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Delay effect in the Nicholson's blowflies model with a nonlinear density-dependent mortality term

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Abstract. This paper is concerned with a class of non-autonomous delayed Nicholson's blowflies model with a nonlinear density-dependent mortality term. Under proper conditions, we prove that the positive equilibrium point is a global attractor of the addressed model with small delays. Moreover, some numerical examples are given to illustrate the feasibility of the theoretical results.

Keywords: Nicholson's blowflies model, nonlinear density-dependent mortality term, time-varying delay, global attractivity.

2010 Mathematics Subject Classification: 34C25, 34K11, 34K25.

1 Introduction

Recently, based on that marine ecologists are currently constructing new fishery models with nonlinear density-dependent mortality rates, the following Nicholson's blowflies model with a nonlinear density-dependent mortality term

$$N'(t) = -D(N(t)) + PN(t-\tau)e^{-N(t-\tau)}, \tag{1.1}$$

was proposed in L. Berezansky et al. [1]. Here function $D(x)$ might have one of the following forms: $D(N) = \frac{aN}{b+N}$ or $D(N) = a - be^{-N}$ with positive constants $a, b > 0$. The detailed biological explanations of the parameters of (1.1) can be found in [1, 13]. Furthermore, (1.1) and its generalized equations have been extensively studied, and this extensive study has produced a lot of progress on the existence and stability of positive equilibrium point, positive periodic solutions, and positive almost periodic solutions, see more details in [2–4, 8, 11–13, 18]. In particular, the author in [9] established several criteria on the global asymptotic stability of zero equilibrium point for the following Nicholson's blowflies model with a nonlinear density-dependent mortality term:

$$N'(t) = -a + be^{-N(t)} + \sum_{j=1}^{m} \beta_j N(t-\tau_j(t))e^{-\gamma_j N(t-\tau_j(t))}, \tag{1.2}$$

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where a, b, β_j, γ_j are positive constants, $\tau_j(t) \geq 0$ is a bounded and continuous function, and $j \in J = \{1, 2, \ldots, m\}$.

On the other hand, the effect of delay on the asymptotic behavior of population models can reveal the essential characteristics of time delay in practical problems, and it has attracted extensive attention in [5, 6, 15, 17]. It is worthy mentioned that there have been few papers concerning the effect of delay on the dynamical behavior of the delayed Nicholson's blowflies model with a nonlinear density-dependent mortality term.

Motivated by the above works, the effect of delay on the dynamical behavior of the delayed Nicholson's blowflies model with a nonlinear density-dependent mortality term attracted our attention. In this paper, we aim to provide a criterion to guarantee that all solutions of (1.2) converge to the positive equilibrium point, which entails that (1.2) is global attractive under the small delays. In fact, one can see the following Remark 2.2 and Remark 3.1 for details.

In what follows, we designate $r = \max_{1\leq j\leq m} \sup_{t\in\mathbb{R}} \tau_j(t)$, $C = C([-r, 0], \mathbb{R})$ be the continuous functions space equipped with the usual supremum norm $\|\cdot\|$, and let $C_+ = C([-r, 0], (0, +\infty))$. If $x(t)$ is continuous and defined on $[-r + t_0, \sigma)$ with $t_0, \sigma \in \mathbb{R}$, then we define $x_t \in C$ where $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-r, 0]$.

It will be always assumed that

$$\gamma^* = \min_{1\leq j\leq m} \gamma_j > 0, \qquad \gamma = \max_{1\leq j\leq m} \gamma_j \geq 1, \qquad \sum_{j=1}^{m} \frac{\beta_j}{\gamma_j a}\frac{1}{e} < 1, \quad \text{and} \quad \ln\frac{b}{a} > \frac{1}{\gamma^*}. \tag{1.3}$$

