 &= b_{00}, & q_3(x,y) &= b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3,\\ q_1(x,y) &= b_{10}x + b_{01}y, & q_2(x,y) &= b_{20}x^2 + 2b_{11}xy + b_{02}y^2.\end{aligned}$$

It is known that on the set **CS** of all cubic differential systems (2.1) acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane [27]. For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **CS**. We can identify the set **CS** of systems (2.1) with a subset of $\mathbb{R}^{20}$ via the map **CS** $\longrightarrow \mathbb{R}^{20}$ which associates to each system (2.1) the 20-tuple $a = (a_{00}, a_{10}, a_{01}, \ldots, a_{03}, b_{00}, b_{10}, b_{01}, \ldots, b_{03})$ of its coefficients and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, \ldots, a_{03}, b_{00}, b_{10}, b_{01}, \ldots, b_{03}, x, y]$.

For the definitions of an affine or GL-comitant or invariant as well as for the definition of a T-comitant and CT-comitant we refer the reader to [27]. Here we shall only construct the necessary T- and CT-comitants associated to configurations of invariant lines for the family of cubic systems mentioned in the statement of Main Theorem.

Let us consider the polynomials

$$\begin{aligned} C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y], \qquad i = 0,1,2,3,\\ D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y], \qquad i = 1,2,3.\end{aligned}$$

As it was shown in [33] the polynomials

$$\{C_0(a,x,y),\ C_1(a,x,y),\ C_2(a,x,y),\ C_3(a,x,y),\ D_1(a),\ D_2(a,x,y)\ D_3(a,x,y)\} \tag{2.2}$$

of degree one in the coefficients of systems (2.1) are GL-comitants of these systems.

Notation 2.1. Let $f, g \in \mathbb{R}[a,x,y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called the transvectant of index k of (f,g) (cf. [16,22])

Here $f(x,y)$ and $g(x,y)$ are polynomials in x and y of the degrees r and s, respectively, and $\boldsymbol{a} \in \mathbb{R}^{20}$ is the 20-tuple formed by all the coefficients of system (2.1).

We remark that the set of GL-invariant polynomials (2.2) could serve as bricks for the construction of any GL-invariant polynomial of an arbitrary degree. More precisely as it was proved in [37] we have the next result.

Theorem 2.2 ([37]). *Any GL-comitant of systems* (2.1) *can be constructed from the elements of the set* (2.2) *by using the operations:* $+$, $-$, $\times$, *and by applying the differential operation* $(f,g)^{(k)}$.

In order to define the needed invariant polynomials it is necessary to construct the following GL-comitants of second degree with respect to the coefficients of the initial systems:

$$\begin{array}{llll}
S_1=(C_0,C_1)^{(1)}, & S_8=(C_1,C_2)^{(2)}, & S_{15}=(C_2,D_2)^{(1)}, & S_{22}=(D_2,D_3)^{(1)},\\
S_2=(C_0,C_2)^{(1)}, & S_9=(C_1,D_2)^{(1)}, & S_{16}=(C_2,C_3)^{(1)}, & S_{23}=(C_3,C_3)^{(2)},\\
S_3=(C_0,D_2)^{(1)}, & S_{10}=(C_1,C_3)^{(1)}, & S_{17}=(C_2,C_3)^{(2)}, & S_{24}=(C_3,C_3)^{(4)},\\
S_4=(C_0,C_3)^{(1)}, & S_{11}=(C_1,C_3)^{(2)}, & S_{18}=(C_2,C_3)^{(3)}, & S_{25}=(C_3,D_3)^{(1)},\\
S_5=(C_0,D_3)^{(1)}, & S_{12}=(C_1,D_3)^{(1)}, & S_{19}=(C_2,D_3)^{(1)}, & S_{26}=(C_3,D_3)^{(2)},\\
S_6=(C_1,C_1)^{(2)}, & S_{13}=(C_1,D_3)^{(2)}, & S_{20}=(C_2,D_3)^{(2)}, & S_{27}=(D_3,D_3)^{(2)}.\\
S_7=(C_1,C_2)^{(1)}, & S_{14}=(C_2,C_2)^{(2)}, & S_{21}=(D_2,C_3)^{(1)}, &
\end{array}$$

We shall use here the following invariant polynomials constructed in [18] and [6–9] to characterize the cubic systems possessing invariant lines of total multiplicity greater than or equal to 8:

$$\begin{aligned}
&\mathcal{D}_1(a)=6S_{24}^3-\left[(C_3,S_{23})^{(4)}\right]^2,\ \mathcal{D}_2(a,x,y)=-S_{23},\\
&\mathcal{D}_3(a,x,y)=(S_{23},\,S_{23})^{(2)}-6C_3(C_3,\,S_{23})^{(4)},\\
&\mathcal{V}_1(a,x,y)=S_{23}+2D_3^2,\quad \mathcal{V}_2(a,x,y)=S_{26},\\
&\mathcal{V}_3(a,x,y)=\ 6S_{25}-3S_{23}-2D_3^2,\quad \mathcal{V}_4(a,x,y)=C_3\left[(C_3,S_{23})^{(4)}\ +\ 36\,(D_3,S_{26})^{(2)}\right],\\
&\mathcal{V}_5(a,x,y)=\ 6T_1(9A_5-7A_6)+2T_2(4T_{16}-T_{17})-3T_3(3A_1+5A_2)+3A_2T_4\\
&\qquad\qquad +36T_5^2-3T_{44},\\
&\mathcal{L}_1(a,x,y)=9C_2\,(S_{24}\ +\ 24S_{27})\ 12D_3\,(S_{20}+8S_{22})-12\,(S_{16},D_3)^{(2)}-3\,(S_{23},C_2)^{(2)}\\
&\qquad\qquad -16\,(S_{19},C_3)^{(2)}+12\,(5S_{20}+24S_{22},C_3)^{(1)},\\
&\mathcal{L}_2(a,x,y)=32\,(13\,S_{19}+33S_{21},D_2)^{(1)}+84\,(9\,S_{11}-2\,S_{14},D_3)^{(1)}\\
&\qquad\qquad -448\,(S_{18},C_2)^{(1)}\ +8D_2\,(12S_{22}+35S_{18}-73S_{20})-56\,(S_{17},C_2)^{(2)}\\
&\qquad\qquad -63\,(S_{23},C_1)^{(2)}+756D_3S_{13}-1944D_1S_{26}+112\,(S_{17},D_2)^{(1)}\\
&\qquad\qquad -378\,(S_{26},C_1)^{(1)}+9C_1\,(48S_{27}-35S_{24}),\\
&\mathcal{L}_6(a,x,y)=2A_3-19A_4,\quad \mathcal{L}_7(a,x,y)=(T_{10},T_{10})^{(2)},\\
&\mathcal{U}_2(a,x,y)=6(S_{23}-3S_{25},S_{26})^{(1)}-3S_{23}(S_{24}-8S_{27})\\
&\qquad\qquad -24S_{26}^2+2C_3(C_3,S_{23})^{(4)}+24D_3(D_3,S_{26})^{(1)}+24D_3^2S_{27},\\
&\mathcal{K}_1(a,x,y)=\left(3223T_2^2T_{140}+2718T_4T_{140}-829T_2^2T_{141},T_{133}\right)^{(10)}/2,\quad \mathcal{K}_2(a,x,y)=T_{74},\\
&\mathcal{K}_3(a,x,y)=Z_1Z_2Z_3,\quad \mathcal{K}_4(a,x,y)=T_{13}-2T_{11},\\
&\mathcal{K}_5(a,x,y)=45T_{42}-T_2T_{14}+2T_2T_{15}+12T_{36}+45T_{37}-45T_{38}+30T_{39},\\
&\mathcal{K}_6(a,x,y)=4T_1T_8(2663T_{14}-8161T_{15})+6T_8(178T_{23}+70T_{24}+555T_{26})\\
&\qquad\qquad +18T_9(30T_2T_8-488T_1T_{11}-119T_{21})+5T_2(25T_{136}+16T_{137})\\
&\qquad\qquad -15T_1(25T_{140}-11T_{141})-165T_{142},
\end{aligned}$$

$$
\begin{aligned}
\mathcal{K}_7(a) &= A_1 + 3A_2, \quad \mathcal{K}_8(a,x,y) = 10A_4T_1 - 3T_2T_{15} + 4T_{36} - 8T_{37},\\
\mathcal{K}_9(a,x,y) &= 3T_1(11T_{15} - 8T_{14}) - T_{23} + 5T_{24},\\
\mathcal{N}_1(a,x,y) &= 4C_2(27D_1D_3 - 8D_2^2) + 2C_2(20S_{15} - 4S_{14} + 39S_{12}) + 18C_1(3S_{21} - D_2D_3)\\
&\quad + 54D_3(3S_4 - S_7) - 288C_3S_9 + 54(S_7, C_3)^{(1)} - 567(S_4, C_3)^{(1)} + 135C_0D_3^2,\\
\mathcal{N}_2(a,x,y) &= 2C_2D_3 - 3C_3D_2, \quad \mathcal{N}_3(a,x,y) = C_2D_3 + 3S_{16},\\
\mathcal{N}_4(a,x,y) &= D_2D_3 + 9S_{21} - 2S_{17}, \quad \mathcal{N}_5(a,x,y) = S_{17} + 2S_{19},\\
\mathcal{N}_6(a,x,y) &= 6C_3(S_{12} + 6S_{11}) - 9C_1(S_{23} + S_{25}) - 8(S_{16}, C_2)^{(1)} - C_3D_2^2,\\
\mathcal{N}_7(a,x,y) &= 6C_3(12S_{11} - S_{12} - 6D_1D_3) - 21C_1S_{23} - 24(S_{16}, C_2)^{(1)} + 3C_1S_{25}\\
&\quad + 4D_2(S_{16} + 2D_2C_3 - C_2D_3), \quad \mathcal{N}_8(a,x,y) = D_2^2 - 4D_1D_3,\\
\mathcal{N}_9(a,x,y) &= C_2^2 - 3C_1C_3, \quad \mathcal{N}_{10}(a,x,y) = 2C_2D_1 + 3S_4, \quad \mathcal{N}_{11}(a) = S_{13},\\
\mathcal{N}_{12}(a,x,y) &= -32D_3^2S_2 - 108D_1D_3S_{10} + 108C_3D_1S_{11} - 18C_1D_3S_{11} - 27S_{10}S_{11}\\
&\quad + 4C_0D_3(9D_2D_3 + 4S_{17}) + 108S_4S_{21},\\
\mathcal{N}_{14}(a,x,y) &= 36D_2D_3(S_8 - S_9) + D_1(108D_2^2D_3 - 54D_3(S_{14} - 8S_{15}))\\
&\quad + 2S_{14}(S_{14} - 22S_{15}) - 8D_2^2(3S_{14} + S_{15}) - 9D_3(S_{14}, C_1)^{(1)} - 16D_2^4,\\
N_{15}(a,x,y) &= 216D_1D_3(63S_{11} - 104D_2^2 - 136S_{15}) + 4536D_3^2S_6 + 4096D_2^4\\
&\quad + 120S_{14}^2 + 992D_2(S_{14}, C_2)^{(1)} - 135D_3\big[28(S_{17}, C_0)^{(1)} + 5(S_{14}, C_1)^{(1)}\big],\\
N_{16}(a,x,y) &= 2C_1D_3 + 3S_{10}, \quad N_{17}(a,x,y) = 6D_1D_3 - 2D_2^2 - (C_3, C_1)^{(2)},\\
N_{18}(a,x,y) &= 2D_2^3 - 6D_1D_2D_3 - 12D_3S_5 + 3D_3S_8,\\
N_{19}(a,x,y) &= C_1D_3(18D_1^2 - S_6) + C_0(4D_2^3 - 12D_1D_2D_3 - 18D_3S_5 + 9D_3S_8) + 6C_2D_1S_8\\
&\quad + 2\big(9D_2D_3S_1 - 4D_2^2S_2 + 12D_1D_3S_2 - 9C_3D_1S_6 - 9D_3(S_4, C_0)^{(1)}\big),\\
N_{20}(a,x,y) &= 3D_2^4 - 8D_1D_2^2D_3 - 8D_3^2S_6 - 16D_1D_3S_{11} + 16D_2D_3S_9,\\
N_{21}(a,x,y) &= 2D_1D_2^2D_3 - 4D_3^2S_6 + D_2D_3S_8 + D_1(S_{23}, C_1)^{(1)},\\
N_{22}(a,x,y) &= T_8, \quad N_{23}(a,x,y) = T_6, \quad N_{24}(a,x,y) = 2T_3T_{74} - T_1T_{136},\\
N_{25}(a,x,y) &= 5T_3T_6 - T_1T_{23}, \; N_{26} = 9T_{135} - 480T_6T_8 - 40T_2T_{74} - 15T_2T_{75},\\
N_{27}(a,x,y) &= 9T_2T_9(2T_{23} - 5T_{24} - 80T_{25}) + 144T_{25}(T_{23} + 5T_{24} + 15T_{26})\\
&\quad - 9(T_{23}^2 - 5T_{24}^2 - 33T_9T_{76}), \quad N_{28}(a,x,y) = T_3 + T_4,\\
W_1(a,x,y) &= 2C_2D_3 - 3C_3D_2,\\
W_2(a,x,y) &= 6C_3(S_{12} + 6S_{11}) - 9C_1(S_{23} + S_{25}) - 8(S_{16}, C_2)^{(1)} - C_3D_2^2,\\
W_3(a,x,y) &= 12D_1C_3 - S_{10}, \quad W_4(a,x,y) = -27S_4 + 4S_7,\\
W_5(a,x,y) &= 3D_1^2C_1 + 4D_1S_2 - 3(S_4, C_0)^{(1)},\\
W_6(a,x,y) &= 2C_2D_1 + 3S_4, \quad W_7(a,x,y) = (S_{10}, D_2)^{(1)},\\
W_8(a,x,y) &= 4C_2(27D_1D_3 - 8D_2^2) + 2C_2(20S_{15} - 4S_{14} + 39S_{12}) + 18C_1(3S_{21} - D_2D_3)\\
&\quad + 54D_3(3S_4 - S_7) - 288C_3S_9 + 54(S_7, C_3)^{(1)} - 567(S_4, C_3)^{(1)} + 135C_0D_3^2,\\
W_9(a,x,y) &= 3S_6D_2^2 + 4S_3D_2^2 - 6D_1D_2S_9,\\
W_{10}(a,x,y) &= 18D_1^2C_2 + 15S_6C_2 - 6D_1C_1D_2 + 4C_0D_2^2 + 27D_1S_4 - 6C_1S_9,\\
W_{11}(a,x,y) &= 9C_0D_3^5 - 6D_3^4(C_1D_2 - S_7) + 4C_2D_3^3(D_2^2 + S_{14} - 2S_{15})\\
&\quad - 12C_3D_3^2[5D_2S_{14} - 4D_2S_{15} - 7(S_{14}, C_2)^{(1)}],\\
W_{12}(a,x,y) &= -480T_6T_8 + 9T_{135} - 40T_2T_{74} - 15T_2T_{75},
\end{aligned}
$$

where

$$Z_1 = 2C_1D_2D_3 - 9C_0(S_{25} + 2D_3^2) + 4C_2(9D_1D_3 + S_{14}) - 3C_3(6D_1D_2 + 5S_8) + 36D_3S_4,$$
$$Z_2 = 12D_1S_{17} + 2D_2(3S_{11} - 2S_{14}) + 6D_3(S_8 - 6S_5) - 9(S_{25}, C_0)^{(1)},$$
$$\begin{aligned} Z_3 = {} & 48D_1^3C_3 + 12D_1^2(C_1D_3 - C_2D_2) + 36D_1(C_0S_{17} - C_3S_6) - 16D_2^2S_2 - 16S_2S_{14} \\ & + 2C_0D_2(3S_{11} + 2S_{14}) + 3D_3(8D_2S_1 + 3C_0S_8 - 2C_1S_6) - 9S_4S_8 \\ & - 216C_3(S_5, C_0)^{(1)}) + 6C_2(D_2S_6 - 4(S_{14}, C_0)^{(1)}) + 54D_1D_2(S_4 + D_3C_0). \end{aligned}$$

