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Extinction and non-extinction of solutions for a nonlocal reaction-diffusion problem

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Abstract

We investigate extinction properties of solutions for the homogeneous Dirichlet boundary value problem of the nonlocal reaction-diffusion equation $u_t - d\Delta u + ku^p = \int_\Omega u^q(x,t)\,dx$ with $p, q \in (0,1)$ and $k, d > 0$. We show that $q = p$ is the critical extinction exponent. Moreover, the precise decay estimates of solutions before the occurrence of the extinction are derived.

Keywords: reaction-diffusion equation; extinction; non-extinction.

AMS Subject Classification (2000): 35K20, 35K55.

1 Introduction and main results

This paper is devoted to the extinction properties of solutions for the following diffusion equation with nonlocal reaction

$$u_t - d\Delta u + ku^p = \int_\Omega u^q(x,t)\,dx, \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

subject to the initial and boundary value conditions

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $p, q \in (0,1)$, $k, d > 0$, $\Omega \subset \mathbb{R}^N (N > 2)$ is an bounded domain with smooth boundary and $u_0(x) \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ is a nonzero non-negative function.

Many physical phenomena were formulated into nonlocal mathematical models ([2, 3, 6, 7]) and there are a large number of papers dealing with the reaction-diffusion equations with nonlocal reactions or nonlocal boundary conditions (see [18, 20, 21, 23] and the references therein). In particular, M. Wang and Y. Wang [23] studied problem (1.1)–(1.3) for $p, q \in [1, +\infty)$ and concluded that: the blow-up occurs for large initial data if $q > p \geq 1$ while all solutions exist globally if $1 \leq q < p$; in case of $p = q$, the issue depends on the comparison of $|\Omega|$ and k. For further studies of problem (1.1)–(1.3) we refer the read to [1, 13, 14, 19, 26] and the references therein. In all the above works, $p, q \in [1, +\infty)$ was assumed.

Extinction is the phenomenon whereby the evolution of some nontrivial initial data $u_0(x)$ produces a nontrivial solution $u(x,t)$ in a time interval $0 < t < T$ and then $u(x,t) \equiv 0$ for all $(x,t) \in \Omega \times [T, +\infty)$. It is an important property of solutions for many evolution equations which have been studied extensively by many researchers. Especially, there are some papers concerning the extinction for the following semilinear parabolic equation for special cases

$$u_t - d\Delta u + ku^p = \lambda u^q, \quad x \in \Omega, \quad t > 0, \tag{1.4}$$

where $p \in (0,1)$ and $q \in (0,1]$. In case $\lambda = 0$, it is well-known that solutions of problem (1.2)–(1.4) vanishes within a finite time. Evans and Knerr [9] established this for the Cauchy problem by constructing a suitable comparison function. Fukuda [10] studied problem (1.2)–(1.4) with $\lambda > 0$ and $q = 1$ and concluded that: when $\lambda < \lambda_1$, the term Δu dominates the term λu so that solutions of problem (1.2)–(1.4) behave the same as those of (1.2)–(1.4) with $\lambda = 0$; when $\lambda > \lambda_1$ and $\int_\Omega u_0 \phi(x) dx > (\lambda - \lambda_1)^{-\frac{1}{1-p}}$, solutions of problem (1.2)–(1.4) grow up to infinity as $t \to \infty$. Here, λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition and $\phi(x) > 0$ in Ω with $\max\limits_{x\in\Omega} \phi(x) = 1$ is the eigenfunction corresponding to the eigenvalue λ_1. Yan and Mu [24] investigated problem (1.2)–(1.4) with $0 < p < q < 1$ and $N > 2(q-p)/(1-p)$ and obtained that the non-negative weak solution of problem (1.2)–(1.4) vanishes in finite time for any initial data provided that k is appropriately large. For papers concerning the extinction for the porous medium equation or the p-Laplacian equation, we refer the reader to [8, 11, 12, 15, 16, 22, 25] and the references therein. Recently, the present author [17] considered the extinction properties of solutions for the homogeneous Dirichlet boundary value problem of the p-Laplacian equation

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) + \beta u^q = \lambda u^r, \quad x \in \Omega, \quad t > 0.$$

But as far as we know, no work is found to deal with the extinction properties of solutions for problem (1.1)–(1.3) which contains a nonlocal reaction term.