Denote $N_t(t_0, \varphi)$ $(N(t; t_0, \varphi))$ as an admissible solution of (1.2) with the admissible initial condition

$$N_{t_0} = \varphi, \qquad \varphi \in C_+ \quad \text{and} \quad \varphi(0) > 0. \tag{1.4}$$

and $[t_0, \eta(\varphi))$ be the maximal right-interval of the existence of $N_t(t_0, \varphi)$. Define a continuous function $F : R \to R$ by setting

$$F(u) = -a + be^{-u} + \sum_{j=1}^{m} \beta_j u e^{-\gamma_j u}.$$

Since

$$F(0) = -a + b > 0, \qquad F(+\infty) = -a < 0,$$

there exists at least one positive constant $\bar{N}$ such that

$$F(\bar{N}) = -a + be^{-\bar{N}} + \sum_{j=1}^{m} \beta_j \bar{N} e^{-\gamma_j \bar{N}} = 0, \tag{1.5}$$

which is a positive equilibrium point of (1.2).

2 Main result

In this section, we establish some sufficient conditions on the global asymptotic stability of positive equilibrium point for (1.2).

Lemma 2.1. *$N(t; t_0, \varphi)$ is positive and bounded on $[t_0, \eta(\varphi))$ and hence $\eta(\varphi) = +\infty$. Moreover,*

$$l = \liminf_{t\to+\infty} N(t; t_0, \varphi) \geq \ln\frac{b}{a} > \frac{1}{\gamma^*}. \tag{2.1}$$

Proof. Let $N(t) = N(t; t_0, \varphi)$. We first claim:

$$N(t) > 0 \quad \text{for all } t \in [t_0, \eta(\varphi)).$$

Suppose, for the sake of contradiction, there exists $t_1 \in (t_0,\ \eta(\varphi))$ such that

$$N(t_1) = 0, \quad N(t) > 0 \quad \text{for all } t \in [t_0, t_1).$$

Then,

$$0 \geq N'(t_1) = -a + be^{-N(t_1)} + \sum_{j=1}^{m} \beta_j N(t_1 - \tau_j(t_1)) e^{-\gamma_j N(t_1 - \tau_j(t_1))} \geq -a + b > 0.$$

This contradiction means that $N(t) > 0$ for all $t \in [t_0, \eta(\varphi))$.

For each $t \in [t_0 - r, \eta(\varphi))$, we define

$$M(t) = \max\left\{\xi : \xi \leq t, x(\xi) = \max_{t_0 - r \leq s \leq t} N(s)\right\}.$$

We now show that $N(t)$ is bounded on $[t_0, \eta(\varphi))$. In the contrary case, observe that $M(t) \to \eta(\varphi)$ as $t \to \eta(\varphi)$, we have

$$\lim_{t \to \eta(\varphi)} N(M(t)) = +\infty.$$

On the other hand,

$$N(M(t)) = \max_{t_0 - r \leq s \leq t} x(s), \quad \text{and so } N'(M(t)) \geq 0, \quad \text{where } M(t) > t_0.$$

Thus, in view of the fact that $\sup_{u \geq 0} ue^{-u} = \frac{1}{e}$, we get

$$\begin{aligned}
0 &\leq N'(M(t)) \\
&= -a + be^{-x(M(t))} + \sum_{j=1}^{m} \beta_j N(M(t) - \tau_j(M(t))) e^{-\gamma_j N(M(t) - \tau_j(M(t)))} \\
&= a\left[-1 + \frac{b}{a} e^{-N(M(t))} + \sum_{j=1}^{m} \frac{\beta_j}{\gamma_j a} \gamma_j N(M(t) - \tau_j(M(t))) e^{-\gamma_j N(M(t) - \tau_j(M(t)))}\right] \\
&\leq a\left[-1 + \frac{b}{a} e^{-N(M(t))} + \sum_{j=1}^{m} \frac{\beta_j}{\gamma_j a} \frac{1}{e}\right], \quad \text{where } M(t) > t_0.
\end{aligned}$$

Letting $t \to \eta(\varphi)$ leads to

$$0 \leq -1 + \sum_{j=1}^{m} \frac{\beta_j}{\gamma_j a} \frac{1}{e},$$

which contradicts to the assumption (1.3). This implies that $x(t)$ is bounded on $[t_0, \eta(\varphi))$. From Theorem 2.3.1 in [7], we easily obtain $\eta(\varphi) = +\infty$.

