Electronic Journal of Qualitative Theory of Differential Equations
2015, No. 44, 1–6; doi: 10.14232/ejqtde.2015.1.44 http://www.math.u-szeged.hu/ejqtde/

On bifurcations of a system of cubic differential equations with an integrating multiplier singular along a second-order curve

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Received 7 April 2015, appeared 17 July 2015
Communicated by Ivan Kiguradze

Abstract. We establish necessary and sufficient conditions for existence of an integrating multiplier of a special form for systems of two cubic differential equations of the first order. We further study bifurcations of such systems with the change of parameters of their integrating multipliers.

Keywords: integrating multiplier, cubic differential equations, bifurcations, limit cycles.

2010 Mathematics Subject Classification: 34C05, 34C07, 34C23, 37G10, 37G15.

1 Introduction

In the present work, we study the following system of differential equations:

$$\begin{cases} \dfrac{dx}{dt} = \sum\limits_{j+k=0}^{3} a_{jk}x^j y^k \equiv P(x,y), \\ \dfrac{dy}{dt} = \sum\limits_{j+k=0}^{3} b_{jk}x^j y^k \equiv Q(x,y), \end{cases} \tag{1.1}$$

that admits an integrating multiplier

$$\mu(x,y) = \exp\left(\frac{R(x,y)}{Z(x,y)}\right), \tag{1.2}$$

where $R(x,y)$ and $Z(x,y)$ are polynomials of the second degree.

The problem of existence of a first integral and a Darboux-type integrating multiplier for the system (1.1) was considered in [2,5]. The problem of existence of limit cycles for the system (1.1) with the integrating multiplier (1.2) was solved in [3,4]. It is known [3] that the system (1.1) with analytical right-hand sides and the integrating multiplier (1.2), where $R(x,y)$ and $Z(x,y)$ are analytical functions at every x and y, does not have limit cycles on the plane. From this perspective, the systems with topological structure changing along with the parameters in (1.1) and (1.2) are of interest.

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2 Results

We address the problem of existence of an integrating multiplier in the case, when $Z(x,y)$ defines a second-order curve:

$$Z(x,y) \equiv Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \tag{2.1}$$

Since the system (1.1) is invariant with respect to nondegenerate linear substitution of variables, under an appropriate substitution it is possible to transform the equation (2.1) into one of the following four types.

1. An irreducible curve of the second order:
$$Z(x,y) \equiv x^2 + ay^2 + 2y = 0; \tag{2.2}$$
2. A pair of crossing straight lines, which are real ($a < 0$), imaginary ($a > 0$), or coinciding ($a = 0$):
$$Z(x,y) \equiv x^2 + ay^2 = 0; \tag{2.3}$$
3. A pair of parallel straight lines, which are real ($a < 0$), imaginary ($a > 0$), or coinciding ($a = 0$):
$$Z(x,y) \equiv x^2 - a = 0; \tag{2.4}$$
4. An ellipse, which is real ($a < 0$), imaginary ($a > 0$), or again a pair of imaginary crossing straight lines ($a = 0$):
$$Z(x,y) \equiv x^2 + y^2 - a = 0. \tag{2.5}$$

For each of these cases, we find a system (1.1) that possesses the integrating multiplier (1.2) and study bifurcations of such systems at the change of parameter a. Moreover, in spite of substantial changes in the topological structure, the system (1.1) remains acyclic.

It is known [4] that the system (1.1) admits the integrating multiplier (1.2), where $R(x,y)$ and $Z(x,y)$ are polynomials, if and only if there exists a polynomial $M(x,y)$ such that:

$$Z'_x \cdot P + Z'_y \cdot Q \equiv Z \cdot M, \tag{2.6}$$
$$R'_x \cdot P + R'_y \cdot Q - R \cdot M \equiv -Z \cdot \left(P'_x + Q'_y\right). \tag{2.7}$$

Using (2.6) and (2.7), we proved in [1] the following statement:

