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Existence of positive solutions for boundary value problems of fractional functional differential equations

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Abstract

This paper deals with the existence of positive solutions for a boundary value problem involving a nonlinear functional differential equation of fractional order α given by $D^\alpha u(t) + f(t, u_t) = 0$, $t \in (0,1)$, $2 < \alpha \leq 3$, $u'(0) = 0$, $u'(1) = bu'(\eta)$, $u_0 = \phi$. Our results are based on the nonlinear alternative of Leray-Schauder type and Krasnosel'skii fixed point theorem.

Keywords: Fractional derivative; Boundary value problem; Functional differential equation; Positive solution; Fixed point theorem

1. Introduction

Recently, an increasing interest in studying the existence of solutions for boundary value problems of fractional order functional differential equations has been observed [1-3]. Fractional differential equations describe many phenomena in various fields of science and engineering such as physics, mechanics, chemistry, control, engineering, etc. In view of the importance and applications of fractional differential equations, a significant progress has been made in this direction by exploring several aspects of these equations, see, for instance, [4-21].

For $\tau > 0$, we denote by C_τ the Banach space of all continuous functions $\psi : [-\tau, 0] \to R$ endowed with the sup-norm

$$\|\psi\|_{[-\tau,0]} := \sup\{|\psi(s)| : s \in [-\tau, 0]\}.$$

If $u : [-\tau, 1] \to R$, then for any $t \in [0,1]$, we denote by u_t the element of C_τ defined by

$$u_t(\theta) = u(t + \theta), \quad \text{for } \theta \in [-\tau, 0].$$

In this paper we investigate a fractional order functional differential equation of the form

$$D^\alpha u(t) + f(t, u_t) = 0, \quad t \in (0,1), \quad 2 < \alpha \leq 3, \tag{1.1}$$

where D^α is the standard Riemann-Liouville fractional order derivative, $f(t, u_t) : [0,1] \times C_\tau \to R$ is a

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continuous function, associated with the boundary condition

$$u'(0) = 0, \quad u'(1) = bu'(\eta), \tag{1.2}$$

and the initial condition

$$u_0 = \phi, \tag{1.3}$$

where $0 < \eta < 1$, $1 < b < \frac{1}{\eta^{\alpha-2}}$, and ϕ is an element of the space

$$C_\tau^+(0) := \{\psi \in C_\tau : \psi(s) \geq 0,\ s \in [-\tau, 0],\ \psi(0) = 0\}.$$

To the best of the authors knowledge, no one has studied the existence of positive solutions for problem (1.1)-(1.3). The aim of this paper is to fill the gap in the relevant literatures. The key tools in finding our main results are the nonlinear alternative of Leray-Schauder type and Krasnosel'skii fixed point theorem.

2. Preliminaries

First of all, we recall some definitions of fractional calculus [16-19].

Definition 2.1. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to R$ is given by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.2. The Riemann-Liouville fractional integral of order α of a function $f : (0, \infty) \to R$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

provided that the integral exists.

The following lemma is crucial in finding an integral representation of the boundary value problem (1.1)-(1.3).

Lemma 2.1 [11]. *Suppose that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$. Then*

$$I^\alpha D^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}.$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$.

Lemma 2.2. *Let $0 < \eta < 1$ and $b \neq \frac{1}{\eta^{\alpha-2}}$. If $g \in C[0,1]$, then the boundary value problem*

$$D^\alpha u(t) + g(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u'(0) = 0, \quad u'(1) = bu'(\eta), \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)g(s)ds + \frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)g(s)ds, \tag{2.3}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & s \le t, \\ t^{\alpha-1}(1-s)^{\alpha-2}, & t \le s, \end{cases} \tag{2.4}$$

and

$$H(t,s) := \frac{\partial G(t,s)}{\partial t} = \frac{\alpha-1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-2}(1-s)^{\alpha-2} - (t-s)^{\alpha-2}, & s \le t, \\ t^{\alpha-2}(1-s)^{\alpha-2}, & t \le s. \end{cases} \tag{2.5}$$

