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Oscillation criteria for two-dimensional system of non-linear ordinary differential equations

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Abstract. New oscillation criteria are established for the system of non-linear equations

$$u' = g(t)|v|^{\frac{1}{\alpha}}\operatorname{sgn} v, \qquad v' = -p(t)|u|^{\alpha}\operatorname{sgn} u,$$

where $\alpha > 0$, $g : [0, +\infty[\ \to [0, +\infty[$, and $p : [0, +\infty[\ \to \mathbb{R}$ are locally integrable functions. Moreover, we assume that the coefficient g is non-integrable on $[0, +\infty]$. Among others, presented oscillatory criteria generalize well-known results of E. Hille and Z. Nehari and complement analogy of Hartman–Wintner theorem for the considered system.

Keywords: two dimensional system of non-linear differential equations, oscillatory properties.

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1 Introduction

On the half-line $\mathbb{R}_+ = [0, +\infty[$, we consider the two-dimensional system of nonlinear ordinary differential equations

$$\begin{aligned} u' &= g(t)|v|^{\frac{1}{\alpha}}\operatorname{sgn} v, \\ v' &= -p(t)|u|^{\alpha}\operatorname{sgn} u, \end{aligned} \tag{1.1}$$

where $\alpha > 0$ and p, $g : \mathbb{R}_+ \to \mathbb{R}$ are locally Lebesgue integrable functions such that

$$g(t) \geq 0 \quad \text{for a. e. } t \geq 0 \tag{1.2}$$

and

$$\int_0^{+\infty} g(s)ds = +\infty. \tag{1.3}$$

By a solution of system (1.1) on the interval $J \subseteq [0, +\infty[$ we understand a pair (u, v) of functions $u, v : J \to \mathbb{R}$, which are absolutely continuous on every compact interval contained in J and satisfy equalities (1.1) almost everywhere in J.

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It was proved by Mirzov in [10] that all non-extendable solutions of system (1.1) are defined on the whole interval $[0,+\infty[$. Therefore, when we are speaking about a solution of system (1.1), we assume that it is defined on $[0,+\infty[$.

Definition 1.1. A solution (u,v) of system (1.1) is called *non-trivial* if $|u(t)|+|v(t)| \neq 0$ for $t \geq 0$. We say that a non-trivial solution (u,v) of system (1.1) is *oscillatory* if its each component has a sequence of zeros tending to infinity, and *non-oscillatory* otherwise.

In [10, Theorem 1.1], it is shown that a certain analogue of Sturm's theorem holds for system (1.1) if the function g is nonnegative. Especially, under assumption (1.2), if system (1.1) has an oscillatory solution, then any other its non-trivial solution is also oscillatory.

Definition 1.2. We say that system (1.1) is *oscillatory* if all its non-trivial solutions are oscillatory.

Oscillation theory for ordinary differential equations and their systems is a widely studied and well-developed topic of the qualitative theory of differential equations. As for the results which are closely related to those of this section, we should mention [2, 4–9, 11–13]. Some criteria established in these papers for the second order linear differential equations or for two-dimensional systems of linear differential equations are generalized to the considered system (1.1) below.

Many results (see, e.g., survey given in [2]) have been obtained in oscillation theory of the so-called "half-linear" equation

$$\Big(r(t)|u'|^{q-1}\operatorname{sgn} u'\Big)' + p(t)|u|^{q-1}\operatorname{sgn} u = 0 \tag{1.4}$$

(alternatively this equation is referred as "equation with the scalar q-Laplacian"). Equation (1.4) is usually considered under the assumptions $q > 1$, $p, r : [0,+\infty[\to \mathbb{R}$ are continuous and r is positive. One can see that equation (1.4) is a particular case of system (1.1). Indeed, if the function u, with properties $u \in C^1$ and $r|u'|^{q-1}\operatorname{sgn} u' \in C^1$, is a solution of equation (1.4), then the vector function $(u, r|u'|^{q-1}\operatorname{sgn} u')$ is a solution of system (1.1) with $g(t) := r^{\frac{1}{1-q}}(t)$ for $t \geq 0$ and $\alpha := q-1$.

Moreover, the equation

$$u'' + \frac{1}{\alpha}\,p(t)|u|^{\alpha}|u'|^{1-\alpha}\operatorname{sgn} u = 0 \tag{1.5}$$

is also studied in the existing literature under the assumptions $\alpha \in \,]0,1]$ and $p : \mathbb{R}_+ \to \mathbb{R}$ is a locally integrable function. It is mentioned in [6] that if u is a so-called proper solution of (1.5) then it is also a solution of system (1.1) with $g \equiv 1$ and vice versa. Some oscillations and non-oscillations criteria for equation (1.5) can be found, e.g., in [6, 7].

Finally, we mention the paper [1], where a certain analogy of Hartman–Wintner's theorem is established (origin one can find in [3, 14]), which allows us to derive oscillation criteria of Hille–Nehari's type for system (1.1).

Let

$$f(t) := \int_0^t g(t)ds \quad \text{for } t \geq 0.$$

In view of assumptions (1.2) and (1.3), there exists $t_g \geq 0$ such that $f(t) > 0$ for $t > t_g$ and $f(t_g) = 0$. We can assume without loss of generality that $t_g = 0$, since we are interested in behaviour of solutions in the neighbourhood of $+\infty$, i.e., we have

$$f(t) > 0 \quad \text{for } t > 0 \tag{1.6}$$

and, moreover,

$$\lim_{t\to+\infty} f(t) = +\infty. \tag{1.7}$$

For any $\lambda \in [0,\alpha[$, we put

$$c_\alpha(t;\lambda) := \frac{\alpha-\lambda}{f^{\alpha-\lambda}(t)} \int_0^t \frac{g(s)}{f^{\lambda-\alpha+1}(s)} \left(\int_0^s f^\lambda(\xi)p(\xi)d\xi \right) ds \quad \text{for } t > 0.$$

Now, we formulate an analogue (in a suitable form for us) of the Hartman–Wintner's theorem for the system (1.1) established in [1].

Theorem 1.3 ([1, Corollary 2.5 (with $\nu = 1-\alpha+\lambda$)]). *Let conditions* (1.2) *and* (1.3) *hold,* $\lambda < \alpha$, *and either*

$$\lim_{t\to+\infty} c_\alpha(t;\lambda) = +\infty,$$

or

$$-\infty < \liminf_{t\to+\infty} c_\alpha(t;\lambda) < \limsup_{t\to+\infty} c_\alpha(t;\lambda).$$

Then system (1.1) *is oscillatory.*

One can see that two cases are not covered by Theorem 1.3, namely, the function $c_\alpha(t;\lambda)$ has a finite limit and $\liminf_{t\to+\infty} c_\alpha(t;\lambda) = -\infty$. The aim of this Section is to find oscillation criteria for system (1.1) in the first mentioned case. Consequetly, in what follows, we assume that

$$\lim_{t\to+\infty} c_\alpha(t;\lambda) =: c_\alpha^*(\lambda) \in \mathbb{R}. \tag{1.8}$$

2 Main results

In this section, we formulate main results and theirs corollaries.