Theorem 2.1 ([1]). *The system (1.1) admits the integrating multiplier (1.2), where $Z(x,y)$ is of the form (2.2) and $R(x,y)$ is a second-degree polynomial, if and only if there exists a nondegenerate linear substitution of the variables that transforms (1.1) into the form:*

$$\begin{cases} \dfrac{dx}{dt} = a_{00} - 2b_{20}x + (aa_{00} - 2)\,y + a_{20}x^2 - 2ab_{20}xy + y^2\,(a_{21} - aa_{20} - 3a) \\ \qquad + a_{30}x^3 + a_{21}x^2y - aa_{30}xy^2 - a^2y^3, \\ \dfrac{dy}{dt} = -a_{00}x - 2b_{20}y + b_{20}x^2 + 2xy\,(a_{20} + 2) - y^2\,(2a_{30} + ab_{20}) \\ \qquad + x^3 + a_{30}x^2y + xy^2\,(a_{21} + 2a) - aa_{30}y^3. \end{cases} \tag{2.8}$$

Thus the system (2.8) has an integrating multiplier

$$\mu(x,y) = \exp\left(\frac{2a_{00} - 4y\,(a_{20} + 1) - 4b_{20}x - 4a_{30}xy - 2y^2\,(a_{21} + 2)}{x^2 + ay^2 + 2y}\right)$$

for all $a \in \mathbb{R}$.

Below we prove a similar statement that addresses the form (2.3).

Theorem 2.2. *The system* (1.1) *has the integrating multiplier* (1.2)*, where* $Z(x,y)$ *is of the form* (2.3) *and* $R(x,y)$ *is a second-degree polynomial, if and only if there exists a nondegenerate linear substitution of variables that transforms the system* (1.1) *into the form:*

$$\begin{cases} \dfrac{dx}{dt} = -ab_{10}y + a_{20}x^2 - 2ab_{20}xy - aa_{20}y^2 + a_{30}x^3 + a_{21}x^2y - aa_{30}xy^2 - 2a^2y^3, \\ \dfrac{dy}{dt} = b_{10}x + b_{20}x^2 + 2a_{20}xy - ab_{20}y^2 + 2x^3 + a_{30}x^2y + (a_{21} + 4a)\, xy^2 - aa_{30}y^3. \end{cases} \tag{2.9}$$

Thus the system (2.9) *has an integrating multiplier:*

$$\mu(x,y) = \exp\left(\frac{-2a_{30}xy - (a_{21} + 2a)\, y^2 - 2b_{20}x - 2a_{20}y - b_{10}}{x^2 + ay^2}\right) \tag{2.10}$$

for all $a \in \mathbb{R}$.

Proof. From (1.1), (2.3), and (2.6), it follows that $M(x,y)$ is a second-degree polynomial: $M(x,y) = mx^2 + nxy + ky^2 + rx + sy + t$. Then we can rewrite the identity (2.6) in the form:

$$2x \cdot \sum_{j+k=0}^{3} a_{jk}x^jy^k + 2ay \cdot \sum_{j+k=0}^{3} b_{jk}x^jy^k \equiv \left(x^2 + ay^2\right)\left(mx^2 + nxy + ky^2 + rx + sy + t\right).$$

So we obtain equalities for the coefficients of the homogeneous polynomials in the left and right-hand sides:

$$\begin{gathered} 2a_{30} = m, \quad 2a_{21} + 2ab_{30} = n, \quad 2a_{12} + 2ab_{21} = k + am, \quad 2a_{03} + 2ab_{12} = an; \\ 2ab_{03} = ak, \quad 2a_{20} = r, \quad 2a_{11} + 2ab_{20} = s, \quad 2a_{02} + 2ab_{11} = ar, \quad 2ab_{02} = as; \\ 2a_{10} = t, \quad 2a_{01} + 2ab_{10} = 0, \quad 2ab_{01} = at, \quad 2a_{00} = 0, \quad 2ab_{00} = 0. \end{gathered}$$

From here we consecutively find:

$$\begin{gathered} a_{00} = b_{00} = 0, \quad t = 2a_{10}, \quad b_{01} = a_{10}, \quad a_{01} = -ab_{10}, \quad r = 2a_{20}, \quad m = 2a_{30}; \\ b_{02} = a_{11} + ab_{20}, \quad s = 2\,(a_{11} + ab_{20})\,, \quad a_{02} = a\,(a_{20} - b_{11})\,, \quad k = 2b_{03}; \\ n = 2a_{21} + 2ab_{30}, \quad a_{03} = a\,(a_{21} + ab_{30} - b_{12})\,, \quad b_{03} = a_{12} + ab_{21} - aa_{30}. \end{gathered}$$

