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Periodic solutions for an impulsive semi-ratio-dependent predator–prey model with patches and time delays

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Abstract. In this paper, we study the existence of a positive periodic solution for a two-species semi-ratio-dependent predator-prey system with time delays and impulses in a two-patch environment. By using the method of coincidence degree theorem, a set of easily verifiable conditions are obtained for the existence of at least one strictly positive periodic solution for the system. In particular, our result generalizes some known criteria.

Keywords: predator-prey model, impulse, periodic solution, semi-ratio-dependent, patch, delay.

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1 Introduction

In recent years, with the application of the theory of differential equations in mathematical ecology, a lot of mathematical models have been proposed in the study of population dynamics [2–4, 6, 8–11, 15–23]. One of the famous models for the dynamics of populations is the so-called semi-ratio-dependent predator-prey system with functional response [4, 7, 9, 15, 17], for example

$$\begin{aligned} x' &= x(a - bx) - g(x)y, \\ y' &= y\left(d - f\frac{y}{x}\right), \end{aligned} \tag{1.1}$$

where x and y stand for the population of the prey and predator, respectively, $g(x)$ is the predator functional response to prey.

In equation (1.1), it has been assumed that the prey grows logistically with growth rate a and carrying capacity a/b in the absence of predation. The predator consumes the prey according to the functional response $g(x)$ and grows logistically with growth rate d and carrying

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capacity $x(t)/f$ proportional to the population size of prey. The parameter f is a measure of the food quality that the prey provides for conversion into predator birth.

The form of the predator equation in (1.1) was first proposed by Leslie [9]. The functional response $g(x)$ in (1.1) can be classified into five types, including the Leslie–Gower model, the Holling–Tanner model, the Holling type III model, the Ivlev's functional response and so on. For more detail see reference [18].

We note that any biological or environmental parameters are naturally subject to fluctuation in time. Cushing [2] pointed out that it is necessary and important to consider models with periodic ecological parameters or perturbations which may be naturally exposed (for example, those due to seasonal effects of weather, food supply, mating habits, hunting or harvesting seasons, etc.). Thus, the assumption of periodicity of the parameters is a way of incorporating the periodicity of the environment. On the other hand, dispersal between patches often occurs in natural ecological environments, and more realistic models should include the dispersal process [8,20,21].

We consider the following systems

$$\begin{aligned} x_1'(t) &= x_1(t)(r_1(t) - a_1(t)x_1(t)) - x_3(t)g(t, x_1(t-\tau_1)) + D_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) &= x_2(t)(r_2(t) - a_2(t)x_2(t)) + D_2(t)(x_1(t) - x_2(t)), \\ x_3'(t) &= x_3(t)\left(r_3(t) - a_3(t)\frac{x_3(t-\tau_2)}{x_1(t-\tau_2)}\right), \end{aligned} \tag{1.2}$$

with initial conditions

$$\begin{aligned} &x_i(\theta) = \phi_i(\theta), \qquad \theta \in [-\tau, 0], \\ &\phi_i(0) > 0, \qquad \phi_i \in C([-\tau, 0), R_+), \qquad i = 1,2,3, \end{aligned}$$

where $x_i(t)$ represents the prey population in the i-th patch ($i = 1,2$), and $x_3(t)$ represents the predator population. $D_i(t)$ denotes the dispersal rate of the prey in the i-th patch ($i = 1,2$). $\tau = \max\{\tau_1, \tau_2\}$.

However, there are numerous examples of evolutionary systems which at certain instants in time are subject to rapid changes. In the simulations of such processes it is frequently convenient and valid to neglect the durations of rapid changes and to assume that the changes can be represented by state jumps. Appropriate mathematical models for processes of the type described above are so-called systems with impulsive effects, see [1]. One note that the research on theory and applications of impulsive differential equations have been many nice works [3,6,10,12–14,22,23]. Because harvest of many a populations are not continuous, the harvest is an annual harvest pulse. To describe a system more accurately, we should consider to use the impulsive differential equation. If we consider the regularly harvest, then (1.2) is revised as the following form:

$$\begin{aligned} x_1'(t) &= x_1(t)(r_1(t) - a_1(t)x_1(t)) - x_3(t)g(t, x_1(t-\tau_1)) + D_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) &= x_2(t)(r_2(t) - a_2(t)x_2(t)) + D_2(t)(x_1(t) - x_2(t)), \\ x_3'(t) &= x_3(t)\left(r_3(t) - a_3(t)\frac{x_3(t-\tau_2)}{x_1(t-\tau_2)}\right), \\ \Delta x_i(t_k) &= b_{ik}x_i(t_k), \qquad i = 1,2,3,\ k = 1,2,\cdots, \end{aligned} \tag{1.3}$$

where $b_{ik}x_i(t_k)$ ($i = 1,2,3$) represents the population $x_i(t)$ at t_k regular harvest pulse. Through this paper, for system (1.3) the following conditions are assumed.

(C_1) $r_i(t)$, $a_i(t)$ $(i = 1, 2, 3)$, $D_1(t)$ and $D_2(t)$ are positive continuous ω-periodic functions; τ_1 and τ_2 are positive constants.

(C_2) $g(t, x)$ is a continuous ω-periodic function with respect to the first variable and is differentiable with respect to the second variable, and $g(t; 0) = 0$, $g(t, x) > 0$ for any $t \in \mathbb{R}$, $x > 0$.

(C_3) There exists a positive constant c_0 such that $g(t, x) \le c_0$ for any $t \in \mathbb{R}$, $x > 0$.

(C_4) $-1 < b_{ik} \le 0$, $i = 1, 2, 3$ for all $k \in N$ and there exists a positive integer q such that $t_{k+q} = t_k + \omega$, $b_{i(k+q)} = b_{ik}$, $i = 1, 2, 3$ and $t_k - \tau_1,\ t_k - \tau_2 \neq t_m$.

In the following, we shall use the notations.

$$\bar{f} = \frac{1}{\omega}\int_0^{\omega} f(s)\,ds, \qquad f^L = \min_{t\in[0,\omega]} f(t), \qquad f^M = \max_{t\in[0,\omega]} f(t).$$

Without loss of generality, we shall assume $t_k \neq 0, \omega$ and $[0, \omega] \cap \{t_k\} = \{t_1, t_2, \ldots, t_q\}$.

The existence of positive periodic solution of (1.2) is investigated in [4], and the following result is obtained.

Theorem 1.1. *In addition to* (C_1)–(C_3), *assume further that the following hold:*

(H_2) $r_i(t) - D_i(t) > 0,\ i = 1, 2,$

(H_3) $a_3^L(\bar{r}_1 - \overline{D}_1) - c_0\bar{r}_3 > 0.$

Then system (1.2) *has at least one positive ω-periodic solution with strictly positive components.*

The proof in [4] shows that Theorem 1.1 has room for improvement.

The organization of this paper is as follows. In the next section, we establish some simple criteria for the existence of a positive periodic solution of system (1.3). We also note that our results improve Theorem A as $b_{ik} \equiv 0$, because our results do not need the condition (H_2). Finally, we give some applications to show our results.

2 Existence of periodic solution

In this section, by using continuation theorem which was proposed in [5] by Gaines and Mawhin, we will establish the existence conditions of at least one positive periodic solution of system (1.3). To do so, we need to make some preparations.

Let X, Z be real Banach spaces, $L\colon \operatorname{Dom} L \subset X \to Z$ be a Fredholm mapping of index zero (index $L = \dim \operatorname{Ker} L - \operatorname{codim} \operatorname{Im} L$), and let $P\colon X \to X$, $Q\colon Z \to Z$ be continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. Denote by L_P the restriction of L to $\operatorname{Dom} L \cap \operatorname{Ker} P$, $K_P\colon \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ the inverse (to L_P), and $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$ an isomorphism of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$.

For convenience, we first introduce Mawhin's continuation theorem [5] as follows.