Theorem 2.1. *Let* $\lambda \in [0,\alpha[$ *and* (1.8) *hold. Let, moreover, the inequality*

$$\limsup_{t\to+\infty} \frac{f^{\alpha-\lambda}(t)}{\ln f(t)} \left(c_\alpha^*(\lambda) - c_\alpha(t;\lambda)\right) > \left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha} \tag{2.1}$$

be satisfied. Then system (1.1) *is oscillatory.*

We introduce the following notations. For any $\lambda \in [0,\alpha[$ and $\mu \in]\alpha,+\infty[$, we put

$$Q(t;\alpha,\lambda) := f^{\alpha-\lambda}(t) \left(c_\alpha^*(\lambda) - \int_0^t p(s)f^\lambda(s)ds \right) \quad \text{for } t > 0,$$

$$H(t;\alpha,\mu) := \frac{1}{f^{\mu-\alpha}(t)} \left(\int_0^t p(s)f^\mu(s)ds \right) \quad \text{for } t > 0,$$

where the number $c_\alpha^*(\lambda)$ is given by (1.8). Moreover, we denote lower and upper limits of the functions $Q(\cdot;\alpha,\lambda)$ and $H(\cdot;\alpha,\mu)$ as follows

$$Q_*(\alpha,\lambda) := \liminf_{t\to+\infty} Q(t;\alpha,\lambda), \qquad H_*(\alpha,\mu) := \liminf_{t\to+\infty} H(t;\alpha,\mu),$$

$$Q^*(\alpha,\lambda) := \limsup_{t\to+\infty} Q(t;\alpha,\lambda), \qquad H^*(\alpha,\mu) := \limsup_{t\to+\infty} H(t;\alpha,\mu).$$

Now we formulate two corollaries of Theorem 2.1.

Corollary 2.2. *Let $\lambda \in [0,\alpha[$, $\mu \in]\alpha,+\infty[$, and (1.8) hold. Let, moreover,*

$$\liminf_{t\to+\infty}\left(Q(t;\alpha,\lambda)+H(t;\alpha,\mu)\right) > \frac{\mu-\lambda}{(\alpha-\lambda)(\mu-\alpha)}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}. \tag{2.2}$$

Then system (1.1) is oscillatory.

Corollary 2.3. *$\lambda \in [0,\alpha[$, $\mu \in]\alpha,+\infty[$, and (1.8) hold. Let, moreover, either*

$$Q_*(\alpha,\lambda) > \frac{1}{\alpha-\lambda}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}, \tag{2.3}$$

or

$$H_*(\alpha,\mu) > \frac{1}{\mu-\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}. \tag{2.4}$$

Then system (1.1) is oscillatory.

Remark 2.4. Oscillation criteria (2.3) and (2.4) coincide with the well-known Hille–Nehari's results for the second order linear differential equations established in [4,12].

Theorem 2.5. *Let $\lambda \in [0,\alpha[$, $\mu \in]\alpha,+\infty[$, and (1.8) hold. Let, moreover,*

$$\limsup_{t\to+\infty}\left(Q(t;\alpha,\lambda)+H(t;\alpha,\mu)\right) > \frac{1}{\alpha-\lambda}\left(\frac{\lambda}{1+\alpha}\right)^{1+\alpha}+\frac{1}{\mu-\alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha}. \tag{2.5}$$

Then system (1.1) is oscillatory.

Now we give two statements complementing Corollary 2.3 in a certain sense.

Theorem 2.6. *Let $\lambda \in [0,\alpha[$, $\mu \in]\alpha,+\infty[$, and (1.8) hold. Let, moreover, inequalities*

$$\frac{\alpha}{\alpha-\lambda}\left(\gamma-\gamma^{\frac{1+\alpha}{\alpha}}\right) \le Q_*(\alpha,\lambda) \le \frac{1}{\alpha-\lambda}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \tag{2.6}$$

and

$$H^*(\alpha,\mu) > \frac{1}{\mu-\alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha}-\gamma-A(\alpha,\lambda) \tag{2.7}$$

be satisfied, where

$$\gamma := \left(\frac{\lambda}{1+\alpha}\right)^{\alpha} \tag{2.8}$$

and $A(\alpha,\lambda)$ is the smallest root of the equation

$$\alpha|x+\gamma|^{\frac{1+\alpha}{\alpha}}-\alpha x+(\alpha-\lambda)Q_*(\alpha,\lambda)-\alpha\gamma=0. \tag{2.9}$$

Then system (1.1) is oscillatory.

Theorem 2.7. *Let $\lambda \in [0,\alpha[$, $\mu \in]\alpha,+\infty[$, and (1.8) hold. Let, moreover, inequalities*

$$\left(\frac{\mu}{1+\alpha}\right)^{\alpha}\frac{\alpha(1+\alpha-\mu)}{(\mu-\alpha)(1+\alpha)} \le H_*(\alpha,\mu) \le \frac{1}{\mu-\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha} \tag{2.10}$$

and

$$Q^*(\alpha,\lambda) > B(\alpha,\mu)+\frac{1}{\alpha-\lambda}\left(\frac{\lambda}{1+\alpha}\right)^{1+\alpha} \tag{2.11}$$

be satisfied, where $B(\alpha,\mu)$ is the greatest root of the equation

$$\alpha|x|^{\frac{1+\alpha}{\alpha}}-\alpha x+(\mu-\alpha)H_*(\alpha,\mu)=0. \tag{2.12}$$

Then system (1.1) is oscillatory.

Finally, we formulate an assertion for the case, when both conditions (2.6) and (2.10) are fulfilled. In this case we can obtain better results than in Theorems 2.6 and 2.7.

Theorem 2.8. *Let $\lambda \in [0, \alpha[$, $\mu \in]\alpha, +\infty[$, and (1.8) hold. Let, moreover, conditions (2.6) and (2.10) be satisfied and*

$$\limsup_{t\to+\infty} \left(Q(t;\alpha,\lambda) + H(t;\alpha,\mu)\right) > B(\alpha,\mu) - A(\alpha,\lambda) + Q_*(\alpha,\lambda) + H_*(\alpha,\mu) - \gamma, \tag{2.13}$$

where the number γ is defined by (2.8), $A(\alpha,\lambda)$ is the smallest root of equation (2.9), and $B(\alpha,\mu)$ is the greatest root of equation (2.12). Then system (1.1) is oscillatory.

Remark 2.9. Presented statements generalize results stated in [2,4–9,11–13] concerning system (1.1) as well as equations (1.4) and (1.5). In particular, if we put $\alpha = 1$, $\lambda = 0$, and $\mu = 2$, then we obtain oscillatory criteria for linear system of differential equations presented in [13]. Moreover, the results of [6] obtained for equation (1.5) are in a compliance with those above, where we put $g \equiv 1$, $\lambda = 0$, and $\mu = 1 + \alpha$. Observe also that Corollary 2.3 and Theorems 2.6 and 2.7 extend oscillation criteria for equation (1.5) stated in [7], where the coefficient p is suppose to be non-negative. In the monograph [2], it is noted that the assumption $p(t) \geq 0$ for t large enough can be easily relaxed to $\int_0^t p(s)ds > 0$ for large t. It is worth mentioning here that we do not require any assumption of this kind.

Finally we show an example, where we can not apply oscillatory criteria from the above mentioned papers, but we can use Theorem 2.1 succesfully.