It follows that the system (1.1) with particular algebraic integral (2.3) has the form:

$$\begin{cases} \dfrac{dx}{dt} = a_{10}x - ab_{10}y + a_{20}x^2 + a_{11}xy + a\,(a_{20} - b_{11})\, y^2 \\ \qquad + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a(a_{21} + ab_{30} - b_{12})y^3 \equiv P(x,y), \\ \dfrac{dy}{dt} = b_{10}x + a_{10}y + b_{20}x^2 + b_{11}xy + (a_{11} + ab_{20})\, y^2 \\ \qquad + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + (a_{12} + ab_{21} - aa_{30})\, y^3 \equiv Q(x,y), \end{cases} \tag{2.11}$$

which further satisfies:

$$M(x,y) = 2\left[a_{30}x^2 + (a_{21} + ab_{30})\, xy + (a_{12} + ab_{21} - aa_{30})\, y^2 + a_{20}x + (a_{11} + ab_{20})\, y + a_{10}\right]. \tag{2.12}$$

The system (2.9) admits an integrating multiplier of the form:

$$\mu(x,y) = \exp\left(\frac{R(x,y)}{x^2 + ay^2}\right),$$

where without loss of generality we may assume that

$$R(x,y) = 2kxy + my^2 + 2nx + 2ry + s, \qquad k^2 + m^2 \neq 0,$$

if and only if the identity (2.7) holds. Substituting (2.11) and (2.12) into (2.7), we get:

$$\begin{aligned} &2\,(ky+n)\,P + 2\,(kx+my+r)\,Q - M\left(2kxy + my^2 + 2nx + 2ry + s\right) \\ &\quad \equiv -\left(x^2+ay^2\right)\left(P'_x + Q'_y\right), \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} P'_x + Q'_y &= 2a_{10} + x\,(2a_{20}+b_{11}) + y\,(3a_{11}+2ab_{20}) \\ &\quad + x^2\,(3a_{30}+b_{21}) + 2xy\,(a_{21}+b_{12}) + y^2(4a_{12}+3ab_{21}-3aa_{30}). \end{aligned}$$

So we get equalities for the coefficients of the homogeneous fourth degree polynomials in the left- and right-hand sides of (2.13):

$$\begin{cases} 2kb_{30} = -3a_{30} - b_{21}, \\ -ka_{30} + kb_{21} + mb_{30} = -a_{21} - b_{12}, \\ k\,(-a_{21}+b_{12}-2ab_{30}) + m\,(b_{21}-a_{30}) = -2a_{12} - 2ab_{21}, \\ ka\,(a_{30}-b_{21}) + m\,(b_{12}-a_{21}-ab_{30}) = -a(a_{21}+b_{12}), \\ 2ak\,(a_{21}-b_{12}+ab_{30}) = -a(4a_{12}+3ab_{21}-3aa_{30}), \end{cases} \tag{2.14}$$

which hold for all $a \in \mathbb{R}$ and $k^2+m^2 \neq 0$. Since one of the coefficients of the system (1.1) can be chosen arbitrarily with the change of parametrization, we let $b_{30} = 2$ (we remark that $b_{30} = 0$ would imply $a_{30} = a_{21} = 0$ and the system (2.11) would not be cubic, hence $b_{30} \neq 0$). Then from (2.13), we get

$$k = -a_{30}, \quad m = -a_{21} - 2a, \quad a_{12} = -aa_{30}, \quad b_{21} = a_{30}, \quad b_{12} = a_{21} + 4a, \quad b_{03} = -aa_{30}.$$

From equalities of the coefficients of the homogeneous third degree polynomials in the left- and right-hand sides of (2.13), we obtain

$$\begin{cases} -2na_{30} + 2kb_{20} + 4r = -2a_{20} - b_{11}, \\ 2k\,(b_{11}-a_{20}) + 2mb_{20} - 2ra_{30} - 2n\,(a_{21}+4a) = -3a_{11} - 2ab_{20}, \\ -2kab_{20} + 2naa_{30} + 2m\,(b_{11}-a_{20}) - 2ra_{21} = -a\,(b_{11}+2a_{20}), \\ 2ka\,(a_{20}-b_{11}) - 4a^2n + 2raa_{30} = -a\,(3a_{11}+2ab_{20}), \end{cases}$$

From here for all $a \in \mathbb{R}$ and $k^2+m^2 \neq 0$, we get $b_{11} = 2a_{20}$, $a_{11} = -2a_{20}$, $r = -a_{20}$, $n = -b_{20}$.