Proof. By Lemma 2.1, the solution of (2.1) can be written as

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds.$$

Using the boundary conditions (2.2), we find that

$$c_2 = c_3 = 0, \quad \text{and}$$

$$c_1 = \frac{1}{(1-b\eta^{\alpha-2})\Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-2}g(s)ds - \int_0^\eta b(\eta-s)^{\alpha-2}g(s)ds\right].$$

Hence, the unique solution of BVP (2.1), (2.2) is

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds + \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{b\eta^{\alpha-2}t^{\alpha-1}}{(1-b\eta^{\alpha-2})\Gamma(\alpha)}\right)\int_0^1 (1-s)^{\alpha-2}g(s)ds \\ &\quad - \frac{bt^{\alpha-1}}{(1-b\eta^{\alpha-2})\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-2}g(s)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1})g(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-2}g(s)ds \\ &\quad + \frac{b\eta^{\alpha-2}t^{\alpha-1}}{(1-b\eta^{\alpha-2})\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}g(s)ds - \frac{bt^{\alpha-1}}{(1-b\eta^{\alpha-2})\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-2}g(s)ds \\ &= \int_0^1 G(t,s)h(s)ds + \frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)g(s)ds. \end{aligned}$$

The proof is complete.

To establish the existence of solutions for (1.1)-(1.3), we need the following known results.

Theorem 2.3 (Nonlinear alternative of Leray-Schauder [22]). *Let E be a Banach space with $C \subset E$*

closed and convex. Assume that U is a relatively open subset of C with $0 \in U$ and $T : \overline{U} \to C$ is completely continuous. Then either

(i) T has a fixed point in $\overline{U}$, or

(ii) there exists $u \in \partial U$ and $\gamma \in (0,1)$ with $u = \gamma T u$.

Theorem 2.4 (Krasnosel'skii [23]). *Let E be a Banach space and let K be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E, with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

$$\|Au\| \le \|u\|, \quad u \in K \cap \partial\Omega_1, \quad \textit{and } \|Au\| \ge \|u\|, \quad u \in K \cap \partial\Omega_2,$$

or

$$\|Au\| \ge \|u\|, \quad u \in K \cap \partial\Omega_1, \quad \textit{and } \|Au\| \le \|u\|, \quad u \in K \cap \partial\Omega_2,$$

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.5. *$G(t,s)$ has the following properties.*

(i) $0 \le G(t,s) \le h(s), \quad t, s \in [0,1],$

where

$$h(s) = \frac{(1-s)^{\alpha-2} - (1-s)^{\alpha-1}}{\Gamma(\alpha)};$$

(ii) $G(t,s) \ge t^{\alpha-1} h(s), \quad$ for $0 \le t, s \le 1$.

Proof. It is easy to check that (i) holds. Next, we prove (ii) holds. If $t \ge s$, then

$$\begin{aligned}
\frac{G(t,s)}{h(s)} &= \frac{t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{(1-s)^{\alpha-2} - (1-s)^{\alpha-1}} \\
&= \frac{t(t-ts)^{\alpha-2} - (t-s)(t-s)^{\alpha-2}}{(1-s)^{\alpha-2} - (1-s)^{\alpha-1}} \\
&\ge \frac{t(t-ts)^{\alpha-2} - (t-s)(t-ts)^{\alpha-2}}{(1-s)^{\alpha-2} - (1-s)^{\alpha-1}} \\
&= \frac{(t-ts)^{\alpha-2}}{(1-s)^{\alpha-2}} = t^{\alpha-2} \ge t^{\alpha-1}.
\end{aligned}$$

If $t \le s$, then

$$\frac{G(t,s)}{h(s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{(1-s)^{\alpha-2} - (1-s)^{\alpha-1}} = \frac{t^{\alpha-1}}{s} \ge t^{\alpha-1}.$$

The proof is complete.