Example 2.10. Let $\alpha = 2$, $g(t) \equiv 1$, $\lambda = 0$, and

$$p(t) := t\cos\left(\frac{t^2}{2}\right) + \frac{1}{(t+1)^3} \quad \text{for } t \geq 0.$$

It is clear that the function p and its integral

$$\int_0^t p(s)ds = \sin\left(\frac{t^2}{2}\right) - \frac{1}{2(t+1)^2} + \frac{1}{2} \quad \text{for } t \geq 0$$

change theirs sign in any neighbourhood of $+\infty$. Therefore neither of results mentioned in Remark 2.9 can be applied.

On the other hand, we have

$$\begin{aligned} c_2(t;0) &= \frac{2}{t^2}\int_0^t s\left(\int_0^s \left(\xi\cos\frac{\xi^2}{2} + \frac{1}{(\xi+1)^3}\right)d\xi\right)ds \\ &= \frac{1}{2} - \frac{2\cos\frac{t^2}{2}}{t^2} + \frac{3}{t^2} - \frac{\ln(t+1)}{t^2} - \frac{1}{t^2(t+1)} \quad \text{for } t > 0 \end{aligned}$$

and thus, the function $c_2(\cdot, 0)$ has the finite limit

$$c_\alpha^*(0) = \lim_{t\to+\infty} c_2(t;0) = \frac{1}{2}.$$

Moreover,

$$\limsup_{t\to+\infty} \frac{t^2}{\ln t}\left(c_\alpha^*(0) - c_2(t;0)\right) = \limsup_{t\to+\infty}\left(\frac{2\cos\frac{t^2}{2} - 3}{\ln t} + \frac{\ln(t+1)}{\ln t} + \frac{1}{(t+1)\ln t}\right) = 1.$$

Consequently, according to Theorem 2.1, system (1.1) is oscillatory.

3 Auxiliary lemmas

We first formulate two lemmas established in [1], which we use in this section.

Lemma 3.1 ([1, Lemma 3.1]). *Let $\alpha > 0$ and $\omega \geq 0$. Then the inequality*

$$\omega x - \alpha|x|^{\frac{1+\alpha}{\alpha}} \leq \left(\frac{\omega}{1+\alpha}\right)^{1+\alpha}$$

is satisfied for all $x \in \mathbb{R}$.

Lemma 3.2 ([1, Lemma 3.2]). *Let $\alpha > 0$. Then*

$$\alpha|x+y|^{\frac{1+\alpha}{\alpha}} \geq \alpha|y|^{\frac{1+\alpha}{\alpha}} + (1+\alpha)x|y|^{\frac{1}{\alpha}}\operatorname{sgn} y \quad \text{for } x, y \in \mathbb{R}.$$

Remark 3.3. One can easily verify (see the proofs of Lemma 4.2 and Corollary 2.5 in [1]) that if (u, v) is a solution of system (1.1) satisfying

$$u(t) \neq 0 \quad \text{for } t \geq t_u \tag{3.1}$$

with $t_u > 0$ and the function $c_\alpha(\cdot;\lambda)$ has a finite limit (1.8), then

$$\begin{aligned} c_\alpha^*(\lambda) = f^\lambda(t_u)\rho(t_u) + \int_0^{t_u} f^\lambda(s)p(s)ds + \frac{\alpha(\gamma - \gamma^{\frac{1+\alpha}{\alpha}})}{\alpha-\lambda} \cdot \frac{1}{f^{\alpha-\lambda}(t_u)} \\ - \int_{t_u}^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds, \end{aligned} \tag{3.2}$$

where the number γ is defined by (2.8),

$$h(t) := \alpha|f^\alpha(t)\rho(t) + \gamma|^{\frac{1+\alpha}{\alpha}} - (1+\alpha)f^\alpha(t)\rho(t)\gamma^{\frac{1}{\alpha}} - \alpha\gamma^{\frac{1+\alpha}{\alpha}} \quad \text{for } t \geq t_u, \tag{3.3}$$

and

$$\rho(t) := \frac{v(t)}{|u(t)|^\alpha \operatorname{sgn} u(t)} - \frac{1}{f^\alpha(t)}\left(\frac{\lambda}{1+\alpha}\right)^\alpha \quad \text{for } t \geq t_u. \tag{3.4}$$

Moreover, according to Lemma 3.2, we have

$$h(t) \geq 0 \quad \text{for } t \geq t_u \tag{3.5}$$

and one can show (see Lemma 4.1 and the proof of Corollary 2.5 in [1]) that

$$\int_{t_u}^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds < +\infty. \tag{3.6}$$

Lemma 3.4. *Let $\lambda \in [0, \alpha[$, (1.8) and (2.6) hold, where the number γ is defined by (2.8). Then every non-oscillatory solution (u, v) of system (1.1) satisfies*

$$\liminf_{t\to+\infty}\left(\frac{f^\alpha(t)v(t)}{|u(t)|^\alpha \operatorname{sgn} u(t)} - \gamma\right) \geq A(\alpha, \lambda), \tag{3.7}$$

where $A(\alpha, \lambda)$ denotes the smallest root of equation (2.9).

Proof. Let (u, v) be a non-oscillatory solution of system (1.1). Then there exists $t_u > 0$ such that (3.1) holds. Define the function ρ by (3.4). Then we obtain from (1.1) that

$$\rho'(t) = -p(t) - \alpha g(t) \left| \rho(t) + \frac{\gamma}{f^{\alpha}(t)} \right|^{\frac{1+\alpha}{\alpha}} + \alpha\gamma \frac{g(t)}{f^{1+\alpha}(t)} \quad \text{for a. e. } t \geq t_u. \tag{3.8}$$

Multiplaying the last equality by $f^{\lambda}(t)$ and integrating it from t_u to t, we get

$$\begin{aligned} \int_{t_u}^{t} f^{\lambda}(s)\rho'(s)ds = & -\alpha \int_{t_u}^{t} g(s) f^{\lambda-1-\alpha}(s) \left| \rho(s) f^{\alpha}(s) + \gamma \right|^{\frac{1+\alpha}{\alpha}} ds \\ & + \alpha\gamma \int_{t_u}^{t} g(s) f^{\lambda-1-\alpha}(s)ds - \int_{t_u}^{t} f^{\lambda}(s)p(s)ds \quad \text{for } t \geq t_u. \end{aligned} \tag{3.9}$$

Integrating the left-hand side of (3.9) by parts, we obtain

$$\begin{aligned} f^{\lambda}(t)\rho(t) = & \left(\alpha\gamma - \alpha\gamma^{\frac{1+\alpha}{\alpha}} \right) \int_{t_u}^{t} g(s) f^{\lambda-1-\alpha}(s)ds - \int_{t_u}^{t} f^{\lambda}(s)p(s)ds \\ & + f^{\lambda}(t_u)\rho(t_u) - \int_{t_u}^{t} g(s) f^{\lambda-1-\alpha}(s)h(s)ds \quad \text{for } t \geq t_u, \end{aligned}$$

where the function h is defined in (3.3). Hence,

$$\begin{aligned} f^{\lambda}(t)\rho(t) = & \, \delta(t_u) - \int_{0}^{t} f^{\lambda}(s)p(s)ds - \int_{t_u}^{t} g(s) f^{\lambda-1-\alpha}(s)h(s)ds \\ & - \frac{\alpha\left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\right)}{\alpha - \lambda} \frac{1}{f^{\alpha-\lambda}(t)} \quad \text{for } t \geq t_u, \end{aligned} \tag{3.10}$$

where

$$\delta(t_u) := f^{\lambda}(t_u)\rho(t_u) + \int_{0}^{t_u} f^{\lambda}(s)p(s)ds + \frac{\alpha\left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\right)}{\alpha - \lambda} \frac{1}{f^{\alpha-\lambda}(t_u)}.$$