The purpose of the present paper is to investigate the extinction properties of solutions for the nonlocal reaction-diffusion problem (1.1)–(1.3). Our results below show that $q = p$ is the critical extinction exponent for the weak solution of problem (1.1)–(1.3): if $0 < p < q < 1$, the non-negative weak solution vanishes in finite time provided that $|\Omega|$ is appropriately small or k is appropriately large; if $0 < q < p < 1$, the weak solution cannot vanish in finite time for any non-negative initial data; if $0 < q = p < 1$, the weak solution cannot vanish in finite time for any non-negative initial data when $k < \int_\Omega \psi^q(x) dx / M^q$ $(\leq |\Omega|)$, while it vanishes in finite time for any initial data u_0 when $k > |\Omega|$. Here $\psi(x)$ is the unique positive solution of the linear elliptic problem

$$-\Delta \psi = 1 \text{ in } \Omega; \quad \psi = 0 \text{ on } \partial\Omega \tag{1.5}$$

and $M = \max\limits_{x\in\Omega} \psi(x)$. This is quite different from that of local reaction case, in which the first eigenvalue of the Dirichlet problem plays a role in the critical case (see [8, 12, 15, 22, 25]).

Moreover, the precise decay estimates of solutions before the occurrence of the extinction will be derived.

We now state our main results.

Theorem 1 *Assume that* $0 < p < q < 1$.

1) If $N < 4(q-p)/[(1-p)(1-q)]$, *the non-negative weak solution of problem* (1.1)–(1.3) *vanishes in finite time provided that the initial data* u_0 (*or* $|\Omega|$) *is appropriately small or* k *is appropriately large.*

2) If $N = 4(q-p)/[(1-p)(1-q)]$, *the non-negative weak solution of problem* (1.1)–(1.3) *vanishes in finite time for any initial data provided that* $|\Omega|$ *is appropriately small or* k *is appropriately large.*

3) If $N > 4(q-p)/[(1-p)(1-q)]$, *the non-negative weak solution of problem* (1.1)–(1.3) *vanishes in finite time for any initial data provided that* $|\Omega|$ *is appropriately small or* k *is appropriately large.*

Moreover, one has

$$\begin{cases} \|u(\cdot,t)\|_2 \le \|u_0\|_2\, e^{-\alpha_1 t}, & t \in [0, T_1), \\ \|u(\cdot,t)\|_2 \le \left[\left(\|u(\cdot,T_1)\|_2^{2-\theta_2} + \dfrac{k_2}{d_1\lambda_1}\right) e^{-(2-\theta_2)d_1\lambda_1\,(t-T_1)} - \dfrac{k_2}{d_1\lambda_1}\right]^{\frac{1}{2-\theta_2}}, & t \in [T_1, T_1^*), \\ \|u(\cdot,t)\|_2 \equiv 0, & t \in [T_1^*, +\infty), \end{cases}$$

for $N < 4(q-p)/[(1-p)(1-q)]$,

$$\begin{cases} \|u(\cdot,t)\|_2 \le \left[\left(\|u_0\|_2^{1-q} + \dfrac{k_1 - |\Omega|^{\frac{3-q}{2}}}{d_1\lambda_1}\right) e^{-(1-q)d_1\lambda_1\, t} - \dfrac{k_1 - |\Omega|^{\frac{3-q}{2}}}{d_1\lambda_1}\right]^{\frac{1}{1-q}}, & t \in [0, T_2^*), \\ \|u(\cdot,t)\|_2 \equiv 0, & t \in [T_2^*, +\infty), \end{cases}$$

for $N = 4(q-p)/[(1-p)(1-q)]$,

$$\begin{cases} \|u(\cdot,t)\|_2 \le \left[\left(\|u_0\|_2^{2-\theta_2} + \dfrac{k_3}{d_3\lambda_1}\right) e^{-(2-\theta_2)d_3\lambda_1\, t} - \dfrac{k_3}{d_3\lambda_1}\right]^{\frac{1}{2-\theta_2}}, & t \in [0, T_3^*), \\ \|u(\cdot,t)\|_2 \equiv 0, & t \in [T_3^*, +\infty), \end{cases}$$

for $N > 4(q-p)/[(1-p)(1-q)]$, *where* d_1, d_3, T_1, T_i^* *and* k_i $(i = 1, 2, 3)$ *are positive constants to be given in the proof,* $\alpha_1 > d_1\lambda_1$ *and*

$$\theta_2 = \frac{2N(1-p) + 4(1+p)}{N(1-p) + 4} \in (1, 2).$$

Remark 1 *One can see from the proof below that the restriction* $N > 4(q-p)/[(1-p)(1-q)]$ *in the case 3) can be extended to* $N > 2(q-p)/(1-p)$. *This has been proved in [24] for the local reaction case.*

Theorem 2 *Assume that* $0 < p = q < 1$.