Let $l = \liminf_{t \to +\infty} N(t)$. By the fluctuation lemma [16, Lemma A.1], there exists a sequence $\{t_p\}_{p \geq 1}$ such that

$$t_p \to +\infty, \quad N(t_p) \to \liminf_{t \to +\infty} N(t), \quad N'(t_p) \to 0 \quad \text{as } p \to +\infty.$$

According to (1.2), we get

$$\begin{aligned} N'(t_p) &= -a + be^{-N(t_p)} + \sum_{j=1}^{m} \beta_j N(t_p - \tau_j(t_p)) e^{-\gamma_j N(t_p - \tau_j(t_p))} \\ &\geq -a + be^{-N(t_p)}, t_p > t_0. \end{aligned}$$

Then, taking limits gives us that

$$l = \liminf_{t\to+\infty} N(t) \geq \ln\frac{b}{a} > \frac{1}{\gamma^*}.$$

This ends the proof of Lemma 2.1. □

Remark 2.2. From Lemma 2.1, it is not difficult to see that (1.2) and (1.4) is uniformly permanent. Moreover, $\bar{N}$ is also a solution of (1.2) and (1.4), and

$$\bar{N} \geq \ln\frac{b}{a} > \frac{1}{\gamma^*}. \tag{2.2}$$

For simplicity, denote $N(t; t_0, \varphi)$ by $N(t)$. Now, we show the global attractivity of $\bar{N}$ by the following three propositions:

Proposition 2.3. *If $x(t) = N(t) - \bar{N}$ is eventually nonnegative, then*

$$\lim_{t\to+\infty} N(t) = \bar{N}.$$

Proof. Clearly, there exists $T > t_0$ such that

$$x(t) = N(t) - \bar{N} \geq 0 \quad \text{for all } t \geq T.$$

In order to prove Proposition 2.3, it suffices to show that $\limsup_{t\to+\infty} x(t) = 0$. Again by way of contradiction, we assume that $\limsup_{t\to+\infty} x(t) > 0$. By the fluctuation lemma [16, Lemma A.1], there exists a sequence $\{t_k\}_{k\geq1}$ such that

$$t_k \to +\infty, \quad x(t_k) \to \limsup_{t\to+\infty} x(t), \quad x'(t_k) \to 0 \quad \text{as } k \to +\infty.$$

In view of (2.2), we can choose $K > T$ to satisfy that

$$\gamma_j N(t) \geq \gamma_j \bar{N} > \gamma_j \frac{1}{\gamma^*} \geq 1, \quad \text{for all } t > K,\ j \in J,$$

which, together with the fact that xe^{-x} decreases on $[1, +\infty)$, implies that

$$\begin{aligned} x'(t_k) &= -a + be^{-N(t_k)} + \sum_{j=1}^{m} \beta_j N(t_k - \tau_j(t_k)) e^{-\gamma_j N(t_k - \tau_j(t_k))} \\ &= -a + be^{-N(t_k)} + \sum_{j=1}^{m} \frac{\beta_j}{\gamma_j} \gamma_j N(t_k - \tau_j(t_k)) e^{-\gamma_j N(t_k - \tau_j(t_k))} \\ &\leq -a + be^{-N(t_k)} + \sum_{j=1}^{m} \frac{\beta_j}{\gamma_j} \gamma_j \bar{N} e^{-\gamma_j \bar{N}}, \qquad t_k > K + r. \end{aligned} \tag{2.3}$$

