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Schwarz symmetric solutions for a quasilinear eigenvalue problem

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Abstract. In the present paper we extend some recent results of R. Filippucci, P. Pucci and Cs. Varga to continuous functionals. As an application we prove the existence of at least three different solutions of a quasilinear eigenvalue problem, for every λ in some interval, which solutions are invariants by Schwarz symmetrization.

Keywords: nonlinear eigenvalue problem, multiple solutions, nonsmooth functional, symmetrization.

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1 Introduction

There is a rich literature on the study of symmetric solutions of the PDE's. A very important paper is due to Gidas, Ni, Nirenberg [9], where they prove symmetry, and some related properties, of positive solutions of second order elliptic equations. Their methods employ various forms of the maximum principle, and a device of moving parallel planes to a critical position. After this work appeared many papers where the solutions of PDE's have different symmetries for e.g. radial symmetry (see e.g. Pacella, Salazar [8], Squassina [11]), axial symmetry or have some symmetry properties with respect to certain group actions.

In articles [13, 14] Van Shaftingen developed an abstract method for the study of symmetrization. In [15] Van Shaftingen and Willem study different symmetry properties (spherical cap, Schwarz, polarization) of least energy positive or nodal solutions of semilinear elliptic problems with Dirichlet or Neumann boundary conditions.

Filippucci, Pucci, Varga in [6] using the symmetric version of Ekeland's variational principle (Van Schaftingen [13]) and the symmetric Mountain Pass theorem (Squassina [10]) establish the existence of two nontrivial (weak) solutions of abstract eigenvalue problems. In order to show the existence of three different symmetric solutions of an abstract eigenvalue problem, they prove a symmetric version of the Pucci and Serrin three critical points theorem. Then, as a consequence of the main results, they show the existence of two nontrivial nonnegative solutions of quasilinear elliptic Dirichlet problems either in a ball of $\mathbb{R}^N$, or in

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an annulus of $\mathbb{R}^N$, both centered at 0. The obtained solutions are invariant by k-spherical cap symmetrization ($1 < k < N$).

In the present paper we extend some of these results to continuous functionals. More precisely, let Ω be a ball in $\mathbb{R}^N$ ($N \geq 3$) and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, which satisfies a natural growth condition, given in Section 2.

Consider the following quasilinear elliptic eigenvalue problem

$$-\sum_{i,j=1}^{N} D_j(a_{ij}(u)D_iu) + \frac{1}{2}\sum_{i,j=1}^{n} a'_{ij}(u)D_iuD_ju = \lambda f(x,u), \tag{1.1}$$

where $\lambda > 0$ is a real parameter, $a_{ij} : \mathbb{R} \to \mathbb{R}$ is of class C^1 with $a_{ij}(x) = a_{ji}(x)$ and by D_i we denote the partial derivative with respect to x_i.

We also assume that there exist $C, \nu > 0$ such that for and all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$ we have

(a_1) $|a_{ij}(s)| \leq C$;

(a_2) $|a'_{ij}(s)| \leq C$;

(a_3) $\sum_{i,j=1}^{N} a_{ij}(s)\xi_i\xi_j \geq \nu|\xi|^2$,

where $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^N$.

Let $\mathcal{E}_\lambda : H_0^1(\Omega) \to \mathbb{R}$,

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{N} a_{ij}(u)D_iuD_judx - \lambda \int_\Omega F(x,u)dx, \tag{1.2}$$

be the corresponding energy functional, where $F(x,t) = \int_0^t f(x,s)ds$.

Under the above conditions the energy functional is continuous. However, we cannot expect that $\mathcal{E}_\lambda$ to be of class C^1 or even locally Lipschitz continuous, so, the classical critical point theory cannot be applied. To overcome this difficulty, we define the derivative of the function only in some special direction. Such techniques has been used for quasilinear problems by several authors (see e.g. Canino [1]; Liu, Guo [3] and the references therein).

The aim of our paper is to prove the existence of at least three different solutions of the quasilinear eigenvalue problem (1.1) for every λ in some interval. Moreover, we prove that the obtained solutions are symmetric invariants by Schwarz symmetrization. A comprehensive survey of results about existence, multiplicity, perturbation from symmetry and concentration phenomena for the quasilinear elliptic equations can be found in the monograph of Squassina [12].

Our paper is organized as follows. In Section 2 we present the necessary symmetrization tools. We begin with the abstract framework of symmetrization following Van Schaftingen [13] and in Section 2.2 we obtain symmetric critical point results for E-differentiable continuous functions $f : X \to \mathbb{R}$, where E is a dense subspace of X.

In order to demonstrate the main results of the present paper, in Section 3 we study first an abstract eigenvalue problem

$$J'(u) = \lambda\mathcal{F}(u), \qquad \mathcal{F}(u) = \int_\Omega F(x,u(x))dx,$$

where $\lambda > 0$ is a real parameter, and we give some information about the symmetry of solutions, when the underlying domain is a ball of $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which satisfies natural growth conditions given in Section 3. In Theorem 3.1 we guarantee three symmetric invariant critical points for the continuous functional f. Using the aforementioned theorem, in Section 4 we justify the existence of three different, Schwartz symmetric critical point of the E-differentiable energy functional $\mathcal{E}_\lambda$ in (1.2).

2 Auxiliary results

2.1 Abstract framework of symmetrization

In this subsection we recall some symmetrizations notions from Van Schaftingen [13] and Squassina [10].

Let us begin with some notion of symmetrizations.

Definition 2.1. The Schwarz symmetrization of a set $A \subset \mathbb{R}^N$ is the unique open ball centered at the origin A^*, such that $\mathcal{L}^N(A^*) = \mathcal{L}^N(A)$, where $\mathcal{L}^N$ denotes the N-dimensional outer Lebesgue measure.

If $\mathcal{L}^N(A) = 0$, then $A^* = \emptyset$, while $A^* = \mathbb{R}^N$, if $\mathcal{L}^N(A) = \infty$.

Definition 2.2. Let $f : A \to \overline{\mathbb{R}}$ a function and $c \in \overline{\mathbb{R}}$. Then we define the following set

$$\{f > c\} = \{x \in A \mid f(x) > c\}.$$

The Schwarz symmetrization of a measurable nonegative function $f : A \to \overline{\mathbb{R}}$ $(A \subset \mathbb{R}^N)$ is the unique function $f^* : A^* \to \overline{\mathbb{R}}$ such that

$$\{f^* > c\} = \{f > c\}^*, \quad \text{for all } c \in \mathbb{R}.$$

Remark 2.3. The function f^* is also characterized by

$$f^*(y) = \sup\{c \in \mathbb{R} : y \in \{f > c\}^*\}.$$

Definition 2.4 (Polarization)**.** A subset H of $\mathbb{R}^N$ is called a polarizer if it is a closed affine half-space of $\mathbb{R}^N$, namely the set of points x which satisfy $\alpha \cdot x \leq \beta$ for some $\alpha \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$ with $|\alpha| = 1$.