1) If $k > |\Omega|$*, the non-negative weak solution of problem* (1.1)–(1.3) *vanishes in finite time for any initial data* u_0*. Moreover, one has*

$$\begin{cases} \|u(\cdot,t)\|_2 \le \left[\left(||u_0||_2^{1-q} + \dfrac{k_4}{d_4\lambda_1}\right) e^{-(1-q)d_4\lambda_1 t} - \dfrac{k_4}{d_4\lambda_1}\right]^{\frac{1}{1-q}}, & t \in [0, T_2^*), \\ \|u(\cdot,t)\|_2 \equiv 0, & t \in [T_2^*, +\infty), \end{cases}$$

where d_4 *and* k_4 *are positive constants to be given in the proof.*

2) If $k < \int_\Omega \psi^q(x)dx/M^q$ $(\le |\Omega|)$*, then the weak solution of problem* (1.1)–(1.3) *cannot vanish in finite time for any non-negative initial data.*

3) If $k = \int_\Omega \psi^q(x)dx/M^q$*, then the weak solution of problem* (1.1)–(1.3) *cannot vanish in finite time for any identically positive initial data.*

Theorem 3 *Assume that* $0 < q < p < 1$*, then the weak solution of* (1.1)–(1.3) *cannot vanish in finite time for any non-negative initial data.*

Remark 2 *One can conclude from Theorems 1–3 that* $q = p$ *is the critical extinction exponent of solutions for problem* (1.1)–(1.3).

The rest of the paper is organized as follows. In Section 2, we will give some preliminary lemmas. We will prove Theorems 1–3 in Section 3-5.

2 Preliminary

Let $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ denote $L^p(\Omega)$ and $W^{1,p}(\Omega)$ norms respectively, $1 \le p \le \infty$. Before proving our main results, we will give some preliminary lemmas which are of crucial importance in the proofs. We first give the following comparison principle, which can be proved as in [22, 23, 25].

Lemma 1 *Suppose that* $\underline{u}(x,t), \overline{u}(x,t)$ *are a subsolution and a supersolution of problem* (1.1)–(1.3) *respectively, then* $\underline{u}(x,t) \le \overline{u}(x,t)$ *a.e. in* Ω_T.

The following inequality problem is often used to derive extinction of solutions (see [22, 25]).

$$\frac{dy}{dt} + \alpha y^k \le 0, \quad t \ge 0; \quad y(0) \ge 0,$$

where $\alpha > 0$ is a constant and $k \in (0,1)$. Due to the nature of our problem, we would like to use the following lemmas which are of crucial importance in the proofs of decay estimates.

Lemma 2 [5] *Let* $y(t)$ *be a non-negative absolutely continuous function on* $[0,+\infty)$ *satisfying*

$$\frac{dy}{dt} + \alpha y^k + \beta y \le 0, \quad t \ge T_0; \quad y(T_0) \ge 0, \tag{2.1}$$

where $\alpha, \beta > 0$ are constants and $k \in (0,1)$. Then we have decay estimate

$$\begin{cases} y(t) \le \left[\left(y^{1-k}(T_0) + \frac{\alpha}{\beta}\right) e^{(k-1)\beta\,(t-T_0)} - \frac{\alpha}{\beta}\right]^{\frac{1}{1-k}}, & t \in [T_0, T_*), \\ y(t) \equiv 0, & t \in [T_*, +\infty), \end{cases}$$

where $T_* = \dfrac{1}{(1-k)\beta} \ln\left(1 + \dfrac{\beta}{\alpha} y^{1-k}(T_0)\right)$.

Lemma 3 [15] *Let $0 < k < m \le 1$, $y(t) \ge 0$ be a solution of the differential inequality*

$$\frac{dy}{dt} + \alpha y^k + \beta y \le \gamma\, y^m, \quad t \ge 0; \quad y(0) = y_0 > 0, \tag{2.2}$$

where $\alpha, \beta > 0, \gamma$ is a positive constant such that $\gamma < \alpha y_0^{k-m}$. Then there exist $\eta > \beta$, such that

$$0 \le y(t) \le y_0\, e^{-\eta t}, \ t \ge 0.$$

Consider the following ODE problem

$$\frac{dy}{dt} + \alpha y^k + \beta y = \gamma\, y^m, \quad t \ge 0; \quad y(0) = y_0 \ge 0; \quad y(t) > 0, \quad t > 0. \tag{2.3}$$

If $\alpha = 0, \beta > 0$ and $\gamma > 0$, we can easily derive that the non-constant solution of this problem is

$$y(t) = \left[\left(y_0^{1-m} - \frac{\gamma}{\beta}\right) e^{-(1-m)\beta\, t} + \frac{\gamma}{\beta}\right]^{\frac{1}{1-m}} > 0, \quad \forall\, t > 0.$$

If $\alpha, \beta, \gamma > 0$, we have

Lemma 4 [17] *Let $\alpha, \beta, \gamma > 0$ and $0 < m < k < 1$. Then there exists at least one non-constant solution of the ODE problem* (2.3).