Lemma 2.1. *Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Assume*

(a) *$Lx \neq \lambda Nx$ for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$,*

(b) *for each* $x \in \operatorname{Ker} L \cap \partial\Omega$, $QNx \neq 0$,

(c) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then $Lx = Nx$ *has at least one solution in* $\bar{\Omega} \cap \operatorname{Dom} L$.

To prove the main conclusion by means of the continuation theorem, we need to introduce some functional spaces.

Let $PC(\mathbb{R}, \mathbb{R}^3) = \{x \colon \mathbb{R} \to \mathbb{R}^3 \mid x \text{ is continuous at } t \neq t_k,\ x(t_k^+),\ x(t_k^-) \text{ exist and } x(t_k^-) = x(t_k),\ k = 1, 2, \ldots,\}$, let $X = \{(u_1(t), u_2(t), u_3(t))^T \in PC(\mathbb{R}, \mathbb{R}^3) : u_i(t + \omega) = u_i(t),\ i = 1, 2, 3\}$ with the norm

$$\|(u_1(t), u_2(t), u_3(t))\|^T = \sum_{i=1}^{3} \sup_{t \in [0,\omega]} |u_i(t)|,$$

and

$$Y = X \times \mathbb{R}^{3q} \quad \text{with the norm} \quad \|u\|_Y = \|x\| + \|y\|, \quad \text{for} \quad u \in Y,\ x \in X,\ y \in \mathbb{R}^{3q},$$

where $|\cdot|$ denotes the Euclidean norm. Then X and Y are Banach spaces.

Theorem 2.2. *In addition to* (C_1)–(C_4), *assume further that the following hold:*

(C_5) $\bar{r}_3\omega + \sum_{k=1}^{q} \ln(1 + b_{3k}) > 0$,

(C_6) $a_3^L(\overline{r_1 - D_1})\omega + a_3^L \sum_{i=1}^{q} \ln(1 + b_{1k}) > c_0\bar{r}_3\omega$.

Then system (1.3) *has at least one positive* ω*-periodic solution.*

Proof. Let

$$u_1(t) = \ln[x_1(t)], \qquad u_2(t) = \ln[x_2(t)], \qquad u_3(t) = \ln[x_3(t)], \tag{2.1}$$

then system (1.3) can be translated to

$$\begin{aligned} u_1'(t) &= r_1(t) - D_1(t) - a_1(t)e^{u_1(t)} - g(t, e^{u_1(t-\tau_1)})e^{u_3(t)-u_1(t)} + D_1(t)e^{u_2(t)-u_1(t)}, \\ u_2'(t) &= r_2(t) - D_2(t) - a_2(t)e^{u_2(t)} + D_2(t)e^{u_1(t)-u_2(t)}, \\ u_3'(t) &= r_3(t) - a_3(t)e^{u_3(t-\tau_2)-u_1(t-\tau_2)}, \\ \Delta u_i(t_k) &= \ln(1 + b_{ik}), \qquad i = 1, 2, 3,\ k = 1, 2, \ldots \end{aligned} \tag{2.2}$$

It is easy to see that if system (2.2) has one ω-periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t))^T$, then $(x_1^*(t), x_2^*(t), y^*(t))^T = (\exp[u_1^*(t)], \exp[u_2^*(t)], \exp[u_3^*(t)])^T$ is a positive ω-periodic solution of (1.3). Therefore, to complete the proof, we need only to prove that (2.2) has one ω-periodic solution.

Let $L \colon \operatorname{Dom} L \subset X \to Y, u \to (u', \Delta u(t_1), \ldots, \Delta u(t_q))$,

$$Nu = \left(\begin{bmatrix} u_1'(t) = r_1(t) - D_1(t) - a_1(t)e^{u_1(t)} - g(t, e^{u_1(t-\tau_1)})e^{u_3(t)-u_1(t)} + D_1(t)e^{u_2(t)-u_1(t)} \\ u_2'(t) = r_2(t) - D_2(t) - a_2(t)e^{u_2(t)} + D_2(t)e^{u_1(t)-u_2(t)}, \\ u_3'(t) = r_3(t) - a_3(t)e^{u_3(t-\tau_2)-u_1(t-\tau_2)} \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} \ln(1 + b_{11}) \\ \ln(1 + b_{21}) \\ \ln(1 + b_{31}) \end{bmatrix}, \begin{bmatrix} \ln(1 + b_{12}) \\ \ln(1 + b_{22}) \\ \ln(1 + b_{32}) \end{bmatrix}, \cdots, \begin{bmatrix} \ln(1 + b_{1q}) \\ \ln(1 + b_{2q}) \\ \ln(1 + b_{3q}) \end{bmatrix} \right).$$

