# PERIODIC AND ASYMPTOTICALLY PERIODIC SOLUTIONS OF NEUTRAL INTEGRAL EQUATIONS

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## 1. INTRODUCTION

Many results have been obtained for periodic solutions of Volterra integral equations (for instance, [1-3] and references cited therein). Here we consider two systems of neutral integral equations

$$x(t) = a(t) + \int_0^t D(t, s, x(s))ds + \int_t^\infty E(t, s, x(s))ds, \ t \in R^+ \tag{1}$$

and

$$x(t) = p(t) + \int_{-\infty}^t P(t, s, x(s))ds + \int_t^\infty Q(t, s, x(s))ds, \ t \in R, \tag{2}$$

where $a$, $p$, $D$, $P$, $E$ and $Q$ are at least continuous. Under suitable conditions, if $\phi$ is a given $R^n$-valued bounded and continuous initial function on $[0, t_0)$ or $(-\infty, t_0)$, then both Eq.(1) and Eq.(2) have solutions denoted by $x(t, t_0, \phi)$ with $x(t, t_0, \phi) = \phi(t)$ for $t < t_0$, satisfying Eq.(1) or Eq.(2) on $[t_0, \infty)$. (cf. Burton-Furumochi [4].) A solution $x(t, t_0, \phi)$ may have a discontinuity at $t_0$.

Concerning our contribution here, we first present two lemmas and then show that if Eq.(1) has an asymptotically $T$-periodic solution, then Eq.(2) has a $T$-periodic solution.

Next, we use Schauder's fixed point theorem to show that Eq.(1) has an asymptotically $T$-periodic solution, thus yielding a $T$-periodic solution of Eq.(2).

We also infer directly that Eq.(2) has $T$-periodic solutions using Schauder's fixed point theorem and growth conditions on $P$ and $Q$.

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Finally, we give two theorems establishing relations between solutions of Equations (1) and (2).

## 2. PRELIMINARIES

Consider the systems of neutral integral equations (1) and (2), where $R^+ := [0, \infty)$, $R := (-\infty, \infty)$, and $a : R^+ \to R^n$, $p : R \to R^n$, $D : \Delta^- \times R^n \to R^n$, $P : \Delta^- \times R^n \to R^n$, $E : \Delta^+ \times R^n \to R^n$, and $Q : \Delta^+ \times R^n \to R^n$ are continuous, and where $\Delta^- := \{(t, s) : s \leq t\}$ and $\Delta^+ := \{(t, s : s \geq t\}$. Throughout this paper suppose that:

$$q(t) := a(t) - p(t) \to 0 \text{ as } \; t \to \infty, \; \text{ and } \; p(t) \; \text{ is } \; T\text{-periodic}, \tag{3}$$

where $q : R^+ \to R^n$, and $T > 0$ is constant,

$$F(t, s, x) := D(t, s, x) - P(t, s, x), \text{ and } P(t + T, s + T, x) = P(t, s, x), \tag{4}$$

where $F : \Delta^- \times R^n \to R^n$,

$$G(t, s, x) := E(t, s, x) - Q(t, s, x), \text{ and } Q(t + T, s + T, x) = Q(t, s, x), \tag{5}$$

where $G : \Delta^+ \times R^n \to R^n$. Moreover, we suppose that for any $J > 0$ there are continuous functions $P_J, F_J : \Delta^- \to R^+$ and $Q_J, G_J : \Delta^+ \to R^+$ such that:

$$P_J(t + T, s + T) = P_J(t, s) \text{ if } s \leq t,$$

$$Q_J(t + T, s + T) = Q_J(t, s) \text{ if } s \geq t,$$

$$|P(t, s, x)| \leq P_J(t, s) \text{ if } s \leq t \text{ and } |x| \leq J,$$

where $| \cdot |$ denotes the Euclidean norm of $R^n$;

$$|Q(t, s, x)| \leq Q_J(t, s) \text{ if } s \geq t \text{ and } |x| \leq J,$$

$$|F(t, s, x)| \leq F_J(t, s) \text{ if } s \leq t \text{ and } |x| \leq J,$$

$$|G(t, s, x)| \leq G_J(t, s) \text{ if } s \geq t \text{ and } |x| \leq J,$$

$$\int_{-\infty}^{t-\tau} P_J(t, s)ds + \int_{t+\tau}^{\infty} (Q_J(t, s) + G_J(t, s))ds \to 0 \; \text{ uniformly for } \; t \in R \; \text{ as } \; \tau \to \infty, \tag{6}$$

and

$$\int_0^t F_J(t, s)ds + \int_t^{\infty} G_J(t, s)ds \to 0 \text{ as } t \to \infty. \tag{7}$$

In this paper, we discuss the existence of periodic and asymptotically periodic solutions of Equations (1) and (2) by using the following theorem.

**Theorem 1 (Schauder's first theorem).** *Let $(C, \|\cdot\|)$ be a normed space, and let $S$ be a compact convex nonempty subset of $C$. Then every continuous mapping of $S$ into $S$ has a fixed point.*

Schauder's second theorem deletes the compactness of $S$ and asks that the map be compact (cf. Smart [5; p. 25]).

## 3. ASYMPTOTICALLY PERIODIC SOLUTIONS OF (1)

For any $t_0 \in R^+$, let $C(t_0)$ be a set of bounded functions $\xi : R^+ \to R^n$ such that $\xi(t)$ is continuous on $R^+$ except at $t_0$, and $\xi(t_0) = \xi(t_0+)$. For any $\xi \in C(t_0)$, define $\|\xi\|_+$ by

$$\|\xi\|_+ := \sup\{|\xi(t)| : t \in R^+\}.$$

Then clearly $\|\cdot\|_+$ is a norm on $C(t_0)$, and $(C(t_0), \|\cdot\|_+)$ is a Banach space. For any $\xi \in C(t_0)$ define a map $H$ on $C(t_0)$ by

$$(H\xi)(t) := \begin{cases} \xi(t), & 0 \leq t < t_0 \\ a(t) + \int_0^t D(t,s,\xi(s))ds + \int_t^\infty E(t,s,\xi(s))ds, & t \geq t_0. \end{cases}$$

Moreover, for any $J > 0$ let $C_J(t_0) := \{\xi \in C(t_0) : \|\xi\|_+ \leq J\}$.

**Definition 1** *A function $\xi : R^+ \to R^n$ is said to be asymptotically $T$-periodic if $\xi = \psi + \mu$ such that $\psi : R \to R^n$ is continuous $T$-periodic, $\mu \in C(t_0)$ for some $t_0 \in R^+$, and $\mu(t) \to 0$ as $t \to \infty$.*

First we have the following lemmas.

