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Qualitative approximation of solutions to difference equations

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Abstract. We present a new approach to the theory of asymptotic properties of solutions to difference equations. Usually, two sequences x, y are called asymptotically equivalent if the sequence $x - y$ is convergent to zero i.e., $x - y \in c_0$, where c_0 denotes the space of all convergent to zero sequences. We replace the space c_0 by various subspaces of c_0. Our approach is based on using the iterated remainder operator. Moreover, we use the regional topology on the space of all real sequences and the 'regional' version of the Schauder fixed point theorem.

Keywords: difference equation, difference pair, prescribed asymptotic behavior, remainder operator, Raabe's test, Gauss's test, Bertrand's test.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{R}$ denote the set of positive integers and the set of real numbers, respectively. In this paper we assume that

$$m \in \mathbb{N}, \qquad f\colon \mathbb{R} \to \mathbb{R}, \qquad \sigma\colon \mathbb{N} \to \mathbb{N}, \qquad \lim \sigma(n) = \infty,$$

and consider difference equations of the form

$$\Delta^m x_n = a_n f(x_{\sigma(n)}) + b_n \tag{E}$$

where $a_n, b_n \in \mathbb{R}$.

Let $p \in \mathbb{N}$. We say that a sequence $x\colon \mathbb{N} \to \mathbb{R}$ is a p-solution of equation (E) if equality (E) is satisfied for any $n \geq p$. We say that x is a solution if it is a p-solution for certain $p \in \mathbb{N}$. If x is a p-solution for any $p \in \mathbb{N}$, then we say that x is a full solution.

In this paper, we present a new approach to the theory of asymptotic properties of solutions. The main concept, in our theory, is an asymptotic difference pair. The idea of the paper

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is based on the following observation. If x is a solution of (E), f is bounded and the sequence a is 'sufficiently small', then $\Delta^m x$ is close to b, and x is close to the set

$$\Delta^{-m}b = \left\{ y \in \mathbb{R}^{\mathbb{N}} : \Delta^m y = b \right\}.$$

This means that

$$x \in \Delta^{-m}b + Z \tag{1.1}$$

where Z is a certain space of 'small' sequences. Usually $Z = c_0$ is the space of all convergent to zero sequences. In this paper we replace c_0 by various subspaces of $\mathbb{R}^{\mathbb{N}}$.

More precisely, assume that A and Z are linear subspaces of $\mathbb{R}^{\mathbb{N}}$ such that $A \subset \Delta^m Z$ and $u\alpha \in A$ for any bounded sequence u and any $\alpha \in A$. If $a \in A$ and x is a solution of (E) such that the sequence $u = f \circ x \circ \sigma$ is bounded, then

$$\Delta^m x = au + b \in A + b \subset \Delta^m Z + b.$$

Hence $\Delta^m x = \Delta^m z + b$ for certain $z \in Z$ and we get $\Delta^m(x - z) = b$. Therefore $x - z \in \Delta^{-m}b$ and we obtain (1.1).

We say that (A, Z) is an asymptotic difference pair of order m (the precise definition is given in Section 3). In the classic case, for example in [7,14,15], we have

$$A = \left\{ a \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} n^{m-1}|a_n| < \infty \right\}, \qquad Z = c_0.$$

In this paper we present some other examples of asymptotic difference pairs. Our purpose is to present some basic properties of such pairs. Next, we use asymptotic difference pairs in the study of asymptotic properties of solutions. For a given asymptotic difference pair (A, Z), assuming $a \in A$, we obtain sufficient conditions under which for any solution x of (E) there exists $y \in \Delta^{-m}b$ such that $x - y \in Z$. Moreover, assuming $Z \subset c_0$ and using the fixed point theorem, we obtain sufficient conditions under which for any $y \in \Delta^{-m}b$ there exists a solution x of (E) such that $y - x \in Z$. Even more, we can 'compute modulo Z' some parts of the set of solutions of (E) (see Theorem 4.9 and Theorem 4.11 in Section 4).

The concept of an asymptotic difference pair is an effect of comparing the results from some previous papers. In those papers, implicitly, some concrete asymptotic difference pairs are used (for details see Section 7). In fact, this paper is a continuation of a cycle of papers [14–20].

In the study of asymptotic properties of solutions to difference equations the Schauder fixed point theorem is often used. This theorem is applicable to convex and compact subsets of Banach spaces. But the space $\mathbb{R}^{\mathbb{N}}$ of all real sequences with usual 'sup' norm is not a normed space. We introduce a topology in $\mathbb{R}^{\mathbb{N}}$, which we call the regional topology. Next, in Theorem 2.6, we present the 'regional version' of the Schauder fixed point theorem. This theorem is applicable to any convex and compact subset Q of $\mathbb{R}^{\mathbb{N}}$ which is ordinary in the sense that $\|x - y\| < \infty$ for all $x, y \in Q$. In fact, the regional topology is the topology of uniform convergence. For more information about the regional topology see [20].

The main technical tool in our investigations is the iterated remainder operator. The fundamental theory of this operator is given in [19]. This approach to the study of asymptotic properties of solutions to difference equations were inspired by the following papers [2–4,6,7,9,11,24–28]. Moreover, the papers [5,8,10,12,13,21–23] were the inspiration to the study of 'continuous' version of the iterated remainder operator. The basic properties of this 'continuous' operator are presented in [20]. Probably, using the 'continuous' version of the iterated remainder operator, some of the results from this paper can be transferred to the theory of ordinary differential equations.

The paper is organized as follows. In Section 2, we introduce notation and terminology. In Section 3, we define asymptotic difference pairs and establish some of their basic properties. In Section 4, we obtain our main results. In Section 5, we present some examples of difference pairs. In our investigations the spaces $\mathrm{A}(t)$ (see (2.1)) play an important role. In Section 6, we obtain some characterizations of $\mathrm{A}(t)$. These results extend some classic tests for absolute convergence of series and extend results from [19]. In Section 7, we present some consequences of our main results. Next we give some remarks.

2 Notation and terminology

Let $\mathbb{Z}$ denote the set of all integers. If $p, k \in \mathbb{Z}$, $p \le k$, then $\mathbb{N}_p$, $\mathbb{N}_p^k$ denote the sets defined by

$$\mathbb{N}_p = \{p, p+1, \dots\}, \qquad \mathbb{N}_p^k = \{p, p+1, \dots, k\}.$$

The space of all sequences $x\colon \mathbb{N} \to \mathbb{R}$ we denote by SQ. Moreover, by BS we denote the Banach space of all bounded sequences $x \in \mathrm{SQ}$ equipped with 'sup' norm. We use the symbols

$$\mathrm{Sol}(E), \qquad \mathrm{Sol}_p(E), \qquad \mathrm{Sol}_\infty(E)$$

to denote the set of all full solutions of (E), the set of all p-solutions of (E), and the set of all solutions of (E) respectively. Note that

$$\mathrm{Sol}(E) \subset \mathrm{Sol}_p(E) \subset \mathrm{Sol}_\infty(E)$$

for any $p \in \mathbb{N}$. For $p \in \mathbb{N}$ we define

$$\mathrm{Fin}(p) = \{x \in \mathrm{SQ} : x_n = 0 \quad \text{for } n \ge p\}.$$

Moreover, let

$$\mathrm{Fin}(\infty) = \mathrm{Fin} = \bigcup_{p=1}^{\infty} \mathrm{Fin}(p).$$

Note that all $\mathrm{Fin}(p)$ are linear subspaces of SQ and

$$0 = \mathrm{Fin}(1) \subset \mathrm{Fin}(2) \subset \mathrm{Fin}(3) \subset \cdots \subset \mathrm{Fin}(\infty).$$

If x, y in SQ, then xy denotes the sequence defined by pointwise multiplication

$$xy(n) = x_n y_n.$$

Moreover, $|x|$ denotes the sequence defined by $|x|(n) = |x_n|$ for every n.

Remark 2.1. A sequence $x \in \mathrm{SQ}$ is a p-solution of (E) if and only if

$$\Delta^m x \in a(f \circ x \circ \sigma) + b + \mathrm{Fin}(p)$$

and, consequently, x is a solution of (E) if and only if

$$\Delta^m x \in a(f \circ x \circ \sigma) + b + \mathrm{Fin}.$$

We use the symbols 'big O' and 'small o' in the usual sense but for $a \in \mathrm{SQ}$ we also regard $\mathrm{o}(a)$ and $\mathrm{O}(a)$ as subspaces of SQ. More precisely, let

$$\mathrm{o}(1) = \{x \in \mathrm{SQ} : \ x \text{ is convergent to zero}\}, \qquad \mathrm{O}(1) = \{x \in \mathrm{SQ} : \ x \text{ is bounded}\}$$

and for $a \in \mathrm{SQ}$ let

$$\mathrm{o}(a) = a\mathrm{o}(1) + \mathrm{Fin} = \{ax : \ x \in \mathrm{o}(1)\} + \mathrm{Fin},$$
$$\mathrm{O}(a) = a\mathrm{O}(1) + \mathrm{Fin} = \{ax : \ x \in \mathrm{O}(1)\} + \mathrm{Fin}.$$

Moreover, let

$$\mathrm{o}(n^{-\infty}) = \bigcap_{s \in \mathbb{R}} \mathrm{o}(n^s) = \bigcap_{k=1}^{\infty} \mathrm{o}(n^{-k}), \qquad \mathrm{O}(n^{\infty}) = \bigcup_{s \in \mathbb{R}} \mathrm{O}(n^s) = \bigcup_{k=1}^{\infty} \mathrm{O}(n^k).$$

Note that if $a_n \neq 0$ for any n, then

$$\mathrm{o}(a) = a\mathrm{o}(1), \qquad \mathrm{O}(a) = a\mathrm{O}(1).$$

For $b \in \mathrm{SQ}$ and $X \subset \mathrm{SQ}$ we define

$$\Delta^{-m} b = \{y \in \mathrm{SQ} : \ \Delta^m y = b\}, \qquad \Delta^{-m} X = \{y \in \mathrm{SQ} : \ \Delta^m y \in X\}.$$

Moreover, let

$$\mathrm{Pol}(m-1) = \Delta^{-m} 0 = \mathrm{Ker}\Delta^m = \{x \in \mathrm{SQ} : \Delta^m x = 0\}.$$

Then $\mathrm{Pol}(m-1)$ is the space of all polynomial sequences of degree less than m.

