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Steady state bifurcations for phase field crystal equations with underlying two dimensional kernel

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Abstract. This paper is concerned with the study of some properties of stationary solutions to phase field crystal equations bifurcating from a trivial solution. It is assumed that at this trivial solution, the kernel of the underlying linearized operator has dimension two. By means of the multiparameter method, we give a second order approximation of these bifurcating solutions and analyse their stability properties. The main result states that the stability of these solutions can be described by the variation of a certain angle in a two dimensional parameter space. The behaviour of the parameter curve is also investigated.

Keywords: phase field crystal equation, bifurcation theory, two dimensional kernel, higher order elliptic equations, stability.

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1 Introduction

During the last decades, *pattern formation equations* have attracted much attention from researchers in applied sciences; see for instance [3, 4, 12, 21]. In materials sciences, pattern formation equations (as Allen–Cahn or Cahn–Hilliard equations) are obtained by phase field methods. In 2004, K. Elder and M. Grant have extended these methods by introducing the so-called *phase field crystal modelling* in order to describe liquid/solid phase transitions in pure materials or alloys [6]. The solid phase, which can be a crystal, is represented by a periodic field whose wavelength accounts for the distance between neighbouring atoms. The liquid state is described by a (spatially) uniform field. We refer the reader to [6, 7, 17–19] for a more comprehensive exposition of the phase field crystal method.

The simplest phase field crystal model is the following sixth order evolution equation:

$$\partial_t u - \partial_{xx}\big(\partial_{xxxx} u + 2\partial_{xx} u + f(u)\big) = 0, \qquad t > 0,\ x \in (0, L). \tag{1.1}$$

Here L is the length of the domain and f is the derivative of a double-well potential. This equation can be viewed as a conservative Swift–Hohenberg equation exactly as the Cahn–Hilliard equation is a conservative version of the Allen–Cahn equation. Performing a linear

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change of variable mapping $(0, L)$ onto $(0, 1)$, equation (1.1) can be rewritten as

$$\partial_t u - \varepsilon\partial_{xx}\big(\varepsilon^2\partial_{xxxx}u + 2\varepsilon\partial_{xx}u + f(u)\big) = 0, \qquad t > 0,\ x \in (0,1), \tag{1.2}$$

with $\varepsilon = 1/L^2$.

This paper focuses on the stationary solutions to (1.2) complemented with initial and boundary conditions (see (2.7)). In order to gain insight properties of stationary solutions, we use a bifurcation approach.

A purpose of this paper is to construct (at least partially) phase diagrams (in the thermodynamical sense) by means of a mathematical analysis of the phase field crystal equation (1.2): see Figures 3.1 and 4.2 below. The parameters ε, M and r involved in the equation (see (2.13) and (2.1)) play the role of thermodynamical state variables. In this respect, it is important to investigate stability of solutions to (1.2). However, since we use perturbation methods, only local stability results will be proved.

It is well known that bifurcations occur only if the kernel of the underlying linearized operator is nontrivial. For the phase field crystal equation (1.2), the case of a one dimensional kernel has been investigated in [16]. In this paper, we focus on two dimensional kernels.

The originality of our approach is the combination of a group theoretic approach (see for instance [9, Chapter XX] or [5]), together with the *multiparameter method* (see [13, Chapter I]). Indeed, the former gives a convenient way to compute bifurcating solutions. However, the persistence of (explicit) solutions of the truncated bifurcation equation (see (4.30)) is not clear and not obvious for the field crystal equation. In general, the proof of persistence can be very computationally intensive: see for instance [2]. To our knowledge, even the sophisticated *path formulation method* (see [15]) do not lead to 3-determinacy in our setting. Therefore we use this approach as a guideline for our computations. More precisely, we use it when we rewrite (4.26) with the help of notation (4.27) and (4.28).

The multiparameter method allows us to prove (quite simply) the existence of bifurcating solutions to the field crystal equation (1.2). This is just a generalisation to higher dimensional kernel of Lyapunov–Schmidt's method. Indeed, for one dimensional kernel, the bifurcation equation reads

$$g(y, \varepsilon) = 0 \quad \text{in } \mathbb{R},$$

where y is a coordinate in the kernel and ε is the bifurcation parameter. By the implicit function theorem, we then get a solution of the form

$$(y, \varepsilon(y)) \quad \text{for } y \simeq 0.$$

In the case of a two dimensional kernel, the bifurcation equation reads

$$g(y, \varepsilon, M) = 0 \quad \text{in } \mathbb{R}^2,$$

where y is a coordinate along a direction in the kernel and ε and M are the bifurcation parameters. Then we get a solution of the form

$$(y, \varepsilon(y), M(y)) \quad \text{for } y \simeq 0.$$

For equation (1.2), the two parameters are ε and the mass M of the initial condition (which is conserved by the dynamics).

The first step is to characterize the parameter values that give rise to two dimensional kernels. The phase diagram of Figure 3.1 features a simple geometric criterion for this (see also Proposition 3.1 for an analytic result).

Then we implement the multiparameter method in order to get bifurcation branches and expansions of solutions to (1.2). According to [13], we have to choose a direction $\binom{\alpha}{\beta}$ in the kernel which will be tangent to a branch of solutions. Let us denote by $y \mapsto v(y)$ this branch, where $y \in \mathbb{R}$, $y \simeq 0$. The parameters ε and M are also parametrized by y; this gives a parameter curve $y \mapsto (\varepsilon(y), M(y))$ in $\mathbb{R}^2$. Theorem 4.1 states an existence result for these pitchfork bifurcation branches and gives second order expansions of $\varepsilon(\cdot)$, $M(\cdot)$ and $v(\cdot)$. In an explicit way, for $y \simeq 0$, the function $x \mapsto v(y)(x)$ is solution to

$$\varepsilon(y)^2\partial_{xxxx}v(y) + 2\varepsilon(y)\partial_{xx}v(y) + f(M(y) + v(y)) = \int_0^1 f(M(y) + v(y))\,\mathrm{d}x \qquad \text{a.e. in } (0,1).$$

We are then led to study two curves: the parameter curve $y \mapsto (\varepsilon(y), M(y))$ and the function valued curve $y \mapsto v(y)$. The former is studied by considering its oriented tangent at $y = 0$. This tangent will be denoted by $T(\alpha)$. We show how $T(\alpha)$ behaves w.r.t. α: see Propositions 4.5, 4.9 and Figure 4.2.

In Proposition 4.10, we state a monotonicity result for $\alpha \mapsto T(\alpha)$. More precisely, in a well identified region of the parameter space, $T(\alpha)$ turns clockwise when α goes from 0 to 1. In a quite surprising way, this monotonicity result is related to the stability of the bifurcating solutions; as we will see now.

The main result of this paper is stated in Theorem 5.3 and concerns the stability of the bifurcating stationary solutions to the phase field crystal equation. If the wave numbers of the interactive modes (i.e. k_* and k_{**} in the sequel) are not consecutive integers then the bifurcating solutions are unstable. This is easily proved. In order to show stability, we use the *principle of reduced stability* from [13, Section I-18] (see also [14]). It allows us to reduce some infinite dimensional eigenvalue problem to a two dimensional one. As evoked above, it appears that the bifurcating solutions are stable exactly when the tangent $T(\alpha)$ turns clockwise. So we connect the issue of stability in the PDE (2.7) with the variation of a one dimensional object (the angle between $T(\alpha)$ and the horizontal axis).

Finally, we use a *truncated bifurcation equation* and symmetries to recover a bifurcation diagram obtained originally in [16] by numerical integration: see Figure 6.1.

2 Equations and functional setting

Let Ω denote the interval $(0,1) \subset \mathbb{R}$ and r be a real number. We define

$$f\colon \mathbb{R} \to \mathbb{R}, \quad u \mapsto (1+r)u + u^3 \tag{2.1}$$

$$V_2 = \{u \in H^2(\Omega) \mid u'(0) = u'(1) = 0\} \tag{2.2}$$

$$V_4 = \{u \in H^4(\Omega) \mid u' = u''' = 0 \text{ on } \partial\Omega\} \tag{2.3}$$

$$\dot{L}^2(\Omega) = \left\{ v \in L^2(\Omega) \;\middle|\; \int_\Omega v\,\mathrm{d}x = 0 \right\}$$

$$\dot{V}_4 = V_4 \cap \dot{L}^2(\Omega), \qquad \dot{V}_2 = V_2 \cap \dot{L}^2(\Omega). \tag{2.4}$$

The space $\dot{V}_4$ is equipped with the bilinear form

$$(u,v)_{\dot{V}_4} = \int_\Omega u^{(4)} v^{(4)}\,\mathrm{d}x,$$

which becomes in turn a Hilbert space since every $v \in \dot{V}_4$ satisfies

$$\|v\|_2 \leq \frac{1}{\sigma_1^2}\|v^{(4)}\|_2, \tag{2.5}$$

where $\|\cdot\|_2$ denotes the standard $L^2(\Omega)$-norm. Indeed,

$$\|v\|_2 \le \frac{1}{\sqrt{\sigma_1}}\|v'\|_2 \le \frac{1}{\sigma_1}\|v''\|_2 \tag{2.6}$$

by Poincaré–Wirtinger and Poincaré's inequalities. Here $\sigma_1 := \pi^2$ denotes the first eigenvalue of the one-dimensional Laplace operator with homogeneous Dirichlet boundary conditions on Ω. Moreover v'' belongs to $\dot V_2$ thus the same estimates give $\|v''\|_2 \le \frac{1}{\sigma_1}\|v^{(4)}\|_2$. Then (2.5) follows. In the same way, if $(u,v)_{\dot V_2} := \int_\Omega u^{(2)}v^{(2)}\,\mathrm{d}x$ then $(\dot V_2, (\cdot,\cdot)_{\dot V_2})$ is a Hilbert space. Of course, $u^{(2)}$ stands for the second derivative of u.

Given initial data $u_0 = u_0(x)$ and a positive parameter ε, the *phase field crystal equation* with homogeneous Neumann boundary condition reads

$$\begin{cases} \partial_t u - \varepsilon\partial_{xx}\big(\varepsilon^2\partial_{xxxx}u + 2\varepsilon\partial_{xx}u + f(u)\big) = 0 & \text{in } \Omega\times(0,\infty) \\ \partial_x u = \partial_{xxx}u = \partial_{xxxxx}u = 0 & \text{on } \partial\Omega\times(0,\infty) \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{2.7}$$

Since every solution $u = u(t,x)$ to (2.7) satisfies

$$\int_\Omega u(x,t)\,\mathrm{d}x = \int_\Omega u_0(x)\,\mathrm{d}x \qquad \forall t > 0,$$

the stationary solutions to the problem above solve

$$u \in M + \dot V_4, \qquad \varepsilon^2 u^{(4)} + 2\varepsilon u^{(2)} + f(u) = \int_\Omega f(u)\,\mathrm{d}x \quad \text{in } L^2(\Omega), \tag{2.8}$$

where $M := \int_\Omega u_0(x)$ is a real parameter.

Introducing the new function v defined by $u = M + v$, (2.8) is equivalent to

$$v \in \dot V_4, \qquad \varepsilon^2 v^{(4)} + 2\varepsilon v^{(2)} + f(M+v) = \int_\Omega f(M+v)\,\mathrm{d}x \quad \text{in } L^2(\Omega). \tag{2.9}$$

The bifurcation problem

We will formulate a bifurcation problem in order to get nontrivial solutions of (2.9). To this end, we will introduce some notation. Let $\varepsilon > 0$, $\varepsilon_* > 0$ and M, M_* be real parameters. We put

$$\delta := (\varepsilon, M), \qquad \delta_* := (\varepsilon_*, M_*), \qquad \mu = (\mu_1,\mu_2) := \delta - \delta_* \in \mathbb{R}^2.$$

Let also

$$L(\delta,\cdot)\colon \dot V_4 \to \dot L^2(\Omega), \qquad v \mapsto \varepsilon^2 v^{(4)} + 2\varepsilon v^{(2)} + f'(M)v \tag{2.10}$$

$$L := L(\delta_*,\cdot). \tag{2.11}$$

In the sequel, $\delta_* = (\varepsilon_*, M_*)$ is the bifurcation point and is fixed; the parameter δ will be close to δ_*. Then we define

$$F\colon (-\varepsilon_*,\infty)\times\mathbb{R}\times\dot V_4 \to \dot L^2(\Omega)$$

through

$$F(\mu_1,\mu_2,v) = (\varepsilon_*+\mu_1)^2 v^{(4)} + 2(\varepsilon_*+\mu_1)v^{(2)} + f(M_*+\mu_2+v) - \int_\Omega f(M_*+\mu_2+v)\,\mathrm{d}x - Lv.$$

With these notations, we will consider the following bifurcation problem

$$\mu \in (-\varepsilon_*, \infty) \times \mathbb{R},\ v \in \dot{V}_4, \qquad Lv + F(\mu, v) = 0 \quad \text{in } \dot{L}^2(\Omega) \tag{2.12}$$

or equivalently

$$\delta = (\varepsilon, M) \in (0, \infty) \times \mathbb{R},\ v \in \dot{V}_4, \qquad Lv + F(\delta - \delta_*, v) = 0 \quad \text{in } \dot{L}^2(\Omega). \tag{2.13}$$

Remark that the equations in (2.9) and (2.13) are equivalent.