**Lemma 1** *If (3)-(7) hold, then for any $t_0 \in R^+$ and any $J > 0$ there is a continuous increasing positive function $\delta = \delta_{t_0,J}(\epsilon) : (0,\infty) \to (0,\infty)$ with*

$$|(H\xi)(t_1) - (H\xi)(t_2)| \leq \epsilon \textit{ if } \xi \in C_J(t_0) \textit{ and } t_0 \leq t_1 < t_2 < t_1 + \delta. \tag{8}$$

**Proof** For any $\xi \in C_J(t_0)$, $t_1$ and $t_2$ with $t_0 \leq t_1 < t_2$ we have

$$|(H\xi)(t_1) - (H\xi)(t_2)|$$
$$\leq |a(t_1) - a(t_2)| + |\int_0^{t_1} D(t_1,s,\xi(s))ds - \int_0^{t_2} D(t_2,s,\xi(s))ds|$$

$$+|\int_{t_1}^{\infty} E(t_1, s, \xi(s))ds - \int_{t_2}^{\infty} E(t_2, s, \xi(s))ds|$$

$$\le |a(t_1) - a(t_2)| + \int_0^{t_1} |P(t_1, s, \xi(s)) - P(t_2, s, \xi(s))|ds \tag{9}$$

$$+ \int_0^{t_1} |F(t_1, s, \xi(s)) - F(t_2, s, \xi(s))|ds + \int_{t_1}^{t_2} F_J(t_2, s)ds$$

$$+ \int_{t_1}^{t_2} Q_J(t_1, s)ds + \int_{t_2}^{\infty} |Q(t_1, s, \xi(s)) - Q(t_2, s, \xi(s))|ds$$

$$+ \int_{t_1}^{t_2} G_J(t_1, s)ds + \int_{t_2}^{\infty} |G(t_1, s, \xi(s)) - G(t_2, s, \xi(s))|ds.$$

Since $a(t)$ is uniformly continuous on $R^+$ from (3), for any $\epsilon > 0$ there is a $\delta > 0$ with

$$|a(t_1) - a(t_2)| \le \frac{\epsilon}{9} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_1. \tag{10}$$

From (6), for the $\epsilon$ there is a $\tau_1 > \max(t_0, 1)$ with

$$\int_{-\infty}^{t-\tau_1} P_J(t, s)ds \le \frac{\epsilon}{27} \text{ if } t \in R. \tag{11}$$

Since $P(t, s, x)$ is uniformly continuous on $U_1 := \{(t, s, x) : t - 2\tau_1 \le s \le t \text{ and } |x| \le J\}$, for the $\epsilon$ there is a $\delta_2$ such that $0 < \delta_2 < 1$ and

$$|P(t_1, s, x) - P(t_2, s, x)| \le \frac{\epsilon}{27\tau_1} \text{ if } (t_1, s, x), (t_2, s, x) \in U_1 \text{ and } |t_1 - t_2| < \delta_2. \tag{12}$$

From (11) and (12), if $\tau_1 \le t_1 < t_2 < t_1 + \delta_2$, then we have

$$\int_0^{t_1} |P(t_1, s, \xi(s)) - P(t_2, s, \xi(s))|ds$$

$$\le \int_{-\infty}^{t_1-\tau_1} P_J(t_1, s)ds + \int_{-\infty}^{t_1-\tau_1} P_J(t_2, s)ds \tag{13}$$

$$+ \int_{t_1-\tau_1}^{t_1} |P(t_1, s, \xi(s)) - P(t_2, s, \xi(s))|ds \le \frac{\epsilon}{9}.$$

On the other hand, if $t_0 \le t_1 < \tau_1$ and $t_1 < t_2 < t_1 + \delta_2$, then from (12) we obtain

$$\int_0^{t_1} |P(t_1, s, \xi(s)) - P(t_2, s, \xi(s))|ds \le \frac{\epsilon}{27},$$

which together with (13), implies

$$\int_0^{t_1} |P(t_1, s, \xi(s)) - P(t_2, s, \xi(s))|ds \le \frac{\epsilon}{9} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_2. \tag{14}$$

Now let $\alpha := \sup\{P_J(t,s) : t-1 \le s \le t\}$. Then, for the $\epsilon$ there is a $\delta_3$ such that $0 < \delta_3 < \min(\epsilon/9\alpha, 1)$ and

$$\int_{t_1}^{t_2} P_J(t_2,s)ds \le \frac{\epsilon}{9} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_3. \tag{15}$$

Next from (7), for the $\epsilon$ there is a $\tau_2 > \max(t_0, 1)$ with

$$\int_0^t F_J(t,s)ds \le \frac{\epsilon}{18} \text{ if } t \ge \tau_2, \tag{16}$$

which yields

$$\begin{aligned} &\int_0^{t_1} |F(t_1,s,\xi(s)) - F(t_2,s,\xi(s))|ds \\ &\le \int_0^{t_1} F_J(t_1,s)ds + \int_0^{t_1} F_J(t_2,s)ds \\ &\le \frac{\epsilon}{9} \text{ if } \tau_2 \le t_1 < t_2. \end{aligned} \tag{17}$$

On the other hand, since $F(t,s,x)$ is uniformly continuous on $U_2 := \{(t,s,x) : 0 \le s \le t \le \tau_2 + 1 \text{ and } |x| \le J\}$, for the $\epsilon$ there is a $\delta_4$ such that $0 < \delta_4 < 1$ and

$$|F(t_1,s,x) - F(t_2,s,x)| \le \frac{\epsilon}{9\tau_2} \text{ if } (t_1,s,x),(t_2,s,x) \in U_2 \text{ and } |t_1 - t_2| < \delta_4,$$

which together with (17), implies

$$\int_0^{t_1} |F(t_1,s,\xi(s)) - F(t_2,s,\xi(s))|ds \le \frac{\epsilon}{9} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_4. \tag{18}$$

Now let $\beta := \sup\{F_J(t,s) : 0 \le s \le t \le \tau_2 + 1\}$. Then, for the $\epsilon$ there is a $\delta_5$ such that $0 < \delta_5 < \min(\epsilon/9\beta, 1)$ and

$$\int_{t_1}^{t_2} F_J(t_2,s)ds \le \frac{\epsilon}{9} \text{ if } t_2 < \tau_2 \text{ and } t_0 \le t_1 < t_2 < t_1 + \delta_5,$$

which together with (16), implies

$$\int_{t_1}^{t_2} F_J(t_2,s)ds \le \frac{\epsilon}{9} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_5. \tag{19}$$

Similarly let $\gamma := \sup\{Q_J(t,s) : t-1 \le s \le t\}$. Then, for the $\epsilon$ there is a $\delta_6$ such that $0 < \delta_6 < \min(\epsilon/9\gamma, 1)$ and

$$\int_{t_1}^{t_2} Q_J(t,s)ds \le \frac{\epsilon}{9} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_6. \tag{20}$$

Next from (6), for the $\epsilon$ there is a $\tau_3 > \max(t_0, 1)$ with

$$\int_{t+\tau_3}^{\infty} Q_J(t,s)ds \le \frac{\epsilon}{27} \text{ if } t \in R. \tag{21}$$

Since $Q(t,s,x)$ is uniformly continuous on $U_3 := \{(t,s,x) : t \le s \le t+2\tau_3 \text{ and } |x| \le J\}$, for the $\epsilon$ there is a $\delta_7$ such that $0 < \delta_7 < 1$ and

$$|Q(t_1,s,x) - Q(t_2,s,x)| \le \frac{\epsilon}{27\tau_3} \text{ if } (t_1,s,x),(t_2,s,x) \in U_3 \text{ and } |t_1 - t_2| < \delta_7,$$

which together with (21), implies

$$\int_{t_2}^{\infty} |Q(t_1,s,\xi(s)) - Q(t_2,s,\xi(s))|ds$$

$$\le \int_{t_2}^{t_2+\tau_3} |Q(t_1,s,\xi(s)) - Q(t_2,s,\xi(s))|ds + \int_{t_2+\tau_3}^{\infty} Q_J(t_1,s)ds + \int_{t_2+\tau_3}^{\infty} Q_J(t_2,s)ds \quad (22)$$

$$\le \frac{\epsilon}{9} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_7.$$

Now from (7), for the $\epsilon$ there is a $\tau_4 > \max(t_0, 1)$ with

$$\int_{t}^{\infty} G_J(t,s)ds \le \frac{\epsilon}{9} \text{ if } t \ge \tau_4. \quad (23)$$