For a subset A of a metric space X and $\varepsilon > 0$ we define an ε-framed interior of A by

$$\mathrm{Int}(A, \varepsilon) = \{x \in X : \overline{\mathrm{B}}(x, \varepsilon) \subset A\}$$

where $\overline{\mathrm{B}}(x, \varepsilon)$ denotes a closed ball of radius ε about x.

We say that a subset U of X is a uniform neighborhood of a subset Z of X, if there exists a positive number ε such that $Z \subset \mathrm{Int}(U, \varepsilon)$. For a positive constant M let

$$|f \leq M| = \{t \in \mathbb{R} : |f(t)| \leq M\}.$$

Let

$$\mathrm{A}(1) := \left\{ a \in \mathrm{SQ} : \sum_{n=1}^{\infty} |a_n| < \infty \right\}.$$

For $t \in [1, \infty)$ we define

$$\mathrm{A}(t) := \left\{ a \in \mathrm{SQ} : \sum_{n=1}^{\infty} n^{t-1} |a_n| < \infty \right\} = (n^{1-t})\mathrm{A}(1). \tag{2.1}$$

Moreover, let

$$A(\infty) = \bigcap_{t \in [1,\infty)} \mathrm{A}(t) = \bigcap_{k=1}^{\infty} \mathrm{A}(k).$$

Obviously any $\mathrm{A}(t)$ is a linear subspace of $\mathrm{o}(1)$ such that

$$\mathrm{O}(1)\mathrm{A}(t) \subset \mathrm{A}(t).$$

Note that if $1 \leq t \leq s$ then

$$\mathrm{A}(\infty) \subset \mathrm{A}(s) \subset \mathrm{A}(t) \subset \mathrm{A}(1).$$

Remark 2.2. If $p \in \mathbb{N}$, $\lambda \in (0,1)$, $t \in [1, \infty)$, $s \in (0, \infty)$, and $\mu > 1$, then

$$0 = \mathrm{Fin}(1) \subset \mathrm{Fin}(p) \subset \mathrm{Fin} \subset \mathrm{o}(\lambda^n) \subset \mathrm{O}(\lambda^n) \subset \mathrm{o}(n^{-\infty}) = \mathrm{A}(\infty),$$

$$\mathrm{A}(\infty) \subset \mathrm{A}(t) \subset \mathrm{A}(1) \subset \mathrm{o}(1) \subset \mathrm{O}(1) \subset \mathrm{o}(n^s) \subset \mathrm{O}(n^s) \subset \mathrm{O}(n^{\infty}) \subset \mathrm{o}(\mu^n) \subset \mathrm{O}(\mu^n).$$

2.1 Unbounded functions

We say that a function $g\colon \mathbb{R} \to \mathbb{R}$ is unbounded at a point $p \in [-\infty, \infty]$ if there exists a sequence $x \in \mathrm{SQ}$ such that $\lim_{n\to\infty} x_n = p$ and the sequence $g \circ x$ is unbounded. Let

$$U(g) = \{p \in [-\infty, \infty] : g \text{ is unbounded at } p\}.$$

A function $g\colon \mathbb{R} \to \mathbb{R}$ is called locally bounded if for any $t \in \mathbb{R}$ there exists a neighborhood U of t such that the restriction $g|U$ is bounded. Note that any continuous function and any monotonic function $g\colon \mathbb{R} \to \mathbb{R}$ are locally bounded.

Remark 2.3. Assume $g\colon \mathbb{R} \to \mathbb{R}$. Then

(a) g is bounded if and only if $U(g) = \emptyset$,

(b) g is locally bounded if and only if $U(g) \subset \{\infty, -\infty\}$.

Example 2.4. Assume $g\colon \mathbb{R} \to \mathbb{R}$, $T = \{t_1, t_2, \ldots, t_n\} \subset \mathbb{R}$. Then

(a) $U(\max(1,t)) = U(t + |t|) = U(e^t) = \{\infty\}$,

(b) $U(\min(1,t)) = U(t - |t|) = U(e^{-t}) = \{-\infty\}$,

(c) if g is a nonconstant polynomial, then $U(g) = \{-\infty, \infty\}$,

(d) if $g(t) = 1/t$ for $t \neq 0$, then $U(g) = \{0\}$,

(e) if $g(t) = ((t - t_1) \cdots (t - t_n))^{-1}$ for $t \notin T$, then $U(g) = T$.

Remark 2.5. Assume $g, h\colon \mathbb{R} \to \mathbb{R}$. Then

$$U(g + h) \subset U(g) \cup U(h), \qquad U(gh) \subset U(g) \cup U(h).$$

This follows from the fact that if g and h are bounded at a point p, then $g + h$ and gh are also bounded at p. Note also that if $U(g) \cap U(h) = \emptyset$, then

$$U(g + h) = U(g) \cup U(h).$$

This is a consequence of the fact that if exactly one of the functions g, h is bounded at a point p, then $g + h$ is unbounded at p.

2.2 Regional topology

For a sequence $x \in \mathrm{SQ}$ we define a generalized norm $\|x\| \in [0, \infty]$ by

$$\|x\| = \sup\{|x_n| : n \in \mathbb{N}\}.$$

We say that a subset Q of SQ is ordinary if $\|x - y\| < \infty$ for any $x, y \in Q$. We regard any ordinary subset Q of SQ as a metric space with metric defined by

$$d(x, y) = \|x - y\|. \tag{2.2}$$

Let $U \subset \mathrm{SQ}$. We say that U is regionally open if $U \cap Q$ is open in Q for any ordinary subset Q of SQ. The family of all regionally open subsets is a topology on SQ which we call the regional topology. We regard any subset of SQ as a topological space with topology induced by the regional topology. The basic properties of regional topology are presented in [20]. We will use the following 'regional' version of the Schauder fixed point theorem.

Theorem 2.6. *Assume Q is an ordinary compact and convex subset of* SQ. *Then any continuous map $F\colon Q \to Q$ has a fixed point.*

Proof. Let $a \in Q$, $W = Q - a$ and $T\colon Q \to W$, $T(x) = x - a$. Since Q is ordinary, we have $W \subset \mathrm{BS}$. Moreover, T is an isometry of Q onto W. Note also that T preserves convexity. Hence W is a compact and convex subset of the Banach space BS. Let

$$H\colon W \to W, \qquad H = T \circ F \circ T^{-1}.$$

Then H is continuous and, by the usual Schauder fixed point theorem, there exist a point $y \in W$ such that $Hy = y$. Let $x = T^{-1}y$. Then

$$x = T^{-1}y = T^{-1}Hy = T^{-1}TFT^{-1}y = FT^{-1}y = Fx. \qquad \square$$

2.3 Remainder operator

In this subsection, we recall from [19] some basic properties of the iterated remainder operator. Let $\mathrm{S}(m)$ denote the set of all sequences $a \in \mathrm{SQ}$ such that the series

$$\sum_{i_1=1}^{\infty}\sum_{i_2=i_1}^{\infty}\cdots\sum_{i_m=i_{m-1}}^{\infty} a_{i_m}.$$

is convergent. For any $a \in \mathrm{S}(m)$ we define the sequence $r^m(a)$ by

$$r^m(a)(n) = \sum_{i_1=n}^{\infty}\sum_{i_2=i_1}^{\infty}\cdots\sum_{i_m=i_{m-1}}^{\infty} a_{i_m}. \tag{2.3}$$

Then $\mathrm{S}(m)$ is a linear subspace of $\mathrm{o}(1)$, $r^m(a) \in \mathrm{o}(1)$ for any $a \in \mathrm{S}(m)$ and

$$r^m\colon \mathrm{S}(m) \to \mathrm{o}(1) \tag{2.4}$$

is a linear operator which we call the iterated remainder operator of order m. The value $r^m(a)(n)$ we denote also by $r^m_n(a)$ or simply $r^m_n a$. If $|a| \in \mathrm{S}(m)$, then $a \in \mathrm{S}(m)$ and $r^m(a)$ is given by

$$r^m(a)(n) = \sum_{j=n}^{\infty}\binom{m-1+j-n}{m-1}a_j.$$

Note that if $m = 1$, then

$$r(a)(n) = r^1(a)(n) = \sum_{j=n}^{\infty} a_j$$

is the n-th remainder of the series $\sum_{j=1}^{\infty} a_j$. The following lemma is a consequence of [19, Lemma 3.1].

Lemma 2.7. *Assume $x, u \in \mathrm{SQ}$ and $p \in \mathbb{N}$. Then*

(001) *if $|x| \in \mathrm{S}(m)$, then $x \in \mathrm{S}(m)$ and $|r^m x| \le r^m|x|$,*

(002) *$|x| \in \mathrm{S}(m)$ if and only if $\sum_{n=1}^{\infty} n^{m-1}|x_n| < \infty$,*

(003) *if $|x| \in \mathrm{S}(m)$, then $r^m_p|x| \le \sum_{n=p}^{\infty} n^{m-1}|x_n|$,*

(004) *$|x| \in \mathrm{S}(m)$ if and only if $x \in \mathrm{A}(m)$,*

(005) $x \in \mathrm{A}(m)$ *if and only if* $\mathrm{O}(x) \subset \mathrm{S}(m)$,

(006) $\Delta^m \mathrm{o}(1) = \mathrm{S}(m), \quad r^m \mathrm{S}(m) = \mathrm{o}(1)$,

(007) *if* $x \in \mathrm{S}(m)$, *then* $\Delta^m r^m x = (-1)^m x$,

(008) *if* $x \in \mathrm{o}(1)$, *then* $r^m \Delta^m x = (-1)^m x$,

(009) *if* Z *is a linear subspace of* $\mathrm{o}(1)$, *then* $r^m \Delta^m Z = Z$,

(010) *if* A *is a linear subspace of* $\mathrm{S}(m)$, *then* $\Delta^m r^m A = A$,

(011) *if* $x \in \mathrm{A}(m)$, $u \in \mathrm{O}(1)$, *then* $|r^m(ux)| \le |u| r^m |x|$,

(012) *if* $ux \in \mathrm{A}(m)$, $\Delta u \ge 0$ *and* $u > 0$, *then* $ur^m|x| \le r^m|ux|$,

(013) *if* $x, y \in \mathrm{S}(m)$ *and* $x_n \le y_n$ *for* $n \ge p$, *then* $r_n^m x \le r_n^m y$ *for* $n \ge p$,

(014) $r^m \mathrm{Fin}(p) = \mathrm{Fin}(p) = \Delta^m \mathrm{Fin}(p)$, $r^m \mathrm{Fin} = \mathrm{Fin} = \Delta^m \mathrm{Fin}$,

(015) *if* $x \ge 0$, *then* $r^m x$ *is nonnegative and nonincreasing.*

3 Asymptotic difference pairs

Let Z be a linear subspace of SQ. We say that a subset W of SQ is Z-invariant if $W + Z \subset W$. We say, that a subset X of SQ is:

asymptotic if X is Fin-invariant,

evanescent if $X \subset \mathrm{o}(1)$,

modular if $\mathrm{O}(1)X \subset X$,

c-stable if X is $\mathrm{o}(1)$-invariant.