Evidently

$$\operatorname{Ker} L = \{u : u(t) = c \in R^3,\ t \in [0,\omega]\},$$
$$\operatorname{Im} L = \left\{ z = (f, a_1, \dots, a_q) \in Y : \int_0^\omega f(s)\,ds + \sum_{k=1}^q a_k = 0 \right\},$$

and

$$\dim \operatorname{Ker} L = 3 = \operatorname{codim} \operatorname{Im} L.$$

So $\operatorname{Im} L$ is closed in Y, L is a Fredholm mapping of index zero. Define

$$Px = \frac{1}{\omega} \int_0^\omega x(t)dt,$$
$$Qz = Q(f, a_1, a_2, \dots, a_q) = \left(\frac{1}{\omega} \left[\int_0^\omega f(s)\,ds + \sum_{k=1}^q a_k \right], 0, \dots, 0 \right).$$

It is easy to show that P and Q are continuous projectors satisfying

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q).$$

Furthermore, through an easy computation, we can find that the inverse $K_P\colon \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ has the form

$$K_P(z) = \int_0^t f(s)\,ds + \sum_{t_k < t} a_k - \frac{1}{\omega} \int_0^\omega \int_0^t f(s)\,ds\,dt - \sum_{k=1}^q a_k.$$

Thus

$$QNu = \left(\begin{pmatrix} \frac{1}{\omega} \int_0^\omega \Big[r_1(t) - D_1(t) - a_1(t) e^{u_1(t)} - g(t, e^{u_1(t-\tau_1)}) e^{u_3(t) - u_1(t)} \\ \qquad + D_1(t) e^{u_2(t) - u_1(t)} \Big]\, dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + b_{1k}), \\ \frac{1}{\omega} \int_0^\omega \Big[r_2(t) - D_2(t) - a_2(t) e^{u_2(t)} + D_2(t) e^{u_1(t) - u_2(t)} \Big]\, dt \\ \qquad + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + b_{2k}), \\ \frac{1}{\omega} \int_0^\omega \Big[r_3(t) - a_3(t) e^{u_3(t-\tau_2) - u_1(t-\tau_2)} \Big]\, dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + b_{3k}), \end{pmatrix}, 0, \dots, 0 \right),$$

and

$$K_P(I - Q)Nu$$
$$= \begin{bmatrix} \int_0^t \Big[r_1(s) - D_1(s) - a_1(s) e^{u_1(s)} - g(t, e^{u_1(s-\tau_1)}) e^{u_3(s) - u_1(s)} \\ \qquad + D_1(s) e^{u_2(s) - u_1(s)} \Big]\, ds + \sum_{t > t_k} \ln(1 + b_{1k}) \\ \int_0^t \Big[r_2(s) - D_2(s) - a_2(s) e^{u_2(s)} + D_2(s) e^{u_1(s) - u_2(s)} \Big]\, ds + \sum_{t > t_k} \ln(1 + b_{2k}) \\ \int_0^t \Big[r_3(s) - a_3(s) e^{u_3(s-\tau_2) - u_1(s-\tau_2)} \Big]\, ds + \sum_{t > t_k} \ln(1 + b_{3k}) \end{bmatrix}$$