Let $\delta := \sup\{Q_J(t,s) : 0 \le t \le s \le \tau_4 + 1\}$. Then, for the $\epsilon$ there is a $\delta_8$ such that $0 < \delta_8 < \min(\epsilon/9\delta, 1)$ and

$$\int_{t_1}^{t_2} G_J(t_1,s)ds \le \frac{\epsilon}{9} \text{ if } t_1 < \tau_4 \text{ and } t_0 \le t_1 < t_2 < t_1 + \delta_8,$$

which together with (23), implies

$$\int_{t_1}^{t_2} G_J(t_1,s)ds \le \frac{\epsilon}{9} \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta_8. \quad (24)$$

Finally from (6), for the $\epsilon$ there is a $\tau_5 > \max(t_0, 1)$ with

$$\int_{t+\tau_5}^{\infty} G_J(t,s)ds \le \frac{\epsilon}{27} \text{ if } t \in R. \quad (25)$$

Since $G(t,s,x)$ is uniformly continuous on $U_4 := \{(t,s,x) : 0 \le t \le s \le t+\tau_5 \text{ and } |x| \le J\}$, for the $\epsilon$ there is a $\delta_9$ such that $0 < \delta_9 < 1$ and

$$|G(t_1,s,x) - G(t_2,s,x)| \le \frac{\epsilon}{27\tau_5} \text{ if } (t_1,s,x),(t_2,s,x) \in U_4 \text{ and } |t_1 - t_2| < \delta_9,$$

which together with (25), implies

$$\int_{t_2}^{\infty} |G(t_1,s,\xi(s)) - G(t_2,s,\xi(s))|ds$$

$$\le \int_{t_2}^{t_2+\tau_5} |G(t_1,s,\xi(s)) - G(t_2,s,\xi(s))|ds + \int_{t_2+\tau_5}^{\infty} G_J(t_1,s)ds + \int_{t_2+\tau_5}^{\infty} G_J(t_2,s)ds \quad (26)$$

$$\leq \frac{\epsilon}{9} \text{ if } t_0 \leq t_1 < t_2 < t_1 + \delta_9.$$

Thus, from (9), (10), (14), (15), (18)-(20), (22), (24) and (26), for the $\delta_* := \min\{\delta_i : 1 \leq i \leq 9\}$ we have (8) with $\delta = \delta_*$. Since we may assume that $\delta_*$ is nondecreasing, we can easily conclude that there is a continuous increasing function $\delta = \delta_{t_0.J} : (0, \infty) \to (0, \infty)$ which satisfies (8).

**Lemma 2** *If (3)-(7) hold, then for any asymptotically $T$-periodic function $\xi(t)$ on $R^+$ such that $\xi(t) = \pi(t) + \rho(t)$, $\xi$, $\rho \in C(t_0)$ for some $t_0 \in R^+$, $\pi(t+T) = \pi(t)$ on $R^+$ and $\rho(t) \to 0$ as $t \to \infty$, the function*

$$I(t) := \int_0^t D(t, s, \xi(s))ds + \int_t^\infty E(t, s, \xi(s))ds, \; t \in R^+$$

*is continuous, asymptotically $T$-periodic, and the $T$-periodic part of $I(t)$ is given by $\int_{-\infty}^t P(t, s, \pi(s))ds + \int_t^\infty Q(t, s, \pi(s))ds$.*

**Proof** By (4)-(7), one can easily check that the functions $d(t) := \int_0^t D(t, s, \xi(s))ds$, $\phi(t) := \int_{-\infty}^t P(t, s, \pi(s))ds$, $e(t) := \int_t^\infty E(t, s, \xi(s))ds$ and $\psi(t) := \int_t^\infty Q(t, s, \pi(s))ds$ belong to the space $C(t_0)$ and that $\phi(t)$ and $\psi(t)$ are $T$-periodic. Therefore, in order to establish the lemma, it is sufficient to show that $d(t) - \phi(t)$ and $e(t) - \psi(t)$ tend to 0 as $t \to \infty$. Let $J > 0$ be a number with $\|\xi\|_+ \leq J$. Then clearly we have $\|\pi\|_+ \leq J$.

First we prove that $d(t) - \phi(t) \to 0$ as $t \to \infty$. From (6), for any $\epsilon > 0$ there is a $\tau_1 > 0$ with

$$\int_{-\infty}^{t-\tau_1} P_J(t, s)ds < \epsilon \text{ if } t \in R.$$

Then, for $t \geq \tau_1$ we have

$$|d(t) - \phi(t)| = |\int_0^t P(t, s, \xi(s))ds - \int_{-\infty}^t P(t, s, \pi(s))ds + \int_0^t F(t, s, \xi(s))ds|$$

$$\leq \int_0^{t-\tau_1} P_J(t, s)ds + \int_{-\infty}^{t-\tau_1} P_J(t, s)ds + \int_{t-\tau_1}^t |P(t, s, \xi(s)) - P(t, s, \pi(s))|ds + \int_0^t F_J(t, s)ds$$

$$< 2\epsilon + \int_{t-\tau_1}^t |P(t, s, \xi(s)) - P(t, s, \pi(s))|ds + \int_0^t F_J(t, s)ds.$$

Since $P(t, s, x)$ is uniformly continuous on $U_1 := \{(t, s, x) : t - \tau_1 \leq s \leq t \text{ and } |x| \leq J\}$, for the $\epsilon$ there is a $\delta_1 > 0$ with

$$|P(t, s, x) - P(t, s, y)| < \frac{\epsilon}{\tau_1} \text{ if } (t, s, x), \; (t, s, y) \in U_1 \text{ and } |x - y| < \delta_1.$$

Moreover, since $\rho(t) \to 0$ as $t \to \infty$, for the $\delta_1$ there is a $\tau_2 > 0$ with

$$|\rho(t)| = |\xi(t) - \pi(t)| < \delta_1 \text{ if } t \geq \tau_2.$$

By (7), we may assume that

$$\int_0^t F_J(t,s)ds < \epsilon \text{ if } t \geq \tau_2.$$

Hence, if $t \geq \tau_1 + \tau_2$, then $|d(t) - \phi(t)| < 4\epsilon$. This proves that $d(t) - \phi(t) \to 0$ as $t \to \infty$.

Next we prove that $e(t) - \psi(t) \to 0$ as $t \to \infty$. From (6), for any $\epsilon > 0$ there is a $\tau_3 > 0$ with

$$\int_{t+\tau_3}^\infty Q_J(t,s)ds < \epsilon \text{ if } t \in R.$$

Then, for $t \geq \tau_3$ we obtain

$$|e(t) - \psi(t)| = |\int_t^\infty Q(t,s,\xi(s))ds + \int_t^\infty G(t,s,\xi(s))ds - \int_t^\infty Q(t,s,\psi(s))ds|$$

$$\leq 2\int_{t+\tau_3}^\infty Q_J(t,s)ds + \int_t^{t+\tau_3} |Q(t,s,\xi(s)) - Q(t,s,\psi(s))|ds + \int_t^\infty G_J(t,s)ds$$

$$< 2\epsilon + \int_t^{t+\tau_3} |Q(t,s,\xi(s)) - Q(t,s,\psi(s))|ds + \int_t^\infty G_J(t,s)ds.$$

Since $Q(t,s,x)$ is uniformly continuous on $U_2 := \{(t,s,x) : t \leq s \leq t + \tau_3 \text{ and } |x| \leq J\}$, for the $\epsilon$ there is a $\delta_2 > 0$ with

$$|Q(t,s,x) - Q(t,s,y)| < \frac{\epsilon}{\tau_3} \text{ if } (t,s,x),\ (t,s,y) \in U_2 \text{ and } |x-y| < \delta_2.$$

Moreover, since $\rho(t) \to 0$ as $t \to \infty$, for the $\delta_2$ there is a $\tau_4 > 0$ with

$$|\rho(t)| = |\xi(t) - \pi(t)| < \delta_2 \text{ if } t \geq \tau_4.$$

By (7), we may assume that

$$\int_t^\infty G_J(t,s)ds < \epsilon \text{ if } t \geq \tau_4.$$

Hence, if $t \geq \tau_3 + \tau_4$, then $|e(t) - \psi(t)| < 4\epsilon$. This proves that $e(t) - \psi(t) \to 0$ as $t \to \infty$.