Given x in $\mathbb{R}^N$ and a polarizer H the reflection of x with respect to the boundary of H is denoted by x_H.

The polarization of a function $u : \mathbb{R}^N \to \mathbb{R}^+$ by a polarizer H is the function $u^H : \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$u^H(x) = \begin{cases} \max\{u(x), u(x_H)\}, & \text{if } x \in H \\ \min\{u(x), u(x_H)\}, & \text{if } x \in \mathbb{R}^N \setminus H. \end{cases} \tag{2.1}$$

The polarization $C^H \subset \mathbb{R}^N$ of a set $C \subset \mathbb{R}^N$ is defined as the unique set which satisfies $\chi_{C^H} = (\chi_C)^H$, where χ denotes the characteristic function. The polarization u^H of a positive function u defined on $C \subset \mathbb{R}^N$ is the restriction to C^H of the polarization of the extension $\tilde{u} : \mathbb{R}^N \to \mathbb{R}^+$ of u by zero outside C. The polarization of a function which may change sign is defined by $u^H := |u|^H$, for any given polarizer H.

In the following we present some crucial abstract symmetrization and polarization results. We begin with the following main assumption.

Let X and V be two real Banach spaces, with $X \subset V$ and let $S \subseteq X$.

Main assumptions. *Let $\mathcal{H}_\star$ be a path-connected topological space and denote by $h : S \times \mathcal{H}_\star \to S$, $(u, H) \mapsto u^H$, the polarization map. Let $\star : S \to S, u \mapsto u^\star$, be any symmetrization map. Assume that the following properties hold.*

1) *The embedding $X \hookrightarrow V$ is continuous;*
2) *h is continuous;*
3) *$(u^\star)^H = (u^H)^\star = u^\star$ and $(u^H)^H = u^H$ for all $u \in S$ and $H \in \mathcal{H}_\star$;*
4) *for all $u \in S$ there exists a sequence $(H)_m \subset \mathcal{H}_\star$ such that $u^{H_1 \dots H_m} \to u^\star$ in V, as $m \to \infty$, where $u^{H_1 \dots H_m} = (((u^{H_1})^{H_2})^{\dots})^{H_m}$;*
5) *$\|u^H - v^H\|_V \le \|u - v\|_V$ for all $u, v \in S$ and $H \in \mathcal{H}_\star$.*

We assume that there exists a Lipschitz continuous map $\Theta : (X, \|\cdot\|_V) \to (S, \|\cdot\|_V)$ with Lipschitz constant $C_\Theta > 0$ such that $\Theta|_S = \mathrm{Id}|_S$ and both maps $h : S \times \mathcal{H}_\star \to S$ and $\star : S \to S$ can be extended to $h : X \times \mathcal{H}_\star \to S$ and $\star : X \to V$ by setting $u = (\Theta(u))^H$ and $u^\star = (\Theta(u))^\star$ for every $u \in X$ and $H \in \mathcal{H}_\star$.

The previous properties, in particular 4) and 5), and the definition of Θ easily yield that

$$\|u^H - v^H\|_V \le C_\Theta \|u - v\|_V, \qquad \|u^\star - v^\star\|_V \le C_\Theta \|u - v\|_V \tag{2.2}$$

for all $u, v \in X$ and for all $H \in \mathcal{H}_\star$.

Now, we describe the set $\mathcal{H}_*$ for the Schwarz symmetrization following the papers [6, 13, 14].

Consider the set of polarizers

$$\mathcal{H} = \{H \text{ closed half space in } \mathbb{R}^N : 0 \in H\}. \tag{2.3}$$

We endow the set $\mathcal{H}$ with a topology. To this aim let $i : \mathbb{R}^N \to \mathbb{R}^N$ be an isometry, that is, $|i(x) - i(y)| = |x - y|$, for every $x, y \in \mathbb{R}^N$ and let $\mathcal{I}$ be the set of isometries on $\mathbb{R}^N$, that is

$$\mathcal{I} = \{i : \mathbb{R}^N \to \mathbb{R}^N : i \text{ is an isometry on } \mathbb{R}^N\}.$$

Definition 2.5. Let $H_1, H_2 \in \mathcal{H}$ and

$$\rho(H_1, H_2) = \inf_{i \in \mathcal{I}} \left\{ \log \left(1 + \sup_{x \in \mathbb{R}^N} \frac{|x - i(x)|}{1 + |x|} + \sup_{x \in i(H_1) \Delta H_2} \frac{1}{1 + |x|} \right) \right\},$$

where Δ is the symmetric difference between two sets.

The distance between H_1 and H_2 is defined by

$$\mathrm{d}(H_1, H_2) = \rho(H_1, H_2) + \rho(H_2, H_1).$$

Remark 2.6. The metric space $(\mathcal{H}, \mathrm{d})$ is a separable, locally compact by Proposition 2.36 of [14].

For any fixed function $u : \mathbb{R}^N \to \mathbb{R}$, we put $u_+ = \max\{u, 0\}$ and $u_- = \min\{u, 0\}$.

We recall now, the notion of extended polarizer as given in Definition 2.17 of [13].

Definition 2.7 (Extended polarizer)**.** The set of polarizers $\mathcal{H}$ is compactified by an addition of two polarizers at infinity defined by $u^{H_{+\infty}} = u_+$ and $u^{H_{-\infty}} = -u_-$, such that $H_n \to H_{+\infty}$ if $\beta_n \to \infty$ and $H_n \to H_{-\infty}$ if $\beta_n \to -\infty$ in the representation of Definition 2.4. The compactified set of polarizers is denoted by $\mathcal{H}^* = \mathcal{H} \cup \{H_{+\infty}, H_{-\infty}\}$ and is homeomorphic with S^N.

Therefore, the set $\mathcal{H}_* = \mathcal{H} \cup \{H_{+\infty}\}$ is homeomorphic with $S^N \setminus \{\text{a point}\}$, which is homeomorphic to $\mathbb{R}^N$. In conclusion, the space $\mathcal{H}_*$ is a path-connected locally compact topological space and so $\mathcal{H}_*$ satisfies all the properties required in the Main assumptions and will be used throughout the paper.

We recall here examples relative to the Schwarz symmetrization proved by Van Schaftingen in [13].