$$
-\begin{bmatrix} \frac{1}{\omega}\int_0^{\omega}\int_0^t \Big[r_1(s)-D_1(s)-a_1(s)e^{u_1(s)}-g(t,e^{u_1(s-\tau_1)})e^{u_3(s)-u_1(s)} \\ \qquad +D_1(s)e^{u_2(s)-u_1(s)}\Big]\,ds+\sum_{k=1}^{q}\ln(1+b_{1k}) \\ \frac{1}{\omega}\int_0^{\omega}\int_0^t \Big[r_2(s)-D_2(s)-a_2(s)e^{u_2(s)}+D_2(s)e^{u_1(s)-u_2(s)}+\Big]\,ds+\sum_{k=1}^{q}\ln(1+b_{2k}) \\ \frac{1}{\omega}\int_0^{\omega}\int_0^t \Big[r_3(s)-a_3(s)e^{u_3(s-\tau_2)-u_1(s-\tau_2)}\Big]\,ds+\sum_{k=1}^{q}\ln(1+b_{3k}) \end{bmatrix}
-\begin{bmatrix} \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^t \Big[r_1(s)-D_1(s)-a_1(s)e^{u_1(s)}-g(t,e^{u_1(s-\tau_1)})e^{u_3(s)-u_1(s)} \\ \qquad +D_1(s)e^{u_2(s)-u_1(s)}\Big]\,ds+\sum_{k=1}^{q}\ln(1+b_{1k}) \\ \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^t \Big[r_2(s)-D_2(s)-a_2(s)e^{u_2(s)}+D_2(s)e^{u_1(s)-u_2(s)}+\Big]\,ds+\sum_{k=1}^{q}\ln(1+b_{2k}) \\ \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^t \Big[r_3(s)-a_3(s)e^{u_3(s-\tau_2)-u_1(s-\tau_2)}\Big]\,ds+\sum_{k=1}^{q}\ln(1+b_{3k}) \end{bmatrix}.
$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Using Lemma 2.4 in [1], it is not difficult to show that $QN(\bar{\Omega})$, $K_p(I-Q)N(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Hence N is L-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset X$.

Now we reach the position to search for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to equation $Lu = \lambda Nu$, $\lambda \in (0,1)$, we have

$$
\begin{aligned}
u_1'(t) &= \lambda\left[r_1(t)-D_1(t)-a_1(t)e^{u_1(t)}-g(t,e^{u_1(t-\tau_1)})e^{u_3(t)-u_1(t)}+D_1(t)e^{u_2(t)-u_1(t)}\right],\\
u_2'(t) &= \lambda\left[r_2(t)-D_2(t)-a_2(t)e^{u_2(t)}+D_2(t)e^{u_1(t)-u_2(t)}\right],\\
u_3'(t) &= \lambda\left[r_3(t)-a_3(t)e^{u_3(t-\tau_2)-u_1(t-\tau_2)}\right],\\
\Delta u_i(t_k) &= \lambda\ln(1+b_{ik}),\ i=1,2,3,\ k=1,2,\ldots
\end{aligned} \tag{2.3}
$$

Since $u_i(t)(i=1,2,3)$ are ω-periodic functions, we need only to prove the result in the interval $[0,\omega]$. Integrating (2.3) over the interval $[0,\,\omega]$ leads to

$$
\begin{aligned}
&\int_0^{\omega} a_1(t)e^{u_1(t)}\,dt+\int_0^{\omega} g(t,e^{u_1(t-\tau_1)})e^{u_3(t)-u_1(t)}\,dt\\
&\quad=\int_0^{\omega}(r_1(t)-D_1(t))\,dt+\int_0^{\omega}D_1(t)e^{u_2(t)-u_1(t)}\,dt+\sum_{k=1}^{q}\ln(1+b_{1k}),
\end{aligned} \tag{2.4}
$$

$$
\int_0^{\omega} a_2(t)e^{u_2(t)}\,dt=\int_0^{\omega}(r_2(t)-D_2(t))\,dt+\int_0^{\omega}D_2(t)e^{u_1(t)-u_2(t)}\,dt+\sum_{k=1}^{q}\ln(1+b_{2k}), \tag{2.5}
$$

and

$$
\int_0^{\omega} a_3(t)e^{u_3(t-\tau_2)-u_1(t-\tau_2)}\,dt=\int_0^{\omega}r_3(t)\,dt+\sum_{k=1}^{q}\ln(1+b_{3k}). \tag{2.6}
$$