We say that a pair (A, Z) of linear subspaces of SQ is a difference asymptotic pair of order m or, simply, m-pair if Z is asymptotic, A is modular and $A \subset \Delta^m Z$. We say that an m-pair (A, Z) is evanescent if Z is evanescent.

Remark 3.1. For any $a \in \mathrm{SQ}$ the spaces $\mathrm{o}(a)$ and $\mathrm{O}(a)$ are asymptotic and modular.

Remark 3.2. If W is an asymptotic subset of SQ, $x \in W$ and x' is a sequence obtained from x by changing finite number of terms, then $x' \in W$. Moreover, a linear subspace Z of SQ is asymptotic if and only if $\mathrm{Fin} \subset Z$.

Remark 3.3. If (A, Z) is an evanescent m-pair, then, using Lemma 2.7 (006), we have

$$A \subset \Delta^m Z \subset \Delta^m \mathrm{o}(1) = \mathrm{S}(m) \subset \mathrm{o}(1).$$

Hence the space A is evanescent.

Remark 3.4. If $a \in \mathrm{SQ}$, then the sequence $\mathrm{sgn} \circ a$ is bounded and $|a| = (\mathrm{sgn} \circ a)a$. Hence, if W is a modular subset of SQ, then $|a| \in W$ for any $a \in W$. In particular, if (A, Z) is an evanescent m-pair and $a \in A$, then

$$|a| \in A \subset \Delta^m Z \subset \Delta^m \mathrm{o}(1) = \mathrm{S}(m).$$

Therefore $A \subset \mathrm{A}(m)$ and, for any $a \in A$, the sequences $r^m a$ and $r^m |a|$ are defined.

Lemma 3.5. *Assume (A, Z) is an m-pair, $a, b \in \mathrm{SQ}$, and $a - b \in A$. Then*

$$\Delta^{-m}a + Z = \Delta^{-m}b + Z.$$

Proof. We have $a - b \in A \subset \Delta^m Z$. Hence there exists $z_0 \in Z$ such that $a - b = \Delta^m z_0$. Let $x \in \Delta^{-m}a$ and $z \in Z$. Then

$$\Delta^m(x - z_0) = \Delta^m x - \Delta^m z_0 = a - (a - b) = b.$$

Therefore $x + z = x - z_0 + z_0 + z \in \Delta^{-m}b + Z$. Thus

$$\Delta^{-m}a + Z \subset \Delta^{-m}b + Z.$$

Since $b - a = -(a - b) \in A$, we have

$$\Delta^{-m}b + Z \subset \Delta^{-m}a + Z.$$

The proof is complete. □

Lemma 3.6. *Assume (A, Z) is an m-pair and $b \in A$. Then*

$$\Delta^{-m}b + Z = \mathrm{Pol}(m - 1) + Z.$$

Proof. This lemma is an immediate consequence of the previous lemma. □

Lemma 3.7. *Assume (A, Z) is an m-pair, $a \in A$, $b, x \in \mathrm{SQ}$ and*

$$\Delta^m x \in \mathrm{O}(a) + b.$$

Then $x \in \Delta^{-m}b + Z$.

Proof. The condition $a \in A$ implies $\mathrm{O}(a) \subset A$. Hence,

$$\Delta^m x - b \in \mathrm{O}(a) \subset A \subset \Delta^m Z.$$

Therefore, there exists $z \in Z$ such that $\Delta^m x - b = \Delta^m z$. Then

$$\Delta^m(x - z) = \Delta^m x - \Delta^m z = b.$$

Thus $x - z \in \Delta^{-m}b$ and we obtain $x = x - z + z \in \Delta^{-m}b + Z$. □

Lemma 3.8 (Comparison test)**.** *Assume A is an asymptotic, modular linear subspace of SQ, $b \in A$, $a \in \mathrm{SQ}$, and $|a_n| \le |b_n|$ for large n. Then $a \in A$.*

Proof. Assume $|a_n| \le |b_n|$ for $n \ge p$. Let

$$h_n = \begin{cases} 0 & \text{if } b_n = 0 \\ a_n/b_n & \text{if } \ b_n \ne 0. \end{cases}$$

Then $h \in \mathrm{O}(1)$. Moreover, if $n \ge p$ and $b_n = 0$, then $a_n = 0$. Hence $a_n = h_n b_n$ for $n \ge p$. Therefore $a - hb \in \mathrm{Fin}(p)$. Let $z = a - hb$. Then

$$a = hb + z \in \mathrm{O}(1)A + \mathrm{Fin} \subset A + A = A.$$

□

4 Solutions

In this section, in Theorems 4.9 and 4.11, we obtain our main results. First we introduce the notion of f-ordinary and f-regular sets. We use these sets in Theorem 4.11. At the end of the section we present some examples of f-regular sets.

We say that a subset W of SQ is f-ordinary if for any $x \in W$ the sequence $f \circ x$ is bounded. We say that a subset W of SQ is f-regular if for any $x \in W$ there exists an index p such that f is continuous and bounded on some uniform neighborhood of the set $x(\mathbb{N}_p)$. For $x \in \mathrm{SQ}$ let

$$\mathrm{L}(x) = \{p \in [-\infty, \infty] : p \text{ is a limit point of } x\}.$$

Lemma 4.1. *If $x \in \mathrm{SQ}$, then*

$$f \circ x \in \mathrm{O}(1) \quad \text{or} \quad \mathrm{L}(x) \cap \mathrm{U}(f) \neq \emptyset.$$

Proof. Assume the sequence $f \circ x$ is unbounded from above. Then there exists a subsequence x_{n_k} such that

$$\lim_{k \to \infty} f(x_{n_k}) = \infty.$$

Let $y_k = x_{n_k}$ and let $p \in L(y)$. There exists a subsequence y_{k_i} such that

$$\lim_{i \to \infty} y_{k_i} = p.$$

Then $\lim_{i \to \infty} f(y_{k_i}) = \infty$ and we obtain $p \in U(f)$. Since y is a subsequence of x, we have $\mathrm{L}(y) \subset \mathrm{L}(x)$. Hence $p \in U(f) \cap L(x)$. Analogously, if the sequence $f \circ x$ is unbounded from below, then $U(f) \cap \mathrm{L}(x) \neq \emptyset$. $\square$

Note that if the sequence $f \circ x$ is bounded, then $f \circ x \circ \sigma$ is also bounded.

Theorem 4.2. *Assume (A, Z) is an m-pair, $a \in A$, and $x \in \mathrm{Sol}_\infty(\mathrm{E})$. Then*

$$x \in \Delta^{-m} b + Z \quad \text{or} \quad \mathrm{L}(x) \cap \mathrm{U}(f) \neq \emptyset.$$

Proof. Assume $\mathrm{L}(x) \cap \mathrm{U}(f) = \emptyset$. Then, by Lemma 4.1, the sequence $f \circ x$ is bounded. Hence the sequence $f \circ x \circ \sigma$ is bounded too. By Remark 2.1,

$$\Delta^m x \in a(f \circ x \circ \sigma) + b + \mathrm{Fin}.$$

Hence

$$\Delta^m x \in a\mathrm{O}(1) + \mathrm{Fin} + b = \mathrm{O}(a) + b.$$

Using Lemma 3.7 we obtain $x \in \Delta^{-m} b + Z$. The proof is complete. $\square$

Corollary 4.3. *Assume (A, Z) is an m-pair, $a \in A$, $B \cup C = \mathbb{R}$, C is closed in $\mathbb{R}$, f is bounded on B, $\mathrm{U}(f) \subset \mathbb{R}$, and x is a solution of (E). Then*

$$x \in \Delta^{-m} b + Z \quad \text{or} \quad \mathrm{L}(x) \cap C \neq \emptyset.$$

Proof. Using the relation $\mathrm{U}(f) \subset \mathbb{R}$ we see that $\mathrm{U}(f) \subset C$. Hence the assertion is a consequence of Theorem 4.2. $\square$

Example 4.4. Assume (A, Z) is an m-pair, $a \in A$, and $f(t) = t^{-1}$ for $t \neq 0$. If x is a solution of (E) such that 0 is not a limit point of x, then, by Theorem 4.2, $x \in \Delta^{-m} b + Z$.

Example 4.5. Assume (A, Z) is an m-pair, $a \in A$, f is continuous and there exists a proper limit $\lim_{t\to\infty} f(t)$. Then, by Theorem 4.2, for any bounded below solution x of (E) we have $x \in \Delta^{-m}b + Z$.