Now we have the following theorem.

**Theorem 2** *If (3)-(7) hold, and if Eq.(1) has an asymptotically $T$-periodic solution with an initial time $t_0$ in $R^+$, then the $T$-periodic extension to $R$ of its $T$-periodic part is a $T$-periodic solution of Eq.(2). In particular, if the asymptotically $T$-periodic solution of Eq.(1) is asymptotically constant, then Eq.(2) has a constant solution.*

**Proof** Let $x(t)$ be an asymptotically $T$-periodic solution of Eq.(1) with an initial time $t_0 \in R^+$ such that $x(t) = y(t) + z(t),\ x,\ y \in C(t_0),\ y(t+T) = y(t)$ on $R^+$ and $z(t) \to 0$ as $t \to \infty$. Then we have

$$y(t) + z(t) = p(t) + q(t) + \int_0^t D(t,s,x(s))ds + \int_t^\infty E(t,s,x(s))ds,\ t \geq t_0. \tag{27}$$

From Lemma 2, taking the $T$-periodic part of the both sides of (27) we obtain

$$y(t) = p(t) + \int_{-\infty}^{t} P(t,s,y(s))ds + \int_{t}^{\infty} Q(t,s,y(s))ds,\ t \geq t_0.$$

From this, it is easy to see that $y(t)$ is a $T$-periodic solution of Eq.(2). The latter part follows easily from the above conclusion.

In order to prove the existence of an asymptotically $T$-periodic solution of Eq.(1) using Schauder's first theorem, we need more assumptions. In addition to (3)-(7), suppose that for some $t_0 \in R^+$ and $J > 0$ the inequality

$$\|a\|_{t_0} + \int_0^t (P_J(t,s) + F_J(t,s))ds + \int_t^{\infty} (Q_J(t,s) + G_J(t,s))ds \leq J \text{ if } t \geq t_0 \tag{28}$$

holds, where $\|a\|_{t_0} := \sup\{|a(t)| : t \geq t_0\}$, and that there are continuous functions $L_J^- : \Delta^- \to R^+$, $L_J^+ : \Delta^+ \to R^+$ and $q_J : [t_0.\infty) \to R^+$ such that $L_J^-(t+T, s+T) = L_J^-(t,s)$ and $L_J^+(t+T, s+T) = L_J^+(t,s)$ satisfying:

$$|P(t,s,x) - P(t,s,y)| \leq L_J^-(t,s)|x-y| \text{ if } (t,s) \in \Delta^-,\ |x| \leq J \text{ and } |y| \leq J; \tag{29}$$

$$|Q(t,s,x) - Q(t,s,y)| \leq L_J^+(t,s)|x-y| \text{ if } (t,s) \in \Delta^+,\ |x| \leq J \text{ and } |y| \leq J; \tag{30}$$

$$q_J(t) \to 0 \text{ as } t \to \infty; \tag{31}$$

and

$$\begin{aligned} &|q(t)| + \int_{-\infty}^{t_0} P_J(t,s)ds + \int_0^{t_0} P_J(t,s)ds + \int_0^t F_J(t,s)ds \\ &+ \int_{t_0}^t L_J^-(t,s)q_J(s)ds + \int_t^{\infty} L_J^+(t,s)q_J(s)ds + \int_t^{\infty} G_J(t,s)ds \\ &\leq q_J(t) \text{ if } t \geq t_0. \end{aligned} \tag{32}$$

Then we have the following theorem.

**Theorem 3** *If (3)-(7) and (28)-(32) hold for some $t_0 \in R^+$ and $J > 0$, then for any continuous initial function $\phi_0 : [0, t_0) \to R^n$ with $\sup\{|\phi_0(s)| : 0 \leq s < t_0\} \leq J$, Eq.(1) has an asymptotically $T$-periodic solution $x(t) = y(t) + z(t)$ such that $x,\ y \in C_J(t_0)$, $y(t+T) = y(t)$ on $R^+$, $x(t)$ satisfies Eq.(1) and $|z(t)| \leq q_J(t)$ on $[t_0, \infty)$, and the $T$-periodic extension to $R$ of $y(t)$ is a $T$-periodic solution of Eq.(2).*

**Proof** Let $S$ be a set of functions $\xi \in C_J(t_0)$ such that $\xi = \pi + \rho,\ \pi \in C_J(t_0),\ \xi(t) = \phi_0(t)$ on $[0, t_0),\ \pi(t+T) = \pi(t)$ on $R^+$ and

$$|\rho(t)| \leq q_J(t) \text{ if } t \geq t_0, \tag{33}$$

and that for the function $\delta = \delta_{t_0,J}(\epsilon)$ in (8), $|\xi(t_1) - \xi(t_2)| \leq \epsilon$ if $t_0 \leq t_1 < t_2 < t_1 + \delta$.

First we prove that $S$ is a compact convex nonempty subset of $C(t_0)$. Since any $\xi \in C_J(t_0)$ such that $\xi(t) = \phi_0(t)$ on $[0, t_0)$ and $\xi(t) \equiv \xi(t_0)$ on $[t_0, \infty)$ is contained in $S$, $S$ is nonempty. Clearly $S$ is a convex subset of $C(t_0)$. In order to prove the compactness of $S$, let $\{\xi_k\}$ be an infinite sequence in $S$ such that $\xi_k = \pi_k + \rho_k,\ \pi_k \in C_J(t_0),\ \pi_k(t+T) = \pi_k(t)$ on $R^+$ and $|\rho_k(t)| \leq q_J(t)$ on $[t_0, \infty)$. From the definition of $S$, if $k,\ l \in N$ and $t_0 \leq t_1 < t_2 < t_1 + \delta$, then we have

$$|\pi_k(t_1) - \pi_k(t_2)| = |\pi_k(t_1 + lT) - \pi_k(t_2 + lT)|$$

$$\leq |\xi_k(t_1 + lT) - \xi_k(t_2 + lT)| + |\rho_k(t_1 + lT) - \rho_k(t_2 + lT)|$$

$$\leq \epsilon + q_J(t_1 + lT) + q_J(t_2 + lT).$$

This implies $|\pi_k(t_1) - \pi_k(t_2)| \leq \epsilon$ by letting $l \to \infty$, where $N$ is the set of positive integers and $\delta = \delta_{t_0,J}(\epsilon)$ is the function in (8). Hence the sets of functions $\{\pi_k\}$ and $\{\rho_k\}$ are uniformly bounded and equicontinuous on $[t_0, \infty)$. Thus, taking a subsequence if necessary, we may assume that the sequence $\{\pi_k\}$ converges to a $\pi \in C_J(t_0)$ uniformly on $R^+$, and the sequence $\{\rho_k\}$ converges to a $\rho \in C(t_0)$ uniformly on any compact subset of $R^+$. Clearly $\pi(t)$ is $T$-periodic on $R^+$, and $\rho(t)$ satisfies (33), and hence the sequence $\{\xi_k\}$ converges to the asymptotically $T$-periodic function $\xi := \pi + \rho$ uniformly on any compact subset of $R^+$ as $k \to \infty$. It is clear that $\xi \in S$. Now we show that $\|\rho_k\|_+ \to 0$ as $k \to \infty$. From (31), for any $\epsilon > 0$ there is a $\tau \geq t_0$ with

$$q_J(t) < \frac{\epsilon}{2} \text{ if } t \geq \tau,$$

which yields

$$|\rho_k(t) - \rho(t)| \leq 2q_J(t) < \epsilon \text{ if } k \in N \text{ and } t \geq \tau. \tag{34}$$