We Taylor expand $F(\mu, v)$ w.r.t. μ and v at $(\mu, v) = (0, 0)$. For this, we write

$$F(\mu, v) = F_1(\mu)v + F_2(\mu)v^2 + F_{03}v^3,$$

where

$$\begin{aligned} F_1(\mu)v &= L(\delta_* + \mu, v) - Lv \\ &= \big((\varepsilon_* + \mu_1)^2 - \varepsilon_*^2\big)v^{(4)} + 2\mu_1 v^{(2)} + \big(f'(M_* + \mu_2) - f'(M_*)\big)v, \end{aligned}$$

so that $v \mapsto F_1(\mu)v$ is a linear operator. Expanding $F_1(\mu)v$ w.r.t. μ, we get

$$F_1(\mu)v = F_{11}(\mu)v + F_{21}(\mu)v$$

with (see (2.1))

$$\begin{aligned} F_{11}(\mu)v &= \mu_1(2\varepsilon_* v^{(4)} + 2v^{(2)}) + 6\mu_2 M_* v \\ F_{21}(\mu)v &= \mu_1^2 v^{(4)} + 3\mu_2^2 v. \end{aligned}$$

Above, $F_2(\mu)\colon \dot{V}_4 \times \dot{V}_4 \to \dot{L}^2(\Omega)$ is a continuous bilinear symmetric operator and $F_2(\mu)v^2$ stands for $F_2(\mu)(v, v)$. We proceed in the same way for $F_2(\mu)v^2$, so that

$$F_2(\mu)v^2 = F_{02}v^2 + F_{12}(\mu)v^2$$

with

$$\begin{aligned} F_{02}(\mu)v^2 &= 3M_* \left(v^2 - \int_\Omega v^2\, \mathrm{d}x\right) \\ F_{12}(\mu)v^2 &= 3\mu_2 \left(v^2 - \int_\Omega v^2\, \mathrm{d}x\right). \end{aligned}$$

The last term is

$$F_{03}v^3 = v^3 - \int_\Omega v^3\, \mathrm{d}x.$$

Solutions to (2.9) are critical point of $E(M + \cdot, \varepsilon)$ in $\dot{V}_4$ where the energy E is defined through

$$E\colon V_2 \times (0, \infty) \to \mathbb{R}, \qquad (u, \varepsilon) \mapsto \int_\Omega \frac{1}{2}(\varepsilon u'' + u)^2 + \frac{r}{2}u^2 + \frac{1}{4}u^4\, \mathrm{d}x. \tag{2.14}$$

3 The linearised equation

For $\delta = (\varepsilon, M) \in (0, \infty) \times \mathbb{R}$, we study the eigenvalue problem (see the previous section and in particular (2.10), for notation)

$$\begin{cases} L(\delta, \varphi) = \lambda\varphi & \text{in } L^2(\Omega) \\ \varphi \in \dot{V}_4 \setminus \{0\}, & \lambda \in \mathbb{R}. \end{cases} \tag{3.1}$$

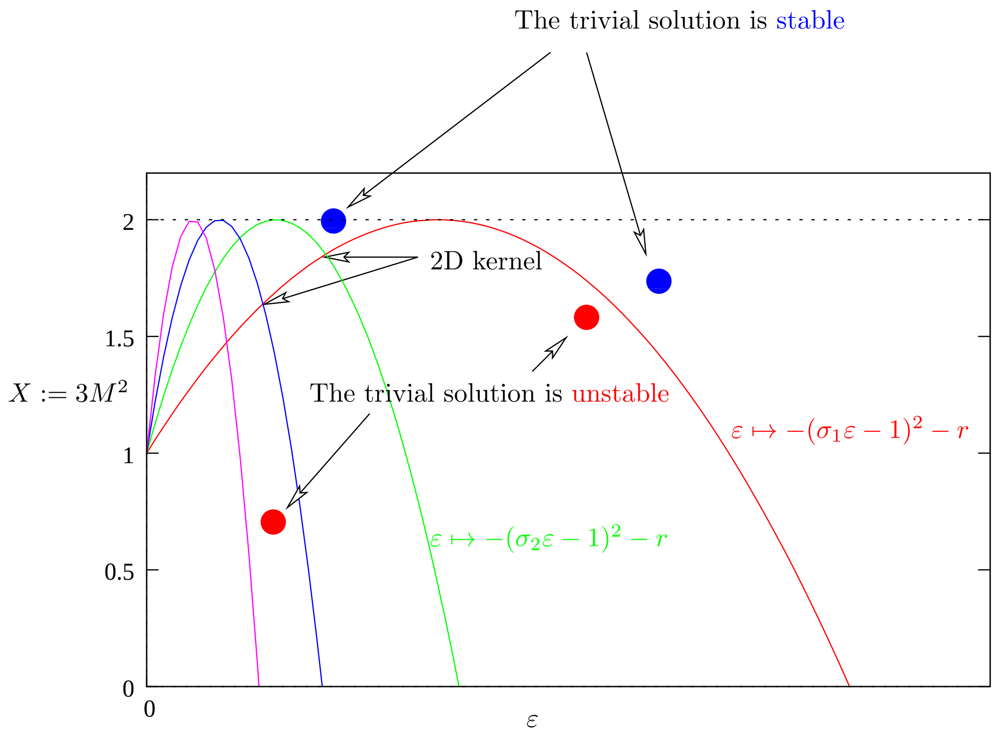

Figure 3.1: Phase diagram for $r = 2$ with four parabolas corresponding to $k = 1, \dots, 4$.

The eigenvalues of (3.1) are

$$\lambda_k := (\varepsilon\sigma_k - 1)^2 + r + 3M^2, \qquad \text{where } \sigma_k := (k\pi)^2, \qquad k = 1, 2, \dots, \tag{3.2}$$

with corresponding eigenfunctions

$$\varphi_k : \Omega \to \mathbb{R}, \qquad x \mapsto \cos(k\pi x).$$

Then 0 is an eigenvalue of (3.1) iff there exists a positive integer k such that

$$3M^2 = -(\varepsilon\sigma_k - 1)^2 - r.$$

That is to say, the point $(\varepsilon, 3M^2)$ is on the parabola given by the function

$$\varepsilon \mapsto -(\varepsilon\sigma_k - 1)^2 - r. \tag{3.3}$$

Thus the operator $L(\cdot, \delta)$ will have a two dimensional kernel iff the point $(\varepsilon, 3M^2)$ lies at the intersection of two such parabolas: see Figure 3.1. If we express this geometric property in an analytical language, we obtain the following result whose proof is straightforward and will be omitted.

Proposition 3.1. *Let $\delta = (\varepsilon, M) \in (0, \infty) \times \mathbb{R}$ and k_*, k_{**} be integers satisfying $1 \leq k_{**} < k_*$. Then*

$$\ker L(\delta, \cdot) = \langle \varphi_*, \varphi_{**} \rangle \iff \begin{cases} (\varepsilon\sigma_{k_*} - 1)^2 + r + 3M^2 = 0, \\ (\varepsilon\sigma_{k_{**}} - 1)^2 + r + 3M^2 = 0. \end{cases}$$

Moreover, if $\ker L(\cdot, \delta) = \langle \varphi_*, \varphi_{**} \rangle$, *then*

$$\varepsilon = \frac{2}{\sigma_{k_*} + \sigma_{k_{**}}}. \tag{3.4}$$

In the statement above, $\varphi_* := \varphi_{k_*}$, $\varphi_{**} := \varphi_{k_{**}}$ and $\langle \varphi_*, \varphi_{**} \rangle$ denotes the real vector space generated by φ_* and φ_{**}.

Stability of the trivial solution

In view of (1.2), the trivial solution $v = 0$ of (2.9) is said to be *linearly stable* if (3.1) has only positive eigenvalues. In Figure 3.1, this corresponds to the case where the point $(\varepsilon, 3M^2)$ is above all parabolas of the form (3.3). If (3.1) has at least one negative eigenvalue, then $v = 0$ is *linearly unstable*.

Besides, the trivial solution is called *neutrally stable* if 0 is an eigenvalue of (3.1) and all the other eigenvalues of (3.1) are positive. In order to have stability of solutions to (2.12) bifurcating from $v = 0$, it is necessary that the trivial solution is neutrally stable. The next result gives a simple criterion for neutrally stability of $v = 0$ in the case of a $2D$ kernel.

Proposition 3.2. *Let $\delta = (\varepsilon, M) \in (0,\infty) \times \mathbb{R}$ and k_*, k_{**} be integers such that $1 \le k_{**} < k_*$ and*

$$\ker L(\delta, \cdot) = \langle \varphi_*, \varphi_{**} \rangle.$$

Then $v = 0$ is neutrally stable iff $k_* = k_{**} + 1$.

Proof. For every $k \ge 1$, we have with (3.2), (3.4)

$$\begin{aligned}
\lambda_k - \lambda_{k_*} &= \varepsilon(\sigma_k - \sigma_{k_*})\big(\varepsilon(\sigma_k + \sigma_{k_*}) - 2\big) \\
&= \frac{2\varepsilon}{\sigma_{k_*} + \sigma_{k_{**}}}(\sigma_k - \sigma_{k_*})(\sigma_k - \sigma_{k_{**}}) \\
&= \frac{2\pi^4\varepsilon}{\sigma_{k_*} + \sigma_{k_{**}}}(k^2 - k_*^2)(k^2 - k_{**}^2). \qquad (3.5)
\end{aligned}$$

If $v = 0$ is neutrally stable and $k \ne k_*,\ k_{**}$, then $\lambda_k > 0 = \lambda_{k_*}$. Hence

$$(k - k_*)(k - k_{**}) > 0, \qquad \forall k \ne k_*,\ k_{**}. \qquad (3.6)$$

The value of $(k - k_*)(k - k_{**})$ at $k = k_{**} + 1$ is $k_{**} + 1 - k_*$. This number is nonpositive since by assumption $k_{**} < k_*$. Thus with (3.6) we get $k_* = k_{**} + 1$.

Conversely, if $k_* = k_{**} + 1$, then (3.5) imply that $\lambda_k - \lambda_{k_*} > 0$ for $k \ne k_*,\ k_{**}$. Thus $v = 0$ is neutrally stable. □

Figure 3.1 points out two values of the parameter δ for which the kernel of $L(\cdot,\delta)$ has dimension two. One of these values corresponds to the case where $k_* = 2$ and $k_{**} = 1$ and lies at the intersection of the green and red parabolas. By Proposition 3.2, the trivial solution is neutrally stable for this value of δ. The other value corresponds to the case where $k_* = 3$ and $k_{**} = 1$. In this situation, $v = 0$ is not neutrally stable. Thus bifurcating solutions will be unstable.

4 Bifurcation with 2D kernel

The case where the kernel of $L(\cdot,\delta_*)$ has dimension one has been investigated in [16]. Here, we focus on the case where this kernel is two dimensional. More precisely, let $\delta_* := (\varepsilon_*, M_*) \in (0,\infty) \times \mathbb{R}$, $p := -r - 3M_*^2$ and k_*, k_{**} be integers satisfying $1 \le k_{**} < k_*$. We assume

$$M_* \ne 0 \qquad (4.1)$$

$$(\varepsilon_*\sigma_{k_*} - 1)^2 + r + 3M_*^2 = (\varepsilon_*\sigma_{k_{**}} - 1)^2 + r + 3M_*^2 = 0 \qquad (4.2)$$

$$\frac{k_*}{k_{**}} \ne 2, \qquad \frac{k_*}{k_{**}} \ne 3. \qquad (4.3)$$

From the geometrical point of view of Figure 3.1, $(\varepsilon_*, 3M_*^2)$ is a specified point at the intersection of two parabolas. According to Proposition 3.1, it follows that

$$\ker L(\cdot, \delta_*) = \langle \varphi_*, \varphi_{**} \rangle.$$

Consequently

$$\varepsilon_* \sigma_{k_*} = 1 + \sqrt{p}, \qquad \varepsilon_* \sigma_{k_{**}} = 1 - \sqrt{p}. \tag{4.4}$$

We will implement the multiparameter method (which is based on the Lyapunov–Schmidt method, see for instance [13, Chapter I]). In view of equation (2.9), the parameters will be ε and M. Moreover we will assume that $\delta = (\varepsilon, M)$ is close to $\delta_* := (\varepsilon_*, M_*)$ so that (see Section 2),

$$\mu := \delta - \delta_*$$

will be close to zero. We recall that we have put

$$\varphi_* := \varphi_{k_*}, \qquad \varphi_{**} := \varphi_{k_{**}}. \tag{4.5}$$

Since the operator L (defined by (2.11)) is self-adjoint with compact resolvent, the set

$$\Big\{ \frac{\varphi_k}{\|\varphi_k\|_2} \mid k \in \mathbb{N} \setminus \{0\} \Big\}$$

is a spectral basis of $\dot{L}^2(\Omega)$. Thus

$$\dim \ker L = \operatorname{codim} R(L),$$

where $R(L) \subset \dot{L}^2(\Omega)$ denotes the range of L, and

$$\dot{L}^2(\Omega) = R(L) \oplus \ker L$$
$$\dot{V}_4 = \Big(R(L) \cap \dot{V}_4 \Big) \oplus \ker L.$$

This decomposition of $\dot{L}^2(\Omega)$, in turn, defines the projection

$$P\colon \dot{L}^2(\Omega) \to \ker L \quad \text{along } R(L). \tag{4.6}$$

Denoting by $(\cdot,\cdot)_2$ the L^2-scalar product, there holds

$$Pv = 2(v, \varphi_*)_2 \varphi_* + 2(v, \varphi_{**})_2 \varphi_{**}, \qquad \forall v \in \dot{L}^2(\Omega). \tag{4.7}$$

We decompose every $v \in \dot{V}_4$ in a unique way into

$$v = u_0 + u_1, \tag{4.8}$$

where $u_0 \in \ker L$, $u_1 \in R(L) \cap \dot{V}_4$. Moreover $(u_0, u_1)_2 = 0$ since L is self-adjoint.

Projection on $R(L)$. Applying $I - P$ to (2.12) and using the notations of Section 2, we obtain

$$(I - P)\big\{ (L + F_1(\mu)) u_1 + F_2(\mu)(u_0 + u_1)^2 + F_{03}(u_0 + u_1)^3 \big\} = 0 \quad \text{ in } \dot{L}^2(\Omega). \tag{4.9}$$

By the implicit function theorem, for (u_0, u_1, μ) close to $(0, 0, 0)$, this equation is equivalent to

$$u_1 = \mathcal{U}(\mu, u_0). \tag{4.10}$$

Moreover, $\mathcal{U}(\mu,0)=0$ and

$$\mathcal{U}(\mu,u_0)=O(u_0^2),\qquad \text{for}\ \ (\mu,u_0)\simeq(0,0). \tag{4.11}$$

This means that there exist C_0, C such that

$$\|\mathcal{U}(\mu,u_0)\|_{\dot V_4}\le C\|u_0\|^2_{\dot V_4},\qquad \forall|\mu|\le C_0,\ \|u_0\|^2_{\dot V_4}\le C_0.$$

Thus

$$\mathcal{U}(\mu,u_0)=a_{02}u_0^2+O(\mu u_0^2+u_0^3), \tag{4.12}$$

where $a_{02}\colon \ker L\times\ker L\to R(L)\cap\dot V_4$ is a continuous bilinear symmetric operator independent of μ and $a_{02}u_0^2:=a_{02}(u_0,u_0)$. The equality (4.12) means that

$$\|\mathcal{U}(\mu,u_0)-a_{02}u_0^2\|_{\dot V_4}\le C\big(|\mu|\|u_0\|^2_{\dot V_4}+\|u_0\|^3_{\dot V_4}\big),\qquad \text{for}\ \ (\mu,u_0)\simeq(0,0).$$

Computation of $a_{02}u_0^2$. For α, $\beta\in\mathbb{R}$, we put $u_0=\alpha\varphi_*+\beta\varphi_{**}$ and $v_2:=a_{02}u_0^2$. At order u_0^2, (4.9) reads

$$(I-P)\big\{(Lv_2+F_{02}u_0^2\big\}=0\qquad \text{in}\ \ \dot L^2(\Omega). \tag{4.13}$$

Since $k_*\neq 2k_{**}$, we get

$$a_{02}u_0^2=v_2=\frac12\big(x_{2k_*}\varphi_{2k_*}+x_{k_*+k_{**}}\varphi_{k_*+k_{**}}+x_{k_*-k_{**}}\varphi_{k_*-k_{**}}+x_{2k_{**}}\varphi_{2k_{**}}\big), \tag{4.14}$$

with

$$\begin{aligned} x_{2k_*}&=-\frac{3M_*}{\lambda_{2k_*}}\alpha^2, & x_{k_*+k_{**}}&=-\frac{6M_*}{\lambda_{k_*+k_{**}}}\alpha\beta,\\ x_{k_*-k_{**}}&=-\frac{6M_*}{\lambda_{k_*-k_{**}}}\alpha\beta, & x_{2k_{**}}&=-\frac{3M_*}{\lambda_{2k_{**}}}\beta^2. \end{aligned} \tag{4.15}$$

Computation of $a_{02}(u_0,\cdot)$. This quantity will be useful later on. Since

$$a_{02}(u_0,\cdot)=\frac12 D_{u_0}v_2=\frac12 D_{u_0}a_{02}u_0^2,$$

we differentiate (4.13) w.r.t. u_0 to get

$$(I-P)\big\{La_{02}(u_0,\cdot)+F_{02}(u_0,\cdot)\big\}=0.$$

Since $u_0=\alpha\varphi_*+\beta\varphi_{**}$,

$$F_{02}(u_0,\varphi_*)=\frac32 M_*\big(\alpha\varphi_{2k_*}+\beta\varphi_{k_*+k_{**}}+\beta\varphi_{k_*-k_{**}}\big).$$