Lemma 2.6. *Let $g \in C^+[0,1] := \{u \in C[0,1], u(t) \geq 0, t \in [0,1]\}$. If $1 < b < \frac{1}{\eta^{\alpha-2}}$, then the unique solution $u(t)$ of BVP (2.1), (2.2) is positive, and satisfying*

$$\min_{t\in[\tau,1]} u(t) \geq \tau^{\alpha-1}\|u\|,$$

where $0 < \tau < 1$.

Proof. Let $g \in C^+[0,1]$, we have from (2.3)-(2.5) that $u(t) \geq 0$. By Lemma 2.3, we get

$$\begin{aligned}
\|u\| = \max_{0\leq t\leq 1} |u(t)| &= \int_0^1 G(t,s)g(s)ds + \frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)g(s)ds \\
&\leq \int_0^1 h(s)g(s)ds + \frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)g(s)ds. \qquad (2.6)
\end{aligned}$$

On the other hand, from Lemma 2.3 and (2.6), we obtain for each $t \in [0,1]$ that

$$\begin{aligned}
u(t) &= \int_0^1 G(t,s)g(s)ds + \frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)g(s)ds \\
&\geq t^{\alpha-1}\int_0^1 h(s)g(s)ds + \frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)g(s)ds \\
&\geq t^{\alpha-1}\left[\int_0^1 h(s)g(s)ds + \frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)g(s)ds\right] \\
&\geq t^{\alpha-1}\|u\|.
\end{aligned}$$

Thus,

$$\min_{t\in[\tau,1]} u(t) \geq \tau^{\alpha-1}\|u\|, \quad 0 < \tau < 1.$$

3. Main results

In the sequel we shall denote by $C_0[0,1]$ the space of all continuous functions $x : [0,1] \to R$ with $x(0) = 0$. This is a Banach space when it is furnished with the usual sup-norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

For each $\phi \in C^+_\tau(0)$ and $x \in C_0[0,1]$ we define

$$x_t(s;\phi) := \begin{cases} x(t+s), & t+s \geq 0, \\ \phi(t+s), & t+s \leq 0, \ s \in [-\tau, 0], \end{cases}$$

and observe that $x_t(s;\phi) \in C_\tau$.

By a solution of the boundary value problem (1.1)-(1.3) we mean a function $u \in C_0[0,1]$ such that $D^\alpha u$ exists on $[0,1]$ and u satisfies boundary condition (1.2) and for a certain ϕ the relation

$$D^\alpha u(t) + f(t, u_t(\cdot;\phi)) = 0$$

holds for all $t \in [0,1]$.

Since $f : [0,1] \times C_\tau \to R$ is a continuous function, set $f(t, u_t(\cdot;\phi)) := g(t)$ in Lemma 2.2, we have by Lemma 2.2 that a function u is a solution of the boundary value problem (1.1)-(1.3) if and only if it satisfies

$$u(t) = \int_0^1 G(t,s) f(s, u_s(\cdot;\phi))ds + \frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})} \int_0^1 H(\eta,s) f(s, u_s(\cdot;\phi))ds, \quad t \in [0,1]. \tag{3.1}$$

We set

$$C_0^+[0,1] = \{u \in C_0[0,1] : u(t) \geq 0, t \in [0,1]\}.$$

Define the cone $P \subset C_0[0,1]$ by

$$P = \left\{ y \in C_0^+[0,1] : \min_{\tau \leq t \leq 1} y(t) \geq \tau^{\alpha-1}\|y\| \right\},$$

where $0 < \tau < 1$.

For $u \in P$, we define the operator T_ϕ as follows:

$$(T_\phi u)(t) = \int_0^1 G(t,s) f(s, u_s(\cdot;\phi))ds + \frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})} \int_0^1 H(\eta,s) f(s, u_s(\cdot;\phi))ds, \quad t \in [0,1]. \tag{3.2}$$

It is easy to know that fixed points of T_ϕ are solutions of the BVP (1.1)-(1.3).