On the other hand, since $\{\rho_k(t)\}$ converges to $\rho(t)$ uniformly on $[0, \tau]$ as $k \to \infty$, for the $\epsilon$ there is a $\kappa \in N$ with

$$|\rho_k(t) - \rho(t)| < \epsilon \text{ if } k \geq \kappa \text{ and } 0 \leq t \leq \tau,$$

which together with (34), implies $\|\rho_k - \rho\|_+ < \epsilon$ if $k \geq \kappa$. This yields $\|\rho_k - \rho\|_+ \to 0$ as $k \to \infty$, and hence, $\|\xi_k - \xi\|_+ \to 0$ as $k \to \infty$. Thus $S$ is compact.

Next we prove that $H$ maps $S$ into $S$ continuously. For any $\xi \in S$ such that $\xi = \pi + \rho,\ \pi \in C_J(t_0),\ \pi(t+T) = \pi(t)$ on $R^+$ and $|\rho(t)| \le q_J(t)$ on $[t_0, \infty)$, let $\phi := H\xi$. Then from (28), for $t \ge t_0$ we have

$$|\phi(t)| \le |a(t)| + \int_0^t (|P(t,s,\xi(s))| + |F(t,s,\xi(s))|)ds + \int_t^\infty (|Q(t,s,\xi(s))| + |G(t,s,\xi(s))|)ds$$

$$\le \|a\|_{t_0} + \int_0^t (P_J(t,s) + F_J(t,s))ds + \int_t^\infty (Q_J(t,s) + G_J(t,s))ds \le J,$$

which together with $\xi \in C_J(t_0)$ and Lemma 1, implies that $\phi \in C_J(t_0)$. Now from Lemma 2, $\phi$ has the unique decomposition $\phi = \psi + \mu,\ \psi \in C_J(t_0),\ \psi(t+T) = \psi(t)$ on $R^+$, and $\mu(t) \to 0$ as $t \to \infty$, where the restriction of $\mu(t)$ on $[t_0, \infty)$ is given by

$$\mu(t) := q(t) - \int_{-\infty}^{t_0} P(t,s,\pi(s))ds$$

$$+ \int_0^{t_0} P(t,s,\xi(s))ds + \int_0^t F(t,s,\xi(s))ds + \int_{t_0}^t (P(t,s,\xi(s)) - P(t,s,\pi(s)))ds$$

$$+ \int_t^\infty (Q(t,s,\xi(s)) - Q(t,s,\pi(s)))ds + \int_t^\infty G(t,s,\xi(s))ds,\ t \ge t_0.$$

Thus from (32), for $t \ge t_0$ we obtain

$$|\mu(t)| \le |q(t)| + \int_{-\infty}^{t_0} P_J(t,s)ds + \int_0^{t_0} P_J(t,s)ds + \int_0^t F_J(t,s)ds$$

$$+ \int_{t_0}^t L_J^-(t,s)q_J(s)ds + \int_t^\infty L_J^+(t,s)q_J(s)ds + \int_t^\infty G_J(t,s)ds \le q_J(t).$$

Moreover, Lemma 1 implies that for the function $\delta = \delta_{t_0,J}(\epsilon)$ in (8) the inequality

$$|\phi(t_1) - \phi(t_2)| \le \epsilon \text{ if } t_0 \le t_1 < t_2 < t_1 + \delta$$

holds. Thus $H$ maps $S$ into $S$. Next we must prove that $H$ is continuous. For any $\xi_i \in S\ (i = 1, 2)$ and $t \ge t_0$ we have

$$|(H\xi_1)(t) - (H\xi_2)(t)|$$

$$\le \int_0^t |D(t,s,\xi_1(s)) - D(t,s,\xi_2(s))|ds + \int_t^\infty |E(t,s,\xi_1(s)) - E(t,s,\xi_2(s))|ds$$

$$\le \int_0^t |P(t,s,\xi_1(s)) - P(t,s,\xi_2(s))|ds + \int_0^t |F(t,s,\xi_1(s)) - F(t,s,\xi_2(s))|ds \qquad (35)$$

$$+ \int_t^\infty |Q(t,s,\xi_1(s)) - Q(t,s,\xi_2(s))|ds + \int_t^\infty |G(t,s,\xi_1(s)) - G(t,s,\xi_2(s))|ds.$$

From (6), for any $\epsilon > 0$ there is a $\tau_1 > t_0$ with

$$\int_{-\infty}^{t-\tau_1} P_J(t,s)ds < \frac{\epsilon}{15} \text{ if } t \in R. \qquad (36)$$

Since $P(t,s,x)$ is uniformly continuous on $U_1 := \{(t,s,x) : t-\tau_1 \le s \le t \text{ and } |x| \le J\}$, for the $\epsilon$ there is a $\delta_1 > 0$ with

$$|P(t,s,x) - P(t,s,y)| < \frac{\epsilon}{15\tau_1} \text{ if } (t,s,x),\ (t,s,y) \in U_1 \text{ and } |x-y| < \delta_1. \tag{37}$$

From (36) and (37), for the $\epsilon$ we obtain

$$\int_0^t |P(t,s,\xi_1(s)) - P(t,s,\xi_2(s))|ds < \frac{\epsilon}{15} \text{ if } t_0 \le t < \tau_1 \text{ and } \|\xi_1 - \xi_2\|_+ < \delta_1, \tag{38}$$

and if $t \ge \tau_1$ and $\|\xi_1 - \xi_2\|_+ < \delta_1$, then we have

$$\int_0^t |P(t,s,\xi_1(s)) - P(t,s,\xi_2(s))|ds$$

$$\le 2\int_{-\infty}^{t-\tau_1} P_J(t,s)ds + \int_{t-\tau_1}^t |P(t,s,\xi_1(s)) - P(t,s,\xi_2(s))|ds < \frac{\epsilon}{5}.$$

This, together with (38), yields

$$\int_0^t |P(t,s,\xi_1(s)) - P(t,s,\xi_2(s))|ds < \frac{\epsilon}{5} \text{ if } \|\xi_1 - \xi_2\|_+ < \delta_1. \tag{39}$$

Next from (7), for the $\epsilon$ there is a $\tau_2 > 0$ with

$$\int_0^t F_J(t,s)ds < \frac{\epsilon}{10} \text{ if } t > \tau_2,$$

which implies

$$\int_0^t |F(t,s,\xi_1(s)) - F(t,s,\xi_2(s))|ds \le 2\int_0^t F_J(t,s)ds < \frac{\epsilon}{5} \text{ if } t > \tau_2. \tag{40}$$

Since $F(t,s,x)$ is uniformly continuous on $U_2 := \{(t,s,x) : 0 \le s \le t \le \tau_2 \text{ and } |x| \le J\}$, for the $\epsilon$ there is a $\delta_2 > 0$ with

$$|F(t,s,x) - F(t,s,y)| < \frac{\epsilon}{5\tau_2} \text{ if } (t,s,x),\ (t,s,y) \in U_2 \text{ and } |x-y| < \delta_2,$$

which yields

$$\int_0^t |F(t,s,\xi_1(s)) - F(t,s,\xi_2(s))ds < \frac{\epsilon}{5} \text{ if } 0 \le t \le \tau_2 \text{ and } \|\xi_1 - \xi_2\|_+ < \delta_2.$$