Example 2.8 (Schwarz symmetrization for non-negative functions)**.** Let $\Omega = B(0,1) \subset \mathbb{R}^N$, $X = W_0^{1,p}(\Omega)$, with $1 < p < N$, $V = L^p \cap L^{p^*}(\Omega)$, with $p^\star = Np/(N-p)$, S be the set of non-negative functions of $W_0^{1,p}(\Omega)$, $*$ denotes the Schwarz symmetrization and $\mathcal{H}_*$ be defined as above. Then the assumptions stated in the Main assumptions are satisfied, see [13].

Example 2.9 (Schwarz symmetrization)**.** Let $\Omega = B(0,1) \subset \mathbb{R}^N$, $X = W_0^{1,p}(\Omega)$, with $1 < p < N$, $V = L^p \cap L^{p^*}(\Omega)$, with $p^\star = Np/(N-p)$, $S = W_0^{1,p}(\Omega)$, $u* = |u|^*$ denotes the Schwarz symmetrization and $\mathcal{H}_*$ be defined as above for Schwarz symmetryzation, but $h(u,H) = |u|^H$ in the Main assumptions. Then all the assumptions stated in the Main assumptions are satisfied, see [13]. In this case $\Theta(u) = \max\{0,u\}$.

2.2 Symmetric critical point results for E-differentiable functions

First, we recall from Guo, Liu [3] some notions and results of nonsmooth critical point theory.

Let X be a Banach space and E be a dense subspace of X. Let $f : X \to \mathbb{R}$ be a continuous functional.

Definition 2.10. A continuous functional f is said to be E-differentiable if

(1) for all $u \in X$ and $\varphi \in E$ the derivative of f in direction φ at u exists and will be denoted by $\langle Df(u), \varphi \rangle$:

$$\langle Df(u), \varphi \rangle = \lim_{t \to 0^+} \frac{f(u+t\varphi) - f(u)}{t};$$

(2) the map $(u, \varphi) \mapsto \langle Df(u), \varphi \rangle$ satisfies:

  (i) $\langle Df(u), \varphi \rangle$ is linear in $\varphi \in E$,

  (ii) $\langle Df(u), \varphi \rangle$ is continuous in u, that is, if a sequence $u_n \to u$ in X, then $\langle Df(u_n), \varphi \rangle \to \langle Df(u), \varphi \rangle$, as $n \to \infty$.

Definition 2.11. The slope of an E-differentiable functional f at $u \in X$, denoted by $|Df(u)|$, is a generalized number in $[0, \infty]$:

$$|Df(u)| = \sup\{\langle Df(u), \varphi \rangle \mid \varphi \in E,\ \|\varphi\| = 1\}.$$

A point $u \in X$ is said to be a critical point of f at level c, if $|Df(u)| = 0$ and $f(u) = c$.

Definition 2.12. Let c be a real number. We say that an E-differentiable functional f satisfies the concrete Palais–Smale condition at level c (shortly $(CPS)_c$) if every sequence $\{u_n\} \subset X$ satisfying $|Df(u_n)| \to 0$ and $f(u_n) \to c$, possesses a strongly convergent subsequence in X.

In the following we recall the notion of the weak slope of a continuous functional from the paper of Canino [1].

Definition 2.13. Let $f : X \to \mathbb{R}$ be a continuous functional and let $u \in X$. We denote by $|df|(u)$ the supremum of the σ's in $[0, \infty[$ such that there exist $\delta > 0$ and a continuous map

$$G : B(u, \delta) \times [0, \delta] \to X$$

such that

$$\forall v \in B(u, \delta),\ \forall t \in [0, \delta] : \|G(v, t) - v\| \leq t,$$
$$\forall v \in B(u, \delta),\ \forall t \in [0, \delta] : f(G(v, t)) \leq f(v) - \sigma t.$$

The extended number $|df|(u)$ is called the weak slope of f at u.

We prove the next important lemma, which is used several times in the following.

Lemma 2.14. *Let f be an E-differentiable functional. Then for every $u \in X$ we have*

$$|df|(u) \geq |Df(u)|.$$

Proof. Let $u \in X$ be fixed.

Case I. If $|Df(u)| = 0$, then the assertion is true.

Case II. Otherwise, if $|Df(u)| > 0$, we can consider an $\varepsilon > 0$ such that $0 < \sigma = |Df(u)| - \varepsilon$. Then by the definition of the slope, there exists $v \in E$, with $\|v\| = 1$ and

$$\sigma < \langle Df(u), v \rangle.$$

Since $\langle Df(u), v \rangle$ is continuous in u, we can choose a $\tilde{\delta} > 0$ such that for every $w \in B(u, \tilde{\delta})$ we have

$$\sigma < \langle Df(w), v \rangle. \tag{2.4}$$

For $\delta = \tilde{\delta}/2$ we define the following continuous map

$$G : B(u, \delta) \times [0, \delta] \to X, \qquad G(w, t) = w - tv.$$

It is trivial that $\|G(w, t) - w\| = t$. On the other hand, since $\langle Df(u), v \rangle$ is linear in v, by (2.4) we have that

$$-\sigma > \langle Df(w), -v \rangle = \lim_{t \searrow 0} \frac{f(w - tv) - f(w)}{t}.$$

After a rearrangement we obtain

$$\lim_{t \searrow 0} \frac{f(w - tv) - f(w) + \sigma t}{t} < 0,$$

which yields that taking a smaller δ if it is necessary, $f(G(w, t)) = f(w - tv) \leq f(w) - \sigma t$, for every $w \in B(u, \delta), \forall t \in [0, \delta]$. Then using the Definition 2.13, we obtain that $|df|(u) \geq \sigma = |Df(u)| - \varepsilon$, and the assertion follows by the arbitrariness of ε. □

Remark 2.15. Using Lemma 2.14, it is easy to verify that if f satisfies the $(CPS)_c$ condition, then f satisfies the $(PS)_c$ condition as well, for every real number c.

Using Lemma 2.14 and the Theorem 3.9 of Squassina in [11], we have the following lemma.

Lemma 2.16. *Let X be a complete metric space and $f : X \to \mathbb{R}$ a continuous functional. Let $\mathbb{D}$ and $\mathbb{S}$ denote the closed unit ball and sphere in $\mathbb{R}^N$, respectively, and $\Gamma_0 \subset C(\mathbb{S}, X)$. Let us define*

$$\Gamma = \{\gamma \in C(\mathbb{D}, X) : \gamma|_{\mathbb{S}} \in \Gamma_0\}.$$

Assume that

$$+\infty > c = \inf_{\gamma \in \Gamma} \sup_{\tau \in \mathbb{D}} f(\gamma(\tau)) > \sup_{\gamma_0 \in \Gamma_0} \sup_{\lambda \in \mathbb{S}} f(\gamma_0(\tau)) = a.$$

Then, for every $\varepsilon \in (0, (c-a)/2)$, every $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_{\tau \in \mathbb{D}} f(\gamma(\tau)) \leq c + \varepsilon,$$

there exists $u \in X$ such that

a) $c - 2\varepsilon \leq f(u) \leq c + 2\varepsilon$;

b) $\mathrm{dist}(u, \gamma(\mathbb{D}) \cap f^{-1}([c - 3\varepsilon, c + 3\varepsilon])) \leq 3\delta$;

c) $|Df(u)| \leq 3\varepsilon/\delta$.