Therefore, in view of relations (3.2) and (3.6), it follows from (3.10) that

$$\begin{aligned} f^{\lambda}(t)\rho(t) = & \, c_{\alpha}^{*}(\lambda) - \int_{0}^{t} f^{\lambda}(s)p(s)ds + \int_{t}^{+\infty} g(s) f^{\lambda-1-\alpha}(s)h(s)ds \\ & - \frac{\alpha\left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\right)}{\alpha - \lambda} \frac{1}{f^{\alpha-\lambda}(t)} \quad \text{for } t \geq t_u. \end{aligned} \tag{3.11}$$

Hence,

$$\begin{aligned} f^{\alpha}(t)\rho(t) = & \, Q(t;\alpha,\lambda) + f^{\alpha-\lambda}(t) \int_{t}^{+\infty} g(s) f^{\lambda-1-\alpha}(s)h(s)ds \\ & - \frac{\alpha\left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\right)}{\alpha - \lambda} \quad \text{for } t \geq t_u. \end{aligned} \tag{3.12}$$

Put

$$m := \liminf_{t \to +\infty} f^{\alpha}(t)\rho(t). \tag{3.13}$$

It is clear that if $m = +\infty$, then (3.7) holds. Therefore, we suppose that

$$m < +\infty.$$

In view of (2.6), (3.5), and (3.13), relation (3.12) yields that

$$m \geq Q_*(\alpha,\lambda) - \frac{\alpha}{\alpha-\lambda}\left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\right) \geq 0. \tag{3.14}$$

If $Q_*(\alpha,\lambda) = \frac{\alpha}{\alpha-\lambda}(\gamma - \gamma^{\frac{1+\alpha}{\alpha}})$, then 0 is a root of equation (2.9). Moreover, in view of Lemma 3.2 and the assumption $\lambda < \alpha$, we see that the function $x \mapsto \alpha|x+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha x - \alpha\gamma^{\frac{1+\alpha}{\alpha}}$ is positive on $]-\infty, 0[$. Consequently, by virtue of notations (3.4), (3.13) and relation (3.14), desired estimate (3.7) holds.

Now suppose that $Q_*(\alpha,\lambda) > \frac{\alpha}{\alpha-\lambda}(\gamma - \gamma^{\frac{1+\alpha}{\alpha}})$. Let $\varepsilon \in]0, Q_*(\alpha,\lambda) - \frac{\alpha}{\alpha-\lambda}(\gamma - \gamma^{\frac{1+\alpha}{\alpha}})[$ be arbitrary. According to (3.14), it is clear that

$$m > \varepsilon. \tag{3.15}$$

Choose $t_\varepsilon \geq t_u$ such that

$$f^\alpha(t)\rho(t) \geq m - \varepsilon \quad \text{and} \quad Q(t;\alpha,\lambda) \geq Q_*(\alpha,\lambda) - \varepsilon \quad \text{for } t \geq t_\varepsilon. \tag{3.16}$$

Then it follows from (3.12) that

$$\begin{aligned} f^\alpha(t)\rho(t) \geq{}& Q_*(\alpha,\lambda) - \varepsilon + f^{\alpha-\lambda}(t)\int_t^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds \\ &- \frac{\alpha\left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\right)}{\alpha-\lambda} \quad \text{for } t \geq t_\varepsilon. \end{aligned} \tag{3.17}$$

On the other hand, the function $x \mapsto \alpha|x+\gamma|^{\frac{1+\alpha}{\alpha}} - (1+\alpha)x\gamma^{\frac{1}{\alpha}} - \alpha\gamma^{\frac{1+\alpha}{\alpha}}$ is non-decreasing on $[0,+\infty[$. Therefore, by virtue of (3.5), (3.15), and (3.16), one gets from (3.17) that

$$f^\alpha(t)\rho(t) \geq Q_*(\alpha,\lambda) - \varepsilon + \frac{\alpha|(m-\varepsilon)+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha\gamma - \lambda(m-\varepsilon)}{\alpha-\lambda} \quad \text{for } t \geq t_\varepsilon,$$

which implies

$$m \geq Q_*(\alpha,\lambda) - \varepsilon + \frac{\alpha|(m-\varepsilon)+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha\gamma - \lambda(m-\varepsilon)}{\alpha-\lambda}.$$

Since ε was arbitrary, the latter relation leads to the inequality

$$\alpha|m+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha m + Q_*(\alpha,\lambda)(\alpha-\lambda) - \alpha\gamma \leq 0. \tag{3.18}$$

One can easily derive that the function $y : x \mapsto \alpha|x+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha x + Q_*(\alpha,\lambda)(\alpha-\lambda) - \alpha\gamma$ is decreasing on $]-\infty, (\frac{\alpha}{1+\alpha})^\alpha - \gamma]$ and increasing on $[(\frac{\alpha}{1+\alpha})^\alpha - \gamma, +\infty[$. Therefore, in view of assumption (2.6), the function y is non-positive at the point $\left(\frac{\alpha}{1+\alpha}\right)^\alpha - \gamma$, which together with (3.4), (3.13), and (3.18) implies desired estimate (3.7). □

Lemma 3.5. *Let $\mu \in]\alpha, +\infty[$ and (2.10) hold. Then every non-oscillatory solution (u,v) of system (1.1) satisfies*

$$\limsup_{t\to+\infty} \frac{f^\alpha(t)v(t)}{|u(t)|^\alpha \operatorname{sgn} u(t)} \leq B(\alpha,\mu), \tag{3.19}$$

where $B(\alpha,\mu)$ is the greatest root of equation (2.12).

Proof. Let (u, v) be a non-oscillatory solution of system (1.1). Then there exists $t_u > 0$ such that (3.1) holds. Define the function ρ by (3.4). Then from (1.1) we obtain the equality (3.8), where the number γ is defined by (2.8).