Hence

$$a_{02}(u_0,\varphi_*)=-\frac32 M_*\left(\frac{\alpha}{\lambda_{2k_*}}\varphi_{2k_*}+\frac{\beta}{\lambda_{k_*+k_{**}}}\varphi_{k_*+k_{**}}+\frac{\beta}{\lambda_{k_*-k_{**}}}\varphi_{k_*-k_{**}}\right). \tag{4.16}$$

In a same way,

$$a_{02}(u_0,\varphi_{**})=-\frac32 M_*\left(\frac{\alpha}{\lambda_{k_*+k_{**}}}\varphi_{k_*+k_{**}}+\frac{\alpha}{\lambda_{k_*-k_{**}}}\varphi_{k_*-k_{**}}+\frac{\beta}{\lambda_{2k_{**}}}\varphi_{2k_{**}}\right). \tag{4.17}$$

Projection on ker L. Since $k_* \neq 2k_{**}$,

$$PF_{02}u_0^2 = 0, \qquad \forall u_0 \in \ker L. \tag{4.18}$$

Then, with u_1 given by (4.10), the *bifurcation equation* reads

$$P\{F_1(\mu)u_0 + 2F_2(\mu)(u_0, u_1) + F_2(\mu)u_1^2 + F_{03}(u_0 + u_1)^3\} = 0$$

or equivalently (see (4.12)),

$$P\{F_1(\mu)u_0 + 2F_{02}(u_0, a_{02}u_0^2) + F_{03}u_0^3 + O(\mu u_0^3 + u_0^4)\} = 0. \tag{4.19}$$

According to Lyapunov–Schmidt's method, every solution (μ, u_0) to the bifurcation equation (4.19) provides a solution to (2.12). In order to solve (4.19), we use the Newton polygon method. Namely, we fix α, β such that $\alpha^2 + \beta^2 = 1$, set

$$\varphi_0 = \alpha\varphi_* + \beta\varphi_{**}, \quad u_0 = y\varphi_0, \qquad \text{for } \ y \in \mathbb{R},\ y \simeq 0 \tag{4.20}$$

and rescale the parameter μ by setting

$$\mu = y^2\tilde{\mu}.$$

If $y \neq 0$, then (4.19) is equivalent to

$$P\{F_{11}(\tilde{\mu})\varphi_0 + 2F_{02}(\varphi_0, a_{02}\varphi_0^2) + F_{03}\varphi_0^3 + O(y)\} = 0.$$

We recast this equation under the form

$$G(\tilde{\mu}, y) = 0. \tag{4.21}$$

There holds in view of (4.4)

$$D_{\tilde{\mu}}G(\tilde{\mu}, 0) = F_{11}(\cdot)\varphi_0 = \begin{pmatrix} 2\frac{p+\sqrt{p}}{\varepsilon_*}\alpha & 6M_*\alpha \\ 2\frac{p-\sqrt{p}}{\varepsilon_*}\beta & 6M_*\beta \end{pmatrix}. \tag{4.22}$$

The above matrix is the matrix of the linear mapping $D_{\tilde{\mu}}G(\tilde{\mu}, 0)\colon \mathbb{R}^2 \to \ker L$, expressed in the canonical basis of $\mathbb{R}^2$ and in the basis $(\varphi_*, \varphi_{**})$ of $\ker L$. If α, β and M_* are nonzero, then $D_{\tilde{\mu}}G(\tilde{\mu}, 0)$ is an isomorphism. Remark that $p \neq 0$ due to (4.2) and $k_* \neq k_{**}$. In order to apply the implicit function theorem, it is enough to find $\tilde{\mu}_0 \neq 0$ such that

$$G(\tilde{\mu}_0, 0) = 0, \qquad \text{with } \ \tilde{\mu}_0 = (\tilde{\mu}_1, \tilde{\mu}_2) \in \mathbb{R}^2. \tag{4.23}$$

For this, we notice that

$$G(\tilde{\mu}_0, 0) = P\{F_{11}(\tilde{\mu})\varphi_0 + 2F_{02}(\varphi_0, a_{02}\varphi_0^2) + F_{03}\varphi_0^3\}.$$

Moreover, by (4.22),

$$F_{11}(\tilde{\mu})\varphi_0 = 2\alpha\left(\tilde{\mu}_1\frac{p+\sqrt{p}}{\varepsilon_*} + 3M_*\tilde{\mu}_2\right)\varphi_* + 2\beta\left(\tilde{\mu}_1\frac{p-\sqrt{p}}{\varepsilon_*} + 3M_*\tilde{\mu}_2\right)\varphi_{**} \tag{4.24}$$

and, since $k_* \neq 3k_{**}$,

$$\begin{aligned} PF_{02}(\varphi_0, a_{02}\varphi_0^2) &= \frac{3M_*}{4}\left(x_{2k_*}\alpha + (x_{k_*+k_{**}} + x_{k_*-k_{**}})\beta\right)\varphi_* \\ &\quad + \frac{3M_*}{4}\left((x_{k_*+k_{**}} + x_{k_*-k_{**}})\alpha + x_{2k_{**}}\beta\right)\varphi_{**} \\ PF_{03}\varphi_0^3 &= \frac{3}{2}\left(\frac{1}{2}\alpha^3 + \alpha\beta^2\right)\varphi_* + \frac{3}{2}\left(\frac{1}{2}\beta^3 + \alpha^2\beta\right)\varphi_{**}. \end{aligned}$$

If we assume $\alpha \neq 0$ and $\beta \neq 0$, then (4.23) is equivalent to

$$\begin{cases} 2\dfrac{p+\sqrt{p}}{\varepsilon_*}\tilde{\mu}_1 + 6M_*\tilde{\mu}_2 = -\dfrac{3M_*}{2}\left(x_{2k_*} + (x_{k_*+k_{**}} + x_{k_*-k_{**}})\dfrac{\beta}{\alpha}\right) - \dfrac{3}{4}\alpha^2 - \dfrac{3}{2}\beta^2, \\ 2\dfrac{p-\sqrt{p}}{\varepsilon_*}\tilde{\mu}_1 + 6M_*\tilde{\mu}_2 = -\dfrac{3M_*}{2}\left((x_{k_*+k_{**}} + x_{k_*-k_{**}})\dfrac{\alpha}{\beta} + x_{2k_{**}}\right) - \dfrac{3}{4}\beta^2 - \dfrac{3}{2}\alpha^2. \end{cases} \tag{4.25}$$

Since $M_* \neq 0$ and $p \neq 0$, it is clear that (4.25) has a unique solution $(\tilde{\mu}_1, \tilde{\mu}_2)$. Thus for every $y \simeq 0$, we have a bifurcating solution $(\delta(y), v(y))$ of (2.13). Moreover,

$$\delta(y) = (\varepsilon(y), M(y)) = (\varepsilon_* + \tilde{\mu}_1 y^2, M_* + \tilde{\mu}_2 y^2) + O(y^3).$$

Next we would like to compute $(\ddot{\varepsilon}(0), \ddot{M}(0))$ where $\ddot{\varepsilon}(0)$ is the value at $y = 0$ of the second derivative of $\varepsilon(\cdot)$. We readily have

$$\ddot{\varepsilon}(0) = 2\tilde{\mu}_1, \qquad \ddot{M}(0) = 2\tilde{\mu}_2.$$

Hence we obtain from (4.25) the following equations.

$$\begin{cases} \dfrac{p+\sqrt{p}}{\varepsilon_*}\ddot{\varepsilon}(0) + 3M_*\ddot{M}(0) = -\dfrac{3M_*}{2}\left(x_{2k_*} + (x_{k_*+k_{**}} + x_{k_*-k_{**}})\dfrac{\beta}{\alpha}\right) - \dfrac{3}{4}\alpha^2 - \dfrac{3}{2}\beta^2, \\ \dfrac{p-\sqrt{p}}{\varepsilon_*}\ddot{\varepsilon}(0) + 3M_*\ddot{M}(0) = -\dfrac{3M_*}{2}\left((x_{k_*+k_{**}} + x_{k_*-k_{**}})\dfrac{\alpha}{\beta} + x_{2k_{**}}\right) - \dfrac{3}{4}\beta^2 - \dfrac{3}{2}\alpha^2. \end{cases} \tag{4.26}$$

We will rewrite these equations in a more convenient form. For this, we put

$$f_* := \frac{9}{2}\frac{M_*^2}{\lambda_{2k_*}} - \frac{3}{4}, \qquad f_{**} := \frac{9}{2}\frac{M_*^2}{\lambda_{2k_{**}}} - \frac{3}{4}, \tag{4.27}$$

$$C_S := 9M_*^2\left(\frac{1}{\lambda_{k_*+k_{**}}} + \frac{1}{\lambda_{k_*-k_{**}}}\right) - \frac{3}{2}. \tag{4.28}$$

In view of (4.15), equations (4.26) reads

$$\begin{cases} \dfrac{p+\sqrt{p}}{\varepsilon_*}\ddot{\varepsilon}(0) + 3M_*\ddot{M}(0) = (f_* - C_S)\alpha^2 + C_S, \\ \dfrac{p-\sqrt{p}}{\varepsilon_*}\ddot{\varepsilon}(0) + 3M_*\ddot{M}(0) = (-f_{**} + C_S)\alpha^2 + f_{**}. \end{cases} \tag{4.29}$$

In (4.29), the unknown is $(\ddot{\varepsilon}(0), \ddot{M}(0))$. ε_*, M_*, k_*, k_{**} are fixed and α is a parameter ranging in $(0,1)$.

Let us notice that f_*, f_{**} and C_S appear naturally if a group theoretic approach is considered (see for instance [9, Chapter XX] and, [5]). More precisely, if $u_0 = X\varphi_* + Y\varphi_{**}$, that is $X = y\alpha$, $Y = y\beta$, then, to third order, the bifurcation equation has the form

$$\begin{cases} -f_*X^3 - C_SXY^2 + \left((\varepsilon\sigma_{k_*} - 1)^2 + r + 3M^2\right)X = 0, \\ -C_SX^2Y - f_{**}Y^3 + \left((\varepsilon\sigma_{k_{**}} - 1)^2 + r + 3M^2\right)Y = 0. \end{cases} \tag{4.30}$$

However, the drawback of this approach is that the persitence of (explicit) solutions to (4.30) is not proved (and not obvious) when we return to the bifurcation equation. To our knowledge, even recent and sophisticated methods like path formulation do not lead to 3-determinacy in our setting.

Besides, it turns out that this truncated equation has a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry; unlike the bifurcation equation (since the fact that u_0 is a solution to (4.19) does not implies that $-u_0$ is a solution too).

We derive from (4.29)

$$\ddot{\varepsilon}(0) = \frac{3\varepsilon_*}{8\sqrt{p}}(A\alpha^2 + B), \qquad \ddot{M}(0) = \frac{1}{8M_*}(C\alpha^2 + D), \tag{4.31}$$

where A, B, C, D satisfy

$$\frac{3}{8}(\sqrt{p}+1)A + \frac{3}{8}C = f_* - C_S \tag{4.32}$$
$$\frac{3}{8}(\sqrt{p}+1)B + \frac{3}{8}D = C_S \tag{4.33}$$
$$\frac{3}{8}(\sqrt{p}-1)A + \frac{3}{8}C = -f_{**} + C_S \tag{4.34}$$
$$\frac{3}{8}(\sqrt{p}-1)B + \frac{3}{8}D = f_{**}. \tag{4.35}$$

Subtracting (4.35) from (4.33), we obtain

$$\frac{3}{4}B = -f_{**} + C_S. \tag{4.36}$$

Also we obtain

$$D = \frac{8}{3}C_S - (\sqrt{p}+1)B \tag{4.37}$$
$$\frac{3}{4}A = f_* + f_{**} - 2C_S \tag{4.38}$$
$$C = \frac{8}{3}(f_* - C_S) - (\sqrt{p}+1)A. \tag{4.39}$$

In order to express A, B, C, D more simply, we put

$$x := \left(\frac{k_*}{k_{**}}\right)^2.$$

Then, in view of (4.4),

$$\sqrt{p} = \frac{x-1}{x+1}$$

and, since $\varepsilon_*\sigma_{k_*} = 1 + \sqrt{p}$,

$$\begin{aligned}\lambda_{2k_*} &= (4\varepsilon_*\sigma_{k_*} - 1)^2 - p = (4(1+\sqrt{p}) - 1)^2 - p \\ &= 12x\frac{4x-1}{(x+1)^2}.\end{aligned} \tag{4.40}$$

Similarly,

$$\lambda_{2k_{**}} = -12\frac{x-4}{(x+1)^2}, \tag{4.41}$$

$$\lambda_{k_*+k_{**}} = 4\sqrt{x}\frac{(2\sqrt{x}+1)(\sqrt{x}+2)}{(x+1)^2}, \tag{4.42}$$

$$\lambda_{k_*-k_{**}} = -4\sqrt{x}\frac{(2\sqrt{x}-1)(\sqrt{x}-2)}{(x+1)^2}. \tag{4.43}$$

Then

$$A = 2 - M_*^2\frac{(x+1)^2(2x^2-61x+2)}{x(4x-1)(x-4)}, \tag{4.44}$$

$$B = -1 + M_*^2\frac{(x+1)^2(4x-61)}{2(4x-1)(x-4)}, \tag{4.45}$$

$$C = -2\frac{x-1}{x+1} + M_*^2\frac{(x^2-1)(4x^2-57x+4)}{x(4x-1)(x-4)}, \tag{4.46}$$

$$D = -2\frac{x+2}{x+1} - M_*^2\frac{(x+1)(4x^2-x+60)}{(4x-1)(x-4)}. \tag{4.47}$$

Then we can state a bifurcation result whose proof comes from the above analysis.

Theorem 4.1. *Let $\delta_* = (\varepsilon_*, M_*) \in (0,\infty)\times\mathbb{R}$, $(\alpha,\beta)\in(-1,1)^2$ and k_*, k_{**} be integers satisfying $1 \le k_* < k_{**}$. We assume that (4.1)–(4.3) hold and*

$$\alpha^2+\beta^2 = 1, \qquad \alpha \neq 0, \qquad \beta \neq 0. \tag{4.48}$$

Then $(\delta, v) = (\delta_*, 0) \in \mathbb{R}^2 \times \dot{V}_4$ is a bifurcation point for equation (2.13). More precisely, there exist $r_1 > 0$ (close to zero) and smooth functions

$$\delta = (\varepsilon(\cdot), M(\cdot))\colon (-r_1, r_1) \to \mathbb{R}^2, \qquad v\colon (-r_1,r_1) \to \dot{V}_4$$

depending on α, β, k_* and k_{**} such that for every $y \in (-r_1, r_1)$, one has

$$\varepsilon(y)^2 v(y)^{(4)} + 2\varepsilon(y)v(y)^{(2)} + f(M(y)+v(y)) = \int_\Omega f(M(y)+v(y))\,\mathrm{d}x \quad \text{ in } L^2(\Omega).$$

Moreover,

$$\varepsilon(y) = \varepsilon_* + \frac{\ddot{\varepsilon}(0)}{2}y^2 + O(y^3), \tag{4.49}$$

$$M(y) = M_* + \frac{\ddot{M}(0)}{2}y^2 + O(y^3), \tag{4.50}$$

$$v(y) = y\varphi_0 + y^2 v_2 + O(y^3), \tag{4.51}$$

where $\ddot{\varepsilon}(0)$, $\ddot{M}(0)$ are defined through (4.31) and (4.44)–(4.47), $\varphi_0 = \alpha\varphi_* + \beta\varphi_{**}$ and v_2 is given by (4.14), (4.15).