Theorem 2.17 (Existence of a quasi-critical sequence)**.** *Let E be a dense subspace of X and f be a continuous E-differentiable functional defined on the Banach space X. We assume that f possesses two different local minima u_0 and u_1 in X. We define*

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\};$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)).$$

Then there exist a sequence $\{u_n\} \in X \setminus \{u_0, u_1\}$ such that

a) $f(u_n)$ *is finite;*

b) $\lim_{n \to \infty} |Df(u_n)| = 0.$

Proof. Since u_0 and u_1 are distinct local minima of f in X, there exists $r_0 > 0$, with $2r_0 < \|u_0 - u_1\|$, such that

$$f(u_i) \leq f(u), \quad \forall u \in B(u_i, r_0), \quad i = 0, 1.$$

We use the notations $f(u_0) = c_0$ and $f(u_1) = c_1$ and assume, without loss of generality, that $c_0 \geq c_1$. We distinguish two cases.

Case 1. We assume the existence of $r \in (0, r_0)$ such that

$$\inf_{u \in S_r} f(u) > c_0, \quad \text{where} \;\; S_r(u_0) = \{u \in X : \|u - u_0\| = r\}.$$

Since $\gamma(0) = u_0 \in B(u_0, r)$ and $\gamma(1) = u_1 \in X \setminus \overline{B(u_0, r)}$ we have $\gamma([0,1] \cap S_r(u_0)) \neq \emptyset$, for every $\gamma \in \Gamma$. Hence $c > c_0$. Now, we use the Lemma 2.16 with $a = c_0$. So, for every $n > \sqrt{\frac{c-a}{2}}$ and $\delta = \frac{3}{n}$, there exists $u_n \in X$ such that

a) $c - \frac{2}{n^2} \leq f(u_n) \leq c + \frac{2}{n^2}$;

b) $|Df(u_n)| \leq \frac{1}{n}$.

Taking the limits as $n \to \infty$ the assertions of the theorem follow immediately.

Case 2. Now, we have

$$\inf_{u \in S_r(u_0)} f(u) = c_0, \quad \text{for all } r \in (0, r_0).$$

Then for $r \in (0, r_0)$ and for every $n \in \mathbb{N}$, there exist a sequence $\{z_n\} \in X$ with

$$\|z_n - u_0\| = r, \qquad f(z_n) \le c_0 + \frac{1}{n^2}.$$

We fix now $r \in (0, r_0)$ and choose $n > 0$ such that $0 < r - \frac{2}{n} < r + \frac{2}{n} < r_0$. We assume

$$\mathcal{U} = \left\{ u \in X : r - \frac{2}{n} \le \|u - u_0\| \le r + \frac{2}{n} \right\}.$$

Then $\inf_{u \in \mathcal{U}} f(u) = c_0$. Now, we apply the Ekeland variational principle (Theorem 1.1 in I. Ekeland [2]) to $f|_{\mathcal{U}}$ and $u = z_n$. Then there exists a sequence $\{u_n\} \in \mathcal{U}$ such that

(i) $c_0 \le f(u_n) \le f(z_n) \le c_0 + \frac{1}{n^2}$;

(ii) $\|u_n - z_n\| \le \frac{1}{n}$;

(iii) $n[f(v) - f(u_n)] \ge -\|u_n - v\|$ for all $v \in \mathcal{U}$.

The first assertion follows from (i).

Using the relations (ii) and $\|z_n - u_0\| = r$, we get

$$\big| \|u_n - u_0\| - r \big| = \big| \|u_n - u_0\| - \|z_n - u_0\| \big| \le \|u_n - z_n\| \le \frac{1}{n}.$$

Then

$$r - \frac{2}{n} \le r - \frac{1}{n} \le \|u_n - u_0\| \le r + \frac{1}{n} \le r + \frac{2}{n},$$

which means that $u_n \in \operatorname{int} \mathcal{U}$.

Let w be any vector in E, with $||w|| = 1$, $t \in \mathbb{R}^+$ and put $v = u_n + tw$. Clearly if $t > 0$ is small enough, then $v \in \mathcal{U}$ and from (iii) it follows that

$$f(u_n + tw) - f(u_n) \ge -\frac{1}{n}\|w\| t = -\frac{1}{n} t.$$

Then

$$\langle Df(u_\varepsilon), w \rangle \ge -\frac{1}{n}.$$

Since $\langle Df(u), w \rangle$ is linear in $w \in E$, and because $-w \in E$ with $\| -w \| = 1$, we get $\langle Df(u_n), -w \rangle \le \frac{1}{n}$, so $|Df(u_n)| \le \frac{1}{n}$. Taking now the limit when $n \to \infty$, we have $\lim_{n \to \infty} |Df(u_n)| = 0$. □

Lemma 2.18. *Let $(X, V, \star, \mathcal{H}_\star, S)$ satisfy the Main assumptions. Assume that $f : X \to \mathbb{R}$ is a continuous E-differentiable functional bounded from below such that*

$$f(u^H) \le f(u) \quad \text{for all } u \in S \text{ and } H \in \mathcal{H}_\star. \tag{2.5}$$

and for all $u \in X$ there exists $\xi \in S$, with $f(\xi) \le f(u)$.

If f satisfies the $(CPS)_{\inf f}$ condition, then there exists $v \in X$, such that $f(v) = \inf f$ and $v = v^\star$ in V.

Proof. Put $\inf f = d$. For the minimizing sequence $(u_n)_n$ we consider the following sequence:

$$\varepsilon_n = \begin{cases} f(u_n) - d, & \text{if } f(u_n) - d > 0 \\ \dfrac{1}{n}, & \text{if } f(u_n) - d = 0. \end{cases}$$

Then $f(u_n) \leq d + \varepsilon_n$ and $\varepsilon_n \to 0$ as $n \to \infty$. By the Symmetric Ekeland principle II., proved by M. Squassina in [10, Theorem 2.8] and Lemma 2.14, there exists a sequence $(v_n)_n \subset X$ such that:

a) $f(v_n) \leq f(u_n)$;

b) $|Df(u_n)| \to 0$;

c) $\|v_n - v_n^\star\| \to 0$;

Since f satisfies the $(CPS)_d$ condition, there exists $v \in X$ such that $v_n \to v$ in X.