Multiplying (3.8) by $f^\mu(t)$ and integrating it from t_u to t, we obtain

$$\int_{t_u}^{t} f^\mu(s)\rho'(s)ds = -\int_{t_u}^{t} f^\mu(s)p(s)ds - \alpha\int_{t_u}^{t} g(s)f^{\mu-\alpha-1}(s)|\rho(s)f^\alpha(s)+\gamma|^{\frac{1+\alpha}{\alpha}}ds + \alpha\gamma\int_{t_u}^{t} g(s)f^{\mu-\alpha-1}(s)ds \quad \text{for } t \ge t_u.$$

Integrating the left-hand side of the last equality by parts, we get

$$f^\alpha(t)\rho(t) = f^{\alpha-\mu}(t)\int_{t_u}^{t} g(s)f^{\mu-\alpha-1}(s)\left[\mu f^\alpha(s)\rho(s) - \alpha|\rho(s)f^\alpha(s)+\gamma|^{\frac{1+\alpha}{\alpha}}\right]ds + \delta(t_u)f^{\alpha-\mu}(t) - H(t;\alpha,\mu) + \frac{\alpha\gamma}{\mu-\alpha} \quad \text{for } t \ge t_u, \tag{3.20}$$

where

$$\delta(t_u) := f^\mu(t_u)\rho(t_u) + \int_0^{t_u} f^\mu(s)p(s)ds - \frac{\alpha\gamma}{\mu-\alpha}f^{\mu-\alpha}(t_u). \tag{3.21}$$

According to Lemma 3.1, it follows from (3.20) that

$$f^\alpha(t)\rho(t) \le \delta_1(t_u)f^{\alpha-\mu}(t) - H(t;\alpha,\mu) + \frac{1}{\mu-\alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \gamma \quad \text{for } t \ge t_u, \tag{3.22}$$

where

$$\delta_1(t_u) := \delta(t_u) - \frac{f^{\mu-\alpha}(t_u)}{\mu-\alpha}\left(\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma\right). \tag{3.23}$$

Put

$$M := \limsup_{t\to+\infty}\left(f^\alpha(t)\rho(t)+\gamma\right). \tag{3.24}$$

Obviously, if $M = -\infty$ then (3.19) holds. Therefore, suppose that

$$M > -\infty.$$

By virtue of (1.7), inequality (3.22) yields

$$M \le -H_*(\alpha,\mu) + \frac{1}{\mu-\alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha}. \tag{3.25}$$

If $H_*(\alpha,\mu) = \left(\frac{\mu}{1+\alpha}\right)^\alpha \frac{\alpha(1+\alpha-\mu)}{(\mu-\alpha)(1+\alpha)}$, then it is not difficult to verify that $(\frac{\mu}{1+\alpha})^\alpha$ is a root of the equation (2.12) and the function $x \mapsto \alpha|x|^{\frac{1+\alpha}{\alpha}} - \alpha x + (\mu-\alpha)H_*(\alpha,\mu)$ is positive on $](\frac{\mu}{1+\alpha})^\alpha, +\infty[$. Consequently, it follows from (3.24) and (3.25) that (3.19) is satisfied.

Now suppose that

$$H_*(\alpha,\mu) > \left(\frac{\mu}{1+\alpha}\right)^\alpha \frac{\alpha(1+\alpha-\mu)}{(\mu-\alpha)(1+\alpha)}.$$

Using the latter inequality in (3.25), we get

$$M < \left(\frac{\mu}{1+\alpha}\right)^\alpha.$$

Let $\varepsilon \in]0, \left(\frac{\mu}{1+\alpha}\right)^{\alpha} - M[$ be arbitrary and choose $t_\varepsilon \geq t_u$ such that

$$\gamma + f^{\alpha}(t)\rho(t) \leq M + \varepsilon, \quad H(t;\alpha,\mu) \geq H_*(\alpha,\mu) - \varepsilon \quad \text{for } t \geq t_\varepsilon. \tag{3.26}$$

Observe that the function $x \mapsto \mu x - \alpha|x|^{\frac{1+\alpha}{\alpha}}$ is non-decreasing on $]-\infty, \left(\frac{\mu}{1+\alpha}\right)^{\alpha}]$ and thus, using relations (3.26) and $M + \varepsilon < \left(\frac{\mu}{1+\alpha}\right)^{\alpha}$, from (3.20) we get

$$\begin{aligned} f^{\alpha}(t)\rho(t) \leq \delta_2(t_u) f^{\alpha-\mu}(t) - H_*(\alpha,\mu) + \varepsilon + \frac{\alpha\gamma}{\mu-\alpha} - \frac{\mu\gamma}{\mu-\alpha} \\ + f^{\alpha-\mu}(t)\int_{t_u}^{t} g(s) f^{\mu-\alpha-1}(s)\left[\mu\left(M+\varepsilon\right) - \alpha|M+\varepsilon|^{\frac{1+\alpha}{\alpha}}\right] ds \quad \text{for } t \geq t_\varepsilon, \end{aligned}$$

where

$$\delta_2(t) := f^{\mu}(t_u)\rho(t_u) + \int_0^{t_u} f^{\mu}(s)p(s)ds + \gamma f^{\mu-\alpha}(t_u).$$

Consequently,

$$f^{\alpha}(t)\rho(t) + \gamma\delta_3(t_u) f^{\alpha-\mu}(t) - H_*(\alpha,\mu) + \varepsilon + \frac{\mu\left(M+\varepsilon\right) - \alpha|M+\varepsilon|^{\frac{1+\alpha}{\alpha}}}{\mu-\alpha} \quad \text{for } t \geq t_\varepsilon,$$

where

$$\delta_3(t_u) := \delta_2(t_u) - \frac{\mu\left(M+\varepsilon\right) - \alpha|M+\varepsilon|^{\frac{1+\alpha}{\alpha}}}{\mu-\alpha} f^{\mu-\alpha}(t_u),$$

which, by virtue of the assumption $\alpha < \mu$ and condition (1.7) and (3.24), yields that

$$M \leq -H_*(\alpha,\mu) + \varepsilon + \frac{\mu\left(M+\varepsilon\right) - \alpha|M+\varepsilon|^{\frac{1+\alpha}{\alpha}}}{\mu-\alpha}.$$

Since ε was arbitrary, the latter inequality leads to

$$\alpha|M|^{\frac{1+\alpha}{\alpha}} - \alpha M + (\mu-\alpha)H_*(\alpha,\mu) \leq 0. \tag{3.27}$$

One can easily derive that the function $y : x \mapsto \alpha|x|^{\frac{1+\alpha}{\alpha}} - \alpha x + H_*(\alpha,\mu)(\mu-\alpha)$ is decreasing on $]-\infty, (\frac{\alpha}{1+\alpha})^{\alpha}]$ and increasing on $[(\frac{\alpha}{1+\alpha})^{\alpha}, +\infty[$. Therefore, in view of assumption (2.10), the function y is non-positive at the point $\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$, which together with (3.4), (3.24), and (3.27) implies desired estimate (3.19). □

4 Proofs of main results

Proof of Theorem 2.1. Assume on the contrary that system (1.1) is not oscillatory, i.e., there exists a solution (u, v) of system (1.1) satisfying relation (3.1) with $t_u > 0$. Analogously to the proof of Lemma 3.4 we show that equality (3.11) holds, where the functions h, ρ and the number γ are defined by (3.3), (3.4), and (2.8). Moreover, conditions (3.5) and (3.6) are satisfied.