This together with (40), implies

$$\int_0^t |F(t,s,\xi_1(s)) - F(t,s,\xi_2(s))|ds < \frac{\epsilon}{5} \text{ if } \|\xi_1 - \xi_2\|_+ < \delta_2. \tag{41}$$

Next from (6), for the $\epsilon$ there is a $\tau_3 > 0$ with

$$\int_{t+\tau_3}^{\infty} (Q_J(t,s) + G_J(t,s))ds < \frac{\epsilon}{10} \text{ if } t \in R,$$

which implies

$$\int_{t+\tau_3}^{\infty} |Q(t,s,\xi_1(s)) - Q(t,s,\xi_2(s))|ds + \int_{t+\tau_3}^{\infty} |G(t,s,\xi_1(s)) - G(t,s,\xi_2(s))|ds < \frac{\epsilon}{5}. \quad (42)$$

Since $Q(t,s,x)$ is uniformly continuous on $U_3 := \{(t,s,x) : t \leq s \leq t+\tau_3 \text{ and } |x| \leq J\}$, for the $\epsilon$ there is a $\delta_3 > 0$ with

$$|Q(t,s,x) - Q(t,s,y)| < \frac{\epsilon}{5\tau_3} \text{ if } (t,s,x),\ (t,s,y) \in U_3 \text{ and } |x-y| < \delta_3,$$

which yields

$$\int_{t}^{t+\tau_3} |Q(t,s,\xi_1(s)) - Q(t,s,\xi_2(s))|ds < \frac{\epsilon}{5} \text{ if } \|\xi_1 - \xi_2\|_+ < \delta_3. \quad (43)$$

Finally from (7), for the $\epsilon$ there is a $\tau_4 > 0$ with

$$\int_t^{\infty} G_J(t,s)ds < \frac{\epsilon}{10} \text{ if } t > \tau_4,$$

which implies

$$\int_{t}^{t+\tau_4} |G(t,s,\xi_1(s)) - G(t,s,\xi_2(s))|ds \leq 2\int_t^{\infty} G_J(t,s)ds < \frac{\epsilon}{5} \text{ if } t > \tau_4. \quad (44)$$

Since $G(t,s,x)$ is uniformly continuous on $U_4 := \{(t,s,x) : 0 \leq t \leq s \leq t+\tau_4 \text{ and } |x| \leq J\}$, for the $\epsilon$ there is a $\delta_4 > 0$ with

$$|G(t,s,x) - G(t,s,y)| < \frac{\epsilon}{5\tau_4} \text{ if } (t,s,x),\ (t,s,y) \in U_4 \text{ and } |x-y| < \delta_4,$$

which yields

$$\int_{t}^{t+\tau_4} |G(t,s,\xi_1(s)) - G(t,s,\xi_2(s))|ds < \frac{\epsilon}{5} \text{ if } 0 \leq t \leq \tau_4 \text{ and } \|\xi_1 - \xi_2\|_+ < \delta_4.$$

This, together with (44), implies

$$\int_{t}^{t+\tau_4} |G(t,s,\xi_1(s)) - G(t,s,\xi_2(s))|ds < \frac{\epsilon}{5} \text{ if } \|\xi_1 - \xi_2\|_+ < \delta_4. \quad (45)$$

Thus, from (35), (39), (41)-(43) and (45), for the $\delta := \min(\delta_1, \delta_2, \delta_3, \delta_4)$ we obtain

$$\|H\xi_1 - H\xi_2\|_+ < \epsilon \text{ if } \xi_1,\ \xi_2 \in S \text{ and } \|\xi_1 - \xi_2\|_+ < \delta,$$

and hence $H$ is continuous.

Now, applying Theorem 1, $H$ has a fixed point in $S$, which is a desired asymptotically $T$-periodic solution of Eq.(1). The latter part is a direct consequence of Theorem 2.

Now we show two examples of a linear equation and a nonlinear equation.

**Example 1.** Consider the scalar linear equation

$$x(t) = p(t) + \rho e^{-t}$$

$$+\int_0^t (e^{8(s-t)} + \frac{1}{8}e^{-t-s})x(s)ds + \int_t^\infty (e^{8(t-s)} + \frac{1}{8}e^{-t-s})x(s)ds,\ t \in R^+, \tag{46}$$

where $p : R \to R$ is a continuous $T$-periodic function, and $\rho$ is a constant with $\|p\| + |\rho| > 0$ and $3\|p\| \geq 7|\rho|$, and where $\|p\| := \sup\{|p(t)| : t \in R\}$. Eq.(46) is obtained from Eq.(1) taking $n = 1$, $a(t) = p(t) + \rho e^{-t}$, $q(t) = \rho e^{-t}$, $D(t,s,x) = (e^{8(s-t)} + e^{-t-s}/8)x$, $P(t,s,x) = e^{8(s-t)}x$, $F(t,s,x) = G(t,s,x) = e^{-t-s}x/8$, $E(t,s,x) = (e^{8(t-s)} + e^{-t-s}/8)x$, and $Q(t,s,x) = e^{8(t-s)}x$. For $J := 8(\|p\| + |\rho|)/5$, we can take the following functions as $P_J,\ Q_J,\ F_J,\ G_J,\ L_J^-$ and $L_J^+$:

$$P_J(t,s) := Je^{8(s-t)} \text{ if } (t,s) \in \Delta^-;$$

$$Q_J(t,s) := Je^{8(t-s)} \text{ if } (t,s) \in \Delta^+;$$

$$F_J(t,s) := \frac{J}{8}e^{-t-s} \text{ if } (t,s) \in \Delta^-;$$

$$G_J(t,s) := \frac{J}{8}e^{-t-s} \text{ if } (t,s) \in \Delta^+;$$

$$L_J^-(t,s) := e^{8(s-t)} \text{ if } (t,s) \in \Delta^-;$$

and

$$L_J^+(t,s) := e^{8(t-s)} \text{ if } (t,s) \in \Delta^+.$$

It is easy to see that the above functions satisfy (3)-(7), (29) and (30). Moreover (28) holds with $t_0 = 0$ for the $J$, since we have $\|a\|_+ \leq \|p\| + |\rho|$, $\int_0^t P_J(t,s)ds < J/8$, $\int_0^t F_J(t,s)ds + \int_t^\infty G_J(t,s)ds \leq J/8$ and $\int_t^\infty Q_J(t,s)ds \leq J/8$ on $R^+$. Now define a function $q_J : R^+ \to R^+$ by

$$q_J(t) := \frac{J}{t+1},\ t \in R^+.$$

Clearly (31) holds. We show that (32) holds with $t_0 = 0$. It is easy to see that for any $t \in R^+$ we have

$$|q(t)| + \int_{-\infty}^0 P_J(t,s)ds + \int_0^t F_J(t,s)ds$$

$$+\int_0^t L_J^-(t,s)q_J(s)ds + \int_t^\infty L_J^+(t,s)q_J(s)ds + \int_t^\infty G_J(t,s)ds$$

$$\leq (|\rho| + \frac{J}{4})e^{-t} + J\int_0^t \frac{e^{8(s-t)}}{s+1}ds + \frac{J}{8(t+1)} \leq (|\rho| + \frac{J}{4})e^{-t} + \frac{9J}{16(t+1)}$$

$$\leq \frac{7J}{16}e^{-t} + \frac{9J}{16(t+1)} \leq q_J(t);$$

that is, (32) holds with $t_0 = 0$. Thus by Theorem 3, Eq.(46) has an asymptotically $T$-periodic solution $x(t) = y(t) + z(t)$ such that $x,\ y \in C_J := C_J(0),\ y(t+T) = y(t)$ and $|z(t)| \le q_J(t)$ on $R^+$, and the $T$-periodic extension to $R$ of $y(t)$ is a $T$-periodic solution of the equation

$$x(t) = p(t) + \int_{-\infty}^{t} e^{8(s-t)}x(s)ds + \int_{t}^{\infty} e^{8(t-s)}x(s)ds,\ t \in R.$$