Here the polynomials

$$A_1 = S_{24}/288, \quad A_2 = S_{27}/72, \; A_3 = (72D_1A_2 + \big(S_{22}, D_2\big)^{(1)})/24,$$
$$A_4 = \big[9D_1(S_{24} - 288A_2) + 4\big(9S_{11} - 2S_{14}, D_3\big)^{(2)} + 8\big(3S_{18} - S_{20} - 4S_{22}, D_2\big)^{(1)}\big]/2^7/3^3,$$
$$A_5 = \big(S_{23}, C_3\big)^{(4)}/2^7/3^5, \quad A_6 = \big(S_{26}, D_3\big)^{(2)}/2^5/3^3$$

are affine invariants, whereas the polynomials

$$T_1 = C_3, \quad T_2 = D_3, \quad T_3 = S_{23}/18, \quad T_4 = S_{25}/6, \quad T_5 = S_{26}/72,$$
$$\begin{aligned} T_6 = {} & \big[3C_1(D_3^2 - 9T_3 + 18T_4) - 2C_2(2D_2D_3 - S_{17} + 2S_{19} - 6S_{21}) \\ & + 2C_3(2D_2^2 - S_{14} + 8S_{15})\big]/2^4/3^2, \end{aligned}$$
$$T_8 = \big[5D_2(D_3^2 + 27T_3 - 18T_4) + 20D_3S_{19} + 12\big(S_{16}, D_3\big)^{(1)} - 8D_3S_{17}\big]/5/2^5/3^3,$$
$$\begin{aligned} T_9 = {} & \big[9D_1(9T_3 - 18T_4 - D_3^2) + 2D_2(D_2D_3 - 3S_{17} - S_{19} - 9S_{21}) + 18\big(S_{15}, C_3\big)^{(1)} \\ & - 6C_2(2S_{20} - 3S_{22}) + 18C_1S_{26} + 2D_3S_{14}\big]/2^4/3^3, \; T_{10} = \big(S_{23}, D_3\big)^{(1)}/2^5/3^3, \end{aligned}$$
$$\begin{aligned} T_{11} = {} & \big[\big(D_3^2 - 9T_3 + 18T_4, C_2\big)^{(2)} - 6\big(D_3^2 - 9T_3 + 18T_4, D_2\big)^{(1)} - 12\big(S_{26}, C_2\big)^{(1)} \\ & + 12D_2S_{26} + 432(A_1 - 5A_2)C_2\big]/2^7/3^4, \end{aligned}$$
$$\begin{aligned} T_{13} = {} & \big[27(T_3, C_2)^{(2)} - 18(T_4, C_2)^{(2)} + 48D_3S_{22} - 216(T_4, D_2)^{(1)} + 36D_2S_{26} \\ & - 1296C_2A_1 - 7344C_2A_2 + (D_3^2, C_2)^{(2)}\big]/2^7/3^4, \end{aligned}$$
$$T_{14} = \big[\big(8S_{19} + 9S_{21}, D_2\big)^{(1)} - D_2(8S_{20} + 3S_{22}) + 18D_1S_{26} + 1296C_1A_2\big]/2^4/3^3,$$
$$\begin{aligned} T_{15} = {} & 8\big(9S_{19} + 2S_{21}, D_2\big)^{(1)} + 3\big(9T_3 - 18T_4 - D_3^2, C_1\big)^{(2)} - 4\big(S_{17}, C_2\big)^{(2)} \\ & + 4\big(S_{14} - 17S_{15}, D_3\big)^{(1)} - 8\big(S_{14} + S_{15}, C_3\big)^{(2)} + 432C_1(5A_1 + 11A_2) \\ & + 36D_1S_{26} - 4D_2(S_{18} + 4S_{22})\big]/2^6/3^3, \end{aligned}$$
$$T_{16} = \big(S_{23}, D_3\big)^{(2)}/2^6/3^3, \; T_{17} = \big(S_{26}, D_3\big)^{(1)}/2^5/3^3,$$
$$T_{21} = \big(T_8, C_3\big)^{(1)}, \quad T_{23} = \big(T_6, C_3\big)^{(2)}/6, \quad T_{24} = \big(T_6, D_3\big)^{(1)}/6,$$
$$\begin{aligned} T_{25} = {} & (15552A_2C_1C_3 + D_3^2D_2^2 - 81D_2^2T_3 - 54D_2^2T_4 + 12D_3D_2S_{17} + 8D_3D_2S_{19} \\ & + 16\big((C_2, D_3)^{(1)}\big)^2 - 5184C_1D_3T_5 + 2592C_2D_2T_5 - 72C_3D_2S_{20})/2^6/3^4, \end{aligned}$$
$$T_{26} = \big(T_9, C_3\big)^{(1)}/4, \quad T_{30} = \big(T_{11}, C_3\big)^{(1)}, \quad T_{31} = \big(T_8, C_3\big)^{(2)}/24,$$
$$T_{32} = \big(T_8, D_3\big)^{(1)}/6, \quad T_{36} = \big(T_6, D_3\big)^{(2)}/12, \quad T_{37} = \big(T_9, C_3\big)^{(2)}/12,$$
$$T_{38} = \big(T_9, D_3\big)^{(1)}/12, \quad T_{39} = \big(T_6, C_3\big)^{(3)}/2^4/3^2, \quad T_{42} = \big(T_{14}, C_3\big)^{(1)}/2,$$
$$T_{44} = \big((S_{23}, C_3)^{(1)}, D_3\big)^{(2)}/5/2^6/3^3,$$

$$
\begin{aligned}
T_{74} = {} & \big[27C_0(9T_3-18T_4-D_3^2)^2 + C_1\big(-62208T_{11}C_3 - 3(9T_3-18T_4-D_3^2)\\
& \times(2D_2D_3 - S_{17} + 2S_{19} - 6S_{21})\big) + 20736T_{11}C_2^2 + C_2(9T_3-18T_4-D_3^2)\\
& \times(8D_2^2 + 54D_1D_3 - 27S_{11} + 27S_{12} - 4S_{14} + 32S_{15}) - 54C_3(9T_3-18T_4-D_3^2)\\
& \times(2D_1D_2 - S_8 + 2S_9) - 54D_1(9T_3-18T_4-D_3^2)S_{16}\\
& - 576T_6(2D_2D_3 - S_{17} + 2S_{19} - 6S_{21})\big]/2^8/3^4,\\
T_{75} = {} & 512C_3(D_2^3D_3 + D_2D_3S_{14} + D_2^2S_{19} + S_{15}S_{19} - 3D_2^2S_{21}) - 648C_0D_3^4\\
& + 144(9C_2D_1 + 2C_1D_2)D_3^3 + +216C_3(1728T_{13}C_1 - 9216T_8D_1 - 79S_8S_{25})\\
& - 64D_2^2(6C_2D_3^2 - 2D_3S_{16}C_2S_{23}) + 384C_2D_2(C_3S_{18} - 2D_3S_{19} - 3C_3S_{20} + 5D_3S_{21})\\
& - 36C_2D_3^2(3S_{11} - 18S_{12} + 8S_{14} - 32S_{15}) + 216C_0D_3^2(S_{23} - 12S_{25})\\
& + 72D_3^2(3C_3S_8 + 54D_1S_{16} + C_1S_{17} - 2C_1S_{19}) - 288C_2^2D_3(S_{18} - 3S_{20})\\
& - 1728(648T_8S_{10} + 11C_3S_9S_{25}) + 96C_2S_{17}(S_{17} + 4S_{21}) - 32C_3S_{14}(3S_{17} + 2S_{19})\\
& - 90C_2S_{12}(31S_{23} - 102S_{25}) + 3456C_3D_1D_2(S_{23} - 3S_{25}) - 216D_1S_{16}(S_{23} - 30S_{25})\\
& - 12C_3(32S_{14}S_{21} - 3S_8S_{23} - 696S_9S_{23}) + 96D_3(S_{14}S_{16} - 8S_{15}S_{16} - 2C_1D_2S_{23})\\
& - 128D_2S_{16}(S_{17} - 6S_{19} + 6S_{21}) - 216C_2D_1D_3(11S_{23} - 42S_{25}) - 12C_1S_{23}(3S_{17}\\
& + 10S_{19} - 360S_{21}) + 72C_1S_{25}(43S_{17} - 78S_{19} - 138S_{21}) + 54C_0(S_{23} - 6S_{25})\\
& \times(S_{23} + 6S_{25}) - 2C_2S_{23}(9S_{11} + 16S_{14} - 96S_{15}) + 12C_2(24S_{16}S_{20} - 8S_{16}S_{18}\\
& - 32S_{19}^2 + 711S_{11}S_{25}) + 48C_3D_2\big[\big(3S_{23} - 15D_3^2, C_1\big)^{(1)} - 4\big(S_{14} + 4S_{15}, C_3\big)^{(1)}\big]\\
& + 9C_2D_3\big[9\big(2S_{25} - S_{23}, C_1\big)^{(1)} + 16\big(S_{14}, C_3\big)^{(1)}\big] + 48S_{16}\big(S_{14}, C_3\big)^{(1)}\\
& + 9S_{16}\big(5S_{23} - 274D_3^2 + 54S_{25}, C_1\big)^{(1)},\\
T_{76} = {} & \big((T_6, C_3)^{(2)}, C_3\big)^{(1)}/36,\\
T_{133} = {} & (T_{74}, C_3)^{(1)}, \quad T_{135} = \big(T_{75}, C_3\big)^{(1)},\\
T_{136} = {} & \big(T_{74}, C_3\big)^{(2)}/24, \quad T_{137} = \big(T_{74}, D_3\big)^{(1)}/6, \quad T_{140} = \big(T_{74}, D_3\big)^{(2)}/12,\\
T_{141} = {} & \big(T_{74}, C_3\big)^{(3)}/36, \quad T_{142} = \big((T_{74}, C_3)^{(2)}, C_3\big)^{(1)}/72
\end{aligned}
$$

are T-comitants of cubic systems (2.1). We note that these polynomials are the elements of the polynomial basis of T-comitants up to degree six constructed by Iu. Calin [11].

Next we consider the differential operator $\mathcal{L} = x\cdot\mathbf{L}_2 - y\cdot\mathbf{L}_1$ constructed in [3] and acting on $\mathbb{R}[a,x,y]$, where

$$
\begin{aligned}
\mathbf{L}_1 = {} & 3a_{00}\frac{\partial}{\partial a_{10}} + 2a_{10}\frac{\partial}{\partial a_{20}} + a_{01}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{02}\frac{\partial}{\partial a_{12}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{21}} + a_{20}\frac{\partial}{\partial a_{30}}\\
& + 3b_{00}\frac{\partial}{\partial b_{10}} + 2b_{10}\frac{\partial}{\partial b_{20}} + b_{01}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{02}\frac{\partial}{\partial b_{12}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{21}} + b_{20}\frac{\partial}{\partial b_{30}},\\
\mathbf{L}_2 = {} & 3a_{00}\frac{\partial}{\partial a_{01}} + 2a_{01}\frac{\partial}{\partial a_{02}} + a_{10}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{20}\frac{\partial}{\partial a_{21}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{12}} + a_{02}\frac{\partial}{\partial a_{03}}\\
& + 3b_{00}\frac{\partial}{\partial b_{01}} + 2b_{01}\frac{\partial}{\partial b_{02}} + b_{10}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{20}\frac{\partial}{\partial b_{21}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{12}} + b_{02}\frac{\partial}{\partial b_{03}}.
\end{aligned}
$$

Using this operator and the affine invariant $\mu_0 = \mathrm{Resultant}_x\big(p_3(a,x,y), q_3(a,x,y)\big)/y^9$ we construct the following polynomials: $\mu_i(a,x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0)$, $i = 1,\ldots,9$, where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

These polynomials are in fact comitants of systems (2.1) with respect to the group $GL(2,\mathbb{R})$ (see [3]). The polynomial $\mu_i(a,x,y)$, $i \in \{0,1,\dots,9\}$ is homogeneous of degree 6 in the coefficients of systems (2.1) and homogeneous of degree i in the variables x and y. The geometrical meaning of these polynomial is revealed in the next lemma.

Lemma 2.3 ([2, 3]). *Assume that a cubic system (S) with coefficients $\tilde{a} \in \mathbb{R}^{12}$ belongs to the family (2.1).*

(i) The total multiplicity of all finite singularities of this system equals $9-k$ if and only if for every $i \in \{0,1,\dots,k-1\}$ we have $\mu_i(\tilde{a},x,y) = 0$ in the ring $\mathbb{R}[x,y]$ and $\mu_k(\tilde{a},x,y) \neq 0$. In this case the factorization $\mu_k(\tilde{a},x,y) = \prod_{i=1}^{k}(u_i x - v_i y) \neq 0$ over $\mathbb{C}$ indicates the coordinates $[v_i : u_i : 0]$ of those finite singularities of the system (S) which "have gone" to infinity. Moreover the number of distinct factors in this factorization is less than or equal to four (the maximum number of infinite singularities of a cubic system) and the multiplicity of each one of the factors $u_i x - v_i y$ gives us the number of the finite singularities of the system (S) which have coalesced with the infinite singular point $[v_i : u_i : 0]$.

(ii) The system (S) is degenerate (i.e. $\gcd(p,q) \neq \mathrm{const}$) if and only if $\mu_i(\tilde{a},x,y) = 0$ in $\mathbb{R}[x,y]$ for every $i \in \{0,1,\dots,9\}$.

3 Preliminary results: the classification theorem for the family of systems in $\mathbb{CSL}_8$

As it was mentioned in Section 2 our work is based on the results of the papers [4, 6–9] where the classification theorems according to the configurations of invariant straight lines for different subfamilies (i.e. systems with either 4 or 3 or 2 or 1 infinite distinct singularities) of systems in $\mathbb{CSL}_8$ were proved. More precisely, our results could be described as follows:

- In [6] the investigation of the subfamily of cubic systems in $\mathbb{CSL}_8$ possessing 4 distinct infinite singularities was done. As a result it was proved that a system in this class could possess only one of the 17 configurations *Config. 8.1–Config. 8.17* given in Figure 3.1.

- The subfamily of cubic systems in $\mathbb{CSL}_8$ possessing 3 distinct infinite singularities was considered in [7]. It was proved that a system in this class could possess only one of the 5 configurations *Config. 8.18–Config. 8.22* presented in Figure 3.1.

- In the articles [5] and [8] the subfamily of cubic systems in $\mathbb{CSL}_8$ with 2 distinct infinite singularities was investigated. This class contains cubic systems which could possess one of the 25 configurations *Config. 8.23–Config. 8.47* given in Figure 3.1.

- And finally in [9] were examined the cubic systems in $\mathbb{CSL}_8$ possessing a single infinite singular point (which is real). It was detected exactly 8 configurations *Config. 8.48–Config. 8.51* (see Figure 3.1) which could possess a cubic system belonging to this class.

We join here all these results (formulated as Main Theorems in the above mentioned articles) in the following classification theorem.

Theorem 3.1. *Assume that a cubic system (1.1) is non-degenerate, i.e. $\sum_{i=0}^{9} \mu_i^2 \neq 0$. Then this system belongs to the family $\mathbb{CSL}_8$, i.e. it possesses invariant straight lines of total multiplicity 8 (including the line at infinity with its own multiplicity), if and only if one of the sets of the conditions Cond. 1–Cond. 51* *given in Table 3.1 is satisfied. In addition, this system possesses exactly one of the 51 configurations Config. 8.j ($j \in \{1, \ldots 51\}$) of invariant straight lines shown in Figure 3.1. Furthermore the quotient set under the action of the affine group and time rescaling on $\mathbb{CSL}_8$ is formed by:*

(i) a discrete set of 22 orbits;

(ii) a set of 29 one-parameter families of orbits. A system of representatives of the quotient set is given in Table 4.1 (column 1).

4 Main results

In this section we state and prove the main results of this article.