Theorem 4.6. *Assume (A, Z) is an m-pair, $a \in A$, and $W \subset \mathrm{SQ}$ is f-ordinary. Then*

$$W \cap \mathrm{Sol}_\infty(\mathrm{E}) \subset \Delta^{-m}b + Z.$$

Proof. Let $x \in W \cap \mathrm{Sol}_\infty(\mathrm{E})$. By Remark 2.1,

$$\Delta^m x \in a(f \circ x \circ \sigma) + b + \mathrm{Fin}.$$

Since $x \in W$, we have $f \circ x = \mathrm{O}(1)$. Hence $f \circ x \circ \sigma = \mathrm{O}(1)$ and

$$\Delta^m x \in a\mathrm{O}(1) + \mathrm{Fin} + b = \mathrm{O}(a) + b.$$

Now, the assertion follows from Lemma 3.7. □

Theorem 4.7. *Assume (A, Z) is an evanescent m-pair, $a \in A$, $M > 0$, $p \in \mathbb{N}$,*

$$y \in \Delta^{-m}b, \quad R = Mr^m|a|, \quad (y \circ \sigma)(\mathbb{N}_p) \subset \mathrm{Int}(|f \le M|, R_p), \tag{4.1}$$

and f is continuous on $|f \le M|$. Then $y \in \mathrm{Sol}_p(\mathrm{E}) + Z$.

Proof. For $x \in \mathrm{SQ}$ let

$$x^* = f \circ x \circ \sigma.$$

Moreover, let $\rho \in \mathrm{SQ}$,

$$\rho_n = \begin{cases} 0 & \text{for } n < p, \\ R_n & \text{for } n \ge p, \end{cases} \qquad S = \{x \in \mathrm{SQ} : |x - y| \le \rho\}. \tag{4.2}$$

By Lemma 2.7 (015), the sequence R is nonincreasing. Hence $\rho_n \le R_p$ for any n.

Assume $x \in S$. Then

$$|x_{\sigma(n)} - y_{\sigma(n)}| \le \rho_{\sigma(n)} \le R_p$$

for any n. By (4.1), $x_{\sigma(n)} \in \overline{\mathrm{B}}(y_{\sigma(n)}, R_p) \subset |f \le M|$ for $n \ge p$. Hence $|x^*_n| \le M$ for $n \ge p$. Therefore the sequence x^* is bounded. Since A is a modular space, we have $ax^* \in A$. By Remark 3.3, $A \subset \mathrm{S}(m)$. Hence the sequence $r^m(ax^*)$ is defined for any $x \in S$. Let

$$H\colon S \to \mathrm{SQ}, \qquad H(x)(n) = \begin{cases} y_n & \text{for } n < p, \\ y_n + (-1)^m r^m_n(ax^*) & \text{for } n \ge p. \end{cases} \tag{4.3}$$

If $x \in S$ and $n \ge p$, then, using Lemma 2.7 (001) and (013), we get

$$|H(x)(n) - y_n| = |r^m_n(ax^*)| \le r^m_n|ax^*| \le r^m_n|Ma| = Mr^m_n|a| = R_n = \rho_n.$$

Hence $HS \subset S$. We will show that H is continuous. By Remark 3.4, $A \subset \mathrm{A}(m)$. Therefore, by Lemma 2.7 (004) and (002),

$$\sum_{n=1}^{\infty} n^{m-1}|a_n| < \infty.$$

Let $\varepsilon > 0$. There exist $q > p$ and $\alpha > 0$ such that

$$2M \sum_{n=q}^{\infty} n^{m-1}|a_n| < \varepsilon \quad \text{and} \quad \alpha \sum_{n=p}^{q} n^{m-1}|a_n| < \varepsilon. \tag{4.4}$$

Let

$$W = \bigcup_{n=p}^{q} [y_{\sigma(n)} - R_p, y_{\sigma(n)} + R_p].$$

Then W is compact and, by (4.1), $W \subset |f \leq M|$. Hence f is uniformly continuous on W. Choose a positive δ such that for $s, t \in W$ the condition $|s - t| < \delta$ implies $|f(s) - f(t)| < \alpha$. Assume $x, z \in S$, $\|x - z\| < \delta$ and let $u = a(x^* - z^*)$. Then, using Lemma 2.7 (001) and (015), we have

$$\|Hx - Hz\| = \sup_{n \geq 1} |H(x)(n) - H(z)(n)| = \sup_{n \geq p} |r_n^m u| \leq \sup_{n \geq p} r_n^m |u| = r_p^m |u|.$$

Hence, by Lemma 2.7 (003),

$$\|Hx - Hz\| \leq \sum_{n=p}^{\infty} n^{m-1}|u_n| \leq \sum_{n=p}^{q} n^{m-1}|u_n| + \sum_{n=q}^{\infty} n^{m-1}|u_n|.$$

Note that $|u_n| \leq \alpha|a_n|$ for $n \in \mathbb{N}_p^q$. Moreover, $|x^*n| \leq M$ and $|z^*n| \leq M$ for $n \geq q$. Hence, by (4.4), we get

$$\|Hx - Hz\| \leq \alpha \sum_{n=p}^{q} n^{m-1}|a_n| + 2M \sum_{n=q}^{\infty} n^{m-1}|a_n| < \varepsilon + \varepsilon.$$

Therefore H is continuous. Obviously the set S is ordinary and convex. We will show that S is compact. Note that, by (4.1) and (2.4), we have $R = Mr^m|a| = \mathrm{o}(1)$. Hence, by (4.2), $\rho = \mathrm{o}(1)$. Let

$$T = \{x \in \mathrm{BS} : |x| \leq \rho\}.$$

Then T is a closed subset of BS. Choose an $\varepsilon > 0$. Then there exists an index q such that $\rho_n < \varepsilon$ for $n \geq q$. For $n = 1, \ldots, q$ let G_n denote a finite ε-net for the interval $[-\rho_n, \rho_n]$ and let

$$G = \{x \in T : x_n \in G_n \text{ for } n \leq q \text{ and } x_n = 0 \text{ for } n > q\}.$$

Then G is a finite ε-net for T. Hence T is a complete and totally bounded metric space and so, T is compact. Let $F\colon T \to S$ be given by $F(x)(n) = x_n + y_n$. Then F is an isometry of T onto S. Hence S is compact. By Theorem 2.6, there exists a sequence $x \in S$ such that $Hx = x$. Then, by (4.3), for $n \geq p$, we have

$$x_n = y_n + (-1)^m r_n^m (ax^*). \tag{4.5}$$

Hence $\Delta^m x_n = \Delta^m y_n + \Delta^m r_n^m((-1)^m ax^*)$ for $n \geq p$. Using the fact that $y \in \Delta^{-m} b$ and Lemma 2.7 (007), we obtain

$$\Delta^m x_n = b_n + a_n x_n^* = b_n + a_n f(x_{\sigma(n)})$$

for $n \geq p$. Thus

$$x \in \mathrm{Sol}_p(\mathrm{E}).$$

By (4.5), $y - x + (-1)^m r^m(ax^*) \in \mathrm{Fin}(p)$. Since $ax^* \in A$, we have

$$y - x \in r^m A + \mathrm{Fin}. \tag{4.6}$$

Using the definition of an evanescent m-pair and Lemma 2.7 (009), we have

$$r^m A \subset r^m \Delta^m Z = Z.$$

Now, by (4.6), $y - x \in Z + \mathrm{Fin}$. By Remark 3.2, $Z + \mathrm{Fin} = Z$. Hence $y \in x + Z$. □

Corollary 4.8. *Assume (A, Z) is an evanescent m-pair, $a \in A$, $y \in \Delta^{-m} b$ and $\{y\}$ is f-regular. Then*

$$y \in \mathrm{Sol}_\infty(\mathrm{E}) + Z.$$

Proof. There exist a positive M and $\delta > 0$ such that

$$(y \circ \sigma)(\mathbb{N}) \subset \mathrm{Int}(|f \leq M|, \delta).$$

Let $R = M r^m |a|$. Then $R = \mathrm{o}(1)$ and $R_p < \delta$ for certain p. Hence

$$\mathrm{Int}(|f \leq M|, \delta) \subset \mathrm{Int}(|f \leq M|, R_p)$$

and, by Theorem 4.7, $y \in \mathrm{Sol}_p(\mathrm{E}) + Z$. □

The next theorem is our first main result. We assume that f is continuous and bounded. This assumption is very strong but our result is also strong.

Theorem 4.9. *Assume (A, Z) is an evanescent m-pair, $a \in A$, $p \in \mathbb{N}$, and f is continuous and bounded. Then*

$$\mathrm{Sol}(\mathrm{E}) + Z = \mathrm{Sol}_p(\mathrm{E}) + Z = \mathrm{Sol}_\infty(\mathrm{E}) + Z = \Delta^{-m} b + Z.$$

Proof. Choose M such that $|f| \leq M$. Then $|f \leq M| = \mathbb{R}$. Hence

$$\mathrm{Int}(|f \leq M|, \delta) = \mathbb{R}$$

for any positive δ. By Theorem 4.7 we have

$$\Delta^{-m} b \subset \mathrm{Sol}_p(\mathrm{E}) + Z$$

for any p. For a given $p \in \mathbb{N}$ we obtain

$$\Delta^{-m} b + Z \subset \mathrm{Sol}(\mathrm{E}) + Z \subset \mathrm{Sol}_p(\mathrm{E}) + Z \subset \mathrm{Sol}_\infty(\mathrm{E}) + Z.$$

On the other hand, by Theorem 4.6, taking $W = \mathrm{SQ}$ we obtain

$$\mathrm{Sol}_\infty(\mathrm{E}) + Z \subset \Delta^{-m} b + Z.$$

The proof is complete. □

Lemma 4.10. *Assume Z is a linear subspace of a linear space X, $D, S, W \subset X$, W is Z-invariant, $W \cap S \subset D + Z$ and $W \cap D \subset S + Z$. Then*

$$W \cap S + Z = W \cap D + Z.$$

Proof. Assume $w \in W \cap S$. Since $W \cap S \subset D + Z$, we have $w \in W \cap (D + Z)$. Hence, there exist $d \in D$ and $z \in Z$ such that $w = d + z$. Since W is Z-invariant, we obtain

$$d = w - z \in W + Z \subset W.$$

Hence $w = d + z \in (W \cap D) + Z$. Therefore $W \cap S \subset W \cap D + Z$ and we obtain

$$W \cap S + Z \subset W \cap D + Z + Z = W \cap D + Z.$$

Analogously, we obtain $W \cap D + Z \subset W \cap S + Z$. □

Now we are ready to prove our second main result.