**Example 2.** Corresponding to Eq.(46), consider the scalar nonlinear equation

$$x(t) = p(t) + \rho e^{-t}$$

$$+ \int_0^t (\sigma e^{8(s-t)} + \tau e^{-t-s})x^2(s)ds + \int_t^\infty (\sigma e^{8(t-s)} + \tau e^{-t-s})x^2(s)ds,\ t \in R^+, \qquad (47)$$

where $p : R \to R$ is a continuous $T$-periodic function, and $\rho,\ \sigma$ and $\tau$ are constants such that $\|p\| + |\rho| > 0,\ (|\sigma| + 4|\tau|)(\|p\| + |\rho|) < 1,\ 4|\sigma|J \le 4|\rho||\sigma| + \|p\|(5|\sigma| + 4|\tau|)$, and $9|\sigma|J < 8$, where $J = 2(1 - \sqrt{1 - (|\sigma| + 4|\tau|)(\|p\| + |\rho|)})/(|\sigma| + 4|\tau|)$. Eq.(47) is obtained from Eq.(1) taking $n = 1,\ a(t) = p(t) + \rho e^{-t},\ q(t) = \rho e^{-t},\ D(t,s,x) = \tau e^{-t-s}x^2,\ E(t,s,x) = (\sigma e^{8(t-s)} + \tau e^{-t-s})x^2$, and $Q(t,s,x) = \sigma e^{8(t-s)}x^2$. It is easy to see that $\|p\| + |\rho| + (|\sigma|/4 + |\tau|)J^2 = J$. For this $J$ we can take the following functions as $P_J,\ Q_J,\ F_J,\ G_J,\ L_J^-$ and $L_J^+$;

$$P_J(t,s) := J^2|\sigma|e^{8(s-t)} \text{ if } (t,s) \in \Delta^-;$$

$$Q_J(t,s) := J^2|\sigma|e^{8(t-s)} \text{ if } (t,s) \in \Delta^+;$$

$$F_J(t,s) := J^2|\tau|e^{-t-s} \text{ if } (t,s) \in \Delta^-;$$

$$G_J(t,s) := J^2|\tau|e^{-t-s} \text{ if } (t,s) \in \Delta^+;$$

$$L_J^-(t,s) := 2J|\sigma|e^{8(s-t)} \text{ if } (t,s) \in \Delta^-;$$

and

$$L_J^+(t,s) := 2J|\sigma|e^{8(t-s)} \text{ if } (t,s) \in \Delta^+.$$

It is easy to see that these functions satisfy (3)-(7), (29) and (30). Moreover, by the choice of $J$, (28) holds with $t_0 = 0$, since we have $\|a\|_+ \le \|p\| + |\rho|,\ \int_0^t P_J(t,s)ds \le J^2|\sigma|/8,\ \int_0^t F_J(t,s)ds + \int_t^\infty G_J(t,s)ds \le J^2|\tau|$ and $\int_t^\infty Q_J(t,s)ds \le J|\sigma|^2/8$ on $R^+$. Now define a function $q_J : R^+ \to R^+$ by

$$q_J(t) := \frac{J}{t+1},\ t \in R^+.$$

Then clearly (31) holds. We show that (32) holds with $t_0 = 0$. It is easy to see that for any $t \in R^+$ we have

$$
\begin{aligned}
&|q(t)| + \int_{-\infty}^{0} P_J(t,s)ds + \int_0^t F_J(t,s)ds \\
&+ \int_0^t L_J^-(t,s)q_J(s)ds + \int_t^\infty L_J^+(t,s)q_J(s)ds + \int_t^\infty G_J(t,s)ds \\
&\le (|\rho| + \frac{|\sigma|}{8}J^2 + |\tau|J^2)e^{-t} + 2|\sigma|J^2\int_0^t \frac{e^{8(s-t)}}{s+1}ds + \frac{|\sigma|J^2}{4(t+1)} \le q_J(t),
\end{aligned}
$$

that is, (32) holds with $t_0 = 0$. Thus by Theorem 3, Eq.(47) has an asymptotically $T$-periodic solution $x(t) = y(t)+z(t)$ such that $x,\ y \in C_J,\ y(t+T) = y(t)$ and $|z(t)| \le q_J(t)$ on $R^+$, and the $T$-periodic extension to $R$ of $y(t)$ is a $T$-periodic solution of the equation

$$x(t) = p(t) + \sigma\int_{-\infty}^{t} e^{8(s-t)}x^2(s)ds + \sigma\int_t^\infty e^{8(t-s)}x^2(s)ds,\ t \in R.$$

## 4. PERIODIC SOLUTIONS

Although Theorem 3 assures the existence of $T$-periodic solutions of Eq.(2), we can prove directly the existence of $T$-periodic solutions of Eq.(2) under weaker assumptions than those in Theorem 3 using Schauder's first theorem.

Let $(\mathcal{P}_T, \|\cdot\|)$ be the Banach space of continuous $T$-periodic functions $\xi : R \to R^n$ with the supremum norm. For any $\xi \in \mathcal{P}_T$, define a map $H$ on $\mathcal{P}_T$ by

$$(H\xi)(t) := p(t) + \int_{-\infty}^{t} P(t,s,\xi(s))ds + \int_t^\infty Q(t,s,\xi(s))ds,\ t \in R.$$

Then, by a method similar to the method used in the proof of Lemma 1, we can prove the following lemma which we state without proof.

**Lemma 3** *If (3)-(6) hold with $G(t,s,x) \equiv 0$, then for any $J > 0$ there is a continuous increasing positive function $\delta = \delta_J(\epsilon) : (0,\infty) \to (0,\infty)$ with*

$$|(H\xi)(t_1) - (H\xi)(t_2)| \le \epsilon \ if\ \xi \in \mathcal{P}_T,\ \|\xi\| \le J \ and\ |t_1 - t_2| < \delta. \tag{48}$$

Now we have the following theorem.

**Theorem 4** *In addition to (3)-(6) with $G(t,s,x) \equiv 0$, suppose that for some $J > 0$ the inequality*

$$\|p\| + \int_{-\infty}^{t} P_J(t,s)ds + \int_t^\infty Q_J(t,s)ds \le J \ if\ t \in R \tag{49}$$

*holds. Then Eq.(2) has a $T$-periodic solution $x(t)$ with $\|x\| \le J$.*

**Proof** Let $S$ be a set of functions $\xi \in \mathcal{P}_T$ such that $\|\xi\| \leq J$ and for the function $\delta = \delta_J(\epsilon)$ in (48), $|\xi(t_1) - \xi(t_2)| \leq \epsilon$ if $|t_1 - t_2| < \delta$.

First we can prove that $S$ is a compact convex nonempty subset of $\mathcal{P}_T$ by a method similar to the one used in the proof of Theorem 3.