In this paper, we assume that $0 < \tau < 1$, $\phi \in C_\tau^+(0)$, and we make use of the following assumption:

(H_1) $f : [0,1] \times C_\tau^+(0) \to [0, +\infty)$ is a continuous function.

Lemma 3.1. *Let* (H_1) *holds. Then* $T_\phi : P \to P$ *is completely continuous.*

Proof. By (H_1), we have $(T_\phi u)(t) \geq 0$, for $u \in P$ and $t \in [0,1]$. It follows from (3.2) and Lemma 2.5 that

$$\|T_\phi u\| = \max_{0 \leq t \leq 1} |(T_\phi u)(t)| \leq \int_0^1 h(s) f(s, u_s(\cdot;\phi))ds + \frac{b}{(\alpha-1)(1-b\eta^{\alpha-2})} \int_0^1 H(\eta,s) f(s, u_s(\cdot;\phi))ds.$$

In view of Lemma 2.5, we have

$$(T_\phi u)(t) \geq \int_0^1 t^{\alpha-1} h(s) f(s, u_s(\cdot;\phi))ds + \frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})} \int_0^1 H(\eta,s) f(s, u_s(\cdot;\phi))ds$$

$$\geq t^{\alpha-1}\left[\int_0^1 h(s)f(s,u_s(\cdot;\phi))ds+\frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)f(s,u_s(\cdot;\phi))ds\right]$$
$$\geq t^{\alpha-1}\|T_\phi u\|,\quad t\in[0,1].$$

So,

$$\min_{\tau\leq t\leq 1}(T_\phi u)(t)\geq \tau^{\alpha-1}\|T_\phi u\|,$$

which shows that $T_\phi P\subset P$. Moreover, similar to the proof of Lemma 3.2 in [10], it is easy to check that $T_\phi : P\to P$ is completely continuous.

Lemma 3.2. *If $0<\tau<1$ and $u\in P$, then we have*

$$\|u_t(\cdot;\phi)\|_{[-\tau,0]}\geq \tau^{\alpha-1}\|u\|,\quad t\in[\tau,1].$$

Proof. From the definition of $u_t(s;\phi)$, for $t\geq\tau$, we have

$$u_t(s;\phi)=u(t+s),\quad s\in[-\tau,0].$$

Thus, we get for $u\in P$ that

$$\|u_t(\cdot;\phi)\|_{[-\tau,0]}=\max_{s\in[-\tau,0]}u(t+s)\geq u(t)\geq t^{\alpha-1}\|u\|\geq\tau^{\alpha-1}\|u\|,\quad t\geq\tau.$$

We are now in a position to present and prove our main results.

Theorem 3.3. *Let* (H_1) *holds. Suppose that the following conditions are satisfied:*

(H_2) *there exist a continuous function $a:[0,1]\to[0,+\infty)$ and a continuous, nondecreasing function $F:[0,+\infty)\to[0,+\infty)$ such that*

$$f(t,\psi)\leq a(t)F(\|\psi\|_{[-\tau,0]}),\quad (t,\psi)\in[0,1]\times C_\tau^+(0),$$

(H_3) *there exists $r>\|\phi\|_{[-\tau,0]}$, with*

$$\frac{r}{F(r)}>\int_0^1 h(s)a(s)ds+\frac{b}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)a(s)ds. \tag{3.3}$$

then BVP (1.1)-(1.3) has at least one positive solution.

Proof. We shall apply Theorem 2.3 (the nonlinear alternative of Leray-Schauder type) to prove that T_ϕ has at least one positive solution.