Theorem 4.11. *Assume (A, Z) is an evanescent m-pair, $a \in A$, and $W \subset \mathrm{SQ}$. Then*

(a) *if W is f-ordinary, then $W \cap \mathrm{Sol}_\infty(\mathrm{E}) \subset \Delta^{-m}b + Z$,*

(b) *if W is f-regular, then $W \cap \Delta^{-m}b \subset \mathrm{Sol}_\infty(\mathrm{E}) + Z$,*

(c) *if W is f-regular and Z-invariant, then*

$$W \cap \mathrm{Sol}_\infty(\mathrm{E}) + Z = W \cap \Delta^{-m}b + Z.$$

Proof. Assertion (a) is a special case of Theorem 4.6. (b) is a consequence of Corollary 4.8. Using (a), (b), Lemma 4.10 and the fact that any f-regular set $W \subset \mathrm{SQ}$ is also f-ordinary we obtain (c). □

Remark 4.12. Any subset of an f-regular set is f-regular. If Z is a linear subspace of $\mathrm{o}(1)$, then any c-stable subset W of SQ is also Z-invariant.

Remark 4.13. Assume $W \subset \mathrm{SQ}$ is f-regular and Z is a linear subspace of $\mathrm{o}(1)$. Then the set $W + Z$ is f-regular and Z-invariant.

Example 4.14. Assume f is continuous and bounded on a certain uniform neighborhood of a set $Y \subset \mathbb{R}$. Then the set

$$W = \{y \in \mathrm{SQ} : y(\mathbb{N}) \subset Y\}$$

is f-regular. If $x \in \mathrm{SQ}$ and $z \in \mathrm{o}(1)$, then $\mathrm{L}(x + z) = \mathrm{L}(x)$. Hence the sets

$$W_1 = \{y \in \mathrm{SQ} : \mathrm{L}(y) \subset Y\}, \qquad W_2 = \{y \in \mathrm{SQ} : \lim y_n \in Y\}$$

are f-regular and c-stable.

Example 4.15. If f is bounded, then SQ is f-ordinary and c-stable. Moreover, if f is continuous, then SQ is f-regular.

Example 4.16. If f is locally bounded, then the set $\mathrm{O}(1)$ of all bounded sequences is f-ordinary and c-stable. Moreover, if f is continuous, then $\mathrm{O}(1)$ is f-regular.

Example 4.17. If f is locally bounded, then the set C of all convergent sequences is f-ordinary and c-stable. Moreover, if f is continuous, then C is f-regular.

Example 4.18. Let Z be a linear subspace of $\mathrm{o}(1)$ and $p \in \mathbb{N}$. We say that a sequence $x \in \mathrm{SQ}$ is (p, Z)-asymptotically periodic if there exists a p-periodic sequence y such that $x - y \in Z$. If f is locally bounded, then the set W of all (p, Z)-asymptotically periodic sequences is f-ordinary and Z-invariant. Moreover, if f is continuous, then W is f-regular.

Example 4.19. If $f(t) = e^t$, then the sets

$$W_1 = \left\{x \in \mathrm{SQ} : \limsup_{n\to\infty} x_n < \infty\right\} \quad \text{and} \quad W_2 = \left\{x \in \mathrm{SQ} : \lim_{n\to\infty} x_n = -\infty\right\}$$

are f-regular and c-stable.

Example 4.20. If f is continuous and $\limsup_{t\to\infty} |f(t)| < \infty$, then the sets

$$W_1 = \left\{ x \in \mathrm{SQ} : \liminf_{n\to\infty} x_n > -\infty \right\} \quad \text{and} \quad W_2 = \left\{ x \in \mathrm{SQ} : \lim_{n\to\infty} x_n = \infty \right\}$$

are f-regular and c-stable.

Example 4.21. If $f(t) = t^{-1}$ for $t \neq 0$, then the set $W = \{x \in \mathrm{SQ} : 0 \notin \mathrm{L}(x)\}$ is f-regular and c-stable.

Example 4.22. Assume $g\colon \mathbb{R} \to \mathbb{R}$ is continuous, $T = \{t_1, t_2, \ldots, t_n\} \subset \mathbb{R}$ and

$$f(t) = \frac{g(t)}{(t - t_1)(t - t_2)\ldots(t - t_n)}$$

for $t \notin T$. Then the set $W = \{x \in \mathrm{SQ} : T \cap \mathrm{L}(x) = \emptyset\}$ is f-regular and c-stable.

5 Examples of difference pairs

We say that a subset A of SQ is an m-space, if (A, A) is an m-pair. In this section we present some examples of difference m-pairs and m-spaces. Next we establish some lemmas to justify our examples. Part of those lemmas are a mathematical folklore. We present the proofs of them for the convenience of the reader.

Remark 5.1. Assume that (A, Z) is an m-pair. If Z^* is a linear subspace of SQ such that $Z \subset Z^*$, then (A, Z^*) is an m-pair. Analogously, if A_* is a modular subspace of A, then (A_*, Z) is an m-pair.

Example 5.2. If $a \in \mathrm{A}(m)$, then $(\mathrm{O}(a), r^m\mathrm{O}(a))$ is an evanescent m-pair.

Example 5.3. If X is an asymptotic and modular subspace of $\mathrm{A}(m)$, then $(X, r^m X)$ is an evanescent m-pair.

Example 5.4. Let $s \in (-\infty, -m)$. The following pairs are evanescent m-pairs

$$(\mathrm{o}(n^s), \mathrm{o}(n^{s+m})), \qquad (\mathrm{O}(n^s), \mathrm{O}(n^{s+m})).$$

Example 5.5. If $s \in \mathbb{R}$ and $(s+1)(s+2)\cdots(s+m) \neq 0$, then

$$(\mathrm{o}(n^s), \mathrm{o}(n^{s+m})), \qquad (\mathrm{O}(n^s), \mathrm{O}(n^{s+m}))$$

are m-pairs.

Example 5.6. Let $\lambda \in (0,1)$. The following spaces are evanescent m-spaces

$$\mathrm{Fin}, \qquad \mathrm{o}(\lambda^n), \qquad \mathrm{O}(\lambda^n), \qquad \mathrm{o}(n^{-\infty}).$$

Example 5.7. Let $\lambda \in (1,\infty)$. The following spaces are m-spaces

$$\mathrm{o}(\lambda^n), \qquad \mathrm{O}(\lambda^n), \qquad \mathrm{O}(n^{\infty}).$$

Example 5.8. If $s \in (-\infty, 0]$, then $(\mathrm{A}(m-s), \mathrm{o}(n^s))$ is an evanescent m-pair.

Example 5.9. Assume $s \in (-\infty, m-1]$, and $q \in \mathbb{N}_0^{m-1}$. Then

$$(\mathrm{A}(m-s),\, \mathrm{o}(n^s)), \qquad (\mathrm{A}(m-q),\, \Delta^{-q}\mathrm{o}(1))$$

are m-pairs.

Example 5.10. If $t \in [1, \infty)$, then $(\mathrm{A}(m+t),\ \mathrm{A}(t))$ is an evanescent m-pair.

Note that Example 5.2 is a special case of Example 5.3. Note also that Lemma 2.7 (010) justifies Example 5.3. To justify Examples 5.4 and 5.5 we need the following four lemmas.

Lemma 5.11 (Cesàro–Stolz lemma). *Assume $x, y \in \mathrm{SQ}$, y is strictly monotonic and one of the following conditions is satisfied*

(*a*) $x = \mathrm{o}(1)$ *and* $y = \mathrm{o}(1)$,

(*b*) y *is unbounded.*

Then

$$\liminf \frac{\Delta x}{\Delta y} \le \liminf \frac{x}{y} \le \limsup \frac{x}{y} \le \limsup \frac{\Delta x}{\Delta y}.$$

Proof. If (a) is satisfied, then the assertion is proved in [1]. Assume (b) and y is unbounded from above. Then y is increasing and $\lim y_n = \infty$. Let

$$L = \liminf \frac{\Delta x_n}{\Delta y_n}.$$

If $L = -\infty$, then the inequality

$$\liminf \frac{\Delta x}{\Delta y} \le \liminf \frac{x}{y} \tag{5.1}$$

is obvious. Assume $L > -\infty$. Choose a constant M such that $M < L$. Then there exists an index p such that $\Delta x_n / \Delta y_n \ge M$ for $n \ge p$. We can assume that $y_n > 0$ and $\Delta y_n > 0$ for $n \ge p$. If $n \ge p$, then

$$\begin{aligned} x_n - x_p &= \Delta x_p + \Delta x_{p+1} + \cdots + \Delta x_{n-1} \\ &\ge M(\Delta y_p + \Delta y_{p+1} + \cdots + \Delta y_{n-1}) = M(y_n - y_p). \end{aligned}$$

Hence $x_n \ge M y_n + x_p - M y_p$ and

$$\frac{x_n}{y_n} \ge M + \frac{x_p - M y_p}{y_n}$$

for $n \ge p$. Since $\lim(1/y_n) = 0$, we have

$$\liminf \frac{x_n}{y_n} \ge M.$$

Therefore, we obtain (5.1). Similarly, one can prove the inequality

$$\limsup \frac{x}{y} \le \limsup \frac{\Delta x}{\Delta y}.$$

Replacing y by $-y$ we obtain the result if y is unbounded from below. $\square$

Lemma 5.12. *Assume $x \in \mathrm{SQ}$, $s \in \mathbb{R}$, and $s > -1$ or $x = \mathrm{o}(1)$. Then*

$$\Delta x = \mathrm{o}(n^s) \implies x = \mathrm{o}(n^{s+1}), \qquad \Delta x = \mathrm{O}(n^s) \implies x = \mathrm{O}(n^{s+1}).$$

Proof. If $s = -1$, then, by assumption, $x = \mathrm{o}(1) = \mathrm{o}(n^{s+1})$. Hence the assertion is true for $s = -1$. Assume $s \neq -1$. Note that

$$\frac{\Delta x_n}{\Delta n^{s+1}} = \frac{\Delta x_n}{n^s}\frac{n^s}{\Delta n^{s+1}}.$$

By the proof of [16, Lemma 2.1], the sequence $(n^s/\Delta n^{s+1})$ is convergent. Hence the assertion follows from Lemma 5.11. □