Since the embedding $X \hookrightarrow V$ is continuous by Main assumption 1), we have that $v_n \to v$ in V, and using the second inequality of (2.2) we obtain $v_n^* \to v^*$ in V. In particular,

$$\|v - v^\star\|_V \leq \|v - v_n\|_V + \|v_n - v_n^\star\|_V + \|v_n^\star - v^\star\|_V \to 0.$$

Therefore $v = v^\star$ in V, as stated. □

Theorem 2.19 (Existence of a third symmetric critical point)**.** *We assume that $(X, V, *, \mathcal{H}_*, S = X)$ satisfy the Main assumptions. Let the functional f satisfy the (CPS) condition in X and verify the polarization condition (2.5). Suppose that the local minima u_0 and u_1 of f in X also verify a polarization condition: $u_0^H = u_0$, $u_1^H = u_1$ for all $H \in \mathcal{H}_*$.*

Then f has at least a third critical point v, which is invariant by symmetrization in V, namely $v = v^*$ in V.

Proof. We prove this theorem in two steps.

Step 1. First, we prove the existence of a sequence $\{u_n\} \in X \setminus \{u_0, u_1\}$ such that

i) $f(u_n)$ is finite;

ii) $\lim_{n\to\infty} \|u_n - u_n^*\| = 0$;

iii) $\lim_{n\to\infty} |Df(u_n)| = 0$.

The proof is the same as in Theorem 2.17.

In Case 1, we use the inequality (2.2) and the assumption $X = S$. Then we can replace Lemma 2.16 by its symmetric version [11, Theorem 3.10]. Thus for all $n > \sqrt{\frac{c-a}{2}}$ and $\delta = \frac{3}{n}$, there exists $u_n \in X$ such that

a) $c - \frac{2}{n^2} \leq f(u_n) \leq c + \frac{2}{n^2}$;

b) $\|u_n - u_n^*\|_V \leq 9((1 + C_\Theta)K + 1)/n$;

c) $|Df(u)| \leq \frac{1}{n}$,

where C_Θ and K are some constants. Now, the assertion follows at once.

In Case 2, we choose $\rho := \min\{r^2/4, (r - r_0)^2/4\}$ and for every $r \in (0, r_0)$ and all $n > 1/\sqrt{\rho}$, we define $\tilde{f} = f$ and $\tilde{f} = \infty$ elsewhere. Now, we apply again the Symmetric Ekeland Principle II [Theorem 2.8] of Squassina [10] combined with Lemma 2.14 (instead of Theorem 1.1 in Ekeland [2]), since (2.2) holds and $S = X$. Then there exists a sequence $\{u_n\} \in \mathcal{U}$ such that

a) $c_0 \le f(u_n) \le c_0 + \frac{1}{n^2}$;

b) $\|u_n - u_n^*\|_V \le ((1 + C_\Theta)K + 1)/n$;

c) $|Df(u)| \le \frac{1}{n}$,

so, we proved the claim in this Step 1.

Step 2. Now, we apply the assertion of Step 1 for n sufficiently large.

In Case 1, the obtained sequence $\{u_n\}$ is a (CPS) sequence, so it possesses a subsequence which will be denoted also by $\{u_n\}$, which is convergent to some $v \in X$, with $f(v) = c > a$, $|Df(v)| = 0$ and $v = v^*$ as seen in the proof of Lemma 2.18.

In Case 2, the constructed (CPS) sequence admits a subsequence converging to some $v \in S_r(u_0)$, with $f(v) = c_0 = a = \max\{f(u_0), f(u_1)\}$, $|Df(v)| = 0$ and as in the proof of the Lemma 2.18, $v = v^*$.

In both cases v is a critical point of f, different from u_0 and u_1, and it is invariant by symmetrization in V. □

3 Main result

Let Ω be a ball, $\Omega = B(0, R) = \{x \in \mathbb{R}^N : |x| < R\}$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$.

Let $(X, V, *, \mathcal{H}_*, S = X)$ satisfy the Main assumptions, where X is a reflexive, real Banach space, which verifies the following embedding condition:

(EC) *there exists $p \in]1, N[$, such that the embedding $X \hookrightarrow L^q(\Omega)$ is continuous for $q \in [1, p^*]$ and compact if $q \in [1, p^*[$. We denote the best embedding constant by $C_q > 0$, i.e. $\|u\|_q \le C_q\|u\|$, for all $u \in X$ and $q \in [1, p^*]$.*

Further we assume, that $J : X \to \mathbb{R}$ is a convex functional such that the following properties hold

(J_1) *J is E-differentiable, with $E = C_0^\infty(\Omega)$;*

(J_2) *J is continuous and weakly lower semicontinuous;*

(J_3) *$J(u^H) \le J(u)$ for all $u \in X$ and $H \in \mathcal{H}_*$, where $(X, V, *, \mathcal{H}_*, S = X)$ satisfies the Main assumptions, with $V = L^p(\Omega)$ and $0 \in H$ for all $H \in \mathcal{H}_*$.*

Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following assumptions

(f_1) for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $|f(x, t)| \le \varepsilon|t|^{p-1} + c_\varepsilon$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$;

(f_2) $F(x,t) = F(y,t)$ for a.e. $x, y \in \Omega$, with $|x| = |y|$, and all $t \in \mathbb{R}$;

(f_3) $F(x,t) \le F(x,-t)$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}^-$;

(f_4) there exists a function $v \in X$ such that $\mathcal{F}(v) = \int_\Omega F(x, v(x))dx > 0$.

We say that $u \in X$ *is* a (weak) *solution of the equation*

$$J'(u) = \lambda \mathcal{F}'(u), \tag{3.1}$$

if

$$\langle DJ(u), \varphi \rangle - \lambda \int_\Omega f(x, u(x))\varphi(x)dx = 0 \quad \text{for every } \varphi \in E$$

holds, where $\lambda > 0$ is a fixed real number.

Let $\mathcal{E}_\lambda : X \to \mathbb{R}$,

$$\mathcal{E}_\lambda(u) = J(u) - \lambda \int_\Omega F(x, u(x))dx.$$

be the energy functional associated to the problem (3.1). The critical points of the energy functional $\mathcal{E}_\lambda$ are exactly the (weak) solutions of (3.1).

Before stating and proving the main theorem, we present some auxiliary results from Filippucci et al. [6].