Proof. It is easy to prove that the following algebraic equation

$$\alpha y^k + \beta y = \gamma\, y^m$$

has unique positive solution (denoted by y_*).

We first consider the case $y_0 > 0$. By considering the sign of $y'(t)$ via $y(t)$ at $[0, y_*)$, we see that: if $0 < y_0 < y_*$, then $y(t)$ is increasing with respect to $t > 0$; if $y_0 > y_*$, then $y(t)$ is decreasing with respect to $t > 0$. Therefore, solution with non-negative initial value y_0 remains positive and of course approaches y_* as $t \to +\infty$.

When $y_0 = 0$, we choose a sufficiently small constant $\varepsilon \in (0, y_*)$ and consider the following problem

$$\frac{dz}{dt} + \alpha z^k + \beta z = \gamma\, z^m, \quad t \ge 0; \quad z(0) = \varepsilon > 0; \quad z(t) > 0, \quad t > 0. \tag{2.4}$$

Then problem (2.4) exists at least one non-constant solution $z = z(t)$ satisfying $z'(t) > 0$ for all $t \in \mathbb{R}$. We continue the proof based on the following claim: there is a time $t_0 \in (-\infty, 0)$, such that $z(t_0) = 0$. By setting $y(t) = z(t + t_0)$, $\forall\ t \geq 0$, we get that $y(t)$ is a non-constant solution satisfying (2.3).

We now only need to prove the above mentioned claim. Indeed, if it is not true, then $0 < z(t) < \varepsilon$ for all $t \in (-\infty, 0)$. Since $0 < m < k < 1$ and $z'(t) > 0$ for all $t \in \mathbb{R}$, there is a $t_1 \in (-\infty, 0)$ so that $\alpha z^k + \beta z \leq \frac{\gamma}{2} z^m$ for all $t \in (-\infty, t_1]$, i.e.,

$$\frac{dz}{dt} \geq \frac{\gamma}{2} z^m \quad \text{for all } t \in (-\infty, t_1].$$

Integrating the above inequality on (t, t_1), we get

$$z^{1-m}(t_1) - z^{1-m}(t) > \frac{\gamma}{2}(1-m)(t_1 - t),$$

which causes a contradiction as $t \to -\infty$.

Lemma 5 [4] *(Gagliardo-Nirenberg) Let $\beta \geq 0$, $N > p \geq 1$, $\beta + 1 \leq q$, and $1 \leq r \leq q \leq (\beta+1)Np/(N-p)$, then for u such that $|u|^\beta u \in W^{1,p}(\Omega)$, we have*

$$\|u\|_q \leq C \|u\|_r^{1-\theta} \left\| \nabla \left(|u|^\beta u \right) \right\|_p^{\theta/(\beta+1)}$$

with $\theta = (\beta+1)(r^{-1} - q^{-1})/\{N^{-1} - p^{-1} + (\beta+1)r^{-1}\}$, where C is a constant depending only on N, p and r.

3 The case $0 < p < q < 1$: proof of Theorem 1

Multiplying (1.1) by u and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + d\|\nabla u\|_2^2 = \int_\Omega u\, dx \int_\Omega u^q(y,t)\, dy - k\|u\|_{p+1}^{p+1}. \tag{3.1}$$

By Hölder inequality, we have

$$\int_\Omega u\, dx \int_\Omega u^q(y,t)\, dy \leq |\Omega|^{\frac{2s-1-q}{s}} \|u\|_s^{q+1}, \tag{3.2}$$

where $s \geq 1$ to be determined later. we substitute (3.2) into (3.1) to get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + d\|\nabla u\|_2^2 = |\Omega|^{\frac{2s-1-q}{s}} \|u\|_s^{q+1} - k\|u\|_{p+1}^{p+1}. \tag{3.3}$$

1) For the case $N < 4(q-p)/[(1-p)(1-q)]$, we set $s = 2$ in (3.3). By lemma 5, one can get

$$\|u\|_2 \leq C_1(N,p)\|u\|_{p+1}^{1-\theta_1} \|\nabla u\|_2^{\theta_1}, \tag{3.4}$$

where

$$\theta_1 = \left(\frac{1}{p+2} - \frac{1}{2}\right)\left(\frac{1}{N} - \frac{1}{2} + \frac{1}{p+1}\right)^{-1} = \frac{N(1-p)}{2(p+1) + N(1-p)}.$$