Main Theorem. *Consider a non-degenerate cubic system (2.1), i.e. the condition $\sum_{i=0}^{9} \mu_i^2 \neq 0$ holds and assume that it belongs to the family $\mathbb{CSL}_8$. More precisely we assume that this system possesses one of the configurations Config. 8.j ($j = 1, \ldots, 51$) (see Figure 3.1), i.e. the corresponding set of the conditions Cond. j given in Table 3.1 is satisfied. Then:*

(A) *this system is integrable and it has the first integral $\mathcal{F}_j$ of generalized Darboux type (1.3)and the corresponding rational integrating factor $\mathcal{R}_j \in \mathbb{R}(x, y)$ ($j = 1, \ldots, 51$) as it is indicated in Table 4.1 (column 3). This table also lists the corresponding invariant lines and their multiplicities, see column 2;*

(B) *the phase portrait of this system corresponds to one of the 52 phase portraits P. 8.1–P. 8.5, P. 8.6(a), P. 8.6(b), P. 8.7–P. 8.51 (see Figure 4.1) if and only if the associated affine invariant conditions given in Table 4.1 (column 4) are satisfied.*

(C) *Among the 52 phase portraits given in Figure 4.1 there are exactly 30 topologically distinct phase portraits as at is indicated in Diagram 4.1 using the geometric invariants defined in Remark 4.2.*

Corollary 4.1. *All the systems in $\mathbb{CSL}_8$ have elementary real first integrals. We only list below in Table 4.2 all real first integrals which correspond to those in the column 3 of Table 4.1 which are given there in complex form.*

Remark 4.2. In order to distinguish topologically the phase portraits of the systems we obtained, we use the following geometric invariants:

- The number $IS^{\mathbb{R}}$ of real infinite singularities.
- The number $FS^{\mathbb{R}}$ of real finite singularities.
- The number Sep^f of separatrices beginning or ending at a finite singularity.
- The number Sep^∞ of separatrices beginning or ending at an infinite singularity.
- The number $FSep$ of separatrices connecting finite singularities.
- The number SC of separatrix connections.
- The maximum number ES^∞ of elliptic sectors in the vicinity of an infinite singularity.

Proof. **(A)** The expressions for the integrating factors and the first integrals presented in Table 4.1 (see column 3) follow, after some easy calculations, by using Theorems 3.1 and 1.4.

(B) We split the proof of this statement in three parts in accordance with the following three groups of the configurations (see Figure 3.1) :

(α) *Configurations 8.1– 8.17*: $x = 2$ the corresponding phase portraits are constructed in [36];

(β) *Configurations 8.18–8.23, 26, 31, 32, 33, 36, 38, 42, 47–51*: the corresponding canonical systems do not depend on parameters;

Configuration of $\mathbb{ISL}s$	The necessary and sufficient conditions		Notation
Config. 8.1	$\mathcal{D}_1 > 0,\ \mathcal{D}_2 > 0,\ \mathcal{D}_3 > 0,$ $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0,\ \mathcal{K}_2 \neq 0$	$\mathcal{K}_3 > 0$	Cond. 1
Config. 8.2		$\mathcal{K}_3 < 0$	Cond. 2
Config. 8.3		$\mathcal{K}_3 = 0$	Cond. 3
Config. 8.4	$\mathcal{D}_1 > 0,\ \mathcal{D}_2 > 0,\ \mathcal{D}_3 > 0,$ $\mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0,\ \ \mathcal{D}_4 \neq 0$	$\mathcal{L}_1 \neq 0$ *and* $\mathcal{K}_7 > 0$	Cond. 4
Config. 8.5		$\mathcal{L}_1 \neq 0$ *and* $\mathcal{K}_7 < 0$	Cond. 5
Config. 8.6		$\mathcal{L}_1 = 0$	Cond. 6
Config. 8.7	$\mathcal{D}_1 > 0,\ \mathcal{D}_2 > 0,\ \mathcal{D}_3 > 0,$ $\mathcal{V}_3 = \mathcal{K}_2 = \mathcal{K}_4 = \mathcal{K}_8 = 0,\ \ \mathcal{D}_4 \neq 0$	$\mathcal{K}_9 > 0$	Cond. 7
Config. 8.8		$\mathcal{K}_9 < 0$	Cond. 8
Config. 8.9		$\mathcal{K}_9 = 0$	Cond. 9
Config. 8.10	$\mathcal{D}_1 < 0,\ \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0,\ \mathcal{K}_2 \neq 0$	$\mathcal{K}_3 > 0$	Cond. 10
Config. 8.11		$\mathcal{K}_3 < 0$	Cond. 11
Config. 8.12		$\mathcal{K}_3 = 0$	Cond. 12
Config. 8.13	$\mathcal{D}_1 < 0,\ \ \mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0,\ \ \mathcal{D}_4 \neq 0$	$\mathcal{L}_1 \neq 0$	Cond. 13
Config. 8.14		$\mathcal{L}_1 = 0$	Cond. 14
Config. 8.15	$\mathcal{D}_1 < 0,\ \ \mathcal{V}_3 = \mathcal{K}_2 = \mathcal{K}_4 = \mathcal{K}_8 = 0,\ \ \mathcal{D}_4 \neq 0$	$\mathcal{K}_9 > 0$	Cond. 15
Config. 8.16		$\mathcal{K}_9 < 0$	Cond. 16
Config. 8.17		$\mathcal{K}_9 = 0$	Cond. 17
Config. 8.18	$\mathcal{D}_3 \neq 0,\ \ \mathcal{D}_1 = \mathcal{V}_4 = \mathcal{V}_5 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0$	$\mathcal{K}_7 \neq 0,\ \mathcal{L}_1 \neq 0$	Cond. 18
Config. 8.19		$\mathcal{K}_7 \neq 0,\ \mathcal{L}_1 = 0$	Cond. 19
Config. 8.20		$\mathcal{K}_7 = \mathcal{V}_5 = \mathcal{L}_6 = 0, \mathcal{L}_1 \neq 0$	Cond. 20
Config. 8.21	$\mathcal{D}_1 =\ \ \mathcal{V}_3 = \mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0,\ \ \mathcal{L}_7 \neq 0, \mathcal{D}_3 \neq 0$	$\mathcal{K}_9 > 0$	Cond. 21
Config. 8.22		$\mathcal{K}_9 < 0$	Cond. 22
Config. 8.23	$\mathcal{D}_2\, N_2 N_3 \neq 0, \mathcal{D}_1 = \mathcal{D}_3 = \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_4 = N_5 = N_6 = N_7 = 0$		Cond. 23

Table 3.1: Conditions for the realization of the configurations

(γ) *Configurations* 8.24, 8.25, 27–30, 34, 35, 37, 39, 40, 41, 43–46: each one of the corresponding canonical systems depends of one parameter.

<table>
<tr><th>Configuration of $\mathbb{ISL}s$</th><th colspan="2">The necessary and sufficient conditions</th><th>Notation</th></tr>
<tr><td>Config. 8.24</td><td rowspan="4">$\mathcal{D}_2N_2N_9\neq0,\ \mathcal{D}_1=\mathcal{D}_3=\mathcal{D}_4=0,$ $N_3=\mathcal{V}_1=\mathcal{V}_3=\mathcal{K}_5=N_1=N_4=N_6=N_8=0$</td><td>$N_{11}<0$</td><td>Cond. 24</td></tr>
<tr><td>Config. 8.25</td><td>$N_{10}>0, N_{11}>0$</td><td>Cond. 25</td></tr>
<tr><td>Config. 8.26</td><td>$N_{10}=0$</td><td>Cond. 26</td></tr>
<tr><td>Config. 8.27</td><td>$N_{10}<0$</td><td>Cond. 27</td></tr>
<tr><td>Config. 8.28</td><td rowspan="3">$\mathcal{D}_2N_2N_{13}\neq0,\ \mathcal{D}_1=\mathcal{D}_3=\mathcal{D}_4=0,$ $N_3=\mathcal{V}_1=\mathcal{V}_3=\mathcal{K}_5=N_1=N_5=N_8=N_{12}=0$</td><td>$N_{15}<0$</td><td>Cond. 28</td></tr>
<tr><td>Config. 8.29</td><td>$N_{14}<0, N_{15}>0$</td><td>Cond. 29</td></tr>
<tr><td>Config. 8.30</td><td>$N_{14}>0$</td><td>Cond. 30</td></tr>
<tr><td>Config. 8.31</td><td rowspan="2">$\mathcal{D}_2N_{10}N_{16}\neq0,\ \mathcal{D}_1=\mathcal{D}_3=\mathcal{D}_4=0,$ $N_2=N_3=\mathcal{V}_1=\mathcal{V}_3=\mathcal{K}_5=N_1=N_{17}=N_{18}=0$</td><td>$N_{10}<0$</td><td>Cond. 31</td></tr>
<tr><td>Config. 8.32</td><td>$N_{10}>0$</td><td>Cond. 32</td></tr>
<tr><td>Config. 8.33</td><td colspan="2">$\mathcal{D}_2N_{16}\neq0,\quad N_2=N_3=\mathcal{V}_1=\mathcal{V}_3=\mathcal{K}_5=N_1=N_{10}=N_{17}=N_{18}=0$</td><td>Cond. 33</td></tr>
<tr><td>Config. 8.34</td><td rowspan="5">$\mathcal{D}_2N_{18}\neq0,\ \mathcal{D}_1=\mathcal{D}_3=\mathcal{D}_4=0,$ $N_2=N_3=\mathcal{V}_1=\mathcal{V}_3=\mathcal{K}_5=N_1=N_{16}=N_{19}=0$</td><td>$N_{21}<0$</td><td>Cond. 34</td></tr>
<tr><td>Config. 8.35</td><td>$N_{20}>0, N_{21}>0$</td><td>Cond. 35</td></tr>
<tr><td>Config. 8.36</td><td>$N_{20}=0$</td><td>Cond. 36</td></tr>
<tr><td>Config. 8.37</td><td>$N_{20}<0$</td><td>Cond. 37</td></tr>
<tr><td>Config. 8.38</td><td>$N_{21}=0$</td><td>Cond. 38</td></tr>
<tr><td>Config. 8.39</td><td rowspan="2">$\mathcal{D}_2\mathcal{V}_3\mathcal{K}_6\neq0,$ $\mathcal{V}_1=\mathcal{L}_1=\mathcal{L}_2=N_{22}=N_{23}=N_{24}=0$</td><td>$\mu_6<0$</td><td>Cond. 39</td></tr>
<tr><td>Config. 8.40</td><td>$\mu_6>0$</td><td>Cond. 40</td></tr>
<tr><td>Config. 8.41</td><td rowspan="3">$\mathcal{D}_2\mathcal{V}_3N_{24}\neq0,\ \mathcal{D}_1=\mathcal{D}_3=\mathcal{D}_4=0,$ $\mathcal{V}_1=\mathcal{L}_1=\mathcal{L}_2=N_{22}=N_{23}=\mathcal{K}_6=0$</td><td>$\mu_6<0$</td><td>Cond. 41</td></tr>
<tr><td>Config. 8.42</td><td>$\mu_6=0$</td><td>Cond. 42</td></tr>
<tr><td>Config. 8.43</td><td>$\mu_6>0$</td><td>Cond. 43</td></tr>
<tr><td>Config. 8.44</td><td rowspan="4">$\mathcal{D}_2\mathcal{V}_1\mathcal{V}_3\neq0,\ \mathcal{D}_1=\mathcal{D}_3=\mathcal{D}_4=0,$ $\mathcal{V}_5=\mathcal{U}_2=\mathcal{K}_4=\mathcal{K}_5=\mathcal{K}_6=N_{24}=N_{25}=N_{26}=N_{27}=0$</td><td>$\mu_6<0$</td><td>Cond. 44</td></tr>
<tr><td>Config. 8.45</td><td>$\mu_6>0, N_{28}<0$</td><td>Cond. 45</td></tr>
<tr><td>Config. 8.46</td><td>$\mu_6>0, N_{28}>0$</td><td>Cond. 46</td></tr>
<tr><td>Config. 8.47</td><td>$\mu_6=0$</td><td>Cond. 47</td></tr>
<tr><td>Config. 8.48</td><td colspan="2">$\mathcal{D}_1=\mathcal{D}_2=\mathcal{D}_3=\mathcal{V}_1=\mathcal{L}_2=N_{23}=W_1=W_2=W_3=W_4=0$</td><td>Cond. 48</td></tr>
<tr><td>Config. 8.49</td><td colspan="2">$\mathcal{D}_1=\mathcal{D}_2=\mathcal{D}_3=\mathcal{V}_1=\mathcal{L}_2=N_{23}=W_1=W_2=N_{16}=W_5=0,\ W_6\neq0$</td><td>Cond. 49</td></tr>
<tr><td>Config. 8.50</td><td colspan="2">$\mathcal{D}_1=\mathcal{D}_2=\mathcal{D}_3=\mathcal{V}_1=\mathcal{L}_2=N_3=W_7=W_8=W_9=W_{10}=0,\ N_{23}\neq0$</td><td>Cond. 50</td></tr>
<tr><td>Config. 8.51</td><td colspan="2">$\mathcal{D}_1=\mathcal{D}_2=\mathcal{D}_3=\mathcal{V}_5=\mathcal{K}_4=\mathcal{K}_5=\mathcal{K}_8=\mathcal{K}_9=N_2=\mathcal{K}_6=W_{11}=W_{12}=0,$ $\mathcal{V}_1\neq0$</td><td>Cond. 51</td></tr>
</table>