Next we prove that $H$ maps $S$ into $S$. For any $\xi \in S$, let $\phi := H\xi$. Then, clearly $\phi(t)$ is $T$-periodic. In addition, from (49) we have

$$|\phi(t)| \leq \|p\| + \int_{-\infty}^{t} P_J(t,s)ds + \int_{t}^{\infty} Q_J(t,s)ds \leq J \text{ if } t \in R,$$

and hence $\|\phi\| \leq J$. Moreover, Lemma 3 implies that for the $\delta$ in (48) we obtain

$$|\phi(t_1) - \phi(t_2)| \leq \epsilon \text{ if } \xi \in \mathcal{P}_T,\ \|\xi\| \leq J \text{ and } |t_1 - t_2| < \delta.$$

Thus $H$ maps $S$ into $S$.

The continuity of $H$ can be proved similarly as in the proof of Theorem 3.

Finally, applying Theorem 1 we can conclude that $H$ has a fixed point $x$ in $S$, which is a $T$-periodic solution of Equation (2) with $\|x\| \leq J$.

**Remark** In addition to the continuity of the map $H$, we can easily prove that $H$ maps each bounded set of $\mathcal{P}_T$ into a compact set of $\mathcal{P}_T$. Thus Theorem 4 can be proved using Schauder's second theorem.

## 5. RELATIONS BETWEEN (1) AND (2)

In Theorem 2, we showed a relation between an asymptotically $T$-periodic solution of Eq.(1) and a $T$-periodic solution of Eq.(2). Moreover, concerning relations between Equations (1) and (2) we have the following theorem.

**Theorem 5** *Under the assumptions (3)-(7), the following five conditions are equivalent:*

(i) *Eq.(2) has a $T$-periodic solution.*

(ii) *For some $q(t)$, $F(t,s,x) \equiv 0$ and $G(t,s,x) \equiv 0$, Eq.(1) has a $T$-periodic solution which satisfies (1) on $R^+$.*

(iii) *For some $q(t)$, $F(t,s,x) \equiv 0$ and $G(t,s,x) \equiv 0$, Eq.(1) has an asymptotically $T$-periodic solution with an initial time in $R^+$.*

(iv) *For some $q(t)$, $F(t,s,x)$ and $G(t,s,x)$, Eq.(1) has a $T$-periodic solution which satisfies (1) on $R^+$.*

(v) *For some $q(t)$, $F(t,s,x)$ and $G(t,s,x)$, Eq.(1) has an asymptotically $T$-periodic solution with an initial time in $R^+$.*

**Proof** First we prove that (i) implies (ii). Let $\pi(t)$ be a $T$-periodic solution of Eq.(2), and let

$$q(t) := \int_{-\infty}^{0} P(t,s,\pi(s))ds, \ t \in R^+.$$

Then, clearly $q(t)$ is continuous and $q(t) \to 0$ as $t \to \infty$. Thus it is easy to see that for the $q(t)$, $F(t,s,x) \equiv 0$ and $G(t,s,x) \equiv 0$, Eq.(1) has a $T$-periodic solution $\pi(t)$, which satisfies (1) on $R^+$.

Next, it is clear that (ii) and (iii) imply (iii) and (v) respectively. Moreover, from Theorem 2, (v) yields (i).

Finally, since it is trivial that (ii) implies (iv), we prove that (iv) yields (ii). Let $\psi(t)$ be a $T$-periodic solution of Eq.(1) with some $q(t)$, $F(t,s,x)$ and $G(t,s,x)$ which satisfies (1) on $R^+$, and let

$$r(t) := \int_{0}^{t} F(t,s,\psi(s))ds + \int_{t}^{\infty} G(t,s,\psi(s))ds, \ t \in R^+.$$

Then, clearly $r(t)$ is continuous and $r(t) \to 0$ as $t \to \infty$. Thus it is easy to see that for $q(t)+r(t)$, $F(t,s,x) \equiv 0$ and $G(t,s,x) \equiv 0$, Eq.(1) has a $T$-periodic solution $\psi(t)$ which satisfies (1) on $R^+$.

In [4, Burton-Furumochi], we discussed a relation between the equation

$$x(t) = a(t) + \int_{0}^{t} P(t,s)x(s)ds + \int_{0}^{t} F(t,s,x(s))ds$$

$$+ \int_{t}^{\infty} Q(t,s)x(s)ds + \int_{t}^{\infty} G(t,s,x(s))ds, \ t \in R^+ \tag{50}$$

and the linear equation

$$x(t) = p(t) + \int_{-\infty}^{t} P(t,s)x(s)ds + \int_{t}^{\infty} Q(t,s)x(s)ds, \ t \in R, \tag{51}$$

where $a$, $p$, $F$ and $G$ satisfy (3)-(7) with $P_J = J|P(t,s)|$ and $Q_J = J|Q(t,s)|$, and where $P : \Delta^- \to R^{n\times n}$ and $Q : \Delta^+ \to R^{n\times n}$ are continuous functions such that $P(t+T,s+T) = P(t,s)$, $Q(t+T,s+T) = Q(t,s)$, $\int_{-\infty}^{t-\tau} |P(t,s)|ds + \int_{t+\tau}^{\infty} |Q(t,s)|ds \to 0$ uniformly for $t \in R$ as $\tau \to \infty$, and $|P| := \sup\{|Px| : |x| = 1\}$. Concerning Equations (50) and (51), we state a theorem. For the proof, see Lemma 4 and Theorem 10 in [4].

**Theorem 6** *Under the above assumptions for Equations (50) and (51), the following hold.*

(i) *If Eq.(50) has an $R^+$-bounded solution with an initial time in $R^+$, then Eq.(51) has an $R$-bounded solution which satisfies (51) on $R$.*

(ii) *If Eq.(51) has an $R$-bounded solution which satisfies (51) on $R$, then Eq.(51) has a $T$-periodic solution.*

Now we have our final theorem concerning relations between Equations (50) and (51).

**Theorem 7** *Under the above assumptions for Equations (50) and (51), the following eight conditions are equivalent:*

(i) *Eq.(51) has a $T$-periodic solution.*

(ii) *For some $q(t)$, $F(t,s,x) \equiv 0$ and $G(t,s,x) \equiv 0$, Eq.(50) has a $T$-periodic solution which satisfies (50) on $R^+$.*

(iii) *For some $q(t)$, $F(t,s,x) \equiv 0$ and $G(t,s,x) \equiv 0$, Eq.(50) has an asymptotically $T$-periodic solution with an initial time in $R^+$.*

(iv) *For some $q(t)$, $F(t,s,x) \equiv 0$ and $G(t,s,x) \equiv 0$, Eq.(50) has an $R^+$-bounded solution with an initial time in $R^+$.*

(v) *For some $q(t)$, $F(t,s,x)$ and $G(t,s,x)$, Eq.(50) has a $T$-periodic solution which satisfies (50) on $R^+$.*

(vi) *For some $q(t)$, $F(t,s,x)$ and $G(t,s,x)$, Eq.(50) has an asymptotically $T$-periodic solution with an initial time in $R^+$.*

(vii) *For some $q(t)$, $F(t,s,x)$ and $G(t,s,x)$, Eq.(50) has an $R^+$-bounded solution with an initial time in $R^+$.*

(viii) *Eq.(51) has an $R$-bounded solution which satisfies (51) on $R$.*

**Proof** The equivalence among (i)-(iii), (v) and (vi) is a direct consequence of Theorem 4. From this and the trivial implication from (iii) to (iv), it is clear that (i) and (iv) imply (iv) and (vii) respectively. Next, from Theorem 5(i), (vii) yields (viii). Finally, from Theorem 5(ii), (viii) implies (i), which completes the proof.

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