$0 < p < 1$ implies that $0 < \theta_1 < 1$. It follows from (3.4) and Young's inequality that

$$\begin{aligned} \|u\|_2^{\theta_2} \leq & C_1(N,p)^{\theta_2} \|u\|_{p+1}^{(1-\theta_1)\theta_2} \|\nabla u\|_2^{\theta_1\theta_2} \\ \leq & C_1(N,p)^{\theta_2} \left(\varepsilon_1 \|\nabla u\|_2^2 + C(\varepsilon_1) \|u\|_{p+1}^{2(1-\theta_1)\theta_2/(2-\theta_1\theta_2)} \right), \end{aligned} \tag{3.5}$$

for $\varepsilon_1 > 0$ and $\theta_2 > 1$ to be determined. We choose $\theta_2 = \frac{2N(1-p)+4(1+p)}{N(1-p)+4}$, then $1 < \theta_2 < 2$ and $2(1-\theta_1)\theta_2/(2-\theta_1\theta_2) = p+1$. Thus, (3.5) becomes

$$\frac{C_1(N,p)^{-\theta_2}}{C(\varepsilon_1)} \|u\|_2^{\theta_2} - \frac{\varepsilon_1}{C(\varepsilon_1)} \|\nabla u\|_2^2 \leq \|u\|_{p+1}^{p+1}. \tag{3.6}$$

We substitute (3.6) into (3.3) to get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \left(d - \frac{k\varepsilon_1}{C(\varepsilon_1)} \right) \|\nabla u\|_2^2 + \frac{kC_1(N,p)^{-\theta_2}}{C(\varepsilon_1)} \|u\|_2^{\theta_2} \leq |\Omega|^{\frac{3-q}{2}} \|u\|_2^{q+1}.$$

We choose ε_1 small enough such that $d_1 := d - \frac{k\varepsilon_1}{C(\varepsilon_1)} > 0$. Once ε_1 is fixed, we set $k_1 = \frac{kC_1(N,p)^{-\theta_2}}{C(\varepsilon_1)}$. Then, by Poincare's inequality, we get

$$\frac{d}{dt}\|u\|_2 + k_1 \|u\|_2^{\theta_2 - 1} + d_1\lambda_1 \|u\|_2 \leq |\Omega|^{\frac{3-q}{2}} \|u\|_2^q. \tag{3.7}$$

Since $N < 4(q-p)/[(1-p)(1-q)]$, we further have $0 < \theta_2 - 1 < q$. By Lemma 3, there exists $\alpha_1 > d_1\lambda_1$, such that

$$0 \leq \|u\|_2 \leq \|u_0\|_2 \, e^{-\alpha_1 t}, \; t \geq 0,$$

provided that

$$\|u_0\|_2 < \left(\frac{k_1}{|\Omega|^{\frac{3-q}{2}}} \right)^{\frac{1}{q-\theta_2+1}} = \left(\frac{kC_1(N,p)^{-\theta_2}}{C(\varepsilon_1)|\Omega|^{\frac{3-q}{2}}} \right)^{\frac{1}{q-\theta_2+1}}. \tag{3.8}$$

Furthermore, there exists T_1, such that

$$\begin{aligned} & k_1 - |\Omega|^{\frac{3-q}{2}} \|u\|_2^{q-\theta_2+1} \\ \geq & k_1 - |\Omega|^{\frac{3-q}{2}} \left(\|u_0\|_2 \, e^{-\alpha_1 T_1} \right)^{q-\theta_2+1} := k_2 > 0, \end{aligned} \tag{3.9}$$

holds for $t \in [T_1, +\infty)$. Therefore, when $t \in [T_1, +\infty)$, (3.7) turns to

$$\frac{d}{dt}\|u\|_2 + k_2 \|u\|_2^{\theta_2 - 1} + d_1\lambda_1 \|u\|_2 \leq 0. \tag{3.10}$$

By Lemma 2, we can obtain the desired decay estimate for

$$T_1^* = \frac{1}{(2-\theta_2)d_1\lambda_1} \ln\left(1 + \frac{d_1\lambda_1}{k_2} ||u(\cdot, T_1)||_2^{2-\theta_2} \right). \tag{3.11}$$

2) When $N = 4(q-p)/[(1-p)(1-q)]$, we still choose $s = 2$ in (3.3), and then $\theta_2 - 1 = q$. Thus, (3.7) becomes

$$\frac{d}{dt}\|u\|_2 + \left(k_1 - |\Omega|^{\frac{3-q}{2}} \right) \|u\|_2^q + d_1\lambda_1 \|u\|_2 \leq 0. \tag{3.12}$$

By Lemma 2, we can obtain the desired decay estimate for

$$T_2^* = \frac{1}{(1-q)d_1\lambda_1} \ln \left(1 + \frac{d_1\lambda_1}{k_1 - |\Omega|^{\frac{3-q}{2}}} ||u_0||_2^{1-q} \right), \tag{3.13}$$

provided that $|\Omega| < k_1^{\frac{2}{3-q}} = \left(\frac{kC_1(N,p)^{-\theta_2}}{C(\varepsilon_1)}\right)^{\frac{2}{3-q}}$.