Multiplying of (3.11) by $g(t)f^{\alpha-1-\lambda}(t)$ and integrating it from t_u to t, one gets

$$\begin{aligned}\int_{t_u}^{t} g(s)f^{\alpha-1}(s)\rho(s)ds = c_\alpha^*(\lambda)&\int_{t_u}^{t}\frac{g(s)}{f^{1+\lambda-\alpha}(s)}ds\\ &-\int_{t_u}^{t}\frac{g(s)}{f^{1+\lambda-\alpha}(s)}\left(\int_0^s f^\lambda(\xi)p(\xi)d\xi\right)ds\\ &+\int_{t_u}^{t}\frac{g(s)}{f^{1+\lambda-\alpha}(s)}\left(\int_s^{+\infty} g(\xi)f^{\lambda-1-\alpha}(\xi)h(\xi)d\xi\right)ds\\ &-\frac{\alpha}{\alpha-\lambda}\left(\gamma-\gamma^{\frac{1+\alpha}{\alpha}}\right)\int_{t_u}^{t}\frac{g(s)}{f(s)}ds\quad\text{for } t\ge t_u,\end{aligned}\tag{4.1}$$

Observe that

$$\begin{aligned}&\int_{t_u}^{t}\frac{g(s)}{f^{1+\lambda-\alpha}(s)}\left(\int_s^{+\infty} g(\xi)f^{\lambda-1-\alpha}(\xi)h(\xi)d\xi\right)ds\\ &\quad= -\frac{f^{\alpha-\lambda}(t)}{\alpha-\lambda}\int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds+\frac{1}{\alpha-\lambda}\int_{t_u}^{t}\frac{g(s)}{f(s)}h(s)ds\\ &\qquad-\frac{f^{\alpha-\lambda}(t_u)}{\alpha-\lambda}\int_{t_u}^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds\quad\text{for } t\ge t_u.\end{aligned}$$

Hence, it follows from (4.1) that

$$\begin{aligned}f^{\alpha-\lambda}(t)\left(c_\alpha^*(\lambda)-c_\alpha(t;\lambda)\right) = &\int_{t_u}^{t}\frac{g(s)}{f(s)}\left[(\alpha-\lambda)f^\alpha(s)\rho(s)-h(s)+\alpha\left(\gamma-\gamma^{\frac{1+\alpha}{\alpha}}\right)\right]ds\\ &+f^{\alpha-\lambda}(t_u)\left[c_\alpha^*(\lambda)-c_\alpha(t_u;\lambda)+\int_{t_u}^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds\right]\\ &-f^{\alpha-\lambda}(t)\int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds\quad\text{for } t\ge t_u.\end{aligned}\tag{4.2}$$

On the other hand, according to (2.8), (3.3), and Lemma 3.1 with $\omega := \alpha$, the estimate

$$\begin{aligned}&(\alpha-\lambda)f^\alpha(s)\rho(s)-h(s)+\alpha\left(\gamma-\gamma^{\frac{1+\alpha}{\alpha}}\right)\\ &\qquad=\alpha\left(f^\alpha(s)\rho(s)+\gamma\right)-\alpha|f^\alpha(s)\rho(s)+\gamma|^{\frac{1+\alpha}{\alpha}}\le\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{\alpha}}\end{aligned}\tag{4.3}$$

holds for $s\ge t_u$. Moreover, in view of (1.2), (1.6), and (3.5), it is clear that

$$f^{\alpha-\lambda}(t)\int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds\ge 0\quad\text{for } t\ge t_u.$$

Consequently, by virtue of the last inequality and (4.3), it follows from (4.2) that

$$\begin{aligned}f^{\alpha-\lambda}(t)\left[c_\alpha^*(\lambda)-c_\alpha(t;\lambda)\right]\le&\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{\alpha}}\ln\frac{f(t)}{f(t_u)}\\ &+f^{\alpha-\lambda}(t_u)\left[c_\alpha^*(\lambda)-c_\alpha(t_u;\lambda)+\int_{t_u}^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds\right]\quad\text{for } t\ge t_u.\end{aligned}$$

Hence, in view of (1.7), we get

$$\limsup_{t\to+\infty}\frac{f^{\alpha-\lambda}(t)}{\ln f(t)}\left[c_\alpha^*(\lambda)-c_\alpha(t;\lambda)\right]\le\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{\alpha}},$$

which contradicts (2.1). ∎

Proof of Corollary 2.2. Observe that for $t > 0$, we have

$$\frac{f^{\alpha-\lambda}(t)}{\ln f(t)}\left(c_\alpha^*(\lambda) - c_\alpha(t;\lambda)\right) = \frac{\alpha-\lambda}{\ln f(t)}\int_0^t \frac{g(s)}{f(s)}Q(s;\alpha,\lambda)ds \tag{4.4}$$

and

$$Q(t;\alpha,\lambda) + H(t;\alpha,\mu) = (\mu-\lambda)f^{\alpha-\mu}(t)\int_0^t g(s)f^{\mu-\alpha-1}(s)Q(s;\alpha,\lambda)ds. \tag{4.5}$$

Moreover, it is easy to show that

$$\int_0^t \frac{g(s)}{f(s)}Q(s;\alpha,\lambda)ds = f^{\alpha-\mu}(t)\int_0^t g(s)f^{\mu-\alpha-1}(s)Q(s;\alpha,\lambda)ds \\ + (\mu-\alpha)\int_0^t g(s)f^{\alpha-\mu-1}(s)\left(\int_0^s g(\xi)f^{\mu-\alpha-1}(\xi)Q(\xi;\alpha,\lambda)d\xi\right)ds \quad \text{for } t > 0. \tag{4.6}$$

On the other hand, by virtue of (2.2), from relation (4.5) one gets

$$\liminf_{t\to+\infty} f^{\alpha-\mu}(t)\int_0^t g(s)f^{\mu-\alpha-1}(s)Q(s;\alpha,\lambda)ds > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{(\alpha-\lambda)(\mu-\alpha)}.$$

Therefore, in view of relation (1.7), it follows from (4.6) that

$$\liminf_{t\to+\infty}\frac{1}{\ln f(t)}\int_0^t \frac{g(s)}{f(s)}Q(s;\alpha,\lambda)ds > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\alpha-\lambda}. \tag{4.7}$$

Now, equality (4.4) and inequality (4.7) guarantee the validity of condition (2.1) and thus, the assertion of the corollary follows from Theorem 2.1. □

Proof of Corollary 2.3. If assumption (2.3) holds, then it follows from (4.4) that condition (2.1) is satisfied and thus, the assertion of the corollary follows from Theorem 2.1.

Let now assumption (2.4) be fulfilled. Observe that

$$\int_0^t f^\alpha(s)p(s)ds = H(t;\alpha,\mu) + (\mu-\alpha)\int_0^t \frac{g(s)}{f(s)}H(s;\alpha,\mu)ds \quad \text{for } t \geq 0t > 0.$$

Therefore, in view of (2.4), we obtain

$$\liminf_{t\to+\infty}\frac{1}{\ln f(t)}\int_0^t f^\alpha(s)p(s)ds > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{4.8}$$

On the other hand, it is clear that

$$c_\alpha'(t;\lambda) = \frac{-(\alpha-\lambda)^2 g(t)}{f^{1+\alpha-\lambda}}\int_0^t g(s)f^{\alpha-\lambda-1}(s)\left(\int_0^s f^\lambda(\xi)p(\xi)d\xi\right)ds \\ + \frac{(\alpha-\lambda)g(t)}{f(t)}\int_0^t f^\lambda(s)p(s)ds \\ = \frac{(\alpha-\lambda)g(t)}{f^{\alpha-\lambda+1}(t)}\int_0^t f^\alpha(s)p(s)ds \quad \text{for } t > 0.$$