Lemma 3.1 ([6, Lemma 3.4]). *Let F be a Carathéodory function, satisfying (f_2) and (f_3). Then, for all $H \in \mathcal{H}_\star$*

$$\int_\Omega F(x, u(x))dx \le \int_\Omega F(x, u^H(x))dx$$

for every $u \in X$, with $F(\cdot, u(\cdot)) \in L^1(\Omega)$. If furthermore F satisfies

$(f_3)'$ $\qquad F(x,t) = F(x,-t)$ *for a.e. $x \in \Omega$ and all $t \in \mathbb{R}^-$*

in place of (f_3), then

$$\int_\Omega F(x, u(x))dx = \int_\Omega F(x, u^H(x))dx$$

for every $u \in X$, with $F(\cdot, u(\cdot)) \in L^1(\Omega)$.

We use the following notation

$$X_{\mathcal{H}_*} = \{u \in X \mid u^H = u, \text{ for all } H \in \mathcal{H}_*\}.$$

Lemma 3.2 ([6, Proposition 3.9]).

i) For every $r \in \mathcal{F}(X) \setminus \sup_{u \in X} \mathcal{F}(u)$, the $\mathcal{F}^{-1}([r, \infty)) \cap X_{\mathcal{H}_\star}$ is a non-empty, weakly closed subset of X and also $\mathcal{F}^{-1}(I^r) \cap X_{\mathcal{H}_\star}$ is non-empty, where $I^r = (r, \infty)$. Moreover,

$$\varphi_1(r) = \inf_{u \in \mathcal{F}^{-1}(I^r) \cap X_{\mathcal{H}_\star}} \frac{\inf\limits_{v \in \mathcal{F}^{-1}(r)} J(v) - J(u)}{r - \mathcal{F}(u)} \tag{3.2}$$

is well-defined.

ii) If furthermore $(f_3)'$ holds, then for all $r \in \mathcal{F}(X) \setminus \inf_{u\in X} \mathcal{F}(u)$ also the set $\mathcal{F}^{-1}((-\infty, r]) \cap X_{\mathcal{H}_\star}$ is a non-empty, weakly closed subset of X and $\mathcal{F}^{-1}(I_r) \cap X_{\mathcal{H}_\star}$ is non-empty, where now $I_r = (-\infty, r)$. Furthermore,

$$\varphi_2(r) = \sup_{u\in\mathcal{F}^{-1}(I_r)\cap X_{\mathcal{H}_\star}} \frac{\inf\limits_{v\in\mathcal{F}^{-1}(r)} J(v) - J(u)}{r - \mathcal{F}(u)} \tag{3.3}$$

is well-defined.

The next two lemmas provide us two different symmetric local minima of the energy functional $\mathcal{E}_\lambda$.

Lemma 3.3. *Let (J_1)–(J_3), (f_1)–(f_4) hold and let f be non constant. Assume that $\mathcal{E}_\lambda$ is coercive, bounded below and there exists a real number r, with $r \in \mathcal{F}(X) \setminus \sup_{u\in X} \mathcal{F}(u)$. Then the infimum of $\mathcal{E}_\lambda = J - \lambda\mathcal{F}$ in $\mathcal{F}^{-1}([r, \infty)) \cap X_{\mathcal{H}_\star}$ is attained at some point u_0, provided that $\lambda \in \mathbb{R}^+$ satisfies the inequality*

$$\lambda > \varphi_1(r).$$

Moreover, u_0 is a local minimizer of $\mathcal{E}_\lambda$ in X, $u_0^H = u_0$ for every $H \in \mathcal{H}_\star$, and $\mathcal{F}(u_0) > r$. If (f_3) is replaced by the stronger condition $(f_3)'$, then the result continues to hold for all $\lambda \in \mathbb{R}$, with $\lambda > \varphi_1(r)$.

Lemma 3.4. *Let (J_1)–(J_3), (f_1)–(f_3) and $(f_3)'$ hold, and let f be non constant. Assume that $\mathcal{E}_\lambda$ is coercive, bounded below and there exists a real number r, with $r \in \mathcal{F}(X) \setminus \inf_{u\in X} \mathcal{F}(u)$. Then the infimum of $\mathcal{E}_\lambda = J - \lambda\mathcal{F}$ in $\mathcal{F}^{-1}((-\infty, r]) \cap X_{\mathcal{H}_\star}$ is attained at some point u_1, provided that $\lambda \in \mathbb{R}$ satisfies the inequality*

$$\lambda < \varphi_2(r).$$

Moreover, u_1 is a local minimizer of $\mathcal{E}_\lambda$ in X, $u_1^H = u_1$ for every $H \in \mathcal{H}_\star$, and $\mathcal{F}(u_1) < r$.

Now, we can state the main result of this section, which extends the Theorem 3.12 of [6].

Theorem 3.5. *Let the functionals $J, \mathcal{F}$ and $\mathcal{E}_\lambda$ satisfy the (J_1)–(J_3) and (f_1)–(f_4) conditions. We assume in addition that*

$(\mathcal{E}_1)$ *$\mathcal{F}(u^H) \geq \mathcal{F}(u)$, for every $u \in X$ and $H \in \mathcal{H}_*$;*

$(\mathcal{E}_2)$ *$\mathcal{E}_\lambda = J - \lambda\mathcal{F}$ is coercive in X, for all $\lambda \in I$;*

$(\mathcal{E}_3)$ *$\mathcal{E}_\lambda$ satisfy the $(CPS)_c$ condition, for every $\lambda \in \mathbb{R}$.*

Assume also that there exists $r \in \mathbb{R}$ such that

(i) $\inf\limits_{u\in X} \mathcal{F}(u) < r < \sup\limits_{u\in X} \mathcal{F}(u)$;

(ii) $\varphi_1(r) < \varphi_2(r)$;

Then $\mathcal{E}_\lambda$ has at least three critical points in X, for every $\lambda \in (\varphi_1(r), \varphi_2(r))$, which are symmetric invariant in V.

Proof. By Lemmas 3.3 and 3.4 we have two different local minima u_0 and u_1 in X for the energy functional E_λ for every $\lambda \in (\varphi_1(r), \varphi_2(r))$ and these minima are also in $X_{\mathcal{H}_*}$. From the assumptions $(\mathcal{E}_2)$ and $(\mathcal{E}_3)$ we have that $\mathcal{E}_\lambda$ is coercive and satisfies the $(CPS)_c$ condition for all $\lambda \in \mathbb{R}$. Furthermore $(\mathcal{E}_1)$ implies that $\mathcal{E}_\lambda(u^H) \leq \mathcal{E}_\lambda(u)$, for all $u \in X$, $H \in \mathcal{H}_*$ and $\lambda \in (\varphi_1(r), \varphi_2(r))$. So, we can apply Theorem 2.19, which ensures the existence of a third invariant critical point u_2 for $\mathcal{E}_\lambda$, with $\lambda \in (\varphi_1(r), \varphi_2(r))$. □

4 Application

Let Ω be a ball in $\mathbb{R}^N$, $X = W_0^{1,2}(\Omega)$ and let $E = C_0^\infty(\Omega)$, which is dense in X.