3) For the case $N > 4(q-p)/[(1-p)(1-q)]$, we back to (3.3). By lemma 5, one can get

$$\|u\|_s \le C_2(N,p)\|u\|_{p+1}^{1-\theta_3}\|\nabla u\|_2^{\theta_3}, \tag{3.14}$$

where

$$\theta_3 = \left(\frac{1}{p+1} - \frac{1}{s}\right)\left(\frac{1}{N} - \frac{1}{2} + \frac{1}{p+1}\right)^{-1} = \frac{2N(s-p-1)}{s[2(p+1)+N(1-p)]}.$$

If $N > 2$, one further needs $p+1 < s < 2N/(N-2)$. The choice of s implies that $0 < \theta_3 < 1$. It follows from (3.14) and Young's inequality that

$$\begin{aligned}\|u\|_s^{q+1} \le& C_2(N,p)^{q+1}\|u\|_{p+1}^{(1-\theta_3)(q+1)}\|\nabla u\|_2^{\theta_3(q+1)} \\ \le& C_2(N,p)^{q+1}\left(\varepsilon_2\|\nabla u\|_2^2 + C(\varepsilon_2)\|u\|_{p+1}^{2(1-\theta_3)(q+1)/[2-\theta_3(q+1)]}\right),\end{aligned} \tag{3.15}$$

for $\varepsilon_2 > 0$ to be determined later. We choose $s = \frac{N(q+1)(1-p)}{N(1-p)-2(q-p)}$, then $\theta_3 = \frac{2(q-p)}{(q+1)(1-p)}$ and $2(1-\theta_3)(q+1)/[2-\theta_3(q+1)] = p+1$. We substitute (3.15) into (3.3) to get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \left(d - \varepsilon_2 C_2(N,p)^{q+1}|\Omega|^{\frac{2s-1-q}{s}}\right)\|\nabla u\|_2^2 + \left(k - C(\varepsilon_2)C_2(N,p)^{q+1}|\Omega|^{\frac{2s-1-q}{s}}\right)\|u\|_{p+1}^{p+1} \le 0.$$

We choose ε_2 small enough such that $d_2 := d - \varepsilon_2 C_2(N,p)^{q+1}|\Omega|^{\frac{2s-1-q}{s}} > 0$. Once ε_2 is fixed, we set $k_0 = C(\varepsilon_2)C_2(N,p)^{q+1}|\Omega|^{\frac{2s-1-q}{s}}$. When $k > k_0 = C(\varepsilon_2)C_2(N,p)^{q+1}|\Omega|^{\frac{2s-1-q}{s}}$, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + d_2\|\nabla u\|_2^2 + (k-k_0)\|u\|_{p+1}^{p+1} \le 0. \tag{3.16}$$

We note (3.6) holds provided that $0 < q < 1$ and is independent of the relation of N and $4(q-p)/[(1-p)(1-q)]$. So, we substitute (3.6) into (3.16) to get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \left(d_2 - \frac{(k-k_0)\varepsilon_1}{C(\varepsilon_1)}\right)\|\nabla u\|_2^2 + \frac{(k-k_0)C_1(N,p)^{-\theta_2}}{C(\varepsilon_1)}\|u\|_2^{\theta_2} \le 0.$$

We recall that $\theta_2 = \frac{2N(1-p)+4(1+p)}{N(1-p)+4} \in (1,2)$. We choose ε_1 small enough such that $d_3 := d_2 - \frac{(k-k_0)\varepsilon_1}{C(\varepsilon_1)} > 0$. Once ε_1 is fixed, we set $k_3 = \frac{(k-k_0)C_1(N,p)^{-\theta_2}}{C(\varepsilon_1)}$. Thus, we get

$$\frac{d}{dt}\|u\|_2 + k_3\|u\|_2^{\theta_2-1} + d_3\lambda_1\|u\|_2 \le 0.$$

By Lemma 2, we can obtain the desired decay estimate for

$$T_3^* = \frac{1}{(2-\theta_2)d_1\lambda_1}\ln\left(1 + \frac{d_3\lambda_1}{k_3}\|u_0\|_2^{2-\theta_2}\right). \tag{3.17}$$

4 The case $0 < p = q < 1$: proof of Theorem 2

In this section, we consider the case $0 < p = q < 1$.