4.1 Sign of $\ddot{\varepsilon}(0)$ and $\ddot{M}(0)$

For every $\alpha \in (0,1)$, Theorem 4.1 gives us the *parameter curve*

$$y \mapsto \delta(y) = (\varepsilon(y), M(y)).$$

The tangent to this curve at $y = 0$ is given by

$$\ddot{\delta}(0) = (\ddot{\varepsilon}(0), \ddot{M}(0))$$

provided that this vector do not vanish. In the sequel, we will compute the signs of $\ddot{\varepsilon}(0)$ and $\ddot{M}(0)$ in order to have informations on the profile of the above curve near $y = 0$. For instance, if $\ddot{\varepsilon}(0)$ and $\ddot{M}(0)$ are positive, then $\delta(y)$ belongs to the first quadrant of the plane (ε, M).

We denote by $A = A(x, M_*)$ the function defined on $(1,\infty) \setminus \{4\} \times \mathbb{R}$ by the left-hand side of (4.44). In the same way, we define the functions $B = B(x, M_*)$, $C = C(x, M_*)$ and $D = D(x, M_*)$ with (4.45)–(4.47).

Sign of $\ddot{\varepsilon}(0)$

Under the assumptions and notations of Theorem 4.1, it follows from (4.31) that $\ddot{\varepsilon}(0)$ and $A\alpha^2 + B$ have the same sign. So for every $(x, M, \alpha) \in (1,\infty) \setminus \{4\} \times \mathbb{R} \times (-1,1)$, we will compute the sign of $A(x,M)\alpha^2 + B(x,M)$. Taking advantage of the monotonicity of $A(x,M)\alpha^2 + B(x,M)$ w.r.t. α^2, we will look at the sign of $B(x,M)$ and $(A+B)(x,M)$. For this, we introduce the so-called *cancellation functions* of B and $A+B$, namely

$$B_0 \colon (1,\infty) \setminus \{61/4\} \to \mathbb{R}, \qquad x \mapsto 2\frac{(4x-1)(x-4)}{(x+1)^2(4x-61)} \tag{4.52}$$

$$(A+B)_0 \colon (1,\infty) \to \mathbb{R}, \qquad x \mapsto -2\frac{x(4x-1)(x-4)}{(x+1)^2(61x-4)}. \tag{4.53}$$

These functions satisfy

$$M^2 = B_0(x) \iff B(x,M) = 0$$
$$M^2 = (A+B)_0(x) \iff (A+B)(x,M) = 0.$$

The sign of $B(x,M)$ is given in the following result.

Lemma 4.2. *If $x \in (1,4) \cup (\frac{61}{4}, \infty)$, then $B_0(x) > 0$ and*

$$M^2 < B_0(x) \implies B(x,M) < 0$$
$$B_0(x) < M^2 \implies B(x,M) > 0.$$

If $x \in (4, \frac{61}{4})$, then $B(x,M) < 0$ for every M.

The statement of Lemma 4.2 and those of the seven forthcoming results are easily proved; thus their proof will be omitted. Regarding the sign of $A+B$, we have the lemma below.

Lemma 4.3. *If $x \in (1,4)$ then $(A+B)_0(x) > 0$ and*

$$M^2 < (A+B)_0(x) \implies (A+B)(x,M) > 0$$
$$(A+B)_0(x) < M^2 \implies (A+B)(x,M) < 0.$$

If $x \in (4,\infty)$ then $(A+B)(x,M) > 0$ for every M.

Moreover, the cancellation functions are ordered or are simultaneously negative.

Lemma 4.4. *For every $x \in (1,4)$,*

$$0 < (A+B)_0(x) < B_0(x).$$

For every $x \in (4, \frac{61}{4})$, one has $(A+B)_0(x) < 0$ and $B_0(x) < 0$.
For every $x \in (\frac{61}{4}, \infty)$, one has $(A+B)_0(x) < 0 < B_0(x)$.

Then we can give the sign of $\ddot{\varepsilon}(0)$.

Proposition 4.5. *Under the assumptions and notations of Theorem 4.1, the sign of*

$$\ddot{\varepsilon}(0) = \frac{3\varepsilon_*}{8\sqrt{p}}(A(x,M_*)\alpha^2 + B(x,M_*))$$

is as follows.

(i) If $x \in (1,4)$, then

- *(a) if $M_*^2 < (A+B)_0(x)$, then $\alpha \mapsto \ddot{\varepsilon}(0)$ is increasing on $(0,1)$ and changes its sign on $(0,1)$;*
- *(b) if $(A+B)_0(x) \leq M_*^2 \leq B_0(x)$, then $\ddot{\varepsilon}(0) < 0$ for every $|\alpha| \in (0,1)$;*
- *(c) if $B_0(x) \leq M_*^2$, then $\alpha \mapsto \ddot{\varepsilon}(0)$ is decreasing on $(0,1)$ and changes its sign on $(0,1)$.*

(ii) If $x \in (4, \frac{61}{4})$, then $\alpha \mapsto \ddot{\varepsilon}(0)$ is increasing on $(0,1)$ and changes its sign on $(0,1)$.

(iii) If $x \in (\frac{61}{4}, \infty)$, then

- *(a) if $M_*^2 < B_0(x)$, then $\alpha \mapsto \ddot{\varepsilon}(0)$ is increasing on $(0,1)$ and changes its sign on $(0,1)$;*
- *(b) if $B_0(x) \leq M_*^2$, then $\ddot{\varepsilon}(0) > 0$ for every $|\alpha| \in (0,1)$.*

Sign of $\ddot{M}(0)$

We proceed as above. We define the cancellation functions D_0 and $(C+D)_0$ of D and $C+D$, namely

$$D_0\colon (1,\infty) \to \mathbb{R}, \qquad x \mapsto 2\frac{(x+2)(4x-1)(x-4)}{(x+1)^2(-4x^2+x-60)} \tag{4.54}$$

$$(C+D)_0\colon (1,\infty) \to \mathbb{R}, \qquad x \mapsto 2\frac{(2x+1)x(4x-1)(x-4)}{(x+1)^2(-60x^2+x-4)}. \tag{4.55}$$

The signs of $D(x,M)$ and $(C+D)(x,M)$ are easily determined with the use of the cancellation functions. More precisely, the following lemmas hold.

Lemma 4.6. *If $x \in (1,4)$, then $D_0(x) > 0$ and*

$$M^2 < D_0(x) \Longrightarrow D(x,M) < 0$$
$$D_0(x) < M^2 \Longrightarrow D(x,M) > 0.$$

If $x \in (4,\infty)$, then $D_0(x) < 0$ and $D(x,M) < 0$ for every M.

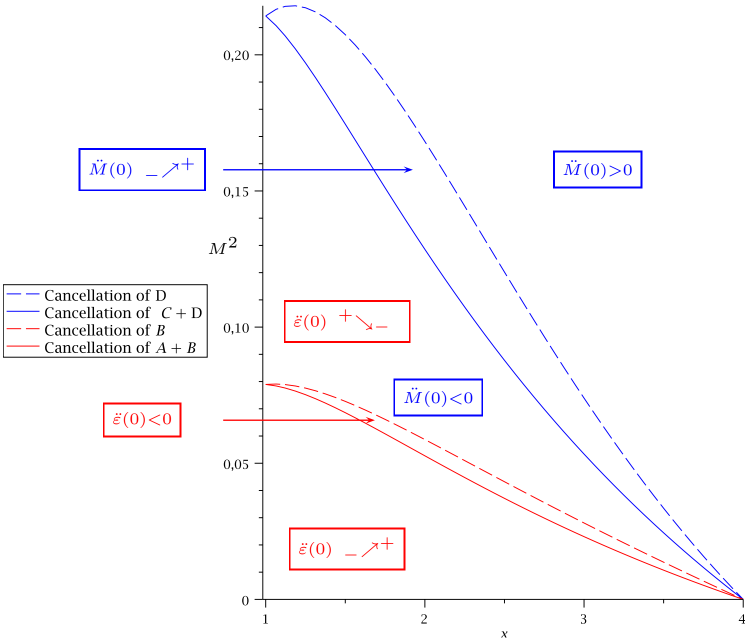

Figure 4.1: The sign of $\ddot{\varepsilon}(0)$ and $\ddot{M}(0)$ in the parameter space (x, M^2) with $x := (k_*/k_{**})^2$.

Lemma 4.7. *If $x \in (1,4)$, then $(C+D)_0(x) > 0$ and*

$$M^2 < (C+D)_0(x) \Longrightarrow (C+D)(x,M) < 0$$
$$(C+D)_0(x) < M^2 \Longrightarrow (C+D)(x,M) > 0.$$

If $x \in (4,\infty)$, then $(C+D)_0(x) < 0$ and $(C+D)(x,M) < 0$ for every M.

Lemma 4.8. *For every $x \in (1,4)$,*

$$0 < (A+B)_0(x) < B_0(x) < (C+D)_0(x) < D_0(x).$$

For every $x \in (4,\infty)$, one has $(C+D)_0(x) < 0$ and $D_0(x) < 0$.

Proposition 4.9. *Under the assumptions and notations of Theorem 4.1, the sign of*

$$\ddot{M}(0) = \frac{1}{8M_*}\big(C(x,M_*)\alpha^2 + D(x,M_*)\big)$$

is as follows.

(i) If $x \in (1,4)$, then

  (a) if $M_*^2 \le (C+D)_0(x)$, then $\ddot{M}(0) < 0$ for every $|\alpha| \in (0,1)$;

  (b) if $(C+D)_0(x) < M_*^2 < D_0(x)$, then $\alpha \mapsto \ddot{M}(0)$ is increasing on $(0,1)$ and changes its sign on $(0,1)$;

  (c) if $D_0(x) \le M_*^2$, then $\ddot{M}(0) > 0$ for every $|\alpha| \in (0,1)$.

(ii) If $x \in (4,\infty)$, then $\ddot{M}(0) < 0$ for every $|\alpha| \in (0,1)$.

The result of Propositions 4.5 and 4.9 are illustrated by Figure 4.1 for x ranging in $(1,4)$.

4.2 Variations of $\alpha \mapsto \frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}$

We will prove a monotonicity result for the function $\alpha \mapsto \frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}$. In the parameter space (ε, M), $\frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}$ is the tangent of the angle between the unit vector $(1,0)$ and $(\ddot{\varepsilon}(0), \ddot{M}(0))$. We recall that $(\ddot{\varepsilon}(0), \ddot{M}(0))$ is tangent to the curve $(\varepsilon(\cdot), M(\cdot))$ at $y = 0$. As we will see later on, the monotonicity of $\alpha \mapsto \frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}$ is related to the stability of bifurcating solutions given by Theorem 4.1.

If $A(x, M_*)\alpha^2 + B(x, M_*) \neq 0$, then

$$\frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)} = \frac{\sqrt{p}}{3M_*\varepsilon_*}\frac{C\alpha^2 + D}{A\alpha^2 + B} \tag{4.56}$$

and

$$\frac{\partial}{\partial X}\frac{CX + D}{AX + B} = \frac{BC - AD}{(AX + B)^2}.$$

Thus it is enough to compute the sign of $BC - AD$. For this, we write

$$BC - AD = B(C + D) - D(A + B).$$

The previous results give informations on the signs of B, $C + D$, D and $A + B$. Thus the following assertions hold.

(i) If $x \in (1, 4)$, then

 (a) if $M_*^2 < (A + B)_0(x)$ or $D_0(x) < M_*^2$, then $BC - AD > 0$;

 (b) $B_0(x) \leq M_*^2 < (C + D)_0(x)$, then

$$BC - AD < 0. \tag{4.57}$$

(ii) If $x \in (4, \frac{61}{4})$, then $BC - AD > 0$.

(iii) If $x \in (\frac{61}{4}, \infty)$ and $M_*^2 < B_0(x)$, then $BC - AD > 0$.

In the other cases, we have to push the analysis a little bit further.

(i) If $x \in (1, 4)$, then there remain to consider two cases. The first one is when $(A + B)_0(x) < M_*^2 < B_0(x)$. Let us denote by M_1 the positive number satisfying $M_1^2 = (A + B)_0(x)$. We have

$$(BC - AD)(x, M_1) = B(C + D)(x, M_1).$$

Moreover $B(x, M_1) < 0$ (according to Lemmas 4.2 and 4.4) and $(C + D)(x, M_1) < 0$ (due to Lemmas 4.7 and 4.8). Thus

$$(BC - AD)|_{M^2 = (A+B)_0(x)} > 0. \tag{4.58}$$

Besides, by (4.57),

$$(BC - AD)|_{M^2 = B_0(x)} < 0. \tag{4.59}$$

Moreover,

$$\frac{1}{2}(BC - AD)(x, M) = 3 + \frac{(x+1)^2 b_1(x)}{x(4x-1)(x-4)}M^2 + \frac{(x+1)^4 b_2(x)}{4x^2(4x-1)^2(x-4)^2}M^4, \tag{4.60}$$

with

$$b_1(x) = -2x^2 + 121x - 2$$
$$b_2(x) = x(4x^2 + 3583x + 4).$$

In particular, $(BC - AD)(x, M)$ is a quadratic polynomial w.r.t. the variable $X := M^2$. With (4.58), (4.59), we deduce that, for every $x \in (1, 4)$, there exists a unique $J_1(x) \in ((A+B)_0(x), B_0(x))$ such that

$$(BC - AD)\left(x, \sqrt{J_1(x)}\right) = 0. \tag{4.61}$$

The second case we will consider is when $(C + D)_0(x) < M_*^2 < D_0(x)$. Arguing as above, we prove that there exists a unique $J_2(x) \in ((C + D)_0(x), D_0(x))$ such that

$$(BC - AD)(x, \sqrt{J_2(x)}) = 0. \tag{4.62}$$

Then the sign of $BC - AD$ follows easily by using (4.60), $J_1(x)$ and $J_2(x)$.

(ii) If $x \in (\frac{61}{4}, \infty)$ and $B_0(x) < M^2$, then in view of (4.60), the discriminant $\Delta(x)$ of the polynomial

$$X \mapsto 3 + \frac{(x+1)^2 b_1(x)}{x(4x-1)(x-4)} X + \frac{(x+1)^4 b_2(x)}{4x^2(4x-1)^2(x-4)^2} X^2 \tag{4.63}$$

satisfies

$$\Delta(x) = \frac{4(x+1)^4}{x^2(4x-1)^2(x-4)^2}(x^4 - 124x^3 + 975x^2 - 124x + 1).$$

We check that there exists $x_3 \in (115, 116)$ such that

$$\begin{aligned} \Delta(x) < 0 &\quad \text{on } \left(\tfrac{61}{4}, x_3\right) \\ \Delta(x) > 0 &\quad \text{on } (x_3, \infty) \\ \Delta(x_3) = 0. & \end{aligned} \tag{4.64}$$

From (4.60), (4.64), we immediately obtain that

$$(BC - AD)(x, M) > 0 \quad \forall (x, M^2) \in \left(\tfrac{61}{4}, x_3\right) \times (B_0(x), \infty).$$

There remains to consider the case where $x > x_3$. With straightforward computations, we prove that the polynomial (4.63) has two positive roots $J_1(x) < J_2(x)$ in the interval $(B_0(x), \infty)$. Hence for every $x > x_3$, there holds

$$(BC - AD)(x, M) \begin{cases} < 0 & \text{if } M^2 \in (J_1(x), J_2(x)) \\ > 0 & \text{if } M^2 \in (B_0(x), J_1(x)) \cup (J_2(x), \infty) \end{cases}.$$

We may summarize the above results in the proposition below.