By taking limits, (1.5) and (2.3) lead to

$$0 \leq -a + be^{-(\limsup_{t\to+\infty} x(t) + \bar{N})} + \bar{N} \sum_{j=1}^{m} \beta_j e^{-\gamma_j \bar{N}} < -a + be^{-\bar{N}} + \bar{N} \sum_{j=1}^{m} \beta_j e^{-\gamma_j \bar{N}} = 0,$$

a contradiction. Hence, $\limsup_{t\to+\infty} x(t) = 0$. This completes the proof. □

Proposition 2.4. *If $x(t) = N(t) - \bar{N}$ is eventually non-positive, then*

$$\lim_{t\to+\infty} N(t) = \bar{N}.$$

Proof. Obviously, we can choose $T > t_0$ such that

$$x(t) = N(t) - \bar{N} \le 0 \quad \text{for all } t \ge T.$$

Next, we prove that $\liminf_{t\to+\infty} x(t) = 0$. Otherwise, $\liminf_{t\to+\infty} x(t) < 0$. Again from the fluctuation lemma [16, Lemma A.1], there exists a sequence $\{\bar{t}_k\}_{k\ge1}$ such that

$$\bar{t}_k \to +\infty, \quad x(\bar{t}_k) \to \liminf_{t\to+\infty} x(t), \quad x'(\bar{t}_k) \to 0 \quad \text{as } k \to +\infty.$$

From (2.1), we can choose $K^* > T$ to satisfy that

$$\gamma_j \bar{N} \ge \gamma_j N(t) > \gamma_j \frac{1}{\gamma^*} \ge 1, \quad \text{for all } t > K^*,\ j \in J,$$

which, together with the fact that xe^{-x} decreases on $[1, +\infty)$, implies that

$$\begin{aligned} x'(\bar{t}_k) &= -a + be^{-N(\bar{t}_k)} + \sum_{j=1}^{m} \beta_j N(\bar{t}_k - \tau_j(\bar{t}_k)) e^{-\gamma_j N(\bar{t}_k - \tau_j(\bar{t}_k))} \\ &= -a + be^{-N(\bar{t}_k)} + \sum_{j=1}^{m} \frac{\beta_j}{\gamma_j} \gamma_j N(\bar{t}_k - \tau_j(\bar{t}_k)) e^{-\gamma_j N(\bar{t}_k - \tau_j(\bar{t}_k))} \\ &\ge -a + be^{-N(\bar{t}_k)} + \sum_{j=1}^{m} \frac{\beta_j}{\gamma_j} \gamma_j \bar{N} e^{-\gamma_j \bar{N}},\ t_k > K^* + r. \end{aligned} \tag{2.4}$$

By taking limits, (1.5) and (2.4) give us that

$$0 \ge -a + be^{-(\liminf_{t\to+\infty} x(t) + \bar{N})} + \bar{N} \sum_{j=1}^{m} \beta_j e^{-\gamma_j \bar{N}} > -a + be^{-\bar{N}} + \bar{N} \sum_{j=1}^{m} \beta_j e^{-\gamma_j \bar{N}} = 0,$$

a contradiction and hence $\liminf_{t\to+\infty} x(t) = 0$. This completes the proof. □

Proposition 2.5. *If $x(t) = N(t) - \bar{N}$ oscillates about zero, and*

$$\frac{r}{e} \sum_{j=1}^{m} \beta_j < 1, \qquad \frac{\gamma r a}{1 - \frac{r}{e} \sum_{j=1}^{m} \beta_j} \le 1 \tag{2.5}$$

then $\lim_{t\to+\infty} N(t) = \bar{N}$.

Proof. Set $y(t) = \gamma x(t) = \gamma(N(t) - \bar{N})$, we have

$$y'(t) = -\gamma a + \gamma b e^{-\bar{N} - \frac{1}{\gamma} y(t)} + \gamma \sum_{j=1}^{m} \beta_j \left[\bar{N} + \frac{1}{\gamma} y(t - \tau_j(t)) \right] e^{-\gamma_j \bar{N} - \frac{\gamma_j}{\gamma} y(t - \tau_j(t))}, \qquad t > t_0. \tag{2.6}$$

Let

$$\lambda = \liminf_{t\to+\infty} y(t), \qquad \mu = \limsup_{t\to+\infty} y(t). \tag{2.7}$$

Since $y(t) = \gamma x(t)$ oscillates about zero, one can get that

$$\lambda \le 0 \le \mu.$$

Now, in order to prove Proposition 2.5, it suffices to show that that $\lambda = \mu = 0$.