Lemma 5.13. *Assume $s \in \mathbb{R}$ and $s + 1 \neq 0$. Then*

$$\mathrm{o}(n^s) \subset \Delta\mathrm{o}(n^{s+1}), \qquad \mathrm{O}(n^s) \subset \Delta\mathrm{O}(n^{s+1}).$$

Proof. Assume $z = \mathrm{o}(n^s)$. Choose $x \in \mathrm{SQ}$ such that $z = \Delta x$. If $s > -1$, then, by Lemma 5.12, $x = \mathrm{o}(n^{s+1})$. Let $s < -1$. Then the series $\sum z_n$ is convergent. Let

$$\sigma = \sum_{n=1}^{\infty} z_n, \qquad x_1 = 0, \qquad x_n = z_1 + \cdots + z_{n-1} - \sigma \quad \text{for } n > 1.$$

Then $x = \mathrm{o}(1)$, $\Delta x = z$ and by Lemma 5.12, we have $x = \mathrm{o}(n^{s+1})$. Hence we obtain $\mathrm{o}(n^s) \subset \Delta\mathrm{o}(n^{s+1})$. Analogously $\mathrm{O}(n^s) \subset \Delta\mathrm{O}(n^{s+1})$. □

Lemma 5.14. *Assume $s \in \mathbb{R}$ and $(s+1)(s+2)\cdots(s+m) \neq 0$. Then*

$$\mathrm{o}(n^s) \subset \Delta^m\mathrm{o}(n^{s+m}), \qquad \mathrm{O}(n^s) \subset \Delta^m\mathrm{O}(n^{s+m}).$$

Proof. The assertion is an easy consequence of the previous lemma. □

Lemma 5.14 justify Examples 5.4 and 5.5.

Lemma 5.15. *If $\lambda \in (0,1) \cup (1,\infty)$, then*

$$\mathrm{o}(\lambda^n) \subset \Delta^m\mathrm{o}(\lambda^n), \qquad \mathrm{O}(\lambda^n) \subset \Delta^m\mathrm{O}(\lambda^n).$$

Proof. Let $x, w \in \mathrm{SQ}$ and $\Delta w = x$. Since $\Delta\lambda^n = \lambda^{n+1} - \lambda^n = \lambda^n(\lambda - 1)$, we have

$$\frac{\Delta w_n}{\Delta\lambda^n} = \frac{x_n}{\Delta\lambda^n} = L\frac{x_n}{\lambda^n} \tag{5.2}$$

where, $L = 1/(\lambda - 1)$. Assume $\lambda \in (0,1)$ and $x \in \mathrm{o}(\lambda^n)$. Then the series $\sum_{n=1}^{\infty} x_n$ is convergent. Hence $x \in \mathrm{S}(1) = \Delta\mathrm{o}(1)$ and there exists $w \in \mathrm{o}(1)$ such that $x = \Delta w$. Using (5.2) and the fact that $x \in \mathrm{o}(\lambda^n)$ we have $\Delta w_n/\Delta\lambda^n \to 0$. Moreover, $w_n \to 0$ and $\lambda^n \to 0$. By Lemma 5.11, we obtain $w \in \mathrm{o}(\lambda^n)$. Hence

$$x = \Delta w \in \Delta\mathrm{o}(\lambda^n).$$

Therefore $\mathrm{o}(\lambda^n) \subset \Delta\mathrm{o}(\lambda^n)$ and, by induction,

$$\mathrm{o}(\lambda^n) \subset \Delta^m\mathrm{o}(\lambda^n).$$

If $x \in \mathrm{O}(\lambda^n)$, then the sequence x_n/λ^n is bounded and, by (5.2), the sequence $\Delta w_n/\Delta\lambda^n$ is also bounded. Hence, by Lemma 5.11, $w \in \mathrm{O}(\lambda^n)$ and we obtain $\mathrm{O}(\lambda^n) \subset \Delta\mathrm{O}(\lambda^n)$. Moreover, by induction

$$\mathrm{O}(\lambda^n) \subset \Delta^m\mathrm{O}(\lambda^n).$$

If $\lambda > 1$, then $\lambda^n \to \infty$ and using Lemma 5.11 (b) we obtain the result. □

Lemma 5.16. $\mathrm{o}(n^{-\infty}) \subset \Delta^m \mathrm{o}(n^{-\infty})$.

Proof. Using [19, Lemma 4.8] we have $r\mathrm{A}(k+1) \subset \mathrm{A}(k)$ for any $k \in \mathbb{N}$. Hence $r\mathrm{A}(\infty) \subset r\mathrm{A}(k+1) \subset \mathrm{A}(k)$ and we get

$$r\mathrm{A}(\infty) \subset \bigcap_{k\in\mathbb{N}} \mathrm{A}(k) = \mathrm{A}(\infty).$$

Therefore $r^2\mathrm{A}(\infty) = rr\mathrm{A}(\infty) \subset r\mathrm{A}(\infty) \subset \mathrm{A}(\infty)$ and so on. After m steps we obtain

$$r^m\mathrm{A}(\infty) \subset \mathrm{A}(\infty).$$

By Lemma 2.7 (007), we have $\Delta^m r^m \mathrm{A}(\infty) = \mathrm{A}(\infty)$. Hence

$$\mathrm{A}(\infty) = \Delta^m r^m \mathrm{A}(\infty) \subset \Delta^m \mathrm{A}(\infty).$$

Now the result follows from the equality $\mathrm{o}(n^{-\infty}) = \mathrm{A}(\infty)$. □

Using Lemma 2.7 (014), Lemma 5.15 and Lemma 5.16 we justify Example 5.6. By Lemma 5.14, we have $\mathrm{O}(n^s) \subset \Delta^m \mathrm{O}(n^\infty)$ for any $s > m$. Hence

$$\mathrm{O}(n^\infty) = \bigcup_{s>m} \mathrm{O}(n^s) \subset \Delta^m \mathrm{O}(n^\infty).$$

Therefore, using Lemma 5.15, we obtain Example 5.7.

Lemma 5.17. *If $s \in (-\infty, m-1]$, then $\mathrm{A}(m-s) \subset \Delta^m(\mathrm{o}(n^s))$.*

Proof. Let $a \in \mathrm{A}(m-s)$. Choose $x \in \mathrm{SQ}$ such that $a = \Delta^m x$. By [16, Theorem 2.1] we have $x \in \mathrm{Pol}(m-1) + \mathrm{o}(n^s)$. Hence

$$\begin{aligned} a = \Delta^m x &\in \Delta^m(\mathrm{Pol}(m-1) + \mathrm{o}(n^s)) \\ &= \Delta^m \mathrm{Pol}(m-1) + \Delta^m \mathrm{o}(n^s) = \Delta^m \mathrm{o}(n^s). \end{aligned}$$

□

Lemma 5.18. *If $q \in \mathbb{N}_0^{m-1}$, then $\mathrm{A}(m-q) \subset \Delta^m \Delta^{-q} \mathrm{o}(1)$.*

Proof. Let $a \in \mathrm{A}(m-q)$. Choose $x \in \mathrm{SQ}$ such that $a = \Delta^m x$. By [17, Lemma 3.1 (d)] we have $x \in \mathrm{Pol}(m-1) + \Delta^{-q}\mathrm{o}(1)$. Hence

$$a = \Delta^m x \in \Delta^m(\mathrm{Pol}(m-1) + \Delta^{-q}\mathrm{o}(1)) = \Delta^m \Delta^{-q} \mathrm{o}(1).$$

Using Lemmas 5.17 and 5.18 we obtain Examples 5.8 and 5.9. □

Lemma 5.19. *If $t \in [m+1, \infty)$, then $r^m\mathrm{A}(t) \subset \mathrm{A}(t-m)$.*

Proof. Choose $k \in \mathbb{N}$ such that $k \le t < k+1$. Let $s = t-k$. Then

$$A(t) = n^{1-t}\mathrm{A}(1) = n^{1-(k+s)}\mathrm{A}(1) = n^{-s}n^{1-k}\mathrm{A}(1) = n^{-s}\mathrm{A}(k).$$

Hence, for $a \in \mathrm{A}(t)$ we have $n^s a \in \mathrm{A}(k)$. By Lemma 2.7 (012),

$$a \in \mathrm{A}(k) \quad \text{and} \quad n^s r|a| \le r|n^s a|.$$

Since $|n^s a| \in \mathrm{A}(k)$ and $r(\mathrm{A}(k)) \subset \mathrm{A}(k-1)$, we have

$$r|n^s a| \in \mathrm{A}(k-1).$$

By the comparison test we obtain $n^s r|a| \in \mathrm{A}(k-1)$. Using the inequality $|ra| \leq r|a|$ we have $n^s|ra| \leq n^s r|a|$. By comparison test, $n^s|ra| \in \mathrm{A}(k-1)$. Hence

$$ra \in n^{-s}\mathrm{A}(k-1) = \mathrm{A}(t-1).$$

Therefore

$$r(\mathrm{A}(t)) \subset \mathrm{A}(t-1)$$

and, by induction, we obtain the result. □

Now let $t \in [1, \infty)$. By Lemma 5.19 we have $r^m \mathrm{A}(m+t) \subset \mathrm{A}(t)$. Hence, using Lemma 2.7 (007),

$$\mathrm{A}(m+t) = \Delta^m r^m \mathrm{A}(m+t) \subset \Delta^m \mathrm{A}(t)$$

and we obtain Example 5.10.

6 Absolute summable sequences

In our investigations the spaces $\mathrm{A}(t)$ play an important role. In this section we obtain some characterizations of $\mathrm{A}(t)$. Our results extend some classical tests for absolute convergence of series and extend results from [19].

Lemma 6.1. *Assume $t \in [1, \infty)$ and $s \in \mathbb{R}$. Then $(n^s) \in \mathrm{A}(t) \Leftrightarrow s < -t$.*

Proof. We have

$$(n^s) \in \mathrm{A}(t) \Leftrightarrow (n^s) \in (n^{1-t})\mathrm{A}(1) \Leftrightarrow (n^{t+s-1}) \in \mathrm{A}(1) \Leftrightarrow t+s-1 < -1 \Leftrightarrow s < -t. \qquad \square$$

Lemma 6.2 (Generalized logarithmic test)**.** *Assume $a \in \mathrm{SQ}$, $t \in [1, \infty)$ and*

$$u_n = -\frac{\ln |a_n|}{\ln n}.$$

Then

(1) *if* $\liminf u_n > t$, *then* $a \in \mathrm{A}(t)$,

(2) *if* $u_n \leq t$ *for large* n, *then* $a \notin \mathrm{A}(t)$,

(3) *if* $\limsup u_n < t$, *then* $a \notin \mathrm{A}(t)$,

(4) *if* $\lim u_n = \infty$, *then* $a \in \mathrm{A}(\infty)$.