Let $U=\{u\in P:\|u\|<r\}$, where r is as in (H_3). Assume that there exist $u\in P$ and $\lambda\in(0,1)$ such that $u=\lambda T_\phi u$, we claim that $\|u\|\neq r$. In fact, if $\|u\|=r$, we have

$$u(t)=\lambda\int_0^1 G(t,s)f(s,u_s(\cdot;\phi))ds+\frac{\lambda bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)f(s,u_s(\cdot;\phi))ds$$

$$\leq \int_0^1 h(s)a(s)F(\|u_s(\cdot;\phi)\|_{[-\tau,0]})ds$$

$$+\frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)a(s)F(\|u_s(\cdot;\phi)\|_{[-\tau,0]})ds. \tag{3.4}$$

By the definition of $u_s(\cdot;\phi)$, we easily obtain that

$$\|u_s(\cdot;\phi)\|_{[-\tau,0]} \leq \max\{\|u\|, \|\phi\|_{[-\tau,0]}\} = \max\{r, \|\phi\|_{[-\tau,0]}\} = r. \tag{3.5}$$

Thus by (3.4), (3.5) and the nondecreasing of F, we get that

$$r = \|u\| \leq \int_0^1 h(s)a(s)F(r)ds + \frac{b}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)a(s)F(r)ds.$$

Consequently

$$\frac{r}{F(r)} \leq \int_0^1 h(s)a(s)ds + +\frac{b}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)a(s)ds,$$

which contradict (3.3). Hence $u \notin \partial U$. By Theorem 2.3, T_ϕ has a fixed point $u \in \overline{U}$. Therefore, BVP (1.1)-(1.3) has at least one positive solution.

Theorem 3.4. *Let* (H_1) *holds. Suppose that the following conditions are satisfied:*

(H_4) *There exist constant* $r_2 > \|\phi\|_{[-\tau,0]}$*, as well as continuous function* $p \in C[0,1]$ *and nondecreasing continuous functions* $L : R^+ \to R^+$ *such that*

$$f(t,\psi) \leq p(t)L(\|\psi\|_{[-\tau,0]}), \quad (t,\psi) \in [0,1] \times C_\tau^+(0),\ \|\psi\|_{[-\tau,0]} \leq r_2. \tag{3.6}$$

(H_5) *There exist functions* $\omega : [0,1] \to [0,\tau]$*, continuous* $c : [0,1] \to R^+$*, and nondecreasing* $J : R^+ \to R^+$ *such that*

$$f(t,\psi) \geq c(t)J(\psi(-\omega(t))), \quad (t,\psi) \in [0,1] \times C_\tau^+(0), \tag{3.7}$$

If r_2 *(as in* (H_4)*) satisfying*

$$\frac{L(r_2)}{r_2}\left(\int_0^1 h(s)p(s)ds + \frac{b}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)p(s)ds\right) \leq 1. \tag{3.8}$$

And there exists a constant $r_1 > 0$ $(r_1 < r_2)$ *satisfying*

$$\frac{J(\tau^{\alpha-1}r_1)}{r_1}\int_\tau^1 \tau^{\alpha-1}h(s)c(s)ds \geq 1. \tag{3.9}$$

Then BVP (1.1)-(1.3) has a positive solution.

Proof. If $u \in P$ with $\|u\| = r_2$, then from (3.6), (3.5), and Lemma 2.5, we get for any $t \in [0,1]$ that

$$
\begin{aligned}
(T_\phi u)(t) &= \int_0^1 G(t,s) f(s, u_s(\cdot;\phi))ds + \frac{bt^{\alpha-1}}{(\alpha-1)(1-b\eta^{\alpha-2})} \int_0^1 H(\eta,s) f(s,u_s(\cdot;\phi))ds \\
&\leq \int_0^1 h(s) f(s,u_s(\cdot;\phi))ds + \frac{b}{(\alpha-1)(1-b\eta^{\alpha-2})} \int_0^1 H(\eta,s) f(s,u_s(\cdot;\phi))ds \\
&\leq \int_0^1 h(s)p(s)L(\|u_s(\cdot;\phi)\|_{[-\tau,0]})ds \\
&\quad + \frac{b}{(\alpha-1)(1-b\eta^{\alpha-2})} \int_0^1 H(\eta,s)p(s)L(\|u_s(\cdot;\phi)\|_{[-\tau,0]})ds \\
&\leq L(r_2)\left(\int_0^1 h(s)p(s)ds + \frac{b}{(\alpha-1)(1-b\eta^{\alpha-2})} \int_0^1 H(\eta,s)p(s)ds\right) \\
&\leq r_2. \qquad (3.10)
\end{aligned}
$$

Now if we set

$$\Omega_1 = \{u \in C[0,1] : \|u\| < r_2\},$$

then (3.10) shows that $\|T_\phi u\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$.