Again from the fact that $y(t)$ oscillates about zero, we can choose a strictly monotonically increasing sequence $\{q_n\}_{n\geq 1}$ to satisfy that

$$q_n > r, \quad \lim_{n\to+\infty} q_n = +\infty, \quad y(q_n) = 0 \quad \text{for all } n = 1, 2, \ldots,$$

and such that in each interval (q_n, q_{n+1}) the function $y(t)$ assumes both positive and negative values. For any positive integer n, let $t_n, s_n \in (q_n, q_{n+1})$ such that

$$y(t_n) = \max_{t\in[q_n,\, q_{n+1}]} y(t) > 0, \qquad y(s_n) = \min_{t\in[q_n, q_{n+1}]} y(t) < 0.$$

Then,

$$y'(t_n) = y'(s_n) = 0, \qquad n = 1, 2, \ldots, \tag{2.8}$$

and

$$\lambda = \liminf_{t\to+\infty} y(t) = \liminf_{n\to+\infty} y(s_n), \qquad \mu = \limsup_{t\to+\infty} y(t) = \limsup_{n\to+\infty} y(t_n). \tag{2.9}$$

Subsequently, we assert that for each positive integer n, there is $T_n \in [t_n - r, t_n) \cap [q_n, t_n)$ such that

$$y(T_n) = 0, \quad \text{and} \quad y(t) > 0 \quad \text{for all } t \in (T_n, t_n). \tag{2.10}$$

In the contrary case, given a positive integer n, we have

$$q_n < t_n - r < q_{n+1} \quad \text{and} \quad y(t) > 0 \quad \text{for all } t \in [t_n - r, t_n),$$

which, together with (1.5), (2.2), (2.6), (2.8) and the fact that xe^{-x} decreases on $[1, +\infty)$, tells us that

$$\begin{aligned}
0 &= -\gamma a + \gamma b e^{-\bar N - \frac{1}{\gamma} y(t_n)} + \gamma \sum_{j=1}^{m} \beta_j \left[\bar N + \frac{1}{\gamma} y(t_n - \tau_j(t_n))\right] e^{-\gamma_j \bar N - \frac{\gamma_j}{\gamma} y(t_n - \tau_j(t_n))} \\
&= -\gamma a + \gamma b e^{-\bar N - \frac{1}{\gamma} y(t_n)} + \gamma \sum_{j=1}^{m} \beta_j \frac{1}{\gamma_j} \left[\gamma_j \bar N + \frac{\gamma_j}{\gamma} y(t_n - \tau_j(t_n))\right] e^{-\gamma_j \bar N - \frac{\gamma_j}{\gamma} y(t_n - \tau_j(t_n))} \\
&< -\gamma a + \gamma b e^{-\bar N} + \gamma \sum_{j=1}^{m} \beta_j \frac{1}{\gamma_j} \gamma_j \bar N e^{-\gamma_j \bar N} \\
&= -\gamma a + \gamma b e^{-\bar N} + \gamma \sum_{j=1}^{m} \beta_j \bar N e^{-\gamma_j \bar N} \\
&= 0.
\end{aligned}$$

This is a contradiction and proves (2.10).