1) If $k > |\Omega|$, we choose $s = p + 1$ in (3.3) to get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + d\|\nabla u\|_2^2 + (k - |\Omega|)\|u\|_{p+1}^{p+1} \leq 0. \tag{4.1}$$

We substitute (3.6) into (4.1) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \left(d - \frac{(k-|\Omega|)\varepsilon_1}{C(\varepsilon_1)}\right)\|\nabla u\|_2^2 + \frac{(k-|\Omega|)C_1(N,p)^{-\theta_2}}{C(\varepsilon_1)}\|u\|_2^{\theta_2} \leq 0.$$

We choose ε_1 small enough such that $d_4 := d - \frac{(k-|\Omega|)\varepsilon_1}{C(\varepsilon_1)} > 0$. Once ε_1 is fixed, we set $k_4 = \frac{(k-|\Omega|)C_1(N,p)^{-\theta_2}}{C(\varepsilon_1)}$. Then, by Poincare inequality, we get

$$\frac{d}{dt}\|u\|_2 + k_4\|u\|_2^{\theta_2-1} + d_4\lambda_1\|u\|_2 \leq 0. \tag{4.2}$$

By Lemma 2, we can obtain the desired decay estimate for

$$T_1^* = \frac{1}{(2-\theta_2)d_4\lambda_1}\ln\left(1 + \frac{d_4\lambda_1}{k_4}||u_0||_2^{2-\theta_2}\right). \tag{4.3}$$

2) If $k < \int_\Omega \psi^q(x)dx/M^q$, we define

$$g(t) = \left[\frac{\int_\Omega \psi^q(x)dx - kM^q}{d}\left(1 - e^{-(1-q)\frac{d}{M}t}\right)\right]^{\frac{1}{1-q}},$$

which satisfies the ODE problem

$$g'(t) + \frac{d}{M}g(t) = \frac{\int_\Omega \psi^q(x)dx - kM^q}{M}g^q(t),\ t \geq 0; \quad g(0) = 0.$$

Let $v(x,t) = g(t)\psi(x)$. Then, we have

$$\begin{aligned}
&v_t - d\Delta v - \int_\Omega v^q(x,t)\,dx + kv^p \\
=&g'(t)\psi(x) + dg(t) - g^q(t)\int_\Omega \psi^q\,dx + kg^q(t)\psi^q \\
\leq&g'(t)M + dg(t) - g^q(t)\int_\Omega \psi^q\,dx + kg^q(t)M^q \\
=&0.
\end{aligned}$$

Moreover, $v(x,0) = g(0)\psi(x) = 0 \leq u_0(x)$ in Ω, and $v|_{(\partial\Omega)_t} = 0$. Therefore, we have $u(x,t) \geq v(x,t) > 0$ in $\Omega \times (0,+\infty)$; i.e., $v(x,t)$ is a non-extinction subsolution of problem (1.1)–(1.3).

3) For $k = \int_\Omega \psi^q(x)dx/M^q$, let $w(x,t) = h(t)\psi(x)$, where $h(t)$ satisfies the ODE problem

$$\frac{dh}{dt} + \frac{d}{M}h = 0, \quad t \geq 0; \quad h(0) = h_0 > 0.$$

Then, for any identically positive initial data, we can choose h_0 sufficiently small such that $h_0\psi(x) \leq u_0(x)$. According to Lemma 1, we get that $w(x,t)$ is a non-extinction subsolution of problem (1.1)–(1.3).

5 The case $0 < q < p < 1$: proof of Theorem 3

Let $z(x,t) = j(t)\psi(x)$, where $j(t)$ satisfies the ODE problem

$$\frac{dj}{dt} + kM^{p-1}j^p(t) + \frac{d}{M}j(t) = \frac{\int_\Omega \psi^q(x)dx}{M}j^q(t), \quad t \geq 0; \quad j(0) = 0; \quad j(t) > 0, \quad t > 0.$$

Then, we have

$$\begin{aligned} & z_t - d\Delta z - \int_\Omega z^q(x,t)\,dx + kz^p \\ =& j'(t)\psi(x) + dj(t) - j^q(t)\int_\Omega \psi^q\,dx + kj^p(t)\psi^p \\ \leq& j'(t)M + dj(t) - j^q(t)\int_\Omega \psi^q\,dx + kj^p(t)M^p \\ =& 0. \end{aligned}$$

Moreover, $z(x,0) = j(0)\psi(x) = 0 \leq u_0(x)$ in Ω, and $\upsilon|_{(\partial\Omega)_t} = 0$. Therefore, we have $u(x,t) \geq z(x,t) > 0$ in $\Omega \times (0,+\infty)$ according to Lemma 1, i.e., $z(x,t)$ is a non-extinction subsolution of problem (1.1)–(1.3).