Equalities of the coefficients of the homogeneous second, first, and zeroth degree polynomials in the left- and right-hand sides of (2.13) with the above conditions imply:

$$\begin{cases} a_{30}\,(s+b_{10}) = a_{10}, \\ (a_{21}+2a)\,(s+b_{10}) = 0, \\ a_{30}\,(s+b_{10}) = -a_{10}, \\ 2b_{20}a_{10} - 2a_{20}b_{10} - 2sa_{20} = 0, \\ 2b_{20}a_{10} + 2a_{20}b_{10} + 2sa_{20} = 0, \\ sa_{10} = 0. \end{cases}$$

These equalities hold only if $a_{10} = 0$ and $s = -b_{10}$. It therefore follows that the system (2.11) with an integrating multiplier (1.2) has form (2.9) and admits an integrating multiplier (2.10) for all $a \in \mathbb{R}$. □

Theorems 2.3 and 2.4 below, addressing the forms (2.4) and (2.5), have similar proofs, which we omit.

Theorem 2.3. *The system* (1.1) *has the integrating multiplier* (1.2)*, where* $Z(x,y)$ *is of the form* (2.4) *and* $R(x,y)$ *is a second-degree polynomial, if and only if there exists a nondegenerate linear substitution of the variables that transforms the system* (1.1) *into the form:*

$$\begin{cases} \dfrac{dx}{dt} = \left(x^2 - a\right)\left(a_{20} + a_{30}x + a_{21}y\right), \\ \dfrac{dy}{dt} = ab_{20} + b_{10}x + aa_{30}y + b_{20}x^2 + 2a_{20}xy + 2x^3 + a_{30}x^2y + a_{21}xy^2. \end{cases} \tag{2.15}$$

Thus the system (2.15) *has an integrating multiplier*

$$\mu(x,y) = \exp\left(\frac{-2a_{30}xy - a_{21}y^2 - 2b_{20}x - 2a_{20}y - 2a - b_{10}}{x^2 - a}\right)$$

for all $a \in \mathbb{R}$.

Theorem 2.4. *The system* (1.1) *has the integrating multiplier* (1.2)*, where* $Z(x,y)$ *is of the form* (2.5) *and* $R(x,y)$ *is a second-degree polynomial, if and only if there exists a nondegenerate linear substitution of the variables that transforms the system* (1.1) *into the form:*

$$\begin{cases} \dfrac{dx}{dt} = a_{30}x^3 + a_{21}x^2y - a_{30}xy^2 - 2y^3 + a_{20}x^2 \\ \qquad - 2b_{20}xy - a_{20}y^2 - aa_{30}x - \left[b_{10} + a\left(a_{21} + 2\right)\right]y - aa_{20}, \\ \dfrac{dy}{dt} = 2x^3 + a_{30}x^2y + \left(a_{21} + 4\right)xy^2 - a_{30}y^3 + b_{20}x^2 + 2a_{20}xy - b_{20}y^2 + b_{10}x + aa_{30}y + ab_{20}. \end{cases}$$

Thus the system (2.4) *has an integrating multiplier*

$$\mu(x,y) = \exp\left(\frac{-2a_{30}xy - \left(a_{21} + 2\right)y^2 - 2b_{20}x - 2a_{20}y - b_{10} - 2a}{x^2 + y^2 - a}\right)$$

for all $a \in \mathbb{R}$.

3 Visualization

Change of the topological structure of the systems (2.8), (2.9), (2.15), and (2.4) upon transition of the parameter a through the bifurcation value $a = 0$ can be visually illustrated by using *WInSet* software [6], which can construct a phase portrait of the system with fixed values of the coefficients and the value of a ranging in an interval containing 0. A video example of such phase portrait at `http://youtu.be/Of7C2x37NbI` demonstrates the absence of limit cycles, as expected.

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