Similarly, we can prove that for each positive integer n, there is $S_n \in [s_n - r, s_n) \cap [q_n, s_n)$ such that

$$y(S_n) = 0, \quad \text{and} \quad y(t) < 0 \quad \text{for all } t \in (S_n, s_n). \tag{2.11}$$

For any $\varepsilon > 0$, (2.9) implies that there exists a positive integer n^* such that $\min\{t_{n^*}, s_{n^*}\} - 2r > t_0$, and

$$\lambda - \varepsilon < y(t) < \mu + \varepsilon \quad \text{for all } t > \min\{t_{n^*}, s_{n^*}\} - 2r. \tag{2.12}$$

Thus,

$$y(t - \tau_j(t)) e^{-\frac{\gamma_j}{\gamma} y(t - \tau_j(t))} < \mu + \varepsilon \quad \text{for all } t > \min\{l_{q^*}, s_{q^*}\} - r, \quad j \in I. \tag{2.13}$$

In view of (1.3), (2.2), (2.10), (2.12) and (2.13), integrating (2.6) from T_n to t_n, we find

$$\begin{aligned}
y(t_n) = & -\gamma a(t_n - T_n) + \gamma b \int_{T_n}^{t_n} e^{-\bar{N}-\frac{1}{\gamma}y(t)} dt \\
& + \gamma \sum_{j=1}^{m} \beta_j \int_{T_n}^{t_n} \left[\bar{N} + \frac{1}{\gamma} y(t-\tau_j(t))\right] e^{-\gamma_j \bar{N} - \frac{\gamma_j}{\gamma} y(t-\tau_j(t))} dt \\
= & -\gamma a(t_n - T_n) + \gamma b \int_{T_n}^{t_n} e^{-\bar{N}} e^{-\frac{1}{\gamma}y(t)} dt \\
& + \sum_{j=1}^{m} \beta_j \int_{T_n}^{t_n} \left[\gamma \bar{N} e^{-\gamma_j \bar{N}} e^{-\frac{\gamma_j}{\gamma} y(t-\tau_j(t))} + e^{-\gamma_j \bar{N}} y(t-\tau_j(t)) e^{-\frac{\gamma_j}{\gamma} y(t-\tau_j(t))}\right] dt \\
< & -\gamma a(t_n - T_n) + \gamma b e^{-\bar{N}} e^{-(\lambda-\varepsilon)} (t_n - T_n) \\
& + (t_n - T_n) \sum_{j=1}^{m} \beta_j \left[\gamma \bar{N} e^{-\gamma_j \bar{N}} e^{-(\lambda-\varepsilon)} + e^{-\gamma_j \bar{N}} (\mu+\varepsilon)\right] \\
= & \, \gamma(t_n - T_n) \left[\left(b e^{-\bar{N}} + \sum_{j=1}^{m} \beta_j \bar{N} e^{-\gamma_j \bar{N}}\right) e^{-(\lambda-\varepsilon)} - a\right] + (t_n - T_n) \sum_{j=1}^{m} \beta_j e^{-\gamma_j \bar{N}} (\mu+\varepsilon) \\
< & \, \gamma r a \left[e^{-(\lambda-\varepsilon)} - 1\right] + \frac{r}{e} \sum_{j=1}^{m} \beta_j (\mu+\varepsilon), \qquad n > n^*.
\end{aligned} \tag{2.14}$$

Letting $n \to +\infty$ and $\varepsilon \to 0^+$, (2.5) and (2.14) give us that

$$\mu \le \frac{\gamma r a}{1 - \frac{r}{e}\sum_{j=1}^{m} \beta_j} (e^{-\lambda} - 1) \le e^{-\lambda} - 1. \tag{2.15}$$