Table 3.1 (continuation): Conditions for the realization of the configurations

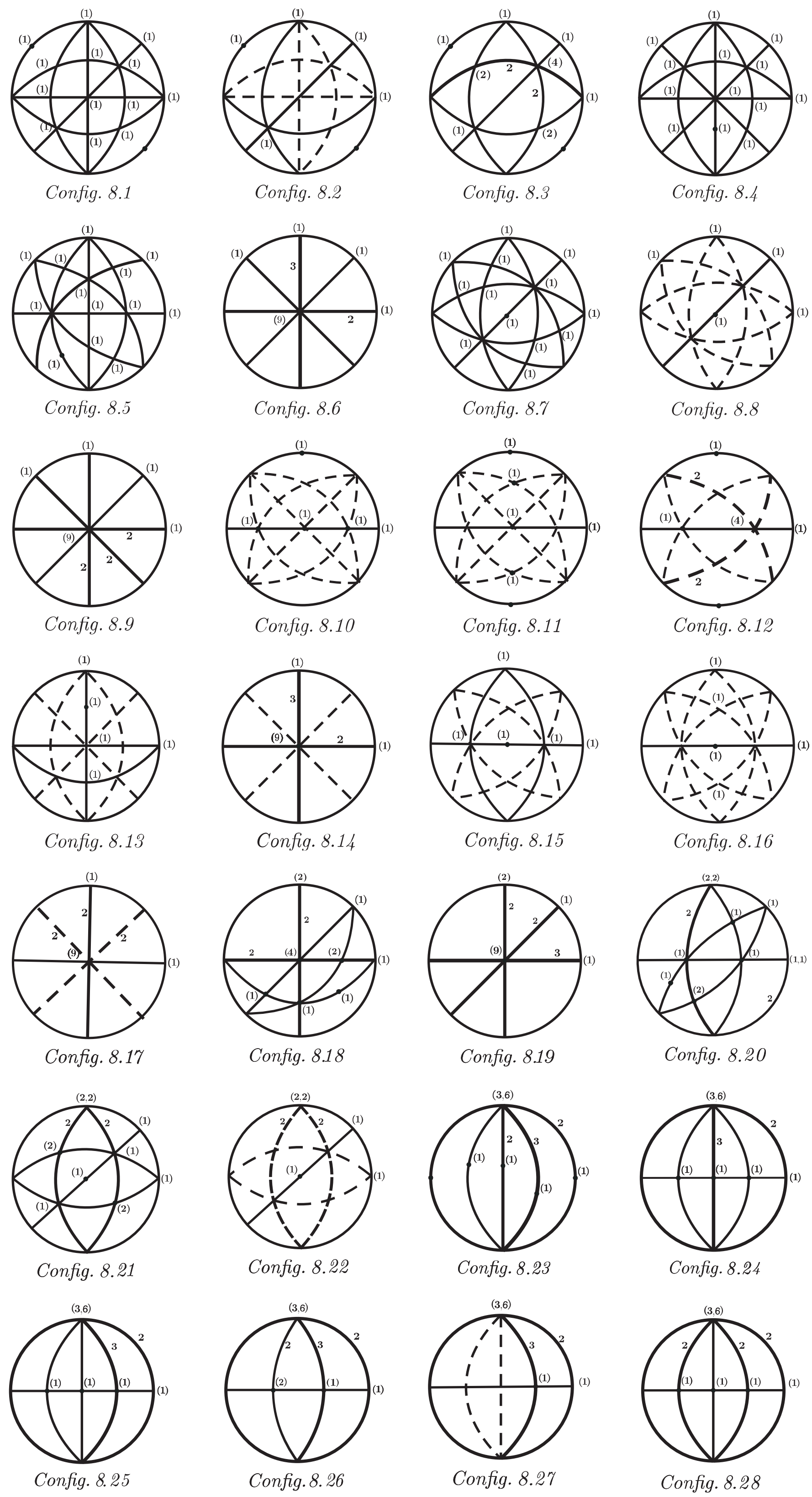

Figure 3.1: Configurations of invariant lines for systems in $\mathbb{CSL}_8$

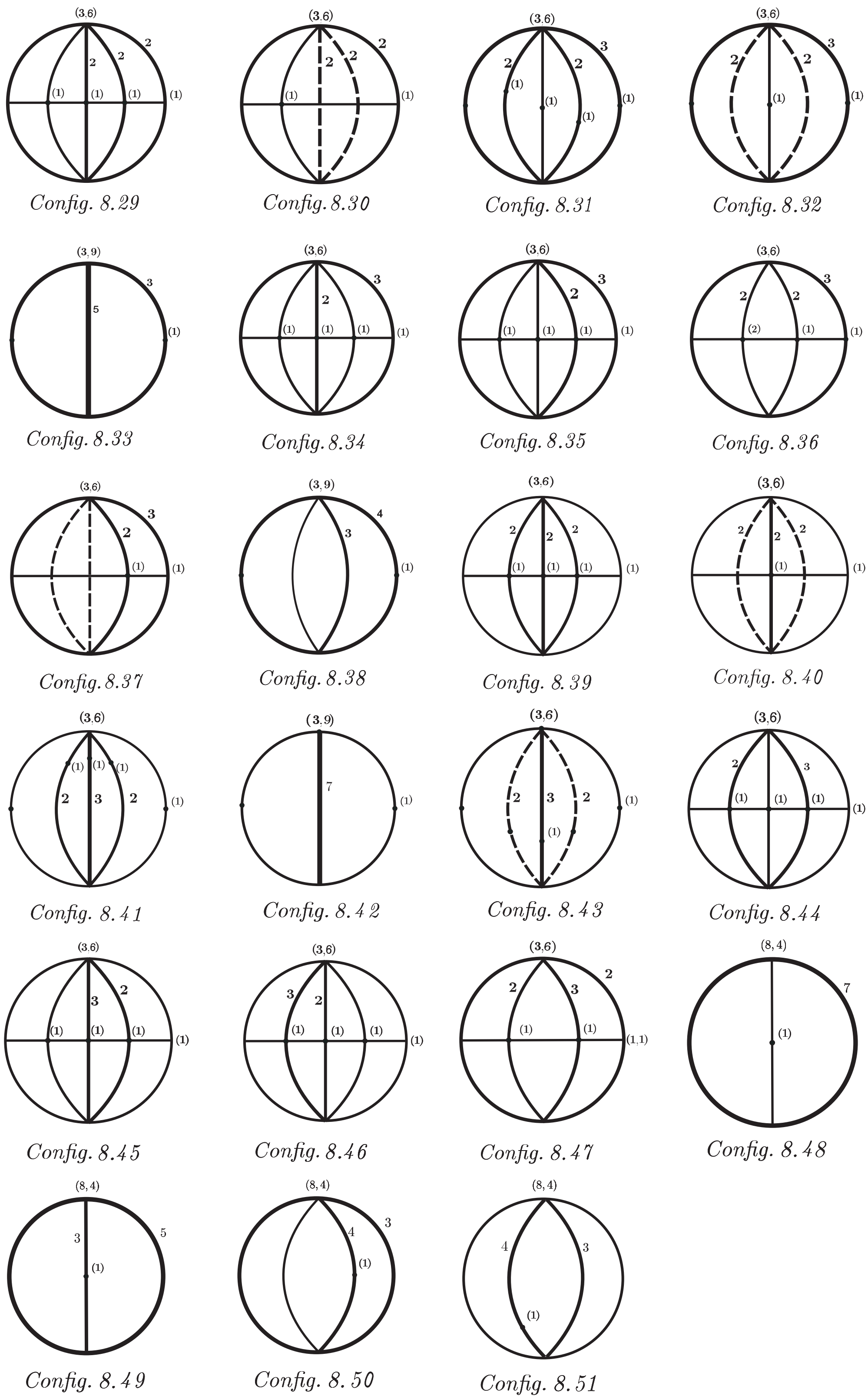

Figure 3.1 (continuation): Configurations of invariant lines for systems in $\mathbf{CSL}_8$

We examine each one of these three groups of configurations.

(α) We remark that *Config. 8.1–Config. 8.17* correspond to systems in $\mathbb{CSL}_8$ which possess 4 distinct infinite singularities (real or/and complex). In the article [36] the authors have detected 17 configurations of invariant lines denoted by $(I.1)$–$(I.17)$ and have determined 18 phase portraits corresponding to these configurations. We mention that the configurations $(I.1)$–$(I.17)$ from [36] coincide with the first 17 configurations *Config. 8.1–Config. 8.17* and we have the following correspondence:

$$\begin{array}{lll}
\textit{Config. 8.1} \Leftrightarrow (I.1); & \textit{Config. 8.2} \Leftrightarrow (I.3); & \textit{Config. 8.3} \Leftrightarrow (I.2); \\
\textit{Config. 8.4} \Leftrightarrow (I.7); & \textit{Config. 8.5} \Leftrightarrow (I.8); & \textit{Config. 8.6} \Leftrightarrow (I.9); \\
\textit{Config. 8.7} \Leftrightarrow (I.12); & \textit{Config. 8.8} \Leftrightarrow (I.16); & \textit{Config. 8.9} \Leftrightarrow (I.13); \\
\textit{Config. 8.10} \Leftrightarrow (I.4); & \textit{Config. 8.11} \Leftrightarrow (I.5); & \textit{Config. 8.12} \Leftrightarrow (I.6); \\
\textit{Config. 8.13} \Leftrightarrow (I.11); & \textit{Config. 8.14} \Leftrightarrow (I.10); & \textit{Config. 8.15} \Leftrightarrow (I.14); \\
\textit{Config. 8.16} \Leftrightarrow (I.17); & \textit{Config. 8.17} \Leftrightarrow (I.15). &
\end{array} \tag{4.1}$$

As it is proved in the article [36], each one of the configurations $(I.1)$–$(I.17)$ leads to a single phase portrait with the exception of configuration $(I.9)$, which leads to 2 topologically distinct phase portraits. We observe that $(I.9)$ corresponds to *Config. 8.6*. So we conclude that the affine invariant conditions provided by Theorem 3.1 for the realization of configuration *Config. 8.j* for $j = 1, \dots, 5, 7, \dots, 17$ guarantees the realization of the corresponding phase portrait $P.8.j$, too. It remains to examine the two phase portraits given by *Config. 8.6* and to determine the necessary and sufficient conditions for their realization.

According to [36] a cubic system possessing the configuration $(I.9)$ could be brought via an affine transformation and time rescaling to the form

$$\dot{x} = x^3, \quad \dot{y} = y^2\big[ax + (1-a)y\big], \quad a \in \mathbb{R} \setminus \{0, 1, 3/2, 2, 3\}. \tag{4.2}$$

As it was proved in [36] these systems have the phase portrait given by Figure 1.9a if $a < 1$ and by Figure 1.9b if $a > 1$ in [36].

On the other hand according to [6] a cubic system possessing the configuration *Config. 8.6* could be brought via an affine transformation and time rescaling to the form

$$\dot{x} = rx^3, \quad \dot{y} = y^2\big[(r-1)x + y\big], \quad (r+2)(r+1)(2r+1)r(r-1) \neq 0. \tag{4.3}$$

Remark 4.3. We observe that the rescaling $(x, y, t) \mapsto (x, -ry, t/r^2)$ brings these systems to the same form with $r_1 = 1/r$. Therefore we may assume $-1 < r < 1$ and $r \notin \{-1/2, 0\}$. Analogously considering systems (4.2) with parameter a we observe that the rescaling $(x, y, t) \mapsto ((a-1)x, y, t/(a-1)^2)$ brings these systems to the same form with $a_1 = a/(a-1)$. So in this case we may consider $0 < a < 2$ and $a \notin \{1, 3/2\}$.

It remains to observe that via the rescaling $(x, y, t) \mapsto (x, -ry, t/r)$ systems (4.3) become

$$\dot{x} = x^3, \quad \dot{y} = y^2\big[(1-r)x + ry\big],$$

which coincide with (4.2) for $r = 1 - a$. Therefore considering the above remark we obtain that the condition $a < 1$ (respectively $a > 1$) for systems (4.2) is equivalent to $r > 0$ (respectively $r < 0$) for systems (4.3). On the other hand evaluating for systems (4.3) the invariant polynomial K_7 we obtain $K_7 = 4r$ and hence $\operatorname{sign}(r) = \operatorname{sign}(K_7)$.

Thus we arrive at the phase portrait $P.8.6(a)$ if $K_7 < 0$ and $P.8.6(b)$ if $K_7 > 0$ (see Figure 4.1). We remark that the phase portrait $P.8.6(a)$ (respectively $P.8.6(b)$ corresponds to Figure 1.9a (respectively Figure 1.9b) from [36].

Canonical form	*Invariant lines and their multiplicities*	*First integrals* ($\mathcal{F}_i$) *Inverse integrating Factors* ($\mathcal{R}_i$)	*Conditions*	*Phase portr.*
1) $\dot{x}=x(x+1)(x-r)$, $\dot{y}=y(y+1)(y-r)$, $0<r\neq 1$	$x(1), x+1(1), y(1), y-r(1), x-y(1), x-r(1), y+1(1)$	$\mathcal{F}_1=\frac{y-r}{x-r}\left(\frac{x}{y}\right)^{r+1}\left(\frac{y+1}{x+1}\right)^r$, $\mathcal{R}_1=\left[xy(x+1)(y+1)\times(x-r)(r-y)\right]^{-1}$	*Cond.1*	*P. 8.1*
2) $\dot{x}=x\left[(x+r)^2+1\right]$, $\dot{y}=y\left[(y+r)^2+1\right]$, $r\neq 0$	$x(1)$, $r+x\pm i(1)$, $x-y(1)$, $y(1)$, $r+y\pm i(1)$	$\mathcal{F}_2=\left(\frac{r+y+i}{r+x+i}\right)^{1+ir}\times\frac{x^2}{y^2}\left(\frac{r+y-i}{r+x-i}\right)^{1-ir}$, $\mathcal{R}_2=\left[xy(1+(r+x)^2)\right]^{-1}\times\left[1+(r+y)^2\right]^{-1}$	*Cond.2*	*P. 8.2*
3) $\dot{x}=x^2(1+x)$, $\dot{y}=y^2(1+y)$	$x(2)$, $x+1(1)$, $x-y(1)$, $y(2)$, $y+1(1)$	$\mathcal{F}_3=\frac{y(x+1)}{x(y+1)}\exp[\frac{x-1}{xy}]$, $\mathcal{R}_3=\left[x^2y^2(x+1)(y+1)\right]^{-1}$	*Cond.3*	*P. 8.3*
4) $\dot{x}=x(x-1)(x+r)$, $\dot{y}=\left[(1-r)x+ry+r\right]\times y(y-1)$, $0<r\neq 1$	$x-1(1)$, $x(1)$, $y(1)$ $x+r(1)$, $x-y(1)$, $y-1(1)$, $x+ry(1)$	$\mathcal{F}_4=\frac{(x-1)(x+r)^r y^{r+1}}{(x-y)(x+ry)^r}$ $\mathcal{R}_4=x\left[(x-1)(r+x)\right]^{-1}\times\left[(x-y)y(x+ry)\right]^{-1}$	*Cond.4*	*P. 8.4*
5) $\dot{x}=x(x-1)(x+r)$, $\dot{y}=\left[(1-r)x+ry+r\right]\times y(y-1)$, $-1\neq r<0$, $(2r+1)(r+2)\neq 0$	$x-1(1)$, $x(1)$, $y(1)$ $x+r(1)$, $x-y(1)$, $y-1(1)$, $x+ry(1)$	$\mathcal{F}_5=\frac{(x-1)(x+r)^r y^{r+1}}{(x-y)(x+ry)^r}$ $\mathcal{R}_5=x\left[(x-1)(r+x)\right]^{-1}\times\left[(x-y)y(x+ry)\right]^{-1}$	*Cond.5*	*P. 8.5*
6) $\dot{x}=rx^3$, $r\neq\pm 1$, $\dot{y}=y^2(rx-x+y)$, $r(2r+1)(r+2)\neq 0$	$x(3)$, $x-y(1)$ y^2, $rx+y(1)$	$\mathcal{F}_6=\frac{(x-y)^r(rx+y)}{(xy)^{1+r}}$ $\mathcal{R}_6=\left[xy(x-y)(rx+y)\right]^{-1}$	*Cond.6* $K_7<0$ $K_7>0$	 *P.8.6(a)* *P.8.6(b)*
7) $\dot{x}=(rx+2y+ry)\times(x^2-1)$, $r\neq 0,\pm 1$ $\dot{y}=(x+2rx+y)\times(y^2-1)$, $r\neq -2,-\frac{1}{2}$	$x\pm 1(1)$, $y\pm 1(1)$, $rx+y\pm(1+r)(1)$, $x-y(1)$	$\mathcal{F}_7=\frac{y+1}{y-1}\left(\frac{x+1}{x-1}\right)^r\left(\frac{rx+y-r-1}{rx+y+r+1}\right)^{r+1}$ $\mathcal{R}_7=(x-y)\left[(x^2-1)(y^2-1)\times((rx+y)^2-(1+r)^2)\right]^{-1}$	*Cond.7*	*P. 8.7*
8) $\dot{x}=(rx+2y+ry)\times(x^2+1)$, $r\neq 0,\pm 1$ $\dot{y}=(x+2rx+y)\times(y^2+1)$, $r\neq -2,-\frac{1}{2}$	$x\pm i(1)$, $y\pm i(1)$, $rx+y\pm i(1+r)(1)$, $x-y(1)$	$\mathcal{F}_8=\frac{y+i}{y-i}\left(\frac{x+i}{x-i}\right)^r\left(\frac{rx+y-i(r+1)}{rx+y+i(r+1)}\right)^{r+1}$ $\mathcal{R}_8=(x-y)\left[(x^2+1)(y^2+1)\times((rx+y)^2+(1+r)^2)\right]^{-1}$	*Cond.8*	*P. 8.8*
9) $\dot{x}=x^2(rx+2y+ry)$, $\dot{y}=y^2(x+2rx+y)$, $r\neq 0,\pm 1,-2,-\frac{1}{2}$	$x(2)$, $x-y(1)$, $y(2)$, $rx+y(2)$	$\mathcal{F}_9=\frac{xy(rx+y)}{(x-y)^2}$ $\mathcal{R}_9=\left[xy(x-y)(rx+y)\right]^{-1}$	*Cond.9*	*P. 8.9*
10) $\dot{x}=\frac{1-r^2}{4}x+x^2-y^2+x^3-3xy^2$, $r\neq 0$ $\dot{y}=\frac{1-r^2}{4}y+2xy+3x^2y-y^3$, $r^2\neq 1,\frac{1}{9}$	$y(1)$, $y\pm ix(1)$, $2y\pm i(1+r+2x)(1)$, $2y\pm i(1-r+2x)(1)$	$\mathcal{F}_{10}=\left(\frac{2y+i(1-r+2x)}{2y-i(1-r+2x)}\right)^{\frac{1+r}{2}}\times\left(\frac{y-ix}{y+ix}\right)^r\left(\frac{2y-i(1+r+2x)}{2y+i(1+r+2x)}\right)^{\frac{1-r}{2}}$, $\mathcal{R}_{10}=(x^2+y^2)^{-1}\times\left[(1+r+2x)^2+4y^2\right]^{-1}\times\left[(1-r+2x)^2+4y^2\right]^{-1}$	*Cond.10*	*P. 8.10*

Table 4.1: First integrals and integrating factors

Remark 4.4. We remark that Theorem 1.1 of [36] states: "...we give the 18 topologically distinct phase portraits on the Poincaré disk ...". However as it follows from Diagram 4.1 among 18 phase portraits given by Configurations $(I.1)$–$(I.17)$ in [36] only 13 are topologically distinct. More precisely, considering the above mentioned correspondence (4.1) we have the following topological equivalence of the phase portraits in [36]:

	Canonical form	*Invariant lines and their multiplicities*	*First integrals* ($\mathcal{F}_i$) *Inverse integrating Factors* ($\mathcal{R}_i$)	*Condi-tions*	*Phase portr.*
11)	$\dot{x} = \frac{1+r^2}{4}x + x^2 - y^2 + x^3 - 3xy^2$, $\dot{y} = \frac{1+r^2}{4}y + 2xy + 3x^2y - y^3$, $r \neq 0$	$y(1)$, $y \pm ix(1)$, $2y + r \pm i(1+2x)(1)$, $2y - r \pm i(1+2x)(1)$	$\mathcal{F}_{11} = \left(\frac{2y-r+i(1+2x)}{2y+r-i(1+2x)}\right)^{\frac{1+r}{2}} \times \left(\frac{y-ix}{y+ix}\right)^r \left(\frac{2y-r-i(1+2x)}{2y+r+i(1+2x)}\right)^{\frac{1-r}{2}}$, $\mathcal{R}_{11} = (x^2+y^2)^{-1} \times \left[(2x+1)^2 + (r+2y)^2\right]^{-1} \times \left[(2x+1)^2 + (r-2y)^2\right]^{-1}$	*Cond.11*	*P. 8.11*
12)	$\dot{x} = x/4 + x^2 - y^2 + x^3 - 3xy^2$, $\dot{y} = y/4 + 2xy + 3x^2y - y^3$	$y(1)$, $y \pm ix(1)$, $2y \pm i(1+2x)(2)$	$\mathcal{F}_{12} = \frac{(y-ix)[2y+i(1+2x)]}{(y+ix)[2y-i(1+2x)]} \times \exp[-\frac{4iy}{(1+2x)^2+4y^2}]$, $\mathcal{R}_{12} = (x^2+y^2)^{-1} \times \left[(2x+1)^2 + 4y^2\right]^{-2}$	*Cond.12*	*P. 8.12*
13)	$\dot{x} = (1+r^2)x \times \left[(x+r)^2 + 1\right]$, $r \neq 0$, $\dot{y} = (1+r^2)^2 y + 2r(1+r^2)xy - rx^3 + y(r^2x^2 - rxy - y^2)$	$x(1)$, $x + r \pm i(1)$, $rx + y + 1 + r^2(1)$ $rx + y(1)$, $y \pm ix(1)$	$\mathcal{F}_{13} = (rx+y)^2 \left(\frac{x+r+i}{x-iy}\right)^{1-ir} \times \left(\frac{x+r-i}{x+iy}\right)^{1+ir}$, $\mathcal{R}_{13} = x\left[(x+r)^2+1\right]^{-1} \times \left[(rx+y)(x^2+y^2)\right]^{-1}$	*Cond.13*	*P. 8.13*
14)	$\dot{x} = (1+r^2)x^3$, $\dot{y} = r^2x^2y - y^3 - rx(x^2+y^2)$, $r \neq 0$	$x(3)$, $x + r \pm iy(1)$, $rx + y(2)$	$\mathcal{F}_{14} = \frac{x^2(rx+y)^2}{x^2+y^2}\left(\frac{x-iy}{x+iy}\right)^{ir}$, $\mathcal{R}_{14} = \left[x(rx+y)(x^2+y^2)\right]^{-1}$	*Cond.14*	*P. 8.14*
15)	$\dot{x} = x(x-1) \times (1 + r^2 - 2x + 2ry)$, $\dot{y} = -(1+r^2)y - rx^3 + (3+r^2)xy - 3x^2y - y^2(2r - rx + y)$, $r \neq 0$	$x(1)$, $rx + y(1)$, $y + r \pm i(x-1)(1)$, $(x-1)(1)$, $y \pm ix(1)$	$\mathcal{F}_{15} = \frac{(x-1)^2}{x^2}\left(\frac{y+ix}{y+r+i(x-1)}\right)^{1+ir} \times \left(\frac{y-ix}{y+r-i(x-1)}\right)^{1-ir}$, $\mathcal{R}_{15} = \frac{(rx+y)(x^2+y^2)^{-1}}{x(x-1)[(x-1)^2(y+r)^2]}$	*Cond.15*	*P. 8.15*
16)	$\dot{x} = 2(1+x^2)(ry - x)$, $\dot{y} = r(r^2+3)x - rx^3 + (1-r^2)y - 3x^2y + rxy^2 - y^3$, $r \neq 0$	$x \pm i(1)$, $rx + y(1)$, $y + 1 \pm i(x+r)(1)$, $y - 1 \pm i(x-r)(1)$	$\mathcal{F}_{16} = \left(\frac{x+i}{x-i}\right)^{2i}\left(\frac{y-1+i(x-r)}{y+1+i(x+r)}\right)^{r-i} \times \left(\frac{y-1-i(x-r)}{y+1-i(x+r)}\right)^{r+i}$ $\mathcal{R}_{16} = \frac{(rx+y)\left[(x-r)^2+(y-1)^2\right]^{-1}}{(1+x^2)\left[(x+r)^2+(y+1)^2\right]}$	*Cond.16*	*P. 8.16*
17)	$\dot{x} = -2x^2(x - ry)$, $\dot{y} = rx^3 - 3x^2y + rxy^2 - y^3$, $r \neq 0$	$x(2)$, $x \pm i(2)$, $rx + y(1)$	$\mathcal{F}_{17} = \frac{x(x^2+y^2)}{(rx+y)^2}$ $\mathcal{R}_{17} = \left[x(rx+y)(x^2+y^2)\right]^{-1}$	*Cond.17*	*P. 8.17*
18)	$\dot{x} = x^3 - 9x^2 - x^2y - xy^2$, $\dot{y} = -y^2(9+y)$	$x(2)$, $x - y(1)$, $y(2)$, $y + 9(1)$ $x - y - 9(1)$	$\mathcal{F}_{18} = \frac{y(x-y-9)}{x} \times \exp[\frac{9y-9x+y^2}{xy}]$, $\mathcal{R}_{18} = \frac{(y+9)}{x^2y^2(x-y-9)}$	*Cond.18*	*P. 8.18*
19)	$\dot{x} = x(x^2 - xy - y^2)$, $\dot{y} = -y^3$	$x(2)$, $y(3)$, $x - y(2)$	$\mathcal{F}_{19} = \frac{y}{y(x-y)}\exp[\frac{-y}{x}]$, $\mathcal{R}_{19} = \left[x^2(x-y)y\right]^{-1}$	*Cond.19*	*P. 8.19*
20)	$\dot{x} = (1-x)x(1+y)$, $\dot{y} = y(1 - x + y - x^2)$	$x(2)$, $x - 1(1)$, $y(1)$, $x - y(1)$, $x - y - 1(1)$	$\mathcal{F}_{20} = \frac{x(x-y-1)}{y}\, e^{\frac{x^2+y-xy}{x}}$, $\mathcal{R}_{20} = \frac{(x-1)x^{-2}}{y(x-y-1)}$	*Cond.20*	*P. 8.20*

Table 4.1 (continuation): First integrals and integrating factors

Canonical form	*Invariant lines and their multiplicities*	*First integrals* ($\mathcal{F}_i$) *Inverse integrating Factors* ($\mathcal{R}_i$)	*Conditions*	*Phase portr.*
21) $\dot{x}=(x^2-1)(x+y)$, $\dot{y}=2x(y^2-1)$	$x\pm 1(2),\ y\pm 1(1),$ $x-y(1)$	$\mathcal{F}_{21}=\frac{(x-1)(y+1)}{(x+1)(y-1)}\,e^{\frac{2(y-x)}{x^2-1}}$, $\mathcal{R}_{21}=\frac{(x-y)(x^2-1)^{-2}}{y^2-1}$	*Cond.21*	*P. 8.21*
22) $\dot{x}=(x^2+1)(x+y)$, $\dot{y}=2x(y^2+1)$	$x\pm i(2),\ y\pm i(1),$ $x-y(1)$	$\mathcal{F}_{22}=\frac{(x-i)(y+i)}{(x+i)(y-i)}\,e^{\frac{2i(y-x)}{x^2+1}}$, $\mathcal{R}_{22}=\frac{(x-y)(x^2+1)^{-2}}{y^2+1}$	*Cond.22*	*P. 8.22*
23) $\dot{x}=(x-1)x(1+x)$, $\dot{y}=x-y+x^2+3xy$	$x(2),\ x+1(1),$ $x-1(3)$	$\mathcal{F}_{23}=x\exp[\frac{y+2xy+x^2(4+y)}{x(1-x)}]$, $\mathcal{R}_{23}=\frac{(x+1)x^{-2}}{(x-1)^2}$	*Cond.23*	*P. 8.23*
24) $\dot{x}=x\big[(x+1)^2-b^2\big]$, $\dot{y}=y(1-b^2+2x)$, $b>1$	$x(3),\ x+1+b(1),$ $y(1),\ x+1-b(1)$	$\mathcal{F}_{24}=\big(\frac{x}{y}\big)^{2b}\,\frac{(x+1-b)^{1-b}}{(x+1+b)^{1+b}}$, $\mathcal{R}_{24}=\big[x\big((x+1)^2-b^2\big)y\big]^{-1}$	*Cond.24*	*P. 8.24*
25) $\dot{x}=x\big[(x+1)^2-b^2\big]$, $\dot{y}=y(1-b^2+2x)$, $0<b<1,\ b\neq 1/3$	$x(3),\ x+1+b(1),$ $y(1),\ x+1-b(1)$	$\mathcal{F}_{25}=\big(\frac{x}{y}\big)^{2b}\,\frac{(x+1-b)^{1-b}}{(x+1+b)^{1+b}}$, $\mathcal{R}_{25}=\big[x\big((x+1)^2-b^2\big)y\big]^{-1}$	*Cond.25*	*P. 8.25*
26) $\dot{x}=x(x+1)^2$, $\dot{y}=y(1+2x)$	$x(3),\ y(1),$ $x+1(2)$	$\mathcal{F}_{26}=\frac{x}{(x+1)y}\,\exp[\frac{x}{x+1}]$, $\mathcal{R}_{26}=\big[x(1+x)^2y\big]^{-1}$	*Cond.26*	*P. 8.26*
27) $\dot{x}=x\big[(x+1)^2+b^2\big]$, $\dot{y}=y(1+b^2+2x)$, $b>0$	$x(3),\ x+1+ib(1),$ $y(1),\ x+1-ib(1)$	$\mathcal{F}_{27}=\big(\frac{x}{y}\big)^{2b}\,\frac{(x+1+ib)^{i-b}}{(x+1-ib)^{i+b}}$, $\mathcal{R}_{27}=\big[x\big((x+1)^2+b^2\big)y\big]^{-1}$	*Cond.27*	*P. 8.27*
28) $\dot{x}=x\big[(x-1)^2-b^2\big]$, $\dot{y}=2y(x-1+b^2)$, $b>1$	$x(1),\ x-1+b(2),$ $y(1),\ x-1-b(2)$	$\mathcal{F}_{28}=\frac{(x-1)^2-b^2}{x^2y}$, $\mathcal{R}_{28}=\big[x\big((x-1)^2-b^2\big)y\big]^{-1}$	*Cond.28*	*P. 8.28*
29) $\dot{x}=x\big[(x-1)^2-b^2\big]$, $\dot{y}=2y(x-1+b^2)$, $0<b<1,\ b\neq 1/3$	$x(1),\ x-1+b(2),$ $y(1),\ x-1-b(2)$	$\mathcal{F}_{29}=\frac{(x-1)^2-b^2}{x^2y}$, $\mathcal{R}_{29}=\big[x\big((x-1)^2-b^2\big)y\big]^{-1}$	*Cond.29*	*P. 8.29*
30) $\dot{x}=x\big[(x-1)^2+b^2\big]$, $\dot{y}=2y(x-1-b^2)$, $b>0$	$x(1),\ x-1+ib(2),$ $y(1),\ x-1-ib(2)$	$\mathcal{F}_{30}=\frac{(x-1)^2+b^2}{x^2y}$, $\mathcal{R}_{30}=\big[x\big((x-1)^2+b^2\big)y\big]^{-1}$	*Cond.30*	*P. 8.30*
31) $\dot{x}=x(x^2-1)$, $\dot{y}=x+2y$	$x(1),\ x+1(2),$ $x-1(2)$	$\mathcal{F}_{31}=\frac{x-1}{x+1}\exp[\frac{2x(1+2xy)}{1-x^2}]$, $\mathcal{R}_3=x(x^2-1)^{-2}$	*Cond.31*	*P. 8.31*
32) $\dot{x}=x(x^2+1)$, $\dot{y}=x-2y$	$x(1),\ x+i(2),$ $x-i(2)$	$\mathcal{F}_{32}=\frac{x-i}{x+i}\exp\big[\frac{i[(x-1)2-4yx^2]}{1+x^2}\big]$, $\mathcal{R}_3=x(x^2+1)^{-2}$	*Cond.32*	*P. 8.32*
33) $\dot{x}=x^3,\ \dot{y}=x+1$	$x(5)$	$\mathcal{F}_{33}=\frac{1+2x+2x^2y}{2x^2}$, $\mathcal{R}_3=x^{-3}$	*Cond.33*	*P. 8.33*
34) $\dot{x}=x\big[(2x+1)^2-b^2\big]$, $\dot{y}=4+(1-b^2)y$, $b>1$	$x(2),\ 2x+1+b(1),$ $2x+1-b(1),$ $(b^2-1)y-4(1)$	$\mathcal{F}_{34}=\frac{x^{2b}(2x+1+b)^{1-b}}{(2x+1+b)^{1-b}(b^2y-y-4)^{2b}}$, $\mathcal{R}_{34}=\big[(b^2-1)y-4\big]^{-1}\times$ $\big[x\big((2x+1)^2-b^2\big)\big]^{-1}$	*Cond.34*	*P. 8.34*