Proposition 4.10. *Under the assumptions and notations of Theorem 4.1, there exists $x_3 \in (115, 116)$ such that for every $x \in (1, 4) \cup (x_3, \infty)$, the polynomial (4.63) has two positive roots $J_1(x) < J_2(x)$. Moreover, the following hold.*

(i) If $M_*^2 \in (J_1(x), J_2(x))$, then the function $\alpha \mapsto \frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}$ is decreasing on the positive intervals of its domain of definition, i.e. on $(0,1)$ if $AB(x, M_*) > 0$ or on $(0, \sqrt{|B/A|})$ and on $(\sqrt{|B/A|}, 1)$ if $AB(x, M_*) < 0$.

(ii) If $M_*^2 \in (0, J_1(x)) \cup (J_2(x), \infty)$, then $(BC - AD)(x, M_*) > 0$ and $\alpha \mapsto \frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}$ is increasing on the positive intervals of its domain of definition.

If $x \in (4, x_3)$, then $\alpha \mapsto \frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}$ is increasing on the positive interval of its domain of definition.

Remark 4.11.
• $J_1(x_3) = J_2(x_3)$.
• We may compute explicitly $J_1(x)$ and $J_2(x)$ thanks to (4.60) (see [1]).
• $\ddot{\varepsilon}(0) = \frac{3\varepsilon_*}{8\sqrt{p}}(A\alpha^2 + B)$ may vanish. This occurs for instance for some $\alpha_0 \in (0,1)$ if $x \in (1,4)$ and $M_*^2 < (A+B)_0(x)$. See Proposition 4.5. In this case, the function $\alpha \mapsto \frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}$ is defined on $(-1,1) \setminus \{-\sqrt{|B/A|}, \sqrt{|B/A|}\}$.
• Fig 4.2 is a phase diagram in the parameter space $(\varepsilon, 3M^2)$. We have chosen some $x \in (1,4)$ and $M_*^2 \in (J_1(x), J_2(x))$. The vector $\vec{u}$ is positively colinear to the tangent to the curve $(\varepsilon(\cdot), 3M^2(\cdot))$ at $y = 0$ for $\alpha = 0.6$. If $\alpha = 0$ or $\alpha = 1$, then $\vec{u} = 0$ since there is no bifurcation branch according to Theorem 4.1. When α goes from 0 to 1, the vector $\vec{u}$ turns clockwise; in accordance with the assertion *(i)* of Proposition 4.10. Its range is depicted by the blue curve.

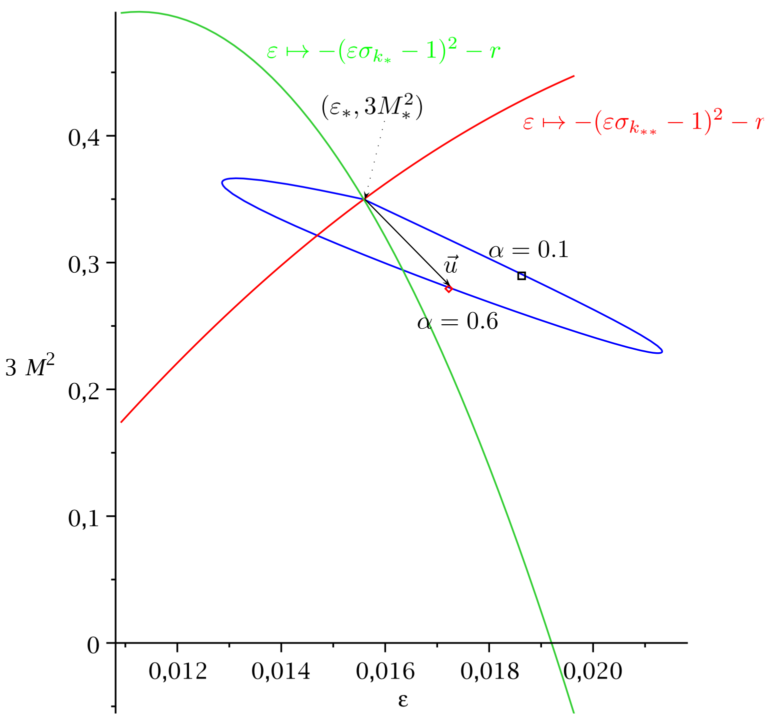

Figure 4.2: Phase diagram for $k_* = 3$, $k_{**} = 2$ (thus $x = 9/4$) and $M_*^2 = 0.4(C+D)_0(x) + 0.6J_2(x)$. See Remark 4.11.

Finally we can combine the above results on the signs of $\ddot{\varepsilon}(0)$, $\ddot{M}(0)$ and the variation of $\frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}$ to obtain a better insight of the behaviour of the curve $y \mapsto \delta(y)$ w.r.t. α. We will only investigate the cases useful in the sequel.

We assume that $x \in (1,4)$ and recall that

$$(A+B)_0(x) < J_1(x) < B_0(x) < (C+D)_0(x) < J_2(x) < D_0(x). \tag{4.65}$$

(a) If $(C+D)_0(x) < M^2 < J_2(x)$, then, due to Proposition 4.5 and (4.65), $\alpha \mapsto \ddot{\varepsilon}(0)$ has a unique zero α_0 in $(0,1)$ and $\alpha_0 = \sqrt{|B/A|}$. Also (see Proposition 4.9), $\alpha \mapsto \ddot{M}(0)$ has a unique zero $\alpha_1 \in (0,1)$ and $\alpha_1 = \sqrt{|D/C|}$.

Let us show that $\alpha_0 < \alpha_1$. By Proposition 4.10 and (4.65), $\alpha \mapsto \frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}$ is decreasing on $(0,\alpha_0)$ and on $(\alpha_0,1)$. Moreover, Lemmas 4.2 and 4.6 imply that

$$\frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}\Big|_{\alpha=0} = \frac{D}{B} < 0. \tag{4.66}$$

Thus

$$\frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)} < 0 \qquad \forall \alpha \in (0,\alpha_0).$$

Hence $\ddot{M}(0) \neq 0$ for every $\alpha \in (0,\alpha_0)$, and consequently, $\alpha_1 \geq \alpha_0$. That is to say $-D/C \geq -B/A$. However, these two numbers are not equal since $M^2 \neq J_1(x), J_2(x)$. Then $\alpha_0 < \alpha_1$. Summing up, we get

$$\begin{aligned}
&\ddot{\varepsilon}(0) \geq 0, && \ddot{M}(0) < 0 && \forall \alpha \in (0,\alpha_0] \\
&\ddot{\varepsilon}(0) < 0, && \ddot{M}(0) < 0 && \forall \alpha \in (\alpha_0,\alpha_1) \\
&\ddot{\varepsilon}(0) < 0, && \ddot{M}(0) > 0 && \forall \alpha \in (\alpha_1,1].
\end{aligned} \tag{4.67}$$

This behaviour may be observed in Figure 4.2.

(b) If $J_2(x) < M^2 < D_0(x)$, then, as above, α_0, α_1 belong to $(0,1)$ and $\alpha_0 \neq \alpha_1$. Moreover, $\alpha \mapsto \frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}$ is increasing on $(0,\alpha_0)$, hence

$$\lim_{\alpha\to\alpha_0,\alpha<\alpha_0} \frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)} = \infty.$$

Since (4.66) still holds, we deduce that $\ddot{M}(0)$ vanishes in $(0,\alpha_0)$. Thus $\alpha_1 < \alpha_0$. Summing up, we get

$$\begin{aligned}
&\ddot{\varepsilon}(0) > 0, && \ddot{M}(0) < 0 && \forall \alpha \in (0,\alpha_1) \\
&\ddot{\varepsilon}(0) \geq 0, && \ddot{M}(0) > 0 && \forall \alpha \in (\alpha_1,\alpha_0] \\
&\ddot{\varepsilon}(0) < 0, && \ddot{M}(0) > 0 && \forall \alpha \in (\alpha_0,1].
\end{aligned} \tag{4.68}$$

5 Properties of bifurcating solutions

5.1 Energy of the bifurcating solutions

In this section, we will compare the energy of the bifurcating solutions $u = M(y) + v(y)$ given by Theorem 4.1 and the energy of the trivial solution $u = M(y)$. Let us recall that, for $(u,\varepsilon) \in V_2 \times (0,\infty)$, the energy of u is given by (2.14). Moreover, $\delta(y) := (\varepsilon(y), M(y))$ for $y \simeq 0$.

Theorem 5.1. *Let $y \mapsto (\delta(y), v(y))$ be a bifurcation branch given by Theorem 4.1. Then*

$$E\big(M(y)+v(y),\varepsilon(y)\big) - E\big(M(y),\varepsilon(y)\big) = \frac{1}{8}G(\alpha)y^4 + O(y^5),$$

where

$$G(\alpha) := (2\sqrt{p}\alpha^2 + p - \sqrt{p})\frac{\ddot{\varepsilon}(0)}{\varepsilon_*} + 3M_*\ddot{M}(0). \tag{5.1}$$

Proof. We put $u(y) := M(y)+v(y)$. For $y \simeq 0$, the derivative of the function S defined through

$$S(y) = E\big(u(y),\varepsilon(y)\big) - E\big(M(y),\varepsilon(y)\big)$$

satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}S(y) = D_uE\big(u(y),\varepsilon(y)\big)\big(\dot{M}(y)+\dot{v}(y)\big) + D_\varepsilon E\big(u(y),\varepsilon(y)\big)\dot{\varepsilon}(y)\\ - D_uE\big(M(y),\varepsilon(y)\big)\dot{M}(y) - D_\varepsilon E\big(M(y),\varepsilon(y)\big)\dot{\varepsilon}(y).\end{aligned} \tag{5.2}$$

Since $u(y)$ and $M(y)$ solve (2.8) with $M = M(y)$ and $\int_0^1 \dot{v}(y)\,\mathrm{d}x = 0$, we have

$$\begin{aligned}D_uE\big(M(y),\varepsilon(y)\big)\dot{M}(y) &= \int_0^1 f\big(M(y)\big)\,\mathrm{d}x\ \dot{M}(y)\\ D_uE\big(u(y),\varepsilon(y)\big)\big(\dot{M}(y)+\dot{v}(y)\big) &= \int_0^1 f\big(u(y)\big)\,\mathrm{d}x\ \dot{M}(y).\end{aligned}$$

Furthermore, by Taylor expansions,

$$\begin{aligned}\int_0^1 f\big(u(y)\big) - f\big(M(y)\big)\,\mathrm{d}x &= 3M(y)\int_0^1 v(y)^2\,\mathrm{d}x + \int_0^1 v(y)^3\,\mathrm{d}x\\ &= 3M_*y^2\int_0^1 \varphi_0^2\,\mathrm{d}x + O(y^3),\end{aligned}$$

in view of (4.50), (4.51). Thus

$$\int_0^1 f\big(u(y)\big) - f\big(M(y)\big)\,\mathrm{d}x\ \dot{M}(y) = \frac{3}{2}M_*\ddot{M}(0)y^3 + O(y^4).$$

Considering, in (5.2), the derivatives w.r.t. ε, we have

$$\begin{aligned}D_\varepsilon E\big(M(y),\varepsilon(y)\big) &= 0\\ D_\varepsilon E\big(u(y),\varepsilon(y)\big) &= \int_0^1 \varepsilon(y)\big(v(y)_{xx}\big)^2 - \big(v(y)_x\big)^2\,\mathrm{d}x.\end{aligned}$$

Thanks to (4.51), we get the following expansions.

$$\begin{aligned}\int_0^1 \big(v(y)_x\big)^2\,\mathrm{d}x &= y^2\int_0^1 (\varphi_{0_x})^2\,\mathrm{d}x + O(y^3)\\ \int_0^1 \big(v(y)_{xx}\big)^2\,\mathrm{d}x &= y^2\int_0^1 (\varphi_{0_{xx}})^2\,\mathrm{d}x + O(y^3).\end{aligned}$$

By (4.49), we have $\dot{\varepsilon}(y) = \ddot{\varepsilon}(0)y + O(y^2)$; and since $\varphi_0 = \alpha\varphi_* + \beta\varphi_{**}$, we infer

$$D_\varepsilon E\big(u(y),\varepsilon(y)\big)\dot{\varepsilon}(y) = y^3\frac{\ddot{\varepsilon}(0)}{2\varepsilon_*}(2\sqrt{p}\alpha^2 + p - \sqrt{p}).$$

By combining the above results, we prove the assertion of the theorem. □

According to Theorem 5.1, it is enough to compute the sign of $G(\alpha)$. Before stating our result, we will introduce some notation. In view of (4.27), (4.40) and (4.41), we have

$$f_* = f_*(x, M) = \frac{3}{8}\frac{M_*^2}{x(4x-1)}(x+1)^2 - \frac{3}{4}, \qquad f_{**} = f_{**}(x, M) = -\frac{3}{8}\frac{M_*^2}{x-4}(x+1)^2 - \frac{3}{4}.$$

Then we denote by $(f_*)_0$ the cancellation function of f_*, i.e.

$$(f_*)_0(x) := 2\frac{x(4x-1)}{(x+1)^2}. \tag{5.3}$$

Similarly

$$(f_{**})_0(x) := -2\frac{x-4}{(x+1)^2}. \tag{5.4}$$

Theorem 5.2. *Let $M_* > 0$, $\alpha \in (0,1)$ and $G(\alpha)$ be given by (5.1).*

(i) If $x \in (1,4)$, then

(a) if $M_*^2 < J_2(x)$, then $G < 0$ on $(0,1)$ and consequently,

$$E\big(u(y), \varepsilon(y)\big) < E\big(M(y), \varepsilon(y)\big) \quad \text{for } \; y \simeq 0;$$

(b) if $J_2(x) < M_*^2 < (f_{**})_0(x)$, then G has two zeros $0 < \alpha_1 < \alpha_2 < 1$ and

$$G < 0 \quad \text{on } \; (0, \alpha_1) \cup (\alpha_2, 1), \qquad G > 0 \quad \text{on } \; (\alpha_1, \alpha_2);$$

(c) if $(f_{**})_0(x) < M_*^2 < (f_*)_0(x)$, then G has one zeros $0 < \alpha_1 < 1$ and

$$G > 0 \quad \text{on } \; (0, \alpha_1), \qquad G < 0 \quad \text{on } \; (\alpha_1, 1);$$

(d) if $(f_*)_0(x) < M_*^2$, then $G > 0$ on $(0,1)$.