For $u \in P$ with $\|u\| = r_1$, we have from (3.7), (3.9), and Lemmas 2.5 and 3.2 that

$$
\begin{aligned}
(T_\phi u)(t) &\geq \int_0^1 G(t,s) f(s,u_s(\cdot;\phi))ds \\
&\geq \int_\tau^1 G(t,s) f(s,u_s(\cdot;\phi))ds \geq \int_\tau^1 \tau^{\alpha-1} h(s) f(s,u_s(\cdot;\phi))ds \\
&\geq \int_\tau^1 \tau^{\alpha-1} h(s)c(s)J(u_s(-\omega(s);\phi))ds \\
&\geq J(\tau^{\alpha-1}\|u\|) \int_\tau^1 \tau^{\alpha-1} h(s)c(s)ds \\
&= J(\tau^{\alpha-1}r_1) \int_\tau^1 \tau^{\alpha-1} h(s)c(s)ds \\
&\geq r_1. \qquad (3.11)
\end{aligned}
$$

Now if we set

$$\Omega_2 = \{u \in C[0,1] : \|u\| < r_1\},$$

then (3.11) shows that $\|T_\phi u\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$.

Hence by the second part of Theorem 2.4, T_ϕ has a fixed point $u \in P \cap (\overline{\Omega}_1 \setminus \Omega_2)$, and accordingly, u is a solution of BVP (1.1)-(1.3).

Set

$$M=\left(\frac{1}{(\alpha-1)\Gamma(\alpha+1)}+\frac{b\eta^{\alpha-2}(1-\eta)}{(\alpha-1)\Gamma(\alpha)(1-b\eta^{\alpha-2})}\right)^{-1}, \tag{3.12}$$

$$m=\frac{(\alpha-1)\Gamma(\alpha+1)}{\tau^{2(\alpha-1)}[\alpha(1-\tau)^{\alpha-1}-(\alpha-1)(1-\tau)^{\alpha}]}. \tag{3.13}$$

Corollary 3.5. *Let* (H_1) *holds. If there exist two positive constants* $r_2>r_1>0$ $(r_2>\|\phi\|_{[-\tau,0]})$ *such that*

(H_6) $f(t,\psi)\le Mr_2,\quad (t,\psi)\in[0,1]\times C_\tau^+(0),\quad \|\psi\|_{[-\tau,0]}\le r_2.$ (3.14)

(H_7) $f(t,\psi)\ge mr_1,\quad (t,\psi)\in[0,1]\times\in C_\tau^+(0),\quad \|\psi\|_{[-\tau,0]}\le r_1.$ (3.15)

Then BVP (1.1)-(1.3) has a positive solution.

Proof. In (H_4), let $p(t)\equiv M$ and $L(x)\equiv x$, then by (3.12) and (3.14), we have that (3.8) holds. In fact, we have

$$\begin{aligned}
&L(r_2)\left(\int_0^1 h(s)p(s)ds+\frac{b}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)p(s)ds\right)\\
&=r_2M\left(\int_0^1 h(s)ds+\frac{b}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)ds\right)\\
&=r_2M\left(\frac{1}{(\alpha-1)\Gamma(\alpha+1)}+\frac{b\eta^{\alpha-2}(1-\eta)}{(\alpha-1)\Gamma(\alpha)(1-b\eta^{\alpha-2})}\right)\\
&=r_2.
\end{aligned}$$