Table 4.1 (continuation): First integrals and integrating factors

Canonical form	*Invariant lines and their multiplicities*	*First integrals* ($\mathcal{F}_i$) *Inverse integrating Factors* ($\mathcal{R}_i$)	*Condi-tions*	*Phase portr.*
35) $\dot{x}=x\big[(2x+1)^2-b^2\big]$, $\dot{y}=4+(1-b^2)y$, $0<b<1,\ b\neq 1/3$	$x(2),\ 2x+1+b(1)$, $2x+1-b(1)$, $(b^2-1)y-4(1)$	$\mathcal{F}_{35}=\frac{x^{2b}(2x+1+b)^{1-b}}{(2x+1+b)^{1+b}(b^2y-y-4)^{2b}}$, $\mathcal{R}_{35}=\big[(b^2-1)y-4\big]^{-1}\times$ $\big[x\big((2x+1)^2-b^2\big)\big]^{-1}$	*Cond.35*	*P. 8.35*
36) $\dot{x}=x(1+2x)^2$, $\dot{y}=4+y$	$x(2),\ 2x+1(2)$, $y+4(1)$	$\mathcal{F}_{36}=\frac{(1+2x)(4+y)}{2x}\exp[\frac{-1}{1+2x}]$, $\mathcal{R}_{36}=\big[x(1+2x)^2(4+y)\big]^{-1}$	*Cond.36*	*P. 8.36*
37) $\dot{x}=x\big[(2x+1)^2+b^2\big]$, $\dot{y}=4+(1+b^2)y$, $b>0$	$x(2),\ 2x+1+ib(1)$, $2x+1-ib(1)$, $(b^2+1)y+4(1)$	$\mathcal{F}_{37}=\frac{x^{2b}(2x+1-ib)^{i-b}}{(2x+1+b)^{i+b}(b^2y+y+4)^{2b}}$, $\mathcal{R}_{37}=\big[(b^2+1)y+4\big]^{-1}\times$ $\big[x\big((2x+1)^2+b^2\big)\big]^{-1}$	*Cond.37*	*P. 8.37*
38) $\dot{x}=x^2(1+x),\ \dot{y}=1$	$x(3),\ x+1(1)$	$\mathcal{F}_{38}=\frac{x}{x+1}\exp[\frac{1+xy}{x}]$, $\mathcal{R}_{38}=\big[x^2(1+x)\big]^{-1}$	*Cond.38*	*P. 8.38*
39) $\dot{x}=x\big[b^2-(2x+1)^2\big]$, $\dot{y}=y(b^2-1-8x)-$ $12x^2y,\ 0<b\neq\frac{1}{3},1$	$x(2),\ 2x+1+b(2)$, $y(1),\ 2x+1-b(2)$	$\mathcal{F}_{39}=x\big[(2x+1)^2-b^2\big]y^{-1}$, $\mathcal{R}_{39}=\big[x\big((2x+1)^2-b^2\big)y\big]^{-1}$	*Cond.39*	*P. 8.39*
40) $\dot{x}=x\big[b^2+(2x+1)^2\big]$, $\dot{y}=y(b^2+1+8x)+$ $12x^2y,\ \ b>0$	$x(2),\ 2x+1+b(2)$, $y(1),\ 2x+1-b(2)$	$\mathcal{F}_{40}=x\big[(2x+1)^2+b^2\big]y^{-1}$, $\mathcal{R}_{39}=\big[x\big((2x+1)^2+b^2\big)y\big]^{-1}$	*Cond.40*	*P. 8.40*
41) $\dot{x}=x(x^2-1)$, $\dot{y}=1-y+3x^2y$	$x(3),\ x+1(2)$, $x-1(2)$	$\mathcal{F}_{41}=\frac{(1-x)^3}{(1+x)^3}\times$ $\exp\big[\frac{2(2-3x^2-2y)}{(1-x)x(1+x)}\big]$, $\mathcal{R}_{41}=\big[(1-x)^2x^2(1+x)^2\big]^{-1}$	*Cond.41*	*P. 8.41*
42) $\dot{x}=x^3,\ \dot{y}=1+3x^2y$	$x(7)$	$\mathcal{F}_{42}=\frac{1+5x^2y}{5x^5},\ \ \mathcal{R}_{42}=x^{-6}$	*Cond.42*	*P. 8.42*
43) $\dot{x}=x(x^2+1)$, $\dot{y}=1+y+3x^2y$	$x(3),\ x+i(2)$, $x-i(2)$	$\mathcal{F}_{43}=\frac{(x+i)^3}{(x-i)^3}\times$ $\exp\big[\frac{i(4+x+6x^2+x^3+4y)}{-x(x^2+1)}\big]$, $\mathcal{R}_{43}=\big[x^2(x^2+1)\big]^{-1}$	*Cond.43*	*P. 8.43*
44) $\dot{x}=x(1+x)\times$ $(2+r+x+rx)$, $\dot{y}=y(2+r+3x)+$ $rx(2+x)y,\ r\neq -3/2$ $-2<r<-1$	$x(3),\ y(1),\ x+1(2)$, $(r+1)x+r+2(1)$	$\mathcal{F}_{44}=\frac{[x(x+1)]^{1+r}}{y^{1+r}(2+r+x+rx)^{2+r}}$, $\mathcal{R}_{44}=\big[xy(x+1)\big]^{-1}$ $\big[(2+r+x+rx)\big]^{-1}$	*Cond.44*	*P. 8.44*
45) $\dot{x}=x(1+x)\times$ $(2+r+x+rx)$, $\dot{y}=xy(3+2r+rx)+$ $(2+r)y,\ -3\neq r<-2$	$x(3),\ y(1),\ x+1(2)$, $(r+1)x+r+2(1)$	$\mathcal{F}_{45}=\frac{[x(x+1)]^{1+r}}{y^{1+r}(2+r+x+rx)^{2+r}}$, $\mathcal{R}_{45}=\big[xy(x+1)\big]^{-1}$ $\big[(2+r+x+rx)\big]^{-1}$	*Cond.45*	*P. 8.45*
46) $\dot{x}=x(1+x)\times$ $(2+r+x+rx)$, $\dot{y}=xy(3+2r+rx)+$ $(2+r)y,\ 0\neq r>-1$	$x(3),\ y(1),\ x+1(2)$, $(r+1)x+r+2(1)$	$\mathcal{F}_{46}=\frac{[x(x+1)]^{1+r}}{y^{1+r}(2+r+x+rx)^{2+r}}$, $\mathcal{R}_{46}=\big[xy(x+1)\big]^{-1}$ $\big[(2+r+x+rx)\big]^{-1}$	*Cond.46*	*P. 8.46*

Table 4.1 (continuation): First integrals and integrating factors

Canonical form	*Invariant lines and their multiplicities*	*First integrals* $(\mathcal{F}_i)$ *Inverse integrating Factors* $(\mathcal{R}_i)$	*Condi-tions*	*Phase portr.*
47) $\dot{x}=x,$ $\dot{y}=y(1+x-x^2)$	$x(3),\ y(1),$ $x+1(3)$	$\mathcal{F}_{47}=\frac{y}{x(1+x)}\exp[x],$ $\mathcal{R}_{47}=\big[x(x+1)y\big]^{-1}$	*Cond.47*	*P. 8.47*
48) $\dot{x}=x,\ \ \dot{y}=-2y-x^3$	$x(1)$	$\mathcal{F}_{48}=x^2(x^3+5y),\ \ \mathcal{R}_{48}=x$	*Cond.48*	*P. 8.48*
49) $\dot{x}=x,\ \dot{y}=y-x^2-x^3$	$x(3)$	$\mathcal{F}_{49}=3x^2+2x^3+6y,\ \mathcal{R}_{49}=1/x$	*Cond.49*	*P. 8.49*
50) $\dot{x}=x(1+x),$ $\dot{y}=y-x^2-x^3$	$x(4),\ x-1(1)$	$\mathcal{F}_{50}=\exp[\frac{x^2+y}{x}]/(x+1),$ $\mathcal{R}_{50}=\big[x^2(1+x)\big]^{-1}$	*Cond.50*	*P. 8.50*
51) $\dot{x}=x^2(1+x),$ $\dot{y}=-1-3x+x^2y-x^3$	$x(3),\ x+1(4)$	$\mathcal{F}_{51}=x\exp[\frac{-1+x(2+y)}{x(1+x)}],$ $\mathcal{R}_{51}=\big[x^2(1+x)^2\big]^{-1}$	*Cond.51*	*P. 8.51*

Table 4.1 (continuation): First integrals and integrating factors

$\widetilde{\mathcal{F}}_2=$	$\exp\big[2r\arctan[\frac{1}{r+x}]-2r\arctan[\frac{1}{r+y}]\big]x^2(1+(r+y)^2)\big[y^2(1+(r+x)^2)\big]^{-1};$
$\widetilde{\mathcal{F}}_8=$	$\frac{[(x-y)^2+(1+r+rx^2+xy)^2]^r}{(1+x^2)^r(1+y^2)[(1+r)^2+(rx+y)^2]^{r+1}}\Big\{\big[(1+r+rxy+y^2)^2-r^2(x-y)^2\big]\times$ $\cos\big[2r\arctan[\frac{x-y}{1+r+rx^2+xy}]\big]+2r(x-y)(1+r+rxy+y^2)\sin\big[2r\arctan[\frac{x-y}{1+r+rx^2+xy}]\big]\Big\};$
$\widetilde{\mathcal{F}}_{10}=$	$\cos\big[(1+r)\,\mathrm{arccot}\,[\frac{2y}{1-r+2x}]+(r-1)\,\mathrm{arccot}\,[\frac{2y}{1+r+2x}]-2r\arctan[\frac{x}{y}]\big];$
$\widetilde{\mathcal{F}}_{11}=$	$\cos\Big[r\arctan[\frac{1+2x}{r-2y}]+2r\arctan[\frac{x}{y}]-r\cdot\arctan[\frac{1+2x}{r+2y}]+\frac{1}{2}\ln\big[\frac{(1+2x)^2+(r+2y)^2}{(1+2x)^2+(r-2y)^2}\big]\Big];$
$\widetilde{\mathcal{F}}_{12}=$	$\Big[\big[(x+2x^2+2y^2)^2-y^2\big]\cos[\frac{4y}{(1+2x)^2+4y^2}]+2y(x+2x^2+2y^2)\sin[\frac{4y}{(1+2x)^2+4y^2}]\Big]\times$ $(x^2+y^2)^{-1}\big[(1+2x)^2+4y^2\big]^{-1};$
$\widetilde{\mathcal{F}}_{13}=$	$\exp\big[2r\arctan[\frac{1}{r+x}]+2r\arctan[\frac{y}{x}]\big]\big[1+(r+x)^2\big](rx+y)^2(x^2+y^2)^{-1};$
$\widetilde{\mathcal{F}}_{14}=$	$x^2(rx+y)^2\exp\big[2r\arctan[\frac{y}{x}]\big](x^2+y^2)^{-1};$
$\widetilde{\mathcal{F}}_{15}=$	$\exp\big[r\arctan[\frac{r+y}{1-x}]+r\arctan[\frac{y}{x}]\big](x-1)\sqrt{x^2+y^2}\,x^{-1}\big[(x-1)^2+(r+y)^2\big]^{-1/2};$
$\widetilde{\mathcal{F}}_{16}=$	$\exp\Big[-4\arctan[\frac{1}{x}]+2\arctan[\frac{1-y}{x-r}]+2\arctan[\frac{1+y}{r+x}]\Big]\Big[\frac{(r-x)^2+(y-1)^2}{(r+x)^2+(1+y)^2}\Big]^r;$
$\widetilde{\mathcal{F}}_{22}=$	$\Big[\big[(xy+1)^2-(x-y)^2\big]\cos[\frac{2(x-y)}{1+x^2}]+2(x-y)(xy+1)\sin[\frac{2(x-y)}{1+x^2}]\Big]\big[(1+x^2)(1+y)^2\big]^{-1};$
$\widetilde{\mathcal{F}}_{27}=$	$\exp\big[-2\arctan[\frac{u}{1+x}]\big]x^{2u}\big[u^2+(1+x)^2\big]^{-u}y^{-2u};$
$\widetilde{\mathcal{F}}_{32}=$	$x(1+2xy)(1+x^2)^{-1}-\arctan[x];$
$\widetilde{\mathcal{F}}_{37}=$	$\exp\big[2\arctan[\frac{u}{1+2x}]\big]x^{2u}\big[u^2+(1+2x)^2\big]^{-u}(4+y+u^2y)^{-2u};$
$\widetilde{\mathcal{F}}_{43}=$	$\Big\{(u^2-x^2)\big[(u^2+x^2)^2-16u^2x^2\big]\cos[\frac{u(4u^2+u^2x+6x^2+x^3+4u^4y)}{x(u^2+x^2)}]-$ $-2ux(u^2-3x^2)(3u^2-x^2)\sin[\frac{u(4u^2+u^2x+6x^2+x^3+4u^4y)}{x(u^2+x^2)}]\Big\}(u^2+x^2)^{-3}.$

Table 4.2: Real First integrals

Figure 1.7 ≅ Figure 1.1; Figure 1.12 ≅ Figure 1.8; Figure 1.13 ≅ Figure 1.9b;
Figure 1.9a ≅ Figure 1.3; Figure 1.5 ≅ Figure 8.

We continue the proof of the Main Theorem.

(β) We point out that each one of the 20 configurations *Config. 8.18–8.23, 26, 31, 32, 33, 36, 38, 41–43, 47–51* corresponds to a system without parameters. The phase portraits of polynomial differential equations are usually presented in the Poincaré disc using the so called Poincaré compactification, see for details Chapter 5 of [15]. The existence of the 8 invariant straight lines taking into account their multiplicities and the knowledge of the real elementary first integrals allows us to draw the 18 phase portraits corresponding to the above mentioned configurations as presented in Figure 4.1.

We note that the study of the phase portraits of systems without parameters can also be done using the algebraic program P4, see for details Chapters 9 and 10 of [15].

(γ) Among the 34 *Config. 8.18–Config. 8.51* there are 14 configurations *Config. 8.24, 8.25, 8.27–8.30, 8.34, 8.35, 8.37, 8.39, 8.40, 8.44–8.46* corresponding to one-parameter families of cubic systems. In what follows we examine each one of these canonical systems applying the following steps:

(i) detect the finite real singularities of the systems and their types;

(ii) examine if the information about the invariant straight lines and the types of finite singularities as well as the types of simple infinite singularities determine univocally the behavior of the trajectories at infinity;

(ii) construct the corresponding phase portraits on the Poincaré disk.

However first we prove the next lemma.

Lemma 4.5. *The one-parameter families of cubic systems which correspond to the configurations Config. 8.24–8.30, 8.34-8.38 possess at infinity exactly one simple and one multiple singularities. Moreover the simple singular point is a star node.*

Proof. According to [8] if a quadratic system possesses one of the mentioned configurations then via an affine transformation it could be brought to one of the following canonical forms, respectively:

- for the configurations *Config. 8.24–8.27* ⇒

$$\dot{x} = x(r + 2x + x^2), \quad \dot{y} = y(r + 2x), \quad r(9r - 8) \neq 0; \tag{4.4}$$

- for the configurations *Config. 8.28–8.30* ⇒

$$\dot{x} = x(r - 2x + x^2), \quad \dot{y} = 2y(x - r), \quad r(r - 1)(9r - 8) \neq 0; \tag{4.5}$$

- for the configurations *Config. 8.34–8.38* ⇒

$$\dot{x} = x(r + x + x^2), \quad \dot{y} = 1 + ry, \quad 9r - 2 \neq 0. \tag{4.6}$$

For each one of the canonical systems (4.4)–(4.6) we calculate $C_3 = x^3y$. Therefore all these 3 families have at infinity two singularities: $N_1(1, 0, 0)$ and $N_2(0, 1, 0)$. Considering Theorem 3.1 (see the configurations under examination presented in Figure 3.1) we conclude that

$N_1(1,0,0)$ is a simple singularity whereas $N_2(0,1,0)$ has multiplicity 9 (six finite singularities coalesced with three infinite ones on the line $Z=0$).

Examining the Jacobian matrix corresponding to the simple singularity $N_1(1,0,0)$ for each one of the above families of systems we detect that it equals $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore this singularity for any of systems (4.4)–(4.6) is a star node and this completes the proof of the lemma. □

γ_1) ***Config. 8.24, 8.25, 8.27.*** According to Lemma 4.5 we consider the family of systems (4.4) which possesses *Config. 8.24* (respectively, *8.25*; *8.27*) if $r<0$ (respectively, $0<r<1$; $r>1$).

We observe that the invariant straight lines $r+2x+x^2=0$ of systems (4.4) are real if $\text{Discrim}[r+2x+x^2,x]=4(1-r)>0$ and they are complex if $1-r<0$ (these invariant lines could not coincide due to $r-1\neq 0$). So in what follows we examine these two cases.

1) *The case* $1-r>0$. We set $1-r=b^2>0$ where b is a new parameter and we get: $r=1-b^2<1$. We may consider $b>0$ due to the change $b\to -b$ and we arrive at the family of systems

$$\dot{x}=x\big[(x+1)^2-b^2\big),\quad \dot{y}=y(1-b^2+2x),\ \ b(b-1)(3b-1)\neq 0. \tag{4.7}$$

These systems possess the following invariant affine lines:

$$\begin{array}{ll} L_1:\ x=0\,(\text{triple}); & L_2:\ x=-(b+1)\,(\text{simple}); \\ L_3:\ x=b-1\,(\text{simple}); & L_4:\ y=0\,(\text{simple}). \end{array} \tag{4.8}$$

In addition the line at infinity $Z=0$ is a double one. All these invariant lines are distinct due to the condition $0<b\neq 1$ and by the relation $L_1<L_2<L_3$ we mean that the invariant line L_2 is located between invariant lines other two parallel lines L_1 and L_3.

We detect that systems (4.7) possess three finite singularities: $M_1(0,0)$, $M_{2,3}(-1\mp b,0)$. For a singularity $M_i(x_i,y_i)$ we denote by $\lambda_1^{(i)}$ and $\lambda_2^{(i)}$ the eigenvalues of the corresponding Jacobian matrix. For the mentioned finite singularities of the above systems we have

$$\lambda_1^{(1)}=1-b^2=\lambda_2^{(1)};\quad \lambda_1^{(2)}=2b(1+b),\ \lambda_2^{(2)}=-(1+b)^2;$$
$$\lambda_1^{(3)}=2b(b-1),\ \lambda_2^{(3)}=-(b-1)^2.$$

Considering the conditions $0<b\neq 1$ we conclude that the singularity $M_1(0,0)$ is a star node and $M_2(-1-b,0)$ is a saddle. On the other hand we observe that the singular point $M_3(-1+b,0)$ is a (stable) node if $0<b<1$ and it is a saddle if $b>1$.