(ii) If $x \in (4, \infty)$, then

(a) if $M_*^2 < (f_*)_0(x)$, then $G < 0$ on $(0,1)$;

(b) if $(f_*)_0(x) < M_*^2$, then G has one zeros $0 < \alpha_1 < 1$ and

$$G < 0 \quad \text{on } \; (0, \alpha_1), \qquad G > 0 \quad \text{on } \; (\alpha_1, 1).$$

Proof. By (4.29), we can write

$$G(\alpha) = 2\sqrt{p}\frac{\ddot{\varepsilon}(0)}{\varepsilon_*}\alpha^2 + (-f_{**} + C_S)\alpha^2 + f_{**}.$$

With (4.31) and (4.36), we get

$$G(\alpha) = \frac{3}{4}A\alpha^4 + \frac{3}{2}B\alpha^2 + f_{**}.$$

Thus we will study the quadratic polynomial $H(\cdot)$ defined through

$$H(X) = \frac{3}{4}AX^2 + \frac{3}{2}BX + f_{**}.$$

We readily have

$$H'(\alpha^2) = 4\sqrt{p}\frac{\ddot{\varepsilon}(0)}{\varepsilon_*} \qquad H(0) = f_{**}. \tag{5.5}$$

Then

$$H(0) > 0 \Longleftrightarrow x \in (1,4) \text{ and } M_*^2 > (f_{**})_0(x). \tag{5.6}$$

Regarding $H(1)$, it turns out that, with (5.1) and (4.29), there holds

$$H(1) = G(1) = (p+\sqrt{p})\frac{\ddot{\varepsilon}(0)}{\varepsilon_*} + 3M_*\ddot{M}(0)\Big|_{\alpha=1} = f_*.$$

Then

$$H(1) > 0 \Longleftrightarrow M_*^2 > (f_*)_0(x). \tag{5.7}$$

Moreover we prove easily that

$$D_0(x) < (f_{**})_0(x) < (f_*)_0(x) \qquad \forall x \in (1,4). \tag{5.8}$$

Hence we are in position to study the different cases appearing in the statement of the theorem and labelled from (i) (a) to (ii) (b).

(i) (a) First we claim that $H(0)$ and $H(1)$ are negative due (5.6), (5.7), (5.8) and (4.65). Thus if $\ddot{\varepsilon}(0)$ has a sign on $(0,1)$, then (5.5) implies that $H < 0$ on $(0,1)$. Otherwise $\ddot{\varepsilon}(0)$ has a unique zero α_0 in $(0,1)$ and, by (5.5), $X := \alpha_0^2$ is the unique critical point of H in $(0,1)$. Moreover by (5.1),

$$H(\alpha_0^2) = G(\alpha_0) = 3M_*\ddot{M}(0)\big|_{\alpha=\alpha_0}. \tag{5.9}$$

If $M_*^2 \le (C+D)_0(x)$, then at $\alpha = \alpha_0$, we have $\ddot{M}(0) < 0$ by Proposition 4.9.

If $(C+D)_0(x) < M_*^2 \le J_2(x)$, then at $\alpha = \alpha_0$, $\ddot{M}(0)$ is also negative by (4.67). Consequently,

$$G(\alpha) < 0 \qquad \forall \alpha \in (0,1).$$

(i) (b) We still have $H(0)$ and $H(1)$ negative. If $J_2(x) < M_*^2 < D_0(x)$, then

$$\ddot{M}(0)\big|_{\alpha=\alpha_0} > 0$$

by (4.68). Thus $H(\alpha_0^2) > 0$ in view of (5.9) and we are able to conclude in this case since $X := \alpha_0^2$ is the unique critical point of H in $(0,1)$.

If $D_0(x) \le M_*^2 < (f_{**})_0(x)$, then $\ddot{M}(0) > 0$ for every $\alpha \in (0,1)$, according to Proposition 4.9. So the assertion follows in this case also.

The other case can be proved easily by using the above methods together with Proposition 4.9. □

5.2 Stability of bifurcation solutions

We refer to the appendix hereafter for the background concerning the one-dimensional phase field crystal equation (2.7). The main result of this paper is the following.

Theorem 5.3. *Under the assumptions and notations of Theorem 4.1, let us suppose that*

$$M_*^2 \neq -\frac{(4x-1)(x-4)}{15(x+1)^2}. \tag{5.10}$$

Then the stability of the stationary solution $v(y)$ to the phase field crystal problem (A.1) is as follows.
If $k_* = k_{**}+1$ (i.e. $x = (1+\frac{1}{k_{**}})^2$) and $M_*^2 \in (J_1(x), J_2(x))$, then $v(y)$ is asymptotically stable in the sense of Lyapunov.
If $k_* = k_{**}+1$ and $M_*^2 \notin [J_1(x), J_2(x)]$, then $v(y)$ is not stable in the sense of Lyapunov.
If $k_* \neq k_{**}+1$, then $v(y)$ is not stable in the sense of Lyapunov.

Remark 5.4.
• For $x \in (1,4)$, $J_1(x)$ and $J_2(x)$ are defined by (4.61) and (4.62). Notice that they are the *cancellation function* of $BC - AD$.
• If $x \in (1,4)$ and $M_*^2 \in (J_1(x), J_2(x))$, then the tangent at $y = 0$ (denoted by $T(\alpha)$) to the parameter curve $y \mapsto \delta(y)$ turns clockwise when α goes from 0 to 1. See Proposition 4.10 and Figure 4.2.
• The above result was unexpected since it connects the stability of bifurcating solutions with the variation of the angle of $T(\alpha)$ with the horizontal axis.
• In view of (4.42), (4.43), C_S defined by (4.28) satisfies

$$C_S = -\frac{45}{2}\frac{(x+1)^2}{(4x-1)(x-4)}M_*^2 - \frac{3}{2}.$$

Thus (5.10) is equivalent to $C_S \neq 0$.
• Let $x \in (1,4)$. Recalling that $(f_{**})_0(x)$ is defined by (5.4), let us suppose that $M^2 \in (J_2(x), (f_{**})_0(x))$. Then $v(y)$ is unstable according to the above result. However, the energy of $v(y)$ may be less than the energy of the trivial solution. More precisely, by Theorem 5.2, there exists $\alpha_1 \in (0,1)$ such that for every $\alpha \in (0, \alpha_1)$ and $y \simeq 0$,

$$E\big(M(y) + v(y), \varepsilon(y)\big) < E\big(M(y), \varepsilon(y)\big).$$

This result is not so common in the literature. Let us recall that $v(y)$ depends on α in the following way:

$$\dot{v}(0) = \alpha\varphi_* + \beta\varphi_{**}.$$

Proof of Theorem 5.3. According to Proposition A.1, it is enough to consider the *constrained Swift–Hohenberg equation* (A.2). As explain in Section 3, if the trivial solution $v = 0$ is not neutrally stable, then $v(y)$ is unstable. By Proposition 3.2, it follows that $v(y)$ is unstable if $k_* \neq k_{**}+1$.

Let us now assume that $k_* = k_{**}+1$. We use the *principle of reduced stability* from [13, Section I-18] (see also [14]). According to this result, it is enough to consider the two-dimensional eigenvalue problem obtained from the linearization of the bifurcation equation (4.19) at $u_0 = y\varphi_0$. That is to say (by differentiating (4.19) w.r.t. u_0), we have to find $\lambda \in \mathbb{R}$ such that the following linear equation set on $\ker L$, namely

$$P\big\{F_1(\mu)\cdot + 2F_{02}(\cdot, a_{02}u_0^2) + 4F_{02}(u_0, a_{02}(u_0,\cdot)) + 3F_{03}(\cdot, u_0^2) + O(\mu u_0^2 + u_0^3)\big\} = \lambda\, \mathrm{Id}_{\ker L} \tag{5.11}$$

has nontrivial solutions. For the bifurcating solution $(\mu(y), v(y))$ of (2.12), we have $u_0 = y\varphi_0$ with $\varphi_0 = \alpha\varphi_* + \beta\varphi_{**}$ and $\mu(y) = y^2\mu_2 + O(y^3)$. Here μ_2 stands for the vector $\frac{1}{2}(\ddot{\varepsilon}(0), \ddot{M}(0))$.

Thus we rescale the eigenvalue λ into $\lambda = y^2\tilde{\lambda}$ and we denote by $\mathcal{A}\colon \ker L \to \ker L$, the linear operator defined by

$$\mathcal{A}w := P\{F_{11}(\mu_2)w + 2F_{02}(w, a_{02}\varphi_0^2) + 4F_{02}(\varphi_0, a_{02}(\varphi_0, w)) + 3F_{03}(w, \varphi_0^2)\}. \tag{5.12}$$

So that (5.11) reads

$$\mathcal{A} + O(y) = \tilde{\lambda}\ \mathrm{Id}_{\ker L}. \tag{5.13}$$

Then eigenvalues $\tilde{\lambda}$ satisfy

$$g(y, \tilde{\lambda}) := \det(\mathcal{A} + O(y) - \tilde{\lambda}\ \mathrm{Id}_{\ker L}) = 0.$$

In view of Lemma 5.5 below, $\mathcal{A}$ is a symmetric and non-diagonal operator since (5.10) is equivalent to $C_S \neq 0$. Hence $\mathcal{A}$ possesses two distinct real eigenvalues $\tilde{\lambda}_1 < \tilde{\lambda}_2$. Then, for $i = 1, 2$,

$$g(0, \tilde{\lambda}_i) = 0, \qquad \frac{\partial}{\partial \tilde{\lambda}} g(0, \tilde{\lambda}_i) = \frac{\mathrm{d}}{\mathrm{d}\tilde{\lambda}} \det(\mathcal{A} - \tilde{\lambda})|_{\tilde{\lambda} = \tilde{\lambda}_i} \neq 0,$$

since $\tilde{\lambda}_i$ is a simple eigenvalue. Therefore, by the implicit function theorem, we get for $y \simeq 0$, two eigenvalues of (5.13), namely

$$\tilde{\lambda}_1(y) = \tilde{\lambda}_1 + O(y), \qquad \tilde{\lambda}_2(y) = \tilde{\lambda}_2 + O(y).$$

The *principle of reduced stability* states that the eigenvalue problem

$$\begin{aligned} &\varepsilon(y)^2 w^{(4)} + 2\varepsilon(y) w^{(2)} + f'(M(y) + v(y))w - \int_\Omega f'(M(y) + v(y))w \, \mathrm{d}x = \lambda w, \\ &w \in \dot{V}_4, \quad w \neq 0, \quad \lambda \in \mathbb{R}, \end{aligned}$$

has two *critical eigenvalues* $\lambda_1(y)$, $\lambda_2(y)$ (i.e. eigenvalues close to zero for $y \simeq 0$) with the following expansions

$$\lambda_1(y) = y^2\tilde{\lambda}_1 + O(y^3), \qquad \lambda_2(y) = y^2\tilde{\lambda}_2 + O(y^3).$$

Hence it remains to compute the sign of the eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of $\mathcal{A}$.

If $M_*^2 \notin [J_1(x), J_2(x)]$, then according to (4.56) and Proposition 4.10, we have

$$(BC - AD)(x, M_*) > 0.$$

So $\det \mathcal{A} = \tilde{\lambda}_1\tilde{\lambda}_2 < 0$ by Lemma 5.7 below. Hence $v(y)$ is unstable.

In the same way, if $M_*^2 \in (J_1(x), J_2(x))$, then $\tilde{\lambda}_1\tilde{\lambda}_2 = \det \mathcal{A} > 0$. Since (see (4.65), (5.8))

$$J_2(x) < (f_{**})_0(x) < (f_*)_0(x),$$

one has

$$f_* < 0, \qquad f_{**} < 0.$$

Thus, with Lemma 5.5,

$$\tilde{\lambda}_1 + \tilde{\lambda}_2 = \mathrm{trace}(\mathcal{A}) = -2(f_*\alpha^2 + f_{**}\beta^2) > 0. \tag{5.14}$$

Therefore $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are positive and $v(y)$ is stable for $y \simeq 0$. This completes the proof of the theorem. □

We now state and prove the lemmas used in the proof of Theorem 5.3.

Lemma 5.5. *Let $\mathcal{A}\colon \ker L \to \ker L$ be the linear operator defined by (5.12). Then the matrix $M(\mathcal{A})$ of $\mathcal{A}$ in the basis $(\varphi_*, \varphi_{**})$ is*

$$M(\mathcal{A}) = 2\begin{pmatrix} -f_*\alpha^2 & -C_S\alpha\beta \\ -C_S\alpha\beta & -f_{**}\beta^2 \end{pmatrix}. \tag{5.15}$$

Remark 5.6. The simple formula (5.15) can be obtained, at least at a formal level, by differentiation starting from (4.30). Since (4.30) has a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry, (5.15) is in accordance with the results of [8, Chapter X]. These results are obtained by using symmetries and universal unfolding theory. Moreover the analogue of $v(y)$ is obtained in [8] as a secondary bifurcation. Here, there is no $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry in (2.12) and $v(y)$ is a primary bifurcating solution in the sense that it bifurcates from the trivial solution.

Proof. We compute the first column of the matrix.

$$\mathcal{A}\varphi_* = P\{F_{11}(\mu_2)\varphi_* + 2F_{02}(\varphi_*, a_{02}\varphi_0^2) + 4F_{02}(\varphi_0, a_{02}(\varphi_0, \varphi_*)) + 3F_{03}(\varphi_*, \varphi_0^2)\}.$$

We have $\mu_2 = \frac{1}{2}(\ddot{\varepsilon}(0), \ddot{M}(0))$, thus with (4.24),

$$PF_{11}(\mu_2)\varphi_* = \left(\frac{p+\sqrt{p}}{\varepsilon_*}\ddot{\varepsilon}(0) + 3M_*\ddot{M}(0)\right)\varphi_*.$$

Due to (4.14), (4.15), we infer

$$2PF_{02}(\varphi_*, a_{02}\varphi_0^2) = -\frac{9}{2}M_*^2\frac{\alpha^2}{\lambda_{2k_*}}\varphi_* - 9M_*^2\alpha\beta\left(\frac{1}{\lambda_{k_*+k_{**}}} + \frac{1}{\lambda_{k_*-k_{**}}}\right)\varphi_{**}.$$

In view of (4.16), we obtain

$$\begin{aligned} &4PF_{02}\left(\varphi_0, a_{02}(\varphi_0, \varphi_*)\right) \\ &\quad = -9M_*^2\left\{\left(\frac{\alpha^2}{\lambda_{2k_*}} + \beta^2\Big(\frac{1}{\lambda_{k_*+k_{**}}} + \frac{1}{\lambda_{k_*-k_{**}}}\Big)\right)\varphi_* + \alpha\beta\left(\frac{1}{\lambda_{k_*+k_{**}}} + \frac{1}{\lambda_{k_*-k_{**}}}\right)\varphi_{**}\right\}. \end{aligned}$$

Besides

$$3PF_{03}(\varphi_*, \varphi_0^2) = \left(\frac{9}{4}\alpha^2 + \frac{3}{2}\beta^2\right)\varphi_* + 3\alpha\beta\varphi_{**}.$$

Let us denote the entries of $M(\mathcal{A})$ by

$$M(\mathcal{A}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then, since $\alpha^2 + \beta^2 = 1$,

$$\begin{aligned} a_{11} = {} & (p+\sqrt{p})\frac{\ddot{\varepsilon}(0)}{\varepsilon_*} + 3M_*\ddot{M}(0) + \frac{3}{4}\alpha^2 + \frac{3}{2} \\ & - 9M_*^2\left\{\left(\frac{3}{2\lambda_{2k_*}} - \left(\frac{1}{\lambda_{k_*+k_{**}}} + \frac{1}{\lambda_{k_*-k_{**}}}\right)\right)\alpha^2 + \frac{1}{\lambda_{k_*+k_{**}}} + \frac{1}{\lambda_{k_*-k_{**}}}\right\}. \end{aligned}$$