Moreover, let $c(t)\equiv m$ and $J(x)\equiv x$ in (H_5), then by (3.13) and (3.15), we have that (3.9) holds. In fact, we have

$$\begin{aligned}
J(\tau^{\alpha-1}r_1)\int_\tau^1\tau^{\alpha-1}h(s)c(s)ds&=\tau^{2(\alpha-1)}r_1m\int_\tau^1 h(s)ds\\
&=\tau^{2(\alpha-1)}r_1m\frac{\alpha(1-\tau)^{\alpha-1}-(\alpha-1)(1-\tau)^{\alpha}}{(\alpha-1)\Gamma(\alpha+1)}=r_1.
\end{aligned}$$

So all conditions of Theorem 3.4 are satisfied. By Theorem 3.4, BVP (1.1)-(1.3) has at least one positive solution.

Having in mind the proof of Theorem 3.4, one can easily conclude the following results.

Theorem 3.6. *Let* (H_1), (H_4), (H_5) *and (3.8) hold. If the function* η *satisfies the condition*

$$\limsup_{x\to0+}\frac{J(x)}{x}>\frac{1}{\tau^{2(\alpha-1)}\int_\tau^1 h(s)c(s)ds}. \tag{3.16}$$

Then BVP (1.1)-(1.3) has a positive solution.

4. Examples

To illustrate our results we present the following examples.

Example 4.1. Consider the boundary value problem of fractional order functional differential equations

$$D^{2.5}u(t) + \frac{1}{2}(1+t^2)u^2\left(t - \frac{1}{3}\right) + 3 = 0, \quad 0 < t < 1, \tag{4.1}$$

$$u'(0) = 0, \quad u'(1) = bu'(\eta), \tag{4.2}$$

$$u(t) = \phi(t), \quad t \in [-\tau, 0], \tag{4.3}$$

where $b = 1.2$, $\eta = \frac{2}{3}$, $\tau = \frac{1}{2}$, and $\phi \in C_\tau^+(0)$ with $\|\phi\|_{[-\tau,0]} < \frac{3}{2}$.

Let $f(t,\psi) = \frac{1}{2}(1+t^2)\psi^2(-\frac{1}{3})+3$, $(t,\psi) \in [0,1]\times C_\tau^+$, and $\alpha = 2.5$. Obviously, $1 < b < \frac{1}{\eta^{\alpha-2}} = 1.2247$. By simple calculation, we obtain that

$$M = \left(\frac{1}{(\alpha-1)\Gamma(\alpha+1)} + \frac{b\eta^{\alpha-2}(1-\eta)}{(\alpha-1)\Gamma(\alpha)(1-b\eta^{\alpha-2})}\right)^{-1} = 3.0864,$$

$$m = \frac{(\alpha-1)\Gamma(\alpha+1)}{\tau^{2(\alpha-1)}[\alpha(1-\tau)^{\alpha-1} - (\alpha-1)(1-\tau)^{\alpha}]} = 64.489.$$

Choosing $r_1 = \frac{1}{22}$, $r_2 = \frac{3}{2} > \|\phi\|_{[-\tau,0]}$, we have

$$\begin{aligned} f(t,\psi) &= \frac{1}{2}(1+t^2)\psi^2\left(-\frac{1}{3}\right) + 3 \le \|\psi\|_{[-\tau,0]} + 3 \le 4.5 \\ &< 4.6026 = 3.0864 \cdot 1.5 = Mr_2, \quad (t,\psi) \in [0,1] \times C_\tau^+ \text{ with } \|\psi\|_{[-\tau,0]} \le \frac{3}{2}, \end{aligned}$$

$$\begin{aligned} f(t,\psi) &= \frac{1}{2}(1+t^2)\psi^2\left(-\frac{1}{3}\right) + 3 \ge 3 > 2.9313 \\ &= 64.489 \cdot \frac{1}{22} = mr_1, \quad (t,\psi) \in [0,1] \times C_\tau^+ \text{ with } \|\psi\|_{[-\tau,0]} \le \frac{1}{22}, \end{aligned}$$

Hence all conditions of Corollary 3.5 are satisfied. By Corollary 3.5, BVP (4.1)-(4.3) has at least one positive solution u such that $\frac{1}{22} \le \|u\| \le \frac{3}{2}$.