We observe that the singular point M_1 (respectively M_2; M_3) is located at the intersection of invariant line $y=0$ with L_1 (respectively L_2; L_3).

If $0<b<1$ then the singular point M_3 (located on the line L_3) is a node. Taking into account that in this case we have $L_2<L_3<L_1$ and the fact that by Lemma 4.5 the infinite singular point $N_1(1,0,0)$ is a node, we obtain the phase portrait given by P. 8.25.

Assume now $b>1$. In this case the singularity M_3 is a saddle and we have $L_2<L_1<L_3$. Since $N_1(1,0,0)$ is a node we arrive in this case at the phase portrait given by P. 8.24.

2) *The case* $1-r<0$. Then we can set $1-r=-b^2<0$ and we get: $r=1+b^2>1$. We may consider again $b>0$ due to the change $b\to -b$ and we arrive at the family of systems

$$\dot{x}=x\big[(x+1)^2+b^2\big),\quad \dot{y}=y(1+b^2+2x),\ \ b\neq 0. \tag{4.9}$$

These systems possess the unique real finite singular point $M_1(0,0)$ which is located at the intersection of the (unique real) invariant lines $x = 0$ and $y = 0$. For this real finite singularity we determine $\lambda_1 = 1 + b^2 = \lambda_2$, i.e. systems (4.9) possess a star node.

As by Lemma 4.5 the infinite singular point $N_1(1,0,0)$ is a star node it is clear that the unique possible phase portrait in this case corresponds to P. 8.27 (see Figure 4.1).

γ_2) ***Config. 8.28, 8.29, 8.30.*** Considering Lemma 4.5 we examine the family of systems (4.5) which possesses *Config. 8.28* (respectively, *8.29*; *8.30*) if $r < 0$ (respectively, $0 < r < 1$; $r > 1$).

Since these systems possess invariant straight lines $r - 2x + x^2 = 0$ and $\text{Discrim}[r - 2x + x^2, x] = 4(1 - r)$ we again consider two cases: $1 - r > 0$ and $1 - r < 0$.

1) *The case* $1 - r > 0$. Then as in the previous case we can set $1 - r = b^2 > 0$, i.e. $r = 1 - b^2$ (where we may consider $b > 0$ due to the change $b \to -b$). This leads to the family of systems

$$\dot{x} = x\big[(x-1)^2 - b^2\big), \quad \dot{y} = 2y(x - 1 + b^2), \;\; b > 0, (b-1)(3b-1) \neq 0, \tag{4.10}$$

which possess the following invariant affine lines:

$$\begin{aligned} &L_1: \; x = 0 \text{ (simple)}; && L_2: \; x = 1 - b \text{ (double)}; \\ &L_3: \; x = 1 + b \text{ (double)}; && L_4: \; y = 0 \text{ (simple)}. \end{aligned} \tag{4.11}$$

In addition the line at infinity $Z = 0$ is a double one. All these invariant lines are distinct due to the condition $0 < b \neq 1$.

On the other hand systems (4.10) possess three finite singularities, located at the intersections these invariant lines: $M_1(0,0)$ and $M_{2,3}(1 \mp b, 0)$. For these singularities we obtain

$$\begin{aligned} &\lambda_1^{(1)} = 1 - b^2, && \lambda_2^{(1)} = -2(1 - b^2); \\ &\lambda_1^{(2)} = 2b(b-1) = \lambda_2^{(2)}; && \lambda_1^{(3)} = 2b(b+1) = \lambda_2^{(3)}. \end{aligned}$$

It is clear that due to the condition $0 < b \neq 1$ the types of these singularities are well determined, and namely: the singularity $M_1(0,0)$ is a saddle whereas $M_2(1 - b, 0)$ and $M_3(1 + b, 0)$ are both star nodes.

On the other hand the position of the invariant lines (and consequently of these singularities) depends on the value of the parameter b. More exactly we have $L_1 < L_2 < L_3$ if $0 < b < 1$ and $L_2 < L_1 < L_3$ if $b > 1$. As a result, considering that the infinite singularity $N_1(1,0,0)$ is a node (see Lemma 4.5) we arrive at the phase portrait given by P. 8.28 if $b > 1$ and by P. 8.29 if $0 < b < 1$.

2) *The case* $1 - r < 0$. Then we can set $1 - r = -b^2 < 0$ and we get: $r = 1 + b^2 > 1$. We may consider again $b > 0$ due to the change $b \to -b$ and we arrive at the family of systems

$$\dot{x} = x\big[(x-1)^2 + b^2\big), \quad \dot{y} = -2y(1 + b^2 - x), \;\; b \neq 0. \tag{4.12}$$

These systems possess the unique real finite singular point $M_1(0,0)$ which is located at the intersection of the (unique real) invariant lines $x = 0$ and $y = 0$. For this real finite singularity we determine $\lambda_1 = 1 + b^2$ and $\lambda_2 = -2(1 + b^2)$, i.e. systems (4.12) possess a saddle at the origin of coordinate.

As by Lemma 4.5 the infinite singular point $N_1(1,0,0)$ is a star node, clearly we obtain the unique possible phase portrait in this case and it corresponds to P. 8.30 (see Figure 4.1).

γ_3) ***Config. 8.34, 8.35, 8.37.*** Taking into account Lemma 4.5 we have to consider systems (4.6), which possess *Config. 8.34* (respectively, *8.35*; *8.37*) if $r < 0$ (respectively, $0 < r < 1/4$; $r > 1/4$).

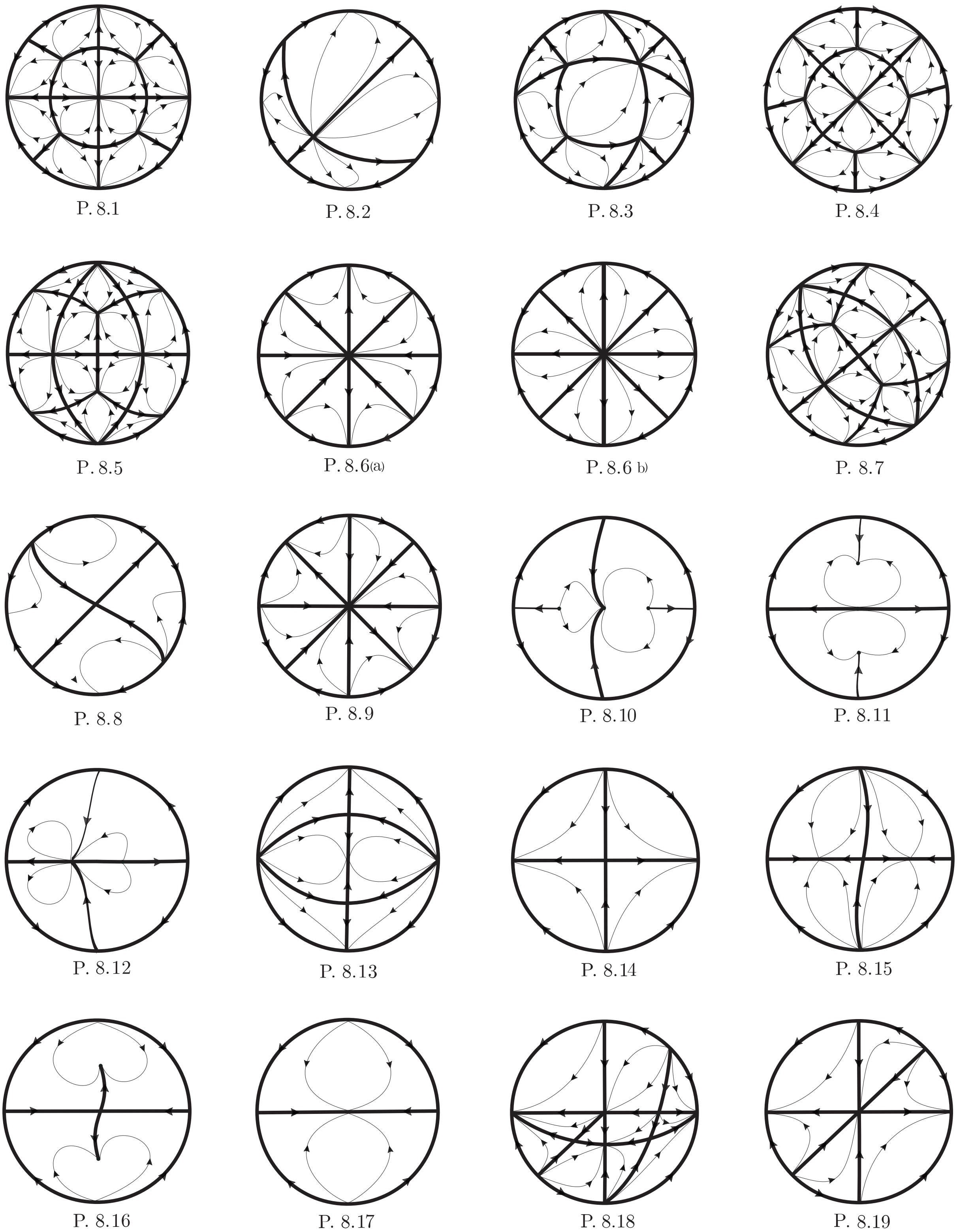

Figure 4.1: Phase portraits of systems in $\mathbb{CSL}_8$

Since the types (real or complex) of the invariant straight lines of systems (4.6) are determined by the sign of $\text{Discrim}[r + x + x^2, x] = 1 - 4r$, we examine two cases: $1 - 4r > 0$ and $1 - 4r < 0$.

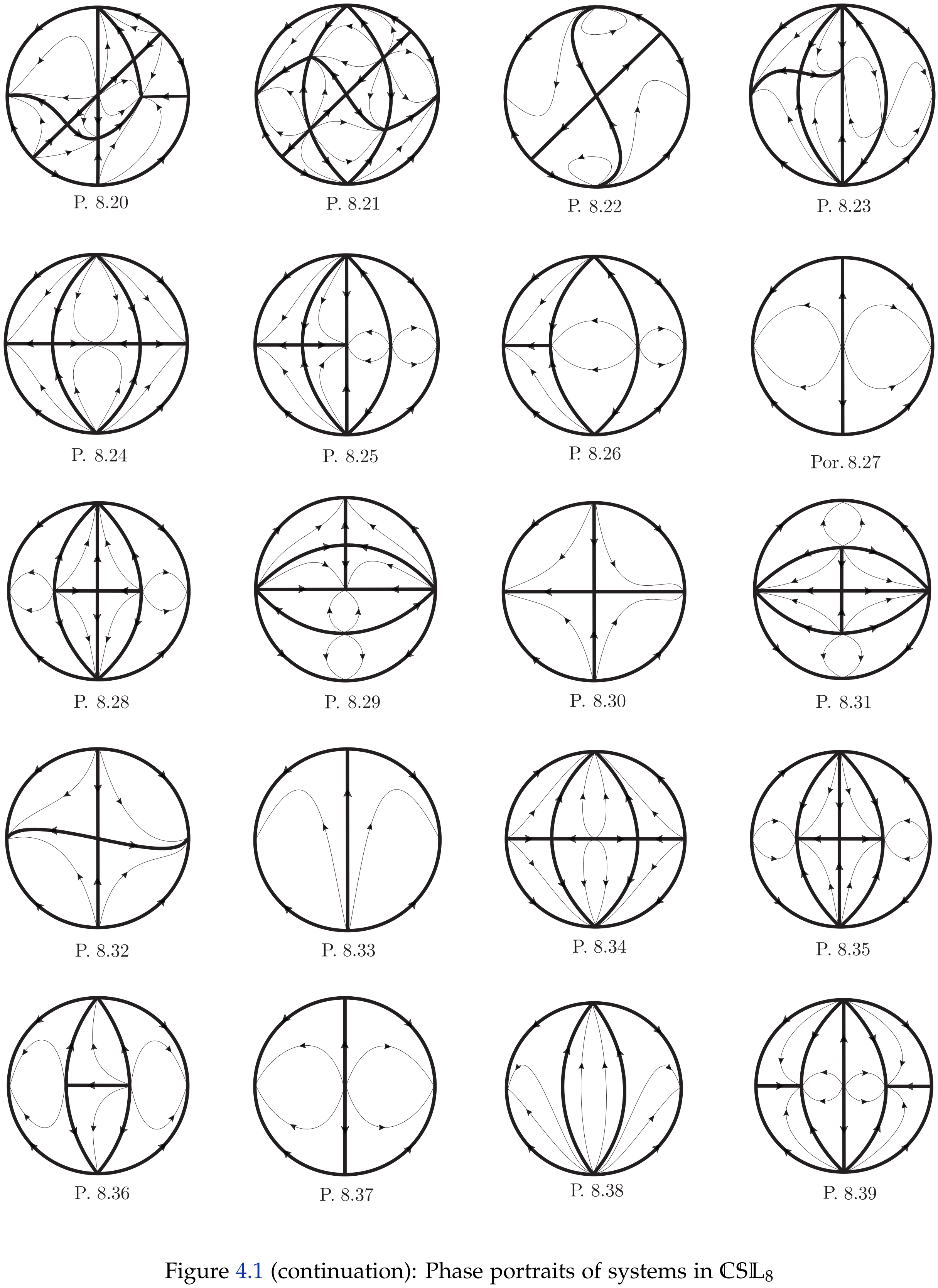

Figure 4.1 (continuation): Phase portraits of systems in $\mathbb{CSL}_8$

1) *The case* $1-4r>0$. We set $1-4r=b^2>0$ where b is a new parameter and we get: $r=(1-b^2)/4<1/4$. We may consider $b>0$ due to the change $b\to -b$ and after an

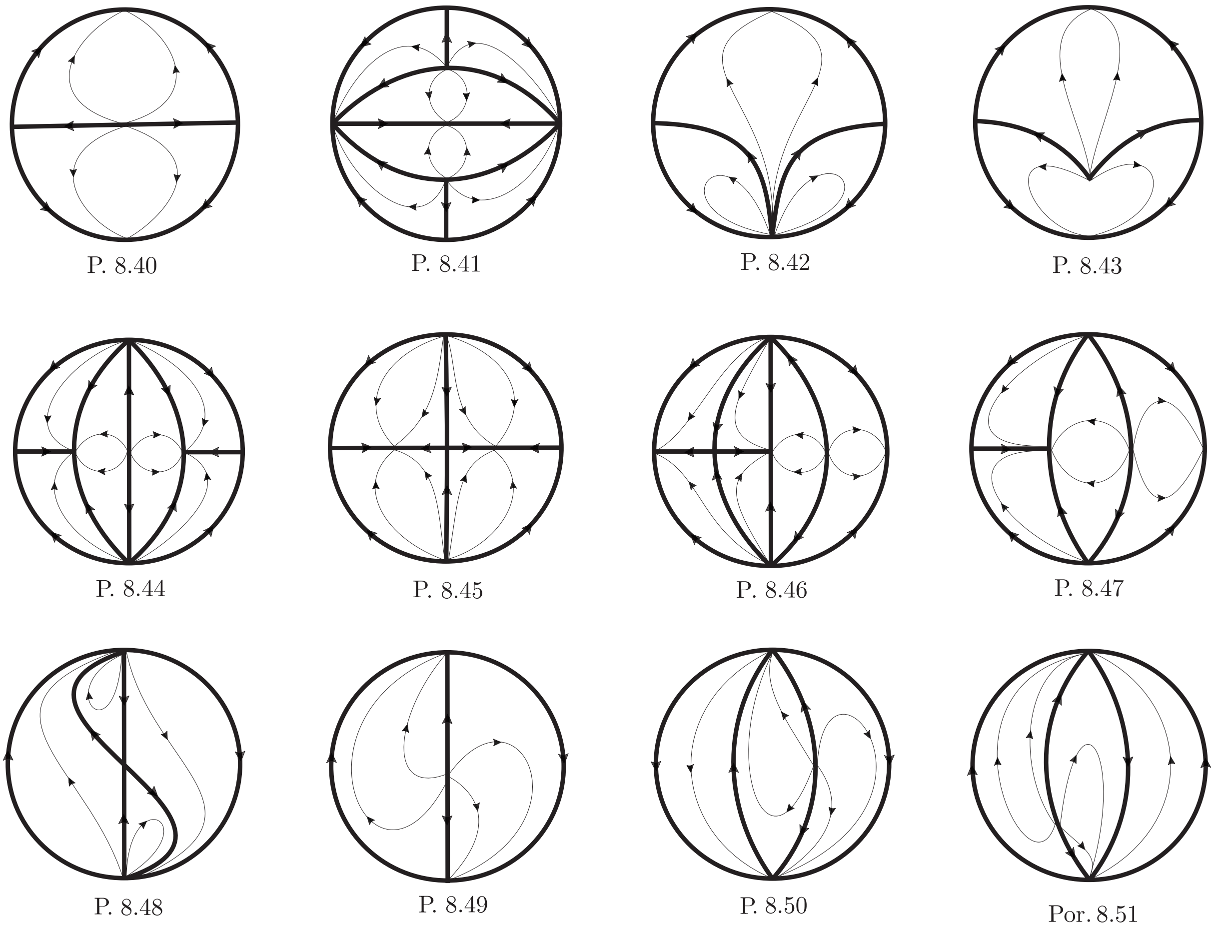

Figure 4.1 (continuation): Phase portraits of systems in $\mathbb{CSL}_8$

additional time rescaling we arrive at the family of systems

$$\dot{x} = x\big[(2x+1)^2 - b^2), \quad \dot{y} = 4 + (1-b^2)y, \;\; 0 < b \neq 1, 1/3. \tag{4.13}$$

These systems possess the following invariant affine lines:

$$\begin{array}{ll} L_1: \; x = 0 \; \text{(double)}; & L_2: \; x = -(b+1)/2 \; \text{(simple)}; \\ L_3: \; x = (b-1)/2 \; \text{(simple)}; & L_4: \; y = 4/(b^2-1) \; \text{(simple)}. \end{array} \tag{4.14}$$

In addition the line at infinity $Z = 0$ is a triple one. All these invariant lines are distinct due to the condition $0 < b \neq 1$.