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References

[1] M. Aassila, The influence of nonlocal nonlinearities on the long time behavior of solutions of diffusion problems, J. Differential Equations **192** (2003), no. 1, 47–69.

[2] J. Bebernes and D. Eberly, *Mathematical Problems From Combustion Theory*, Springer, New York, 1989.

[3] J. Bebernes and A. Bressan, Thermal behavior for a confined reactive gas, J. Differential Equations **44** (1982), no. 1, 118–133.

[4] C. S. Chen, R. Y. Wang, L^∞ estimates of solution for the evolution m-Laplacian equation with initial value in $L^q(\Omega)$, Nonlinear Anal. **48** (2002), no. 4, 607–616.

[5] S. L. Chen, The extinction behavior of the solutions for a class of reaction-diffusion equations, Appl. Math. Mech. (English Ed.) **22** (2001), no. 11, 1352–1356.

[6] W. A. Day, Extensions of a property of the heat equation to linear thermoelasticity and other theories, Quart. Appl. Math. **40** (1982/83), no. 3, 319–330.

[7] W. A. Day, A decreasing property of solutions of parabolic equations with applications to thermoelasticity, Quart. Appl. Math. **40** (1982/83), no. 4, 468–475.

[8] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, New York, 1993.

[9] L. C. Evans and B. F. Knerr, Instantaneous shrinking of the support of nonnegative solutions to certain nonlinear parabolic equations and variational inequalities, Illinois J. Math. **23** (1979), no. 1, 153–166.

[10] I. Fukuda, Extinction and Growing-up of Solutions of Some Nonlinear Parabolic Equations, Transactions of the Kokushikan Univ. Faculty of Engineering, 20 (1987), 1-11.

[11] Y. G. Gu, Necessary and sufficient conditions for extinction of solutions to parabolic equations, Acta Math. Sinica **37** (1994), no. 1, 73–79. (In Chinese)

[12] Y. X. Li and C. H. Xie, Blow-up for p-Laplacian parabolic equations, Electron. J. Differential Equations **2003**, No. 20, 12 pp.

[13] Q. L. Liu, Y. X. Li, H. J. Gao, Uniform blow-up rate for diffusion equations with nonlocal nonlinear source, Nonlinear Anal. **67** (2007), no. 6, 1947–1957.

[14] Q. L. Liu, C. H. Xie, S. L. Chen, Global blowup and blowup rate estimates of solutions for a class of nonlinear non-local reaction-diffusion problems, Acta Math. Sci. Ser. B Engl. Ed. **24** (2004), no. 2, 259–264.

[15] W. J. Liu, M. X. Wang, B. Wu, Extinction and decay estimates of solutions for a class of porous medium equations, J. Inequal. Appl. **2007**, Art. ID 87650, 8 pp.

[16] W. J. Liu, B. Wu, A note on extinction for fast diffusive p-Laplacian with sources, Math. Methods Appl. Sci. **31** (2008), no. 12, 1383–1386.

[17] W. J. Liu, Extinction properties of Solutions for a class of fast diffusive p-Laplacian Equation, preprint.

[18] C. V. Pao, *Nonlinear parabolic and elliptic equations*, Plenum, New York, 1992.

[19] C. Peng and Z. Yang, Blow-up for a degenerate parabolic equation with a nonlocal source, Appl. Math. Comput. **201** (2008), no. 1-2, 250–259.

[20] P. Souplet, Blow-up in nonlocal reaction-diffusion equations, SIAM J. Math. Anal. **29** (1998), no. 6, 1301–1334.

[21] P. Souplet, Uniform blow-up profiles and boundary behavior for diffusion equations with nonlocal nonlinear source, J. Differential Equations **153** (1999), no. 2, 374–406.

[22] Y. Tian and C. L. Mu, Extinction and non-extinction for a p-Laplacian equation with nonlinear source, Nonlinear Anal. **69** (2008), no. 8, 2422–2431.

[23] M. Wang and Y. Wang, Properties of positive solutions for non-local reaction-diffusion problems, Math. Methods Appl. Sci. **19** (1996), no. 14, 1141–1156.

[24] L. Yan and C. L. Mu, Extinction of solutions for a nonlinear parabolic equation, Sichuan Daxue Xuebao **43** (2006), no. 3, 514–516. (In Chinese).

[25] J. X. Yin and C. H. Jin, Critical extinction and blow-up exponents for fast diffusive p-Laplacian with sources, Math. Methods Appl. Sci. **30** (2007), no. 10, 1147–1167.

[26] S. Zheng and L. Wang, Blow-up rate and profile for a degenerate parabolic system coupled via nonlocal sources, Comput. Math. Appl. **52** (2006), no. 10-11, 1387–1402.

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