Furthermore, from (1.3), (2.2), (2.6), (2.11) and (2.12), we obtain

$$\begin{aligned}
y(s_n) = & -\gamma a(s_n - S_n) + \gamma b \int_{S_n}^{s_n} e^{-\bar{N}-\frac{1}{\gamma}y(t)} dt \\
& + \gamma \sum_{j=1}^{m} \beta_j \int_{S_n}^{s_n} \left[\bar{N} + \frac{1}{\gamma} y(t-\tau_j(t))\right] e^{-\gamma_j \bar{N} - \frac{\gamma_j}{\gamma} y(t-\tau_j(t))} dt \\
> & -\gamma a(s_n - S_n) + \gamma b (s_n - S_n) e^{-\bar{N}} e^{-(\mu+\varepsilon)} \\
& + \gamma \sum_{j=1}^{m} \beta_j \int_{S_n}^{s_n} \left[\bar{N} + \frac{1}{\gamma} y(t-\tau_j(t))\right] e^{-\gamma_j \bar{N} - (\mu+\varepsilon)} dt \\
> & -\gamma a(s_n - S_n) + \gamma b (s_n - S_n) e^{-\bar{N}} e^{-(\mu+\varepsilon)} \\
& + (s_n - S_n) \sum_{j=1}^{m} \beta_j \left[\gamma \bar{N} e^{-\gamma_j \bar{N}} e^{-(\mu+\varepsilon)} + e^{-\gamma_j \bar{N}} (\lambda-\varepsilon) e^{-(\mu+\varepsilon)}\right] \\
> & -\gamma a(s_n - S_n) + \gamma b (s_n - S_n) e^{-\bar{N}} e^{-(\mu+\varepsilon)} \\
& + (s_n - S_n) \sum_{j=1}^{m} \beta_j \left[\gamma \bar{N} e^{-\gamma_j \bar{N}} e^{-(\mu+\varepsilon)} + e^{-\gamma_j \bar{N}} (\lambda-\varepsilon)\right] \\
= & \, \gamma(s_n - S_n) \left[(b e^{-\bar{N}} + \sum_{j=1}^{m} \beta_j \bar{N} e^{-\gamma_j \bar{N}}) e^{-(\mu+\varepsilon)} - a\right] + (s_n - S_n) \sum_{j=1}^{m} \beta_j e^{-\gamma_j \bar{N}} (\lambda-\varepsilon) \\
> & \, \gamma r a [e^{-(\mu+\varepsilon)} - 1] + \frac{r}{e} \sum_{j=1}^{m} \beta_j (\lambda-\varepsilon), \qquad n > n^*.
\end{aligned} \tag{2.16}$$

Letting $n \to +\infty$ and $\varepsilon \to 0^+$, (2.5) and (2.16) lead to

$$\lambda \geq \frac{\gamma r a}{1 - \frac{r}{e}\sum_{j=1}^{m}\beta_j}(e^{-\mu} - 1) \geq e^{-\mu} - 1. \tag{2.17}$$

Thus, we have from (2.15) and (2.17) that

$$e^{-\mu} - 1 \leq \lambda \leq \mu \leq e^{-\lambda} - 1.$$

According to the proof in Theorem 4.1 of [17], one can show $\lambda = \mu = 0$. This ends the proof. □

By Propositions 2.3, 2.4 and 2.5, we have the following result.

Theorem 2.6. *Suppose that* (2.5) *holds, then the positive equilibrium point* $\bar{N}$ *of* (1.2) *is a global attractor.*

Remark 2.7. Since

$$\lim_{r\to 0^+} \frac{r}{e}\sum_{j=1}^{m}\beta_j = 0, \qquad \lim_{r\to 0^+} \frac{\gamma r a}{1 - \frac{r}{e}\sum_{j=1}^{m}\beta_j} = 0,$$

then condition (2.5) naturally holds under the sufficiently small delay, and the positive equilibrium point $\bar{N}$ is a global attractor of (1.2) with the small delays. Moreover,

$$\lim_{r\to +\infty} \frac{r}{e}\sum_{j=1}^{m}\beta_j = +\infty$$

implies that condition (2.5) is not satisfied when the delays in (1.2) is sufficiently large.