We detect that systems (4.13) possess three finite singularities: $M_1\big(0, 4/(b^2-1)\big)$ and $M_{2,3}\big((-1 \mp b)/2,\, 4/(b^2-1)\big)$, located at the intersections of the line L_4 with invariant lines L_i $(i = 1, 2, 3)$, respectively. For these singularities we obtain

$$\lambda_1^{(1)} = 1 - b^2 = \lambda_2^{(1)}; \quad \lambda_1^{(2)} = 2b(1+b), \quad \lambda_2^{(2)} = 1 - b^2;$$
$$\lambda_1^{(3)} = 2b(b-1), \quad \lambda_2^{(3)} = 1 - b^2.$$

We observe that

$$\lambda_1^{(2)}\lambda_2^{(2)} = 2b(1+b)^2(1-b), \quad \lambda_1^{(3)}\lambda_2^{(3)} = -2(b-1)^2 b(1+b).$$

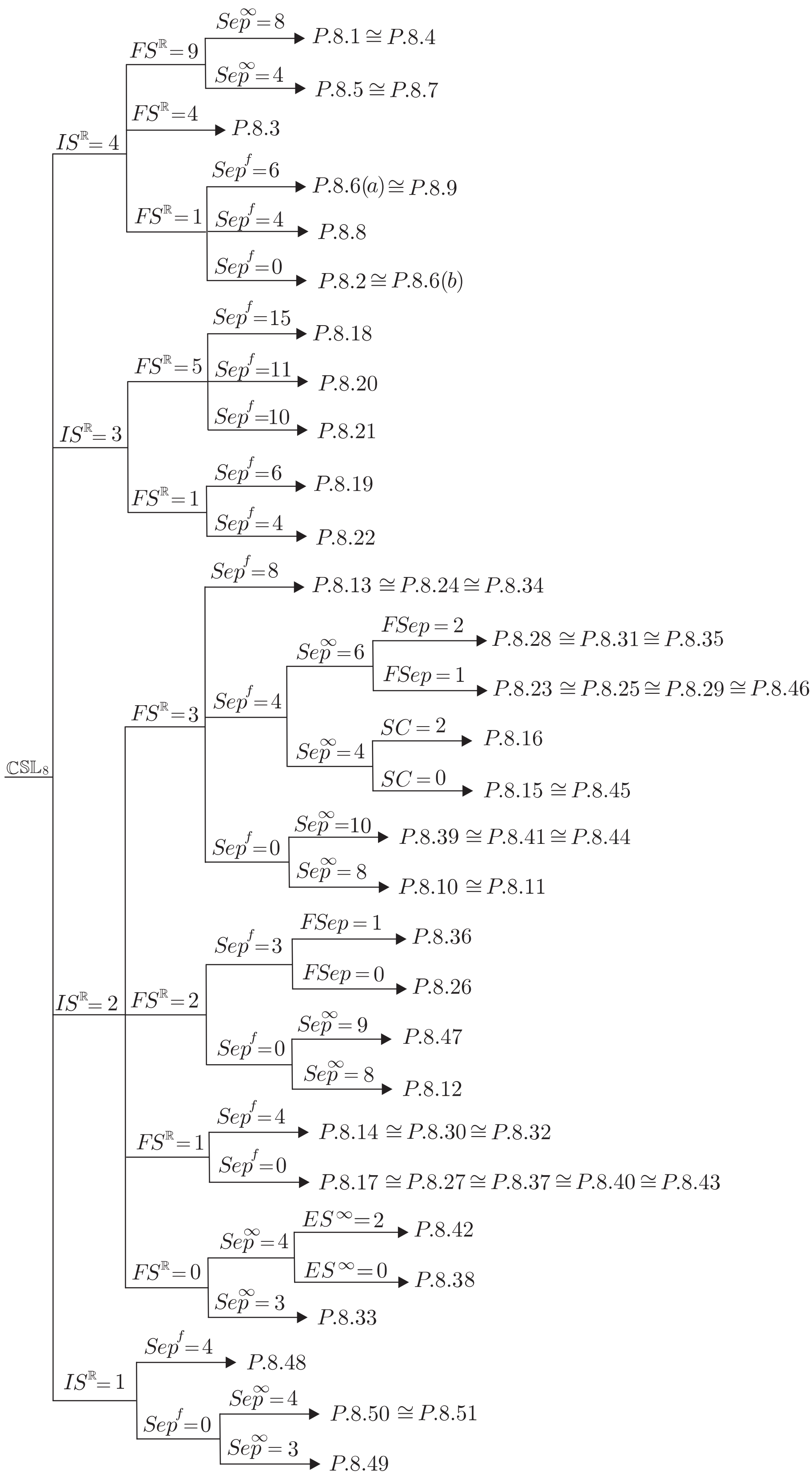

Diagram 4.1: Topologically distinct phase portraits

Therefore considering the condition $0 < b \neq 1$ we conclude that the singularity M_1 is a star node and M_3 is a saddle. On the other hand we observe that the singular point M_2 is a (unstable) node if $0 < b < 1$ and it is a saddle if $b > 1$.

Assume first $0 < b < 1$. Then the singular point M_2 (located on the line L_2) is a node. Taking into account that in this case we have $L_2 < L_3 < L_1$ and the fact that by Lemma 4.5 the singular point $N_1(1,0,0)$ is a node, we obtain the phase portrait given by P. 8.35.

Suppose now $b > 1$. In this case the singularity M_2 is a saddle and we have $L_2 < L_1 < L_3$. Since $N_1(1,0,0)$ is a node we arrive in this case at the phase portrait given by P. 8.34.

2) *The case* $1-4r < 0$. Then we can set $1-4r = -b^2 < 0$, i.e. $r = (1+b^2)/4$ and due to a time rescaling systems (4.6) become

$$\dot{x} = x\big[(2x+1)^2 + b^2\big), \quad \dot{y} = 4 + (1+b^2)y, \quad b > 0. \tag{4.15}$$

These systems possess the unique real finite singular point $M_1\big(0, -4/(b^2+1)\big)$ which is located at the intersection of the (unique real) invariant lines $x = 0$ and $y = -4/(b^2+1)$. For this real finite singularity we determine $\lambda_1 = (1+b^2)/4 = \lambda_2$. So systems (4.15) possess a star node.

Since by Lemma 4.5 the infinite singular point $N_1(1,0,0)$ is a star node and the existence of exactly three real singularities (M_1, $N_1(1,0,0)$ and $N_2(0,1,0)$) of the above systems clearly this leads to the unique possible phase portrait which corresponds to P. 8.37 (see Figure 4.1).

γ_4) ***Config. 8.39.*** According to Table 4.1 we consider the family of systems

$$\begin{aligned} &\dot{x} = x\big[b^2 - (2x+1)^2\big], \quad 0 < b \neq 1,\, 1/3, \\ &\dot{y} = (b^2 - 1 - 8x - 12x^2)y, \end{aligned} \tag{4.16}$$

which possess the following invariant affine lines:

$$\begin{aligned} &L_1: \; x = 0\,(\text{double}); \quad L_2: \; x = -(b+1)/2\,(\text{double}); \\ &L_3: \; x = (b-1)/2\,(\text{double}); \quad L_4: \; y = 0\,(\text{simple}). \end{aligned} \tag{4.17}$$

We also detect that systems (4.16) possess three finite singularities:

On the other hand systems (4.16) possess three finite singularities, located at the intersections of the invariant line L_4 with L_1, L_2 and L_3, respectively: $M_1(0,0)$ and $M_{2,3}((-1\mp b)/2, 0)$. For these singularities we obtain

$$\lambda_1^{(1)} = b^2 - 1 = \lambda_2^{(1)}; \quad \lambda_1^{(2)} = -2b(1+b) = \lambda_2^{(2)}; \;\; \lambda_1^{(3)} = 2b(1-b) = \lambda_2^{(3)}$$

It is clear that due to the condition $0 < b \neq 1$ the types of these singularities are well determined, and namely, all three are star nodes.

Next we examine the infinite singularities of (4.16) for which we have $C_3 = 8x^3y$. Hence at infinity we have two singularities: $N_1(1,0,0)$ and $N_2(0,1,0)$. Considering Theorem 3.1 (see *Config. 8.39* from Figure 3.1) we conclude that $N_1(1,0,0)$ is a simple singularity whereas $N_2(0,1,0)$ has multiplicity 9 (six finite singularities coalesced with three infinite on the line $Z = 0$).

Constructing the corresponding systems at infinity, possessing the point $N_1(1,0,0)$ at the origin of coordinates, we get the systems

$$\dot{u} = 8u + 4uz, \quad \dot{z} = -4z - 4z^2 - z^3 + b^2z^3. \tag{4.18}$$

Evidently the singular point $(0,0)$ of these systems (which corresponds to the infinite singularity $N_1(1,0,0)$ of systems (4.16)) is a saddle for any value of the parameter b.

Thus considering the types of the finite singularities and of the simple infinite singular point $N_1(1,0,0)$ as well as the existence of three four invariant lines (4.17) we arrive at the unique phase portrait which corresponds to P. 8.39 (see Figure 4.1).

γ_5) ***Config. 8.40.*** Considering Table 4.1 we have to examine the family of systems

$$\begin{aligned}\dot{x} &= x\big[b^2 + (2x+1)^2\big], \quad b > 0,\\ \dot{y} &= (b^2 + 1 + 8x + 12x^2)y,\end{aligned} \tag{4.19}$$

which possess the unique real finite singular point $M_1(0,0)$ for which $\lambda_1 = 1 + b^2 = \lambda_2$. Hence this singularity is a star node located at the intersection of unique real invariant affine line $y = 0$ and $x = 0$.

For systems (4.19) we have $C_3 = -8x^3y$ and considering *Config. 8.40* from Figure 3.1 (see Theorem 3.1) we get the same two infinite singularities: $N_1(1,0,0)$ (simple) and $N_2(0,1,0)$ (of multiplicity 9). We detect that $N_1(1,0,0)$ corresponds to the singular point $(0,0)$ of the systems

$$\dot{u} = -8u - 4uz, \quad \dot{z} = 4z + 4z^2 + z^3 + b^2z^3 \tag{4.20}$$

and hence this singular point is a saddle for any value of the parameter b.

Thus considering the existence of exactly three real singularities (M_1, $N_1(1,0,0)$ and $N_2(0,1,0)$) of the above systems as well as of two real invariant lines $x = 0$ and $y = 0$ we get the unique possible phase portrait which corresponds to P. 8.40 (see Figure 4.1).

γ_6) ***Config. 8.44, 8.45, 8.46.*** Considering Table 4.1 we observe that each one of these three configuration corresponds to the same one-parameter family of systems

$$\begin{aligned}\dot{x} &= x(1+x)(2+r+x+rx),\\ \dot{y} &= (2+r+3x+2rx+rx^2)y, \quad r(r+3)(r+2)(3+2r) \neq 0\end{aligned} \tag{4.21}$$

for different value of the parameter r. More exactly we have *Config. 8.44* (respectively, *8.45*; *8.46*) if $-2 < r < -1$ (respectively, $r < -2$; $r > -1$).

The above systems possess the following invariant affine lines:

$$\begin{aligned}&L_1: \ x = 0 \text{ (triple)}; && L_2: \ x = -1 \text{ (double)};\\ &L_3: \ x = -\frac{r+2}{r+1} \text{ (simple)}; && L_4: \ y = 0 \text{ (simple)}.\end{aligned} \tag{4.22}$$

All these invariant lines are distinct due to the condition $(r+2) \neq 0$.

We detect that systems (4.21) possess three finite singularities: $M_1(0,0)$, $M_2(-1,0)$ and $M_3\big(-\frac{r+2}{r+1}, 0\big)$, located at the intersections of the line L_4 with invariant lines L_i $(i = 1,2,3)$, respectively. For these singularities we obtain

$$\begin{aligned}&\lambda_1^{(1)} = 2 + r = \lambda_2^{(1)}; && \lambda_1^{(2)} = -1 = \lambda_2^{(2)};\\ &\lambda_1^{(3)} = \frac{r+2}{r+1}, && \lambda_2^{(3)} = -\frac{(r+2)^2}{(r+1)^2}.\end{aligned}$$

Therefore we conclude that the singularities M_1 and M_2 are star nodes, whereas the type of M_2 depends on the the sign of the expression $(r+2)(r+1)$. More precisely we have a node if $-2 < 3 < -1$ and we have a saddle if either $r < -2$ or $r > -1$.

On the other hand for systems (4.21) we have $C_3 = x^3y$, i.e. at infinity we have two singularities: $N_1(1,0,0)$ and $N_2(0,1,0)$. Considering Theorem 3.1 (see *Config. 8.44 – 8.46* from Figure 3.1) we conclude that $N_1(1,0,0)$ is a simple singularity whereas $N_2(0,1,0)$ has multiplicity 9 (six finite singularities coalesced with three infinite on the line $Z = 0$).

Constructing the corresponding systems at infinity, possessing the point $N_1(1,0,0)$ at the origin of coordinates, we get the systems

$$\dot{u} = u, \quad \dot{z} = (1+r)z + (3+2r)z^2 + 2z^3 + rz^3. \tag{4.23}$$

Evidently the type of the singular point $(0,0)$ of these systems (which corresponds to the infinite singularity $N_1(1,0,0)$ of systems (4.21)) is a node if $r > -1$ and it is a saddle if $r < -1$.

Since the singular points M_1 and M_2 are both star nodes, it remains to examine the corresponding intervals of the variation of the parameter r and to determine the types of the singularities M_3 and $N_1(1,0,0)$. Moreover it is necessary to detect the position of the invariant line L_3 with respect to L_1 and L_2. This information is sufficient to get in unique mode the corresponding phase portrait.

If $r < -2$ then M_3 and $N_1(1,0,0)$ are both saddles and we get $L_2 < L_3 < L_1$. As a result we arrive at the phase portrait given by P. 8.45 (see Figure 4.1).

Assume now $-2 < r < -1$. In this case M_3 is a node and $N_1(1,0,0)$ is a saddle. On the other hand we have $L_2 < L_1 < L_3$ and this leads to the phase portrait which corresponds to P. 8.44.

Admit finally, $r > -1$. Then M_3 is a saddle, $N_1(1,0,0)$ is a node and $L_3 < L_2 < L_1$. In this case we get the phase portrait given by P. 8.46 in Figure 4.1 and this completes the proof of the statement $(\boldsymbol{B})$ of Main Theorem.

$(\boldsymbol{C})$ To prove that only 30 among 52 obtained phase portrait are topologically distinct it is sufficient to evaluate for each one of them the geometrical invariants presented in Remark 4.2. In such a way we get Diagram 4.1 which proves this statement of the theorem and therefore completes the proof of Main Theorem. □

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