Recalling the notation (4.27), (4.28) and using (4.29), we obtain

$$\begin{aligned} a_{11} &= (f_* - C_S)\alpha^2 + C_s + (-3f_* + C_S)\alpha^2 - C_S \\ &= -2f_*\alpha^2. \end{aligned}$$

Regarding a_{21}, we have

$$\begin{aligned} a_{21} &= -18M_*^2\left(\frac{1}{\lambda_{k_*+k_{**}}}+\frac{1}{\lambda_{k_*-k_{**}}}\right)\alpha\beta+3\alpha\beta \\ &= (-2C_S-3)\alpha\beta+3\alpha\beta \\ &= -2C_S\alpha\beta. \end{aligned}$$

This gives the first column of $M(\mathcal{A})$. By using symmetries resulting from the non-resonant condition $x \neq 4, 9$, we may obtain the second column from the first one. More precisely, the second column is obtained by exchanging k_* and k_{**} on one hand, and by exchanging α and β on the other hand. Thus $a_{12} = a_{21}$ and

$$a_{22} = -2f_{**}\beta^2.$$

This completes the proof of the lemma. □

Lemma 5.7. *Let $M_* \in \mathbb{R}$, k_*, k_{**} be positive integers such that $k_{**} < k_*$ and f_*, f_{**}, C_S be defined by (4.27), (4.28). Let A, B, C, D be given by (4.36)–(4.39) and $\mathcal{A}$ be the operator defined through (5.12) (whose matrix is given by (5.15)). Then*

$$\det\mathcal{A} = -\frac{9}{8}\alpha^2\beta^2(BC-AD). \tag{5.16}$$

Remark 5.8. To our knowledge, the relation (5.16) is new. It explains why the statement of the stability of $v(y)$ is quite simple in the sense that the stability is linked to quantities relying on the parameter curve $y \mapsto \delta(y)$. In particular, if

$$\alpha \mapsto \frac{\ddot{M}(0)}{\ddot{\varepsilon}(0)}$$

is increasing, then (5.16) implies that $v(y)$ is unstable.

Proof. By using (4.36)–(4.39), we get

$$\begin{aligned} BC-AD &= B\left(\frac{8}{3}(f_*-C_S)-(\sqrt{p}+1)A\right)-A\left(\frac{8}{3}C_S-(\sqrt{p}+1)B\right) \\ &= \frac{8}{3}(B(f_*-C_S)-AC_S) \\ &= \frac{8}{3}\left(\frac{4}{3}(-f_{**}+C_S)(f_*-C_S)-\frac{4}{3}(f_*+f_{**}-2C_S)C_S\right) \\ &= \frac{32}{9}(-f_*f_{**}+C_S^2). \end{aligned}$$

However by (5.15),

$$\det\mathcal{A} = 4\alpha^2\beta^2(f_*f_{**}-C_S^2).$$

Thus (5.16) follows. □

5.3 Symmetries

We start to state precisely a local uniqueness result for bifurcating branches. In particular, we will emphasize the dependence of these solutions w.r.t. α and β.

Theorem 5.9. *Under the assumptions and notations of Theorem 4.1, there exist*

$$R_1 = R_1(\alpha,\beta) > 0, \qquad \delta_1 = \delta_1(\alpha,\beta) > 0$$

and a smooth function

$$\mu(\cdot,\alpha,\beta)\colon (-\delta_1,\delta_1) \to (-R_1,R_1)$$

such that for every $y \in (-\delta_1,\delta_1)$ and $\mu \in (-R_1,R_1)$, one has

$$PF\big(\mu, y\varphi_0 + \mathcal{U}(\mu,y\varphi_0)\big) = 0 \iff \mu = \mu(y,\alpha,\beta).$$

Remark 5.10.
• We recall that $\varphi_0 := \alpha\varphi_* + \beta\varphi_{**}$.
• δ_1 and R_1 may be chosen independently of α and β provided that α and β remain bounded away from 0.
• In this setting, the bifurcating solution $v(y)$ of Theorem 4.1 will be denoted by $v(y,\alpha,\beta)$.

Let $(\alpha,\beta) \in (-1,1)^2$ satisfy (4.48). For y close to zero, we have four (distinct) solutions

$$\big(\mu(y,\pm\alpha,\pm\beta), v(y,\pm\alpha,\pm\beta)\big)$$

to equation (2.12). In view of the remark above, we may suppose that the numbers $\delta_1(y,\pm\alpha,\pm\beta)$ are equal. So we will denote their common value by δ_m.

The goal of this subsection is to establish relations between these solutions. This is achieved by using a suitable translation of the space variable. Let us write k_* and k_{**} under the form

$$k_* = 2^{r_1}\ell_*, \qquad k_{**} = 2^{r_2}\ell_{**}, \tag{5.17}$$

where r_1, r_2 are nonnegative integers and ℓ_*, ℓ_{**} are positive odd integers. Let us denote by r the minimum of r_1 and r_2.

The above mentioned translation consists, roughly speaking, in the translation $x \mapsto x + 2^{-r}$. To be more specific, for every $v \in L^2(\Omega)$, let us denote by $Jv\colon \mathbb{R} \to \mathbb{R}$ the 2-periodic and even function satisfying

$$Jv = v \qquad \text{a.e. in } [0,1].$$

We put

$$Tv := (Jv)(\cdot + 2^{-r})\big|_{\Omega},$$

which means that Tv is the restriction to Ω of the function $(Jv)(\cdot + 2^{-r})$.

Let

$$\dot{L}^2_{2^{1-r}}(\Omega) = \{v \in \dot{L}^2(\Omega) \mid Jv \text{ is } 2^{1-r} \text{ periodic}\}.$$

Then $L + F(\mu,\cdot)$ maps $\dot{V}_4 \cap \dot{L}^2_{2^{1-r}}(\Omega)$ into $\dot{L}^2_{2^{1-r}}(\Omega)$ and φ_*, φ_{**} belong clearly to $\dot{L}^2_{2^{1-r}}(\Omega)$. Then we infer that Theorem 4.1 still holds if we restrict our analysis to 2^{1-r}-periodic functions. Thus, each bifurcating solution given by Theorem 4.1 satisfies

$$v(y,\alpha,\beta) \in \dot{L}^2_{2^{1-r}}(\Omega).$$

Moreover,

$$T\big(\dot{L}^2_{2^{1-r}}(\Omega)\big) \subset \dot{L}^2_{2^{1-r}}(\Omega)$$
$$T\big(\dot{V}_4 \cap \dot{L}^2_{2^{1-r}}(\Omega)\big) \subset \dot{V}_4 \cap \dot{L}^2_{2^{1-r}}(\Omega)$$

since every $v \in \dot{V}_4 \cap \dot{L}^2_{2^{1-r}}(\Omega)$ has the representation

$$v = \sum_{m\geq 1} x_{m2^r} \varphi_{m2^r} \qquad \text{in } \dot{V}_4.$$

Thus T commutes with $L + F(\mu, \cdot)$.

Since ℓ_* and ℓ_{**} are odd, we have

$$T\varphi_* = (-1)^{2^{r_1-r}} \varphi_* = \begin{cases} -\varphi_* & \text{if } r_1 = r \\ \varphi_* & \text{if } r_1 > r \end{cases}, \qquad T\varphi_{**} = (-1)^{2^{r_2-r}} \varphi_{**} = \begin{cases} -\varphi_{**} & \text{if } r_2 = r \\ \varphi_{**} & \text{if } r_2 > r \end{cases}.$$

We then deduce in a standard way that, for every $y \in (-\delta_m, \delta_m)$,

$$v\big(y, (-1)^{2^{r_1-r}}\alpha, (-1)^{2^{r_2-r}}\beta\big) = Tv(y, \alpha, \beta), \qquad \mu\big(y, (-1)^{2^{r_1-r}}\alpha, (-1)^{2^{r_2-r}}\beta\big) = \mu(y, \alpha, \beta). \quad (5.18)$$

Thus, among the four (distinct) solutions $(\mu, v)(y, \pm\alpha, \pm\beta)$ to (2.12), only two are essentially different. The other ones are obtained through T (since at least one of the numbers $r_1 - r$, $r_2 - r$ vanishes). For instance, if $r = r_1 < r_2$, then

$$v(y, -\alpha, \beta) = Tv(y, \alpha, \beta), \qquad \mu(y, -\alpha, \beta) = \mu(y, \alpha, \beta)$$
$$v(y, -\alpha, -\beta) = Tv(y, \alpha, -\beta), \qquad \mu(y, -\alpha, -\beta) = \mu(y, \alpha, -\beta).$$

If $r = r_1 = r_2$, then $\mu(\cdot, \alpha, \beta)$ is even. Indeed, by Theorem 5.9 and (5.18), we have

$$\mu(-y, \alpha, \beta) = \mu(y, -\alpha, -\beta) = \mu\big(y, (-1)^{2^{r_1-r}}\alpha, (-1)^{2^{r_2-r}}\beta\big)$$
$$= \mu(y, \alpha, \beta).$$

However, if $k_* = 4k_{**}$, then we can prove that $\mu(\cdot, \alpha, \beta)$ is not even.

Let us notice that (2.12) has the trivial symmetry

$$S\colon L^2(\Omega) \to L^2(\Omega), \qquad u \mapsto u(1 - \cdot).$$

This symmetry allows to relate solutions in some cases. However, if k_*, k_{**} are even, then it turns out that each of the four solutions

$$\big(\mu(y, \pm\alpha, \pm\beta), v(y, \pm\alpha, \pm\beta)\big)$$

is invariant under S. So S is useless in what case unlike T (see (5.18)).

6 Rough approximation of the 8-loop

The aim of this section is to recover by means of two analytical approximations the so-called 8-*loops* appearing in [16, Figure 15]. More precisely, we will use the truncated bifurcation equation (4.30) to approximate the bifurcating solutions (μ, v) to (2.12) given by Theorem 4.1. By suitable choices of parameter values, we will reconstruct analytically the first 8-loop on the left of [16, Figure 15], which was obtained by numerical integration.

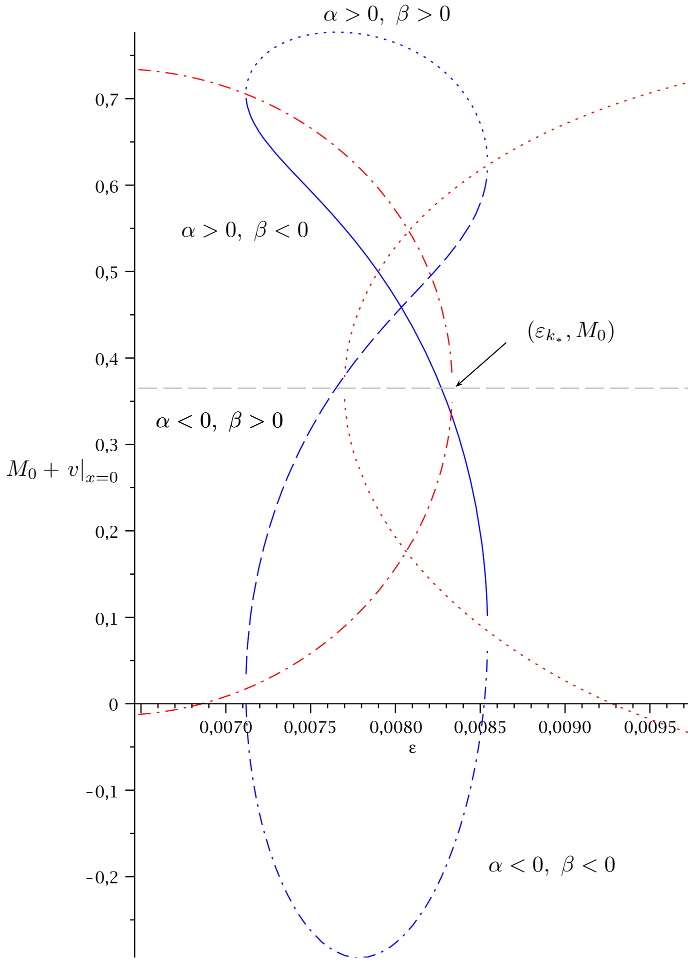

Figure 6.1: The blue curves represent four branches bifurcating from the point (ε_*, M_*) with 2D kernel. The two red curves show bifurcation branches with 1D kernel, fixed mass M_0 and bifurcation points ε_{k_*} (pointed on the figure) and $\bar{\varepsilon}_{k_{**}}$. We have chosen $k_* = 4$, $k_{**} = 3$, $r = -0.5$ and $M_0 = \sqrt{2/15}$. See also Remark 6.1.

Under the assumptions and notations of Theorem 4.1, we set for $\delta = (\varepsilon, M) \in (0, \infty) \times \mathbb{R}$

$$P_*(\delta) := (\varepsilon\sigma_{k_*} - 1)^2 + r + 3M^2,$$
$$P_{**}(\delta) := (\varepsilon\sigma_{k_{**}} - 1)^2 + r + 3M^2.$$

If we neglect the higher order terms in (4.19), we may assume that (X, Y) solves (4.30). This is the first approximation. Thus, if y, α and β are not zero, we have

$$\begin{cases} f_* y^2\alpha^2 + C_S y^2\beta^2 = P_*(\delta), \\ C_S y^2\alpha^2 + f_{**} y^2\beta^2 = P_{**}(\delta). \end{cases} \tag{6.1}$$

In order to clarify things, we would like to highlight that, in (6.1), f_* depends on δ_* and not on δ, since in (4.27), we have

$$\lambda_{2k_*} = (\varepsilon_*\sigma_{2k_*} - 1)^2 + r + 3M_*^2.$$

Our second approximation is a second order approximation of the function $v(y, \delta, \alpha)$, namely

$$v(y, \alpha, \beta) \simeq y\varphi_0 + y^2 a_{02}\varphi_0^2,$$

where $\varphi_0 = \alpha\varphi_* + \beta\varphi_{**}$ and $a_{02}\varphi_0^2$ is given by (4.14) and (4.15). Unlike to our previous analysis, we will no more assume that y is close to zero. This is why the above approximation is said to be *rough*.