3 An example

In this section, we will give an example to verify the correctness of our main results obtained in previous section. Considering the following Nicholson's blowflies model with a nonlinear density-dependent mortality term:

$$N'(t) = -\frac{11}{10} + e^2 e^{-N(t)} + \frac{e^2}{40}N(t-\tau)e^{-N(t-\tau)} + \frac{e^2}{40}N(t-2\tau)e^{-N(t-2\tau)}. \tag{3.1}$$

Obviously,

$$r = 2\tau, \quad a = \frac{11}{10}, \quad b = e^2, \quad \beta_1 = \beta_1 = \frac{e^2}{40}, \quad \gamma_1 = \gamma_2 = 1, \quad \bar{N} = 2.$$

If we choose $\tau = 0.1$, it is straight to check that (3.1) satisfies (1.3) and (2.5). It follows from Theorem 2.6 that the positive equilibrium point 2 is a global attractor of (3.1). Fig. 3.1 supports this result with the numerical solutions of system (3.1) with different initial values. Moreover, if we choose $\tau = 10$, then, (3.1) does not satisfy (2.5), we give the numerical simulations in Fig. 3.2 to show that 2 is no longer a global attractor of (3.1). This implies that a small delay does not affect the asymptotic behavior of system (3.1), and large delay will cause the complex dynamic behavior of this system.

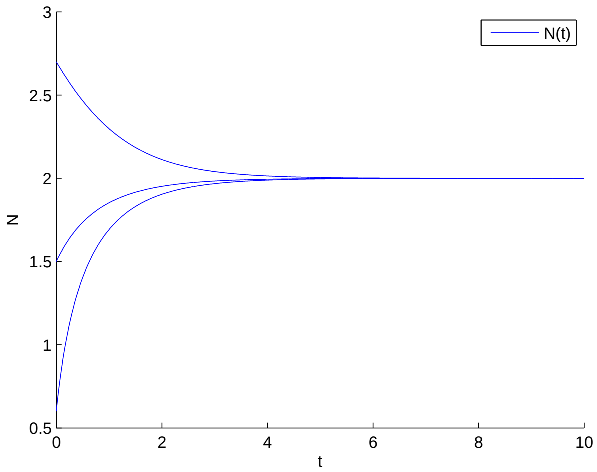

Figure 3.1: Numerical solutions of (3.1) with $\tau = 0.1$ and initial values $\varphi(s) = 0.6,\ 1.5,\ 2.7,\ s \in [-0.2, 0]$.

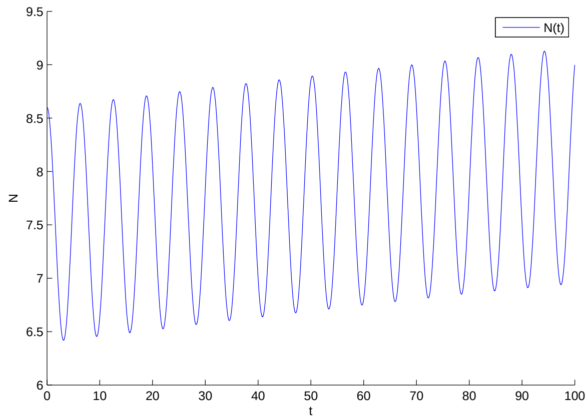

Figure 3.2: Numerical solutions of (3.1) with $\tau = 10$ and initial value $\varphi(s) = 8.6,\ s \in [-20, 0]$.

Remark 3.1. The effect of delay on the asymptotic behavior plays an important role in describing the dynamics of population models [10, 14]. Thus it has been extensively studied by many scholars in recent decades. In this article, we first studied the effect of delay on the asymptotic behavior of Nicholson's blowflies model with a nonlinear density-dependent mortality term. By means of the fluctuation lemma and some differential inequality technique, delay-dependent criteria are obtained for the global attractivity of the considered model. The sufficient condition, which is easily checked in practice, has a wide range of application. This implies that the obtained results in this article are completely new and extend previously known studies to some extent. In addition, the method in this paper can be applied to study the effect of the delay on the asymptotic behavior for some other dynamical systems. Also, it is natural to ask whether the delay affects the dynamical behavior of the addressed systems involving time-varying delays and time-varying coefficients. We leave this as our future work.

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