Proof. If $\liminf u_n > t$, then there exists a number $s > t$ such that $u_n > s$ for large n. Then $|a_n| \leq n^{-s}$ for large n. Hence (1) follows from the comparison test and from the fact that $(n^{-s}) \in \mathrm{A}(t)$. If $u_n \leq t$ for large n, then $|a_n| \geq n^{-t}$ for large n. Hence (2) follows from the fact that $(n^{-t}) \notin \mathrm{A}(t)$. The assertion (3) follows immediately from (2) and (4) is a consequence of (1). □

Lemma 6.3 (Generalized Raabe's test)**.** *Assume* $a \in \mathrm{SQ}$, $t \in [1, \infty)$,

$$u_n = n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right).$$

Then

(1) *if* $\liminf u_n > t$, *then* $a \in \mathrm{A}(t)$,

(2) *if* $u_n \le t$ *for large* n, *then* $a \notin \mathrm{A}(t)$,

(3) *if* $\limsup u_n < t$, *then* $a \notin \mathrm{A}(t)$,

(4) *if* $\lim u_n = \infty$, *then* $a \in \mathrm{A}(\infty)$.

Proof. Let

$$b_n = n^{t-1}a_n, \qquad w_n = n\left(\frac{|b_n|}{|b_{n+1}|} - 1\right).$$

Then

$$\begin{aligned} w_n &= n\left(\frac{n^{t-1}|a_n|}{(n+1)^{t-1}|a_{n+1}|} - 1\right) = n\left(\left(\frac{n}{n+1}\right)^{t-1}\frac{|a_n|}{|a_{n+1}|} - 1\right) \\ &= n\left(\frac{n}{n+1}\right)^{t-1}\left(\frac{|a_n|}{|a_{n+1}|} - \left(\frac{n+1}{n}\right)^{t-1}\right) = n\left(\frac{n}{n+1}\right)^{t-1}\left(\frac{|a_n|}{|a_{n+1}|} - \left(1 + \frac{1}{n}\right)^{t-1}\right). \end{aligned}$$

If $s \in \mathbb{R}$, then using the Taylor expansion of the function $(1+x)^s$ we obtain

$$(1+x)^s = 1 + sx + \mathrm{o}(x) \quad \text{for } x \to 0.$$

Hence

$$\left(1 + \frac{1}{n}\right)^{t-1} = 1 + (t-1)\frac{1}{n} + \mathrm{o}(n^{-1}).$$

Therefore

$$\begin{aligned} w_n = &\left(\frac{n}{n+1}\right)^{t-1} n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) - \left(\frac{n}{n+1}\right)^{t-1}(t - 1 - n\mathrm{o}(n^{-1})) \\ &= c_n u_n - c_n(t - 1 - \mathrm{o}(1)), \qquad c_n = \left(\frac{n}{n+1}\right)^{t-1} \to 1. \end{aligned}$$

Thus

$$\liminf w_n = \liminf u_n - (t-1) = \liminf u_n - t + 1.$$

Hence, if $\liminf u_n > t$, then $\liminf w_n > 1$ and by the usual Raabe's test we obtain $b \in \mathrm{A}(1)$ i.e., $a \in \mathrm{A}(t)$. The assertion (1) is proved. Now, we assume that $u_n \le t$ for large n. Then

$$n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) \le t \quad \text{i.e.,} \quad \frac{|a_n|}{|a_{n+1}|} \le \frac{t}{n} + 1 \quad \text{for large } n.$$

Hence

$$\begin{aligned} w_n &= n\left(\frac{n}{n+1}\right)^{t-1}\left(\frac{|a_n|}{|a_{n+1}|} - \left(1 + \frac{1}{n}\right)^{t-1}\right) \\ &\le n\left(\frac{n}{n+1}\right)^{t-1}\left(\frac{t}{n} + 1 - \left(1 + \frac{1}{n}\right)^{t-1}\right). \end{aligned}$$

It is easy to see that if $t \geq 1$ and $x \in (0,1)$, then $(1+x)^t \geq 1+tx$. Hence

$$\left(1+\frac{1}{n}\right)^t \geq 1+\frac{t}{n}, \quad \text{and} \quad \frac{t}{n}+1-\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{t-1} \leq 0.$$

Therefore

$$\frac{t}{n}+1-\left(1+\frac{1}{n}\right)^{t-1} \leq \frac{1}{n}\left(1+\frac{1}{n}\right)^{t-1} = \frac{1}{n}\left(\frac{n+1}{n}\right)^{t-1}$$

Hence $w_n \leq 1$ for large n and, by the usual Raabe's test, we obtain $b \notin \mathrm{A}(1)$ i.e., $a \notin \mathrm{A}(t)$. The assertion (2) is proved. (3) is an immediate consequence of (2). (4) follows from (1). □

Lemma 6.4 (Generalized Schlömilch's test). *Assume $a \in \mathrm{SQ}$, $t \in [1,\infty)$,*

$$u_n = n \ln \frac{|a_n|}{|a_{n+1}|}.$$

Then

(1) *if* $\liminf u_n > t$, *then* $a \in \mathrm{A}(t)$,

(2) *if* $u_n \leq t$ *for large* n, *then* $a \notin \mathrm{A}(t)$,

(3) *if* $\limsup u_n < t$, *then* $a \notin \mathrm{A}(t)$,

(4) *if* $\lim u_n = \infty$, *then* $a \in \mathrm{A}(\infty)$.

Proof. If $\liminf u_n = b > t$ and $c \in (t,b)$, then $\liminf u_n > c$ for large n. Hence

$$\frac{|a_n|}{|a_{n+1}|} \geq \exp\left(\frac{c}{n}\right).$$

Since $e^x \geq 1+x$ for $x > 0$, we have

$$\frac{|a_n|}{|a_{n+1}|} > 1+\frac{c}{n} \quad \text{and} \quad n\left(\frac{|a_n|}{|a_{n+1}|}-1\right) > c > t$$

for large n. Now, by Raabe's test we obtain (1).

Assume $u_n \leq t$ for large n. Then

$$\ln \frac{|a_n|}{|a_{n+1}|} \leq \frac{t}{n} \quad \text{and} \quad \frac{|a_n|}{|a_{n+1}|} \leq e^{\frac{t}{n}}$$

for large n. Let $b_n = (n-1)^{-t}$. Since

$$e < \left(1+\frac{1}{n-1}\right)^n,$$

we have

$$e^{\frac{t}{n}} < \left(1+\frac{1}{n-1}\right)^t = \left(\frac{n}{n-1}\right)^t = \frac{b_n}{b_{n+1}}.$$

Hence

$$\frac{|a_n|}{|a_{n+1}|} \leq e^{\frac{t}{n}} < \frac{b_n}{b_{n+1}} \quad \text{and} \quad \frac{|a_n|}{b_n} < \frac{|a_{n+1}|}{b_{n+1}}$$

for large n. Hence, there exists a $\lambda > 0$ such that $|a_n|/b_n > \lambda$ for large n. Therefore

$$|a_n| > \lambda b_n > \lambda n^{-t}$$

for large n. Using the fact that $(n^{-t}) \notin \mathrm{A}(t)$ we have $a \notin \mathrm{A}(t)$ and we obtain (2). The assertion (3) is an immediate consequence of (2). (4) follows from (1). □

Lemma 6.5 (Generalized Gauss's test). *Let $a \in \mathrm{SQ}$, $t \in [1, \infty)$, $\lambda, s \in \mathbb{R}$, $s < -1$ and*

$$\frac{|a_n|}{|a_{n+1}|} = 1 + \frac{\lambda}{n} + \mathrm{O}(n^s)$$

Then

(a) *if $\lambda > t$, then $a \in \mathrm{A}(t)$,*

(b) *if $\lambda \le t$, then $a \notin \mathrm{A}(t)$.*

Proof. Let $b_n = n^{t-1} a_n$. Then

$$\begin{aligned}\frac{|b_n|}{|b_{n+1}|} &= \frac{n^{t-1}|a_n|}{(n+1)^{t-1}|a_{n+1}|} = \left(\frac{n}{n+1}\right)^{t-1} \frac{|a_n|}{|a_{n+1}|} \\ &= \left(\frac{n}{n+1}\right)^{t-1} \left(1 + \frac{\lambda}{n} + \mathrm{O}(n^s)\right).\end{aligned}$$

It is easy to see that

$$\left(\frac{n}{n+1}\right)^{t-1} = \left(1 - \frac{1}{n+1}\right)^{t-1} = 1 - \frac{t-1}{n+1} + \mathrm{O}\left(\frac{1}{(n+1)^2}\right) = 1 - \frac{t-1}{n} + \mathrm{O}\left(\frac{1}{n^2}\right).$$

Hence

$$\begin{aligned}\frac{|b_n|}{|b_{n+1}|} &= \left(1 - \frac{t-1}{n} + \mathrm{O}(n^{-2})\right)\left(1 + \frac{\lambda}{n} + \mathrm{O}(n^s)\right) \\ &= 1 + \frac{\lambda}{n} + \mathrm{O}(n^s) - \frac{t-1}{n} - \frac{\lambda(t-1)}{n^2} + \mathrm{O}(n^{s-1}) + \mathrm{O}(n^{-2}) \\ &= 1 + \frac{\lambda - t + 1}{n} + \mathrm{O}(n^{s'}).\end{aligned}$$

For some $s' < -1$. If $\lambda > t$, then $\lambda - t + 1 > 1$ and, by the usual Gauss's test, $b \in \mathrm{A}(1)$. Hence $a \in \mathrm{A}(t)$. Analogously, if $\lambda \le t$, then $\lambda - t + 1 \le 1$ and $b \notin \mathrm{A}(1)$. Therefore $a \notin \mathrm{A}(t)$. □