In order to solve (6.1), we choose a suitable value M_0 of M, close to M_*. The solutions are parametrized by $\varepsilon \simeq \varepsilon_*$ as in [16]. We obtain

$$y^2\alpha^2 = \frac{-f_{**}P_*(\varepsilon, M_0) + C_S P_{**}(\varepsilon, M_0)}{-f_* f_{**} + C_S^2} \tag{6.2}$$

$$y^2 = \frac{P_*(\varepsilon, M_0) - (f_* - C_S)y^2\alpha^2}{C_S}. \tag{6.3}$$

If, in the above equations, $y^2 \in (0,1)$ and $\alpha^2 \in (0,1)$, then we choose w.l.o.g. y, α to be positive and $\beta := \sqrt{1-\alpha^2}$. So what for $\delta = (\varepsilon, M_0)$, we obtain four approximated solutions to (2.8) of the form $M_0 + \tilde{v}(y, \pm\alpha, \pm\beta)$ with

$$\begin{aligned}\tilde{v}(y,\alpha,\beta) &= y(\alpha\varphi_* + \beta\varphi_{**})\\ &\quad - \frac{3M_*}{2}y^2\left(\frac{\alpha^2}{\lambda_{2k_*}}\varphi_{2k_*} + 2\frac{\alpha\beta}{\lambda_{k_*+k_{**}}}\varphi_{k_*+k_{**}} + 2\frac{\alpha\beta}{\lambda_{k_*-k_{**}}}\varphi_{k_*-k_{**}} + \frac{\beta^2}{\lambda_{2k_{**}}}\varphi_{2k_{**}}\right).\end{aligned}$$

Remark 6.1. Let us make some comments on the bifurcation diagram of Figure 6.1.
• The top blue curve is the graph of

$$\varepsilon \mapsto M_0 + \tilde{v}(y,\alpha,\beta)|_{x=0},$$

where the dependence of (y, α, β) w.r.t. ε is given by (6.2), (6.3).
• Following [16, Figure 15], we have chosen $k_* = 4$, $k_{**} = 3$, $r = -0.5$ and $M_0 = \sqrt{2/15}$. Then by (4.4),

$$\begin{aligned}\sqrt{p} &= \frac{7}{25}\\ \varepsilon_* &= \frac{2}{25\pi^2}\\ M_* &= \sqrt{-\frac{1}{3}(r+p)} = \frac{\sqrt{3162}}{150}.\end{aligned}$$

• We have

$$0 < \frac{M_* - M_0}{M_*} < 0.026.$$

So M_0 is close to M_* as required by the theory. Moreover, M_0 must be chosen smaller than M_*. Indeed, one has

$$(x, M_*^2) \simeq (1.8,\ 0.14).$$

Thus, in view of Figure 4.1, we have $\ddot{M}(0) < 0$ for all $|\alpha| \in (0,1)$ and, by (4.50), each bifurcating branch of mass $y \mapsto M(y)$ satisfies

$$M(y) < M_*, \qquad \forall y \simeq 0.$$

This estimate can also be obtained by analytical arguments thanks Proposition 4.9.
• The translation T of subsection 5.3 acts on the approximate solutions $\tilde{v}(y, \pm\alpha, \pm\beta)$ as well. Indeed, we have $0 = r = r_2 < r_1 = 2$. Thus

$$\tilde{v}(y,\alpha,-\beta) = T\tilde{v}(y,\alpha,\beta), \qquad \tilde{v}(y,-\alpha,-\beta) = T\tilde{v}(y,-\alpha,\beta).$$

Thus, in Figure 6.1, the curve corresponding to $\alpha > 0$, $\beta > 0$ can be related to the curve corresponding to $\alpha > 0$, $\beta < 0$ by means of T. Also the curve corresponding to $\alpha < 0$, $\beta < 0$ can be related to the curve corresponding to $\alpha < 0$, $\beta > 0$ by means of T.

• We can see secondary bifurcations between interactive modes solutions and single modes solutions. At the bifurcation point, we have $\alpha = 0$ or $\beta = 0$.

• Let $k_* = 4$, $k_{**} = 3$ and $r = -0.5$. If y is small enough, then $v(y, \alpha, \beta)$ is an asymptotically stable solution to

$$\partial_t v - \varepsilon(y)\partial_{xx}\big(\varepsilon(y)^2\partial_{xxxx}v + 2\varepsilon(y)\partial_{xx}v + f(M(y) + v)\big) = 0.$$

Indeed, (4.65) implies that the graphs of $x \mapsto J_1(x)$ and $x \mapsto J_2(x)$ lie respectively between the red curves and the blue curves of Figure 4.1. Thus in view of the remark above $M_*^2 \in (J_1(x), J_2(x))$. The claim follows then from Theorem 5.3. This stability result is in accordance with the numerical simulations featured in [16, Figure 15].

A Phase field crystal equation and stability

The aim of this appendix is to show that stability for the *phase field crystal equation* (2.7) and for the following *constrained Swift–Hohenberg equation*,

$$\begin{cases} \partial_t u + \varepsilon^2\partial_{xxxx}u + 2\varepsilon\partial_{xx}u + f(u) = \int_\Omega f(u)\,\mathrm{d}x & \text{in } \Omega \times (0, \infty) \\ \partial_x u = \partial_{xxx}u = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

are essentially the same. Notice that, in view of Section 4, a stability analysis will give result concerning the above equation. So we have to fill the gap with the phase field crystal equation.

Since the linearized operator corresponding to (2.7) is not symmetric, we will consider for this equation, *asymptotic stability in the sense of Lyapunov*; see for instance [11, Chapter 3]. We will first define the semigroup associated to (2.7). For simplicity, the derivatives ∂_x and ∂_{xx} will be denoted by ∇ and Δ. Moreover, notice that all of the results below still hold if the interval Ω is replaced by a smooth bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$.

Recalling the notation (2.3), we put

$$\dot V_5 := \{u \in H^5(\Omega) \mid u' = u''' = 0 \text{ on } \partial\Omega\}$$
$$\dot H^{-1}(\Omega) := \text{dual of } H^1(\Omega) \cap \dot L^2(\Omega).$$

Let $v_0 \in \dot V_2$ and $T \in (0, \infty)$. We say that v is a *weak solution to* (2.7) in $\Omega \times (0, T)$ if

$$\begin{aligned} & v \in L^2(0, T, \dot V_5) \cap C([0, T], \dot V_2), \qquad \frac{\mathrm{d}}{\mathrm{d}t}v \in L^2(0, T, \dot H^{-1}(\Omega)) \\ & \frac{\mathrm{d}}{\mathrm{d}t}v - \varepsilon\Delta\big(\varepsilon^2\Delta^2 v + 2\varepsilon\Delta v + f(M + v) - \textstyle\int_\Omega f(M + v)\,\mathrm{d}x\big) = 0 \quad \text{in } L^2(0, T, \dot H^{-1}(\Omega)) \\ & v|_{t=0} = v_0 \quad \text{in } \dot V_2. \end{aligned} \tag{A.1}$$

By implementing the Galerkin scheme, we prove that (A.1) admits a unique solution v (see [10,20]). Moreover, the map $S_{PFC}\colon [0, \infty) \times \dot V_2 \to \dot V_2$, $(t, v_0) \mapsto v(t)$ is a semigroup in the sense of Temam. Thus $S_{PFC}(t)v_0$ stands for $v(t)$.

In the same way but referring to the above *constrained Swift–Hohenberg equation*, the problem

$$
\begin{aligned}
& v \in L^2(0,T,\dot{V}_4) \cap C([0,T],\dot{V}_2), \qquad \frac{\mathrm{d}}{\mathrm{d}t} v \in L^2(0,T,\dot{L}^2(\Omega)) \\
& \frac{\mathrm{d}}{\mathrm{d}t} v + \varepsilon^2 \Delta^2 v + 2\varepsilon \Delta v + f(M+v) - \textstyle\fint_\Omega f(M+v)\,\mathrm{d}x = 0 \quad \text{ in } L^2(0,T,\dot{L}^2(\Omega)) \\
& v|_{t=0} = v_0 \quad \text{ in } \dot{V}_2
\end{aligned}
\tag{A.2}
$$

has a unique (strong) solution and infer also a semigroup denoted by S_{cSH}. Without loss of generality, we may assume $\varepsilon = 1$ and $M = \fint_\Omega u_0\,\mathrm{d}x = 0$. It is clear that (A.1) and (A.2) have the same steady states. Moreover, the energy E – see (2.14) – defined through

$$E(v) = \frac{1}{2}\|\Delta v\|_2^2 - \|\nabla v\|_2^2 + \int_\Omega \frac{r+1}{2} v^2 + \frac{1}{4} v^4\,\mathrm{d}x$$

is a Lyapunov functional for S_{PFC} and S_{cSH}. It is clear for the later. For the former, we test the equation of (A.1) with $(-\Delta^{-1})\frac{\mathrm{d}}{\mathrm{d}t}v$ where $-\Delta\colon H^1(\Omega)\cap \dot{L}^2(\Omega) \to \dot{H}^{-1}(\Omega)$. We find

$$\frac{\mathrm{d}}{\mathrm{d}t} E(v(t)) = -\left\| \frac{\mathrm{d}}{\mathrm{d}t} v \right\|_{\dot{H}^{-1}}^2 \leq 0.$$

Notice that the linearized operator for (A.2) at any stationary solution v_∞ is self adjoint with compact resolvant. Thus its spectrum consists on an increasing sequence of eigenvalues.

The main result of this appendix is the following.

Proposition A.1. *Let v_∞ be a stationary solution to* (A.1)*. If v_∞ is linearly stable for the semigroup S_{cSH} (i.e. the corresponding eigenvalues are positive), then v_∞ is asymptotically stable in the sense of Lyapunov, for the semigroup S_{PFC}.*

If one of these eigenvalues is negative and 0 *is not an eigenvalue, then v_∞ is not stable in the sense of Lyapunov, for the semigroup S_{PFC}.*

Roughly speaking, the above results states that if 0 is not an eigenvalue, then the stationary solution v_∞ has the same stability for the semigroup S_{PFC} and for the semigroup S_{cSH}.

Proof. Let us assume that v_∞ is linearly stable for S_{cSH}. Expanding the energy $E(v)$ for $v \in \dot{V}_2$, $v \simeq v_\infty$, we get

$$E(v) = E(v_\infty) + \frac{1}{2} D^2 E(v_\infty)(v - v_\infty)^2 + O(\|v - v_\infty\|_{\dot{V}_2}^3).$$

With Lemma A.2 below, we deduce that there exist $\varepsilon > 0$ and $r_2 > 0$ such that

$$E(v) \geq E(v_\infty) + 2\varepsilon, \qquad \forall v \in \dot{V}_2,\ \|v - v_\infty\|_{\dot{V}_2} = r_2. \tag{A.3}$$

Since v_∞ is linearly stable, we may assume without loss of generality, that v_∞ is the only one stationary solution in the ball $B(v_\infty, r_2)$ of $\dot{V}_2$ with radius r_2 and center v_∞. Since (A.1) and (A.2) have the same steady states, v_∞ is also the unique stationary solution to (A.1) in $B(v_\infty, r_2)$.

Besides, by continuity of $E(\cdot)$, there exists $r_1 \in (0, r_2)$ such that

$$E(v_0) \leq E(v_\infty) + \varepsilon, \qquad \forall v_0 \in B(v_\infty, r_1).$$

Since the energy is a Lyapunov function for the semigroup S_{PFC}, there holds

$$E(S_{PFC}(t)v_0) \leq E(v_\infty) + \varepsilon, \qquad \forall v_0 \in B(v_\infty, r_1),\ \forall t \geq 0.$$

Hence by (A.3),

$$S_{PFC}(t)v_0 \in B(v_\infty, r_2), \qquad \forall t \geq 0.$$

Moreover, by standard methods (see [20]), we can show that the trajectory $\{S_{PFC}(t)v_0 \mid t \geq 0\}$ is relatively compact in $\dot{V}_2$. Since v_∞ is a isolated stationary solution, we deduce from *LaSalle's invariance principle* (see [11]) that

$$S_{PFC}(t)v_0 \xrightarrow[t\to\infty]{} v_\infty \quad \text{in } \dot{V}_2.$$

Thus v_∞ is asymptotically stable equilibrium of S_{PFC}.

Conversely, since 0 is not in the spectrum of the linearized operator for (A.2) at v_∞, there exists $r_3 > 0$ such that v_∞ is the only one stationary solution to (A.1) in $B(v_\infty, r_3)$.

Since v_∞ is linearly unstable, there exists $v_0 \in \dot{V}_2$ arbitrary close to v_∞ such that

$$E(v_0) < E(v_\infty).$$

By continuity of E, there exists a positive number r_1 depending on v_0 such that

$$E(v) > E(v_0), \qquad \forall v \in B(v_\infty, r_1).$$

Moreover,

$$E(S_{PFC}(t)v_0) \leq E(v_0), \qquad \forall t \geq 0,$$

since E is a Lyapunov function for the semigroup S_{PFC}. Thus

$$S_{PFC}(t)v_0 \notin B(v_\infty, r_1), \qquad \forall t \geq 0.$$

Moreover, *LaSalle's invariance principle* implies that

$$d\big(S_{PFC}(t)v_0, \mathcal{E}\big) := \inf_{w\in\mathcal{E}} \|S_{PFC}(t)v_0 - w\|_{\dot{V}_2} \xrightarrow[t\to\infty]{} 0,$$

where $\mathcal{E}$ denotes the set of all stationary solutions to (A.1). Thus for some positive time t_1 and $w_\infty \in \mathcal{E}$, there holds

$$\|S_{PFC}(t_1)v_0 - w_\infty\|_{\dot{V}_2} \leq \frac{r_3}{2}.$$

However, since v_∞ is the only steady states of (A.1) in $B(v_\infty, r_3)$, we have

$$\|v_\infty - w_\infty\|_{\dot{V}_2} \geq r_3.$$

Thus

$$\|S_{PFC}(t_1)v_0 - v_\infty\|_{\dot{V}_2} \geq \|v_\infty - w_\infty\|_{\dot{V}_2} - \|S_{PFC}(t_1)v_0 - w_\infty\|_{\dot{V}_2} \geq \frac{r_3}{2},$$

which means that v_∞ is not stable in the sense of Lyapunov. This completes the proof of the proposition. □

Lemma A.2. *Let $v_\infty \in \dot{V}_2$. Let us assume that, for some positive constant c, we have*

$$D^2E(v_\infty)v^2 \geq c\|v\|_2^2, \qquad \forall v \in \dot{V}_2, \tag{A.4}$$

where $\|\cdot\|_2$ denote the standard norm in $L^2(\Omega)$. Then the bilinear mapping $D^2E(v_\infty)$ is coercive on $\dot{V}_2$, that is there exists $c_1 > 0$ such that

$$D^2E(v_\infty)v^2 \geq c_1\|v\|_{\dot{V}_2}^2, \qquad \forall v \in \dot{V}_2. \tag{A.5}$$

Proof. Let $\varepsilon > 0$ to be chosen later. Since $v_\infty \in L^\infty(\Omega)$, there exists $C > 0$ such that for every $v \in \dot{V}_2$,

$$\varepsilon \int_\Omega f'(v_\infty) v^2 \, \mathrm{d}x \geq -\varepsilon C \|v\|_2^2.$$

With (A.4),

$$\|\Delta v\|_2^2 - 2\|\nabla v\|_2^2 + (1+\varepsilon) \int_\Omega f'(v_\infty) v^2 \, \mathrm{d}x \geq (c - \varepsilon C)\|v\|_2^2.$$

Then

$$(1+\varepsilon^{-1}) D^2 E(v_\infty) v^2 \geq \|\Delta v\|_2^2 - 2\|\nabla v\|_2^2 + (c\varepsilon^{-1} - C)\|v\|_2^2.$$

We use the interpolation inequality

$$\|\nabla v\|_2^2 \leq \|\Delta v\|_2 \|v\|_2 \leq \frac{1}{4}\|\Delta v\|_2^2 + \|v\|_2^2,$$

to get

$$(1+\varepsilon^{-1}) D^2 E(v_\infty) v^2 \geq \frac{1}{2}\|\Delta v\|_2^2 + (c\varepsilon^{-1} - C - 2)\|v\|_2^2.$$

We obtain (A.5) by choosing ε small enough. □

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