Hence, we have

$$c_\alpha(\tau;\lambda) - c_\alpha(t;\lambda) = (\alpha-\lambda)\int_t^\tau \frac{g(s)}{f^{\alpha-\lambda+1}(s)}\left(\int_0^s f^\alpha(\xi)p(\xi)d\xi\right)ds \quad \tau \geq t > 0$$

and consequently, by virtue of assumption (1.8) and condition (4.8), we get

$$c_\alpha^*(\lambda) - c_\alpha(t;\lambda) = (\alpha-\lambda)\int_t^{+\infty} \frac{g(s)\ln f(s)}{f^{\alpha-\lambda+1}(s)}\left(\frac{1}{\ln f(s)}\int_0^s f^\alpha(\xi)p(\xi)d\xi\right)ds \quad \text{for } t>0. \tag{4.9}$$

In view of (4.8), there exist $\varepsilon > 0$ and $t_\varepsilon > 0$ such that

$$\frac{1}{\ln f(t)}\int_0^t f^\alpha(s)p(s)ds \geq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \varepsilon \quad \text{for} \quad t \geq t_\varepsilon.$$

Hence, it follows from (4.9) that

$$c_\alpha^*(\lambda) - c_\alpha(t;\lambda) \geq (\alpha-\lambda)\left(\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \varepsilon\right)\int_t^{+\infty}\frac{g(s)\ln f(s)}{f^{\alpha-\lambda+1}(s)} \quad \text{for } t \geq t_\varepsilon.$$

Since $\varepsilon > 0$, by virtue of (1.7), from the last relation we derive inequality (2.1). Therefore, the assertion of the corollary follows from Theorem 2.1. □

Proof of Theorem 2.5. Assume on the contrary that system (1.1) is not oscillatory, i.e., there exists a solution (u, v) of system (1.1) satisfying relation (3.1) with $t_u > 0$. Analogously to the proofs of Lemmas 3.4 and 3.5 we derive equalities (3.11) and (3.20), where the numbers γ, $\delta(t_u)$ and the functions h, ρ are given by (2.8), (3.21) and (3.3), (3.4).

It follows from (3.11) and (3.20) that

$$\begin{aligned}
&Q(t;\alpha,\lambda) + H(t;\alpha,\mu)\\
&\quad = -f^{\alpha-\lambda}(t)\int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds\\
&\qquad + \frac{\alpha}{\alpha-\lambda}\left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\right) + \frac{\alpha\gamma}{\mu-\alpha} + \delta(t_u)f^{\alpha-\mu}(t)\\
&\qquad + f^{\alpha-\mu}(t)\int_{t_u}^{t} g(s)f^{\mu-\alpha-1}(s)\left[\mu f^\alpha(s)\rho(s) - \alpha|\rho(s)f^\alpha(s)+\gamma|^{\frac{1+\alpha}{\alpha}}\right]ds
\end{aligned} \tag{4.10}$$

is satisfied for $t \geq t_u$. Moreover, according to Lemma 3.1 with $\omega := \mu$, it is clear that

$$\mu\left(f^\alpha(t)\rho(t)+\gamma\right) - \alpha|\rho(t)f^\alpha(t)+\gamma|^{\frac{1+\alpha}{\alpha}} \leq \left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} \quad \text{for } t \geq t_u. \tag{4.11}$$

Therefore, using (2.8), (3.5), and (4.11) in relation (4.10), we get

$$\begin{aligned}
&Q(t;\alpha,\lambda) + H(t;\alpha,\mu)\\
&\quad \leq \frac{1}{\alpha-\lambda}\left(\frac{\lambda}{1+\alpha}\right)^{1+\alpha} + \frac{1}{\mu-\alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} + \widetilde{\delta}(t_u)f^{\alpha-\mu}(t) \quad \text{for } t \geq t_u,
\end{aligned} \tag{4.12}$$

where

$$\widetilde{\delta}(t_u) := \delta(t_u) - \left[\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma\right]\frac{f^{\mu-\alpha}(t_u)}{\mu-\alpha}.$$

Consequently, by virtue of (1.7), relation (4.12) leads to a contradiction with assumption (2.5). □

Proof of Theorem 2.6. Suppose on the contrary that system (1.1) is not oscillatory. Then there exists a solution (u, v) of system (1.1) satisfying relation (3.1) with $t_u > 0$. Analogously to the proof of Lemma 3.5 one can show that relation (3.22) holds, where the numbers γ, $\delta_1(t_u)$ and the function ρ are given by (2.8), (3.23), and (3.4). On the other hand, according to Lemma 3.4, estimate (3.7) is fulfilled, where $A(\alpha, \lambda)$ is the smallest root of equation (2.9).

Let $\varepsilon > 0$ be arbitrary. Then there exists $t_\varepsilon \geq t_u$ such that

$$f^\alpha(t)\rho(t) \geq A(\alpha, \lambda) - \varepsilon \quad \text{for } t \geq t_\varepsilon.$$

Hence, it follows from (3.22) that

$$H(t; \alpha, \mu) \leq \delta_1(t_u) f^{\alpha-\mu}(t) - A(\alpha, \lambda) + \varepsilon + \frac{1}{\mu - \alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \gamma \quad \text{for } t \geq t_\varepsilon.$$

Since ε was arbitrary, in view of (1.7), from the latter inequality we get

$$H^*(\alpha, \mu) \leq \frac{1}{\mu - \alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \gamma - A(\alpha, \lambda, \gamma),$$

which contradicts assumption (2.7). □

Proof of Theorem 2.7. Assume on the contrary that system (1.1) is not oscillatory, i.e., there exists a solution (u, v) of system (1.1) satisfying relation (3.1) with $t_u > 0$. Analogously to the proof of Lemma 3.4 we show that equality (3.12) holds, where the number γ and the functions h, ρ are defined by (2.8), (3.3), and (3.4).

On the other hand, according to Lemma 3.5, estimate (3.19) is fulfilled, where $B(\alpha, \mu)$ is the greatest root of equation (2.12). Let $\varepsilon > 0$ be arbitrary. Then there exists $t_\varepsilon \geq t_u$ such that

$$f^\alpha(t)\rho(t) + \gamma \leq B(\alpha, \mu) + \varepsilon \quad \text{for } t \geq t_\varepsilon.$$

In view of the last inequality, (1.2), (1.6) and (3.5), it follows from (3.12) that

$$Q(t; \alpha, \lambda) \leq B(\alpha, \mu) + \varepsilon - \gamma + \frac{\alpha}{\alpha - \lambda}\left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\right) \quad \text{for } t \geq t_\varepsilon.$$

Since ε was arbitrary, we get

$$Q^*(\alpha, \lambda) \leq B(\alpha, \mu) + \frac{\gamma^{\frac{1+\alpha}{\alpha}}}{\alpha - \lambda},$$

which contradicts (2.11). □

Proof of Theorem 2.8. Suppose on the contrary that system (1.1) is not oscillatory. Then there exists a solution (u, v) of system (1.1) satisfying relation (3.1) with $t_u > 0$. Put

$$m := A(\alpha, \lambda), \qquad M := B(\alpha, \mu), \tag{4.13}$$

i.e., m denotes the smallest root of equation (2.9) and M is the greatest root of equation (2.12). According to Lemmas 3.4 and 3.5, we have

$$\liminf_{t\to+\infty} f^\alpha(t)\rho(t) \geq m, \qquad \limsup_{t\to+\infty} \left(f^\alpha(t)\rho(t) + \gamma\right) \leq M, \tag{4.14}$$

where the function ρ and the number γ are defined in (3.4) and (2.8).