Lemma 6.6 (Generalized Kummer's test). *Assume a, c are positive sequences,*

$$t \in [1, \infty), \qquad K_n = \frac{c_n a_n}{a_{n+1}} \left(\frac{n}{n+1}\right)^{t-1} - c_{n+1}.$$

Then

(1) *if $\liminf K_n > 0$, then $a \in \mathrm{A}(t)$,*

(2) *if the series $\sum_{n=1}^{\infty} c_n^{-1}$ is divergent and $K_n \le 0$ for large n, then $a \notin \mathrm{A}(t)$,*

(3) *if the series $\sum_{n=1}^{\infty} c_n^{-1}$ is divergent and $\limsup K_n < 0$, then $a \notin \mathrm{A}(t)$.*

Proof. This lemma is an easy consequence of the usual Kummer's test since, by definition of the space $\mathrm{A}(t)$, we have

$$(a_n) \in \mathrm{A}(t) \;\Leftrightarrow\; (n^{t-1})(a_n) \in \mathrm{A}(1).$$

□

Lemma 6.7 (Generalized Bertrand's test). *Assume $a \in \mathrm{SQ}$, $t \in [1,\infty)$ and*

$$\frac{|a_n|}{|a_{n+1}|} = 1 + \frac{t}{n} + \frac{\lambda_n}{n \ln n}.$$

Then

(1) *if* $\liminf \lambda_n > 1$, *then* $a \in \mathrm{A}(t)$,

(2) *if* $\lambda_n \leq 1$ *for large* n, *then* $a \notin \mathrm{A}(t)$,

(3) *if* $\limsup \lambda_n < 1$, *then* $a \notin \mathrm{A}(t)$.

Proof. Let

$$c_n = n \ln n, \qquad u_n = \left(\frac{n}{n+1}\right)^{t-1}, \qquad K_n = u_n \frac{c_n a_n}{a_{n+1}} - c_{n+1}.$$

Then

$$K_n = u_n \left(1 + \frac{t}{n} + \frac{\lambda_n}{n \ln n}\right) n \ln n - c_{n+1} = u_n (n+t) \ln n + u_n \lambda_n - c_{n+1}.$$

Since

$$u_n = \left(\frac{n}{n+1}\right)^{t-1} = \left(1 - \frac{1}{n+1}\right)^{t-1} = 1 + \frac{1-t}{n+1} + \mathrm{O}(n^{-2}),$$

we have

$$\begin{aligned}
K_n = &\left(1 + \frac{1-t}{n+1} + \mathrm{O}(n^{-2})\right)(n+t)\ln n - (n+1)\ln(n+1) + \lambda_n u_n \\
&= (n+t)\ln n + \frac{(n+t)(1-t)\ln n}{n+1} + \mathrm{o}(1) - (n+1)\ln(n+1) + \lambda_n u_n \\
&= (n+1)\ln n + \frac{(n+1)(t-1)\ln n}{n+1} + \frac{(n+t)(1-t)\ln n}{n+1} \\
&\quad + \mathrm{o}(1) - (n+1)\ln(n+1) + \lambda_n u_n = \ln\left(\frac{n}{n+1}\right)^{n+1} + \lambda_n u_n + \mathrm{o}(1) \\
&= \ln\left(1 - \frac{1}{n+1}\right)^{n+1} + \lambda_n u_n + \mathrm{o}(1) = -1 + \lambda_n u_n + \mathrm{o}(1).
\end{aligned}$$

Since $u_n \to 1$, we have

$$\liminf K_n = -1 + \liminf \lambda_n.$$

Hence, by Kummer's test and by the divergence of the series $\sum_{n=1}^{\infty} c_n^{-1}$, we obtain (1) and (3). Since $(1+x)^t \geq 1 + tx$ for $t, x \in [0,\infty)$, we have

$$1 + \frac{t}{n} \leq \left(1 + \frac{1}{n}\right)^t \quad \text{and} \quad n + t \leq n\left(\frac{n+1}{n}\right)^t.$$

Hence

$$u_n(n+t) \leq n\frac{n+1}{n}\left(\frac{n+1}{n}\right)^{t-1} u_n = n+1.$$

Moreover,

$$e < \left(1 + \frac{1}{n}\right)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1} \quad \Rightarrow \quad e^{-1} > \left(\frac{n}{n+1}\right)^{n+1} \quad \Rightarrow \quad \ln\left(\frac{n}{n+1}\right)^{n+1} < -1.$$

Hence

$$K_n = u_n(n+t)\ln n + u_n\lambda_n - c_{n+1} \le (n+1)\ln n + u_n\lambda_n - (n+1)\ln(n+1)$$
$$= u_n\lambda_n + \ln\left(\frac{n}{n+1}\right)^{n+1} < -1 + u_n\lambda_n.$$

Now, using Kummer's test and the fact that $u_n \in (0,1]$, we obtain (2). □

7 Remarks

In this section we present some consequences of our results. Next we give some final remarks.

The first part of Theorem 4.11 we may state in the following form.

Theorem 7.1. *Assume (A, Z) is an evanescent m-pair, $a \in A$, and $W \subset \mathrm{SQ}$. Then*

(a) *if W is f-ordinary, then for any solution x of (E) such that $x \in W$ there exists a sequence y such that $\Delta^m y = b$ and $x - y \in Z$,*

(b) *if W is f-regular, then for any sequence $y \in W$ such that $\Delta^m y = b$ there exists a solution x of (E) such that $y - x \in Z$.*

Using this theorem and Lemma 3.6 we obtain the following theorem.

Theorem 7.2. *Assume (A, Z) is an evanescent m-pair, $a, b \in A$, and $W \subset \mathrm{SQ}$. Then*

(a) *if W is f-ordinary, then for any solution x of (E) such that $x \in W$ there exists a polynomial sequence $\varphi \in \mathrm{Pol}(m-1)$ such that $x - \varphi \in Z$,*

(b) *if W is f-regular, then for any polynomial sequence $\varphi \in \mathrm{Pol}(m-1)$ such that $\varphi \in W$ there exists a solution x of (E) such that $\varphi - x \in Z$.*

Using Theorem 7.1, Example 5.10 and the generalized Raabe's test (Lemma 6.3) we obtain the following theorem.

Theorem 7.3. *Assume $W \subset \mathrm{SQ}$ is f-regular, $t \in [1, \infty)$, and*

$$\liminf n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) > m + t.$$

Then for any $y \in W \cap \Delta^{-m}b$ there exists a solution x of (E) such that

$$\limsup n\left(\frac{|y_n - x_n|}{|y_{n+1} - x_{n+1}|} - 1\right) \ge t.$$

Using Example 5.8 and the generalized Schlömilch's test (Lemma 6.4) we obtain the following theorem.

Theorem 7.4. *Assume $W \subset \mathrm{SQ}$ is f-ordinary, $s \in (-\infty, 0]$, and*

$$\liminf n \ln\frac{|a_n|}{|a_{n+1}|} > m - s.$$

Then for any solution x of (E) such that $x \in W$ there exists $y \in \Delta^{-m}b$ such that

$$x_n - y_n = \mathrm{o}(n^s).$$

Using Example 5.6 we obtain the following theorem.

Theorem 7.5. *Assume $W \subset \mathrm{SQ}$ is f-regular, $\lambda \in (0,1)$, and $a \in \mathrm{o}(\lambda^n)$. Then for any $y \in W \cap \Delta^{-m}b$ there exists a solution x of* (E) *such that*

$$y_n - x_n = \mathrm{o}(\lambda^n).$$

Using other examples of m-pairs one can obtain many other theorems.

Asymptotic difference pairs are used, implicitly, in some papers. The classical case is $(\mathrm{A}(m), \mathrm{o}(1))$, see for example [7,14,15]. The pair $(\mathrm{A}(m-s), \mathrm{o}(n^s))$, for a fixed $s \in (-\infty, 0]$, is used in [16], [17], [18] and [20]. The pair $(\mathrm{A}(m+p), \mathrm{A}(p))$, for a fixed $p \in \mathbb{N}$, is used in [19]. The pair $(\mathrm{A}(m-q), \Delta^{-q}\mathrm{o}(1))$, for a fixed $q \in \mathbb{N}_0^{m-1}$, is used in [17, Theorem 7.5].

Our results may be partially extended to the case of nonautonomous equations. The basic difference is as follows. If $f\colon \mathbb{R} \to \mathbb{R}$, $x \in \mathrm{SQ}$ and the sequence $(f(x_n))$ is bounded, then the sequence $(f(x_{\sigma(n)}))$ is also bounded. On the other hand, if $f\colon \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, $x \in \mathrm{SQ}$, then the boundedness of the sequence $(f(n, x_n))$ does not imply the boundedness of the sequence $(f(n, x_{\sigma(n)}))$.

In some papers the term generalized solution is used instead of our solution and the term solution in place of our full solutions.

The terminology is a matter of taste. A separate question is why study the generalized solution at all. As a kind of motivation we give three examples. These examples are taken from [18].

Example 7.6. Assume $a_n \geq 0$, the series $\sum_{n=1}^{\infty} a_n$ is convergent and there exists an index $p > 1$ such that $a_p = 0$, $a_{p+1} = 1$. Consider the equation

$$\Delta x_n = a_n |x_{n-1}|.$$

Then every number $\lambda \in \mathbb{R}$ is the limit of a certain solution. On the other hand, if x is a full convergent solution, then $\lim x_n \geq 0$.

Example 7.7. Assume $a_n \geq 0$, the series $\sum_{n=1}^{\infty} n a_n$ is convergent and $a_p = 1$ for certain p. Consider the equation

$$\Delta^2 x_n = a_n x_n^2.$$

Then every real constant λ is the limit of a certain solution but if λ is the limit of a full solution, then $\lambda < 2$.

Example 7.8. Assume that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Consider the equation

$$\Delta x_n = a_n x_n + a_n.$$

Then any real constant λ is the limit of a certain solution. Morever, if $a_n \neq -1$ for all $n \in \mathbb{N}$, then any real λ is the limit of a certain full solution. On the other hand, if $a_p = -1$ for certain p and x is a p-solution, then $x_n = -1$ for any $n > p$.

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