We consider the problem (1.1), namely

$$-\sum_{i,j=1}^{N} D_j(a_{ij}(u)D_iu) + \frac{1}{2}\sum_{i,j=1}^{N} a'_{ij}(u)D_iuD_ju = \lambda f(x,u),$$

where a_{ij} satisfy the conditions (a_1)–(a_3). We recall the corresponding energy functional $\mathcal{E}_\lambda$ defined in (1.2), as

$$\mathcal{E}_\lambda(u) = J(u) - \lambda\mathcal{F}(u),$$

where $J(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{N} a_{ij}(u)D_iuD_judx$ and $\mathcal{F}(u) = \int_\Omega F(x,u)dx$, $F(x,t) = \int_0^t f(x,s)ds$.

By the conditions $(a_1) - (a_3)$, the functional $\mathcal{E}_\lambda$ is continuous, E-differentiable and

$$\begin{aligned}\langle D\mathcal{E}_\lambda(u),\varphi\rangle &= \lim_{t\to 0}\frac{\mathcal{E}_\lambda(u+t\varphi)-\mathcal{E}_\lambda(u)}{t}\\ &= \int_\Omega \sum_{i,j=1}^{N} a_{ij}(x)D_iuD_ju\varphi dx + \frac{1}{2}\sum_{i,j=1}^{N} a'_{ij}(u)D_iuD_ju\varphi dx\\ &\quad -\lambda\int_\Omega f(x,u)\varphi dx, \quad \text{for all } u\in W_0^{1,2}(\Omega), \varphi\in C_0^\infty(\Omega),\end{aligned}$$

and

$$|D\mathcal{E}_\lambda(u)| = \sup\{\langle D\mathcal{E}_\lambda(u),\varphi\rangle \mid \varphi\in C_0^\infty(\Omega),\ \|\varphi\|_{C_0^\infty(\Omega)} = 1\}. \tag{4.1}$$

We say that $u \in W_0^{1,2}(\Omega)$ is a weak solution of the quasilinear problem (1.1) if u is a critical point of $\mathcal{E}_\lambda$.

We assume that $f : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfies the conditions (f_1)–(f_4) with $p = 2$ and in addition the following assumptions hold

(f_5) there exists $c > 0$ such that $|f(x,t)| \le c|t|$ for a.e. $x\in\Omega$ and all $t\in\mathbb{R}$;

(f_6) there exist $q \in (2, 2^*)$ and a positive constant $M > 0$, such that for a.e. $x \in \Omega$ and all $t\in\mathbb{R}$, $|F(x,t)| \le M|t|^q$.

Remark 4.1.

a) Condition (f_6) is stronger than (f_5) for $t \in [-1,1]$ while the condition (f_5) is stronger when t is outside of the interval $[-1,1]$. We need both conditions in the proof of our main result (Theorem 4.1).

b) The embedding condition (EC) also remains true for $p = 2$.

Now, we give a concrete example for the function f, which satisfies the conditions (f_1)–(f_6).

Example 4.2. Let $f(x,u) = a(x)\cdot g(u)$, where

(1) a is a nonnegative measurable function such that $a(x) = a(y)$, for a.e. $x,y\in\Omega$ with $|x| = |y|$;

(2) g is defined by

$$g(u) = \begin{cases} |u|^{\alpha-2}u(|u|^{\beta-\alpha}-1), & \text{if } |u| \geq 1 \text{ and } 1<\alpha<\beta<2 \\ |u|^{\gamma-2}u(|u|^{\delta-\gamma}-1), & \text{if } |u|<1 \text{ and } 2<\gamma<\delta. \end{cases}$$

The main result in this section is the following.

Theorem 4.3. *We assume that the conditions (a_1)–(a_3), (f_1)–(f_6) hold.*

Then there exists $\overline{\lambda}>0$, such that for every $\lambda>\overline{\lambda}$, the problem (1.1) *has two nontrivial nonnegative solutions which are invariant by Schwarz symmetrization and also a third solution, which is Schwarz symmetric, but possibly trivial.*

In order to prove this theorem, we use the Theorem 3.5. So, first of all we verify the hypotheses of this theorem.

Lemma 4.4. *We assume that the conditions (a_1)–(a_3) hold. Then we have*

$$J(u^H) \leq J(u), \quad \text{for every } u \in X \text{ and } H \in \mathcal{H}_*. \tag{4.2}$$

Proof. Let us consider the following function $j : \mathbb{R}\times[0,\infty)\to\mathbb{R}$, defined by

$$j(u,|\nabla u|) = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(u)D_iuD_ju. \tag{4.3}$$

The assumptions (a_1)–(a_3) for a_{ij} imply that the assumptions of Corollary 3.3. of [7] are satisfied. Then the required inequality follows immediately. □

Lemma 4.5. *Assume that the (a_3) and (f_1) hold. Then the energy functional $\mathcal{E}_\lambda$ is coercive, for every $\lambda \in \mathbb{R}$.*

Proof. Let $\lambda\in\mathbb{R}$ is fixed and take $\varepsilon \in (0, \frac{\nu}{2\lambda C_2^2})$, where ν is from the condition (a_3) and C_2 is the embedding constant from (EC). Then, by (a_3) and (f_1)

$$\begin{aligned} \mathcal{E}_\lambda(u) &\geq \frac{1}{2}\int_\Omega \nu|\nabla u|^2 - \lambda\int_\Omega(\varepsilon|u|+c_\varepsilon) \\ &\geq \frac{1}{2}\nu\|u\|^2 - \varepsilon\lambda C_2^2\|u\|^2 - c_\varepsilon\lambda|\Omega| \\ &= \left(\frac{1}{2}\nu - \varepsilon\lambda C_2^2\right)\|u\|^2 - c_\varepsilon\lambda|\Omega|. \end{aligned}$$

Hence if $\|u\|\to\infty$, then $\mathcal{E}_\lambda(u)\to\infty$, which means that the energy functional is coercive for every $\lambda\in\mathbb{R}$. □

In order to prove the (CPS) condition of the energy functional, we recall here a lemma of A. Canino (see [1]).