Analogously to the proof of Theorem 2.5 we show that relation (4.10) holds for $t \geq t_u$, where the number $\delta(t_u)$ and the function h are defined by (3.21) and (3.3).

In view of (2.6), one can easily show that the function $y : x \mapsto \alpha|x+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha x + Q_*(\alpha,\lambda)(\alpha-\lambda) - \alpha\gamma$ is positive on $]-\infty, 0[$ and there exists $\bar{x} \in [0,+\infty[$ such that $y(\bar{x}) \leq 0$, which yields that $m \geq 0$.

On the other hand, in view of (2.10), one can easily verify that the function $z : x \mapsto \alpha|x|^{\frac{1+\alpha}{\alpha}} - \alpha x + (\mu-\alpha)H_*(\alpha,\mu)$ is positive on $]\left(\frac{\mu}{1+\alpha}\right)^\alpha, +\infty[$ and there exists $\tilde{x} \leq \left(\frac{\mu}{1+\alpha}\right)^\alpha$ such that $z(\tilde{x}) \leq 0$. Consequently, we have $M \leq \left(\frac{\mu}{1+\alpha}\right)^\alpha$.

We first assume that $m > 0$ and $M < \left(\frac{\mu}{1+\alpha}\right)^\alpha$. Let $\varepsilon \in]0, \min\left\{m, \left(\frac{\mu}{1+\alpha}\right)^\alpha - M\right\}[$ be arbitrary. Then, by virtue of (4.14), there exists $t_\varepsilon \geq t_u$ such that

$$f^\alpha(t)\rho(t) \geq m - \varepsilon, \qquad f^\alpha(t)\rho(t) + \gamma \leq M + \varepsilon \quad \text{for } t \geq t_\varepsilon. \tag{4.15}$$

The function $x \mapsto \alpha|x+\gamma|^{\frac{1+\alpha}{\alpha}} - (1+\alpha)x\gamma^{\frac{1}{\alpha}}$ is non-decreasing on $[0,+\infty[$. Therefore, in view of (3.3) and (4.15), we get

$$f^{\alpha-\lambda}(t)\int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds \geq \frac{\alpha|m-\varepsilon+\gamma|^{\frac{1+\alpha}{\alpha}} - \lambda(m-\varepsilon) - \alpha\gamma^{\frac{1+\alpha}{\alpha}}}{\alpha-\lambda} \tag{4.16}$$

for $t \geq t_\varepsilon$. Moreover, the function $x \mapsto \mu x - \alpha|x|^{\frac{1+\alpha}{\alpha}}$ is non-decreasing on $]-\infty, \left(\frac{\mu}{1+\alpha}\right)^\alpha[$ and thus, in view of (4.15), we obtain

$$\begin{aligned} f^{\alpha-\mu}(t)\int_{t_\varepsilon}^{t} g(s)f^{\mu-\alpha-1}(s)\left[\mu f^\alpha(s)\rho(s) - \alpha|\rho(s)f^\alpha(s)+\gamma|^{\frac{1+\alpha}{\alpha}}\right]ds \\ \leq \frac{\mu(M+\varepsilon) - \alpha|M+\varepsilon|^{\frac{1+\alpha}{\alpha}} - \mu\gamma}{\mu-\alpha} \quad \text{for} \quad t \geq t_\varepsilon. \end{aligned} \tag{4.17}$$

Now it follows from (4.10), (4.16), and (4.17) that

$$\begin{aligned} Q(t;\alpha,\lambda) + H(t;\alpha,\mu) \leq M + \varepsilon + H_*(\alpha,\mu) - (m-\varepsilon) + Q_*(\alpha,\lambda) - \gamma \\ + \frac{\alpha(M+\varepsilon) - \alpha|M+\varepsilon|^{\frac{1+\alpha}{\alpha}} - (\mu-\alpha)H_*(\alpha,\mu)}{\mu-\alpha} \\ - \frac{\alpha|m-\varepsilon+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha(m-\varepsilon) + (\alpha-\lambda)Q_*(\alpha,\lambda) - \alpha\gamma}{\alpha-\lambda} \\ + \delta(t_\varepsilon)f^{\alpha-\mu}(t) \quad \text{for } t \geq t_\varepsilon, \end{aligned} \tag{4.18}$$

where

$$\delta(t_\varepsilon) := \delta(t_u) + \int_{t_u}^{t_\varepsilon} g(s)f^{\mu-\alpha-1}(s)\left[\mu f^\alpha(s)\rho(s) - \alpha|\rho(s)f^\alpha(s)+\gamma|^{\frac{1+\alpha}{\alpha}}\right]ds.$$

Since ε was arbitrary, in view of (1.7) and (4.13), inequality (4.18) yields that

$$\limsup_{t\to+\infty}\left(Q(t;\alpha,\lambda) + H(t;\alpha,\mu)\right) \leq B(\alpha,\mu) - A(\alpha,\lambda,\gamma) + Q_*(\alpha,\lambda) + H_*(\alpha,\mu) - \gamma, \tag{4.19}$$

which contradicts assumption (2.13).

If $m = 0$ then, in view of (3.5), it is clear that

$$-f^{\alpha-\lambda}(t)\int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds \leq 0 = -\frac{\alpha|m+\gamma|^{\frac{1+\alpha}{\alpha}} - \lambda m - \alpha\gamma^{\frac{1+\alpha}{\alpha}}}{\alpha-\lambda} \tag{4.20}$$

for $t \geq t_u$. On the other hand, if $M = \left(\frac{\mu}{1+\alpha}\right)^{\alpha}$ then, using Lemma 3.1 with $\omega := \mu$, one can show that

$$\begin{aligned} &f^{\alpha-\mu}(t) \int_{t_u}^{t} g(s) f^{\mu-\alpha-1}(s) \left[\mu f^{\alpha}(s) \rho(s) - \alpha |\rho(s) f^{\alpha}(s) + \gamma|^{\frac{1+\alpha}{\alpha}} \right] ds \\ &\quad \leq \frac{\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma}{\mu - \alpha} - \frac{f^{\mu-\alpha}(t_u)}{f^{\mu-\alpha}(t)} \left(\frac{\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma}{\mu - \alpha} \right) \\ &\quad = \frac{\mu M - \alpha |M|^{\frac{1+\alpha}{\alpha}} - \mu\gamma}{\mu - \alpha} - \frac{f^{\mu-\alpha}(t_u)}{f^{\mu-\alpha}(t)} \left(\frac{\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma}{\mu - \alpha} \right) \quad \text{for } t \geq t_u. \end{aligned} \tag{4.21}$$

Consequently, if $m = 0$ (resp. $M = \left(\frac{\mu}{1+\alpha}\right)^{\alpha}$), then we derive from (4.10), the inequality (4.19) similarly as above, but we use (4.20) instead of (4.16) (resp. (4.21) instead of (4.17)). □

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