Example 4.2. Consider the boundary value problem of fractional order functional differential equations

$$D^{2.2}u(t) + \sqrt{t}\sin^2 t \cdot u^4\left(t - \frac{1}{4}\right) = 0, \quad 0 < t < 1, \tag{4.4}$$

$$u'(0) = 0, \quad u'(1) = bu'(\eta), \tag{4.5}$$

$$u(t) = \phi(t), \quad t \in [-\tau, 0], \tag{4.6}$$

where $b = 1.1$, $\eta = 0.2$, $\tau = \frac{1}{3}$ and $\phi \in C_\tau^+(0)$ with $\|\phi\|_{[-\tau,0]} < \frac{3}{4}$.

Let $f(t,\psi) = \sqrt{t}\sin^2 t \cdot u^4\left(t - \frac{1}{4}\right)$, $(t,\psi) \in [0,1] \times C_\tau^+$, and $\alpha = 2.2$.

Obviously, $1 < b < \frac{1}{\eta^{\alpha-2}} = 1.3797$, and

$$f(t,\psi) = \sqrt{t}\sin^2 t \cdot \psi^4\left(-\frac{1}{4}\right) \leq a(t)F(\|\psi\|_{[-\tau,0]}),$$

where $a(t) = \sqrt{t}$ and $F(x) = x^4$, which implies that (H_2) holds.

With the aid of computation we have that

$$\begin{aligned}
&\int_0^1 h(s)a(s)ds + \frac{b}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)a(s)ds \\
&\quad = \frac{1}{\Gamma(2.2)}\int_0^1 ((1-s)^{0.2} - (1-s)^{1.2}))s^{0.5}ds \\
&\quad\quad + \frac{1.1}{1.2(1-1.1\cdot 0.2^{0.2})}\left[\int_0^{0.2}\frac{1.2}{\Gamma(2.2)}(0.2^{0.2}\cdot(1-s)^{0.2} - (0.2-s)^{0.2})s^{0.5}ds\right. \\
&\quad\quad \left. + \int_{0.2}^1 \frac{1.2}{\Gamma(2.2)}0.2^{0.2}\cdot(1-s)^{0.2}s^{0.5}ds\right] \\
&\quad \leq \frac{1}{\Gamma(2.2)}\int_0^1 (1-s)^{0.2}s^{0.5}ds - \frac{1}{\Gamma(2.2)}\int_0^1 (1-s)^{1.2}s^{0.5}ds \\
&\quad\quad + \frac{1.1}{1.2(1-1.1\cdot 0.2^{0.2})}\cdot\frac{1.2}{\Gamma(2.2)}0.2^{0.2}\int_0^1 (1-s)^{0.2}s^{0.5}ds \\
&\quad = \frac{1}{\Gamma(2.2)}B(1.5,1.2) - \frac{1}{\Gamma(2.2)}B(1.5,2.2) + \frac{1.1\cdot 0.2^{0.2}}{(1-1.1\cdot 0.2^{0.2})\Gamma(2.2)}B(1.5,1.2) \\
&\quad = 2.1458,
\end{aligned}$$

where $B(\cdot,\cdot)$ is a Beta function. Choosing $r = \frac{3}{4}$, then $r > \|\phi\|_{[-\tau,0]}$, and

$$\begin{aligned}
\frac{r}{F(r)} &= \frac{1}{r^3} = 2.3704 \\
&\quad > 2.1458 = \int_0^1 h(s)a(s)ds + \frac{b}{(\alpha-1)(1-b\eta^{\alpha-2})}\int_0^1 H(\eta,s)a(s)ds,
\end{aligned}$$

that is, condition (H_3) holds. Thus all conditions of Theorem 3.3 are satisfied. By Theorem 3.3, BVP (4.4)-(4.6) has at least one positive solution.

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