Lemma 4.6. *Let (u_h) be a bounded sequence in $H_0^1(\Omega)$ satisfying*

$$\int_\Omega \sum_{i,j=1}^{N} a_{ij}(u_h)D_iu_hD_ju_hv\,dx + \frac{1}{2}\sum_{i,j=1}^{N} a'_{ij}(u_h)D_iu_hD_ju_hv\,dx = \langle\beta_h, v\rangle, \quad \text{for all } v\in C_0^\infty(\Omega)$$

with (β_h) strongly convergent in $H^{-1}(\Omega)$. Then it is possible to extract a subsequence (u_{h_k}) strongly convergent in $H^{-1}(\Omega)$.

Lemma 4.7. *Assuming that* (a_1)*–*(a_3) *and* (f_1) *are true, the energy functional* $\mathcal{E}_\lambda$ *satisfies the* $(CPS)_c$ *condition, for every* λ *and* $c \in \mathbb{R}$*.*

Proof. Let $\{u_n\} \subset W_0^{1,2}(\Omega)$ be an arbitrary Palais–Smale sequence for $\mathcal{E}_\lambda$, i.e.

(a) $\{\mathcal{E}_\lambda(u_n)\}$ is bounded;

(b) $|D\mathcal{E}_\lambda|(u_n) \to 0$.

We have to prove that $\{u_n\}$ contains a strongly convergent subsequence. Since $\mathcal{E}_\lambda$ is coercive, we have that the sequence $\{u_n\}$ is bounded.

By the condition (f_1), the Nemytskii operator $f(x,\cdot)$ is compact operator from $W_0^{1,2}(\Omega)$ into $W^{-1,2}(\Omega)$ (the dual space of $W_0^{1,2}(\Omega)$) (see for example [4]). So, $\{f(x,u_n)\}$ is strongly convergent in $W^{-1,2}(\Omega)$.

Now, using the Lemma 4.6 with the choice: $\beta_n = \lambda f(x,u_n) + D\mathcal{E}_\lambda(u_n)$, we can extract a strongly convergent subsequence of $\{u_n\}$ in $W_0^{1,2}$, which completes the proof. □

Now, we are ready to prove the main theorem of this section.

Proof of Theorem 4.1. By Lemma 4.4, Lemma 4.5, Lemma 4.7, the conditions $(\mathcal{E}_1)$–$(\mathcal{E}_3)$ of Theorem 3.5 are satisfied.

In what follows we verify the conditions (i)–(ii).

Let λ_1 be the first eigenvalue of the problem

$$-\triangle u = \lambda u,$$

in $H_0^1(\Omega)$, that is λ_1 is defined by the Rayleigh quotient

$$\lambda_1 = \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{||u||^2}{||u||_2^2}. \tag{4.4}$$

By [5], this infimum is achieved and $\lambda_1 > 0$.

By (f_4), there exists $v \in H_0^1(\Omega)$ such that $\mathcal{F}(v) > 0$, so the number $\varphi_1(0)$ is well defined by Lemma 3.2 and

$$\bar{\lambda} = \varphi_1(0) = \inf_{u \in \mathcal{F}^{-1}((0,\infty))} \frac{J(u)}{\mathcal{F}(u)}. \tag{4.5}$$

On the other hand by (a_1) and (f_5), we have

$$\frac{J(u)}{\mathcal{F}(u)} \geq \frac{\nu ||u||^2}{c \cdot ||u||_2^2} \geq \frac{\nu}{c} \cdot \lambda_1, \tag{4.6}$$

where λ_1 is the first eigenvalue from (4.4). Hence, by (4.5) and (4.6)

$$\bar{\lambda} = \varphi_1(0) > 0.$$

From the definition of φ_1 in (3.2), for every $u \in \mathcal{F}^{-1}((0,\infty))$, we have

$$\varphi_1(r) \leq \frac{J(u)}{\mathcal{F}(u) - r}, \qquad \forall r \in (0, \mathcal{F}(u)),$$

so

$$\limsup_{r \to 0^+} \varphi_1(r) \leq \varphi_1(0) = \bar{\lambda}. \tag{4.7}$$

Using the embedding condition (EC) and the assumption (f_6), we obtain

$$|\mathcal{F}(u)| \leq M\|u\|_q^q \leq MC_q^q\|u\|^q = \bar{C}\|u\|^q, \tag{4.8}$$

where $\bar{C} = MC_q^q$.

Let $r > 0$ and $u \in \mathcal{F}^{-1}(r)$. Then by (a_1) and (4.8), we have

$$r = \mathcal{F}(u) \leq \bar{C}\left(\|u\|^2\right)^{\frac{q}{2}} \leq \bar{C}\left(\frac{2J(u)}{\nu}\right)^{\frac{q}{2}},$$

so

$$J(u) \geq Kr^{\frac{2}{q}}, \tag{4.9}$$

where $K = \frac{\nu}{2}\bar{C}^{-\frac{2}{q}}$.

Therefore, by the definition of φ_2 in (3.3) and by the fact that $u \equiv 0 \in \mathcal{F}^{-1}(I_r)$, we obtain

$$\varphi_2(r) \geq \frac{1}{r}\inf_{u\in\mathcal{F}^{-1}(r)} J(u) \geq \frac{1}{r}Kr^{\frac{2}{q}} = Kr^{\frac{2}{q}-1}. \tag{4.10}$$

It follows that $\lim_{r\to 0^+} \varphi_2(r) = \infty$, since $q > 2$.

In conclusion, we have proved that

$$\limsup_{r\to 0^+} \varphi_1(r) \leq \varphi_1(0) = \bar{\lambda} < \limsup_{r\to 0^+} \varphi_2(r) = \infty.$$

From here we can conclude that for every integers $n \geq \bar{n} = 2 + [\bar{\lambda}]$ there exists $r_n > 0$ so close to zero that

$$\varphi_1(r_n) < \bar{\lambda} + 1/n < n < \varphi_2(r_n). \tag{4.11}$$

By condition (f_4), there exists $v \in X = W_0^{1,2}(\Omega)$ with $\mathcal{F}(v) > 0$. So, $\emptyset \neq [0, \mathcal{F}(v)] \subset \mathcal{F}(X)$ hence we can assume without loss of generality that r_n defined in (4.11) satisfies the conditions (i) and (ii) of Theorem 3.5. Therefore by Theorem 3.5, the problem (1.1) admits two nontrivial solutions which are invariants by Schwarz symmetrization and a third solution, possibly zero, which is also symmetric invariant in $V = L^2(\Omega)$, for all

$$\lambda \in \bigcup_{n=\bar{n}}^{\infty} (\varphi_1(r_n), \varphi_2(r_n)) \supset \bigcup_{n=\bar{n}}^{\infty} [\bar{\lambda} + 1/n, n] = (\bar{\lambda}, \infty),$$

as claimed. □

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