Noting that

$$
\int_0^{\omega} e^{u_3(t-\tau_2)-u_1(t-\tau_2)}\,dt=\int_0^{\omega}e^{u_3(t)-u_1(t)}\,dt,
$$

and $-1 < b_{3k} \le 0$, we derive from (2.6) that

$$a_3^L \int_0^\omega e^{u_3(t)-u_1(t)}\,dt = a_3^L \int_0^\omega e^{u_3(t-\tau_2)-u_1(t-\tau_2)}\,dt \le \bar{r}_3\omega,$$

which implies

$$\int_0^\omega e^{u_3(t)-u_1(t)}\,dt \le \frac{\bar{r}_3\omega}{a_3^L}.$$

This together with the first equation of (2.3), (2.4), (C_2) and (C_3), yields

$$\begin{aligned}
\int_0^\omega |u_1'(t)|\,dt &< \int_0^\omega \Big(r_1(t) + D_1(t) + a_1(t)e^{u_1(t)} + g(t, e^{u_1(t-\tau_1)})e^{u_3(t)-u_1(t)} \\
&\qquad + D_1(t)e^{u_2(t)-u_1(t)}\Big)\,dt \\
&= 2\int_0^\omega a_1(t)e^{u_1(t)}\,dt + 2\int_0^\omega g(t, e^{u_1(t-\tau_1)})e^{u_3(t)-u_1(t)}\,dt \\
&\quad + 2\int_0^\omega D_1(t)dt - \sum_{k=1}^{q} \ln(1+b_{1k}) \\
&\le 2\int_0^\omega a_1(t)e^{u_1(t)}\,dt + 2c_0\int_0^\omega e^{u_3(t)-u_1(t)}\,dt + 2\bar{D}_1\omega - \sum_{k=1}^{q} \ln(1+b_{1k}) \\
&\le 2\int_0^\omega a_1(t)e^{u_1(t)}\,dt + \frac{2c_0\bar{r}_3\omega}{a_3^L} + 2\bar{D}_1\omega - \sum_{k=1}^{q} \ln(1+b_{1k}).
\end{aligned} \tag{2.7}$$

From (2.3), (2.5) and (2.6), we also have

$$\begin{aligned}
\int_0^\omega |u_2'(t)|\,dt &< \int_0^\omega \Big(r_2(t) + D_2(t) + a_2(t)e^{u_2(t)} + D_2(t)e^{u_1(t)-u_2(t)}\Big)\,dt \\
&= 2\int_0^\omega a_2(t)e^{u_2(t)}\,dt + 2\int_0^\omega D_2(t)\,dt - \sum_{k=1}^{q} \ln(1+b_{2k}),
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
\int_0^\omega |u_3'(t)|\,dt &< \int_0^\omega \Big(r_3(t) + a_3(t)e^{u_3(t-\tau_2)-u_1(t-\tau_2)}\Big)\,dt \\
&= 2\bar{r}_3\omega + \sum_{k=1}^{q} \ln(1+b_{3k}) \le 2\bar{r}_3\omega.
\end{aligned} \tag{2.9}$$

Multiplying the first equation of (2.3) by $e^{u_1(t)}$ and integrating over $[0,\omega]$, we obtain

$$\begin{aligned}
&-\sum_{k=1}^{p} b_{1k}e^{u_1(t_k)} + \int_0^\omega a_1(t)e^{2u_1(t)}\,dt \\
&\quad = \int_0^\omega (r_1(t) - D_1(t))e^{u_1(t)}\,dt + \int_0^\omega D_1(t)e^{u_2(t)}\,dt - \int_0^\omega g(t, e^{u_1(t-\tau_1)}e^{u_3(t)})\,dt \\
&\quad < \int_0^\omega (r_1(t) - D_1(t))e^{u_1(t)}\,dt + \int_0^\omega D_1(t)e^{u_2(t)}\,dt
\end{aligned}$$

Since $-1 < b_{1k} \le 0$, so we have

$$\int_0^\omega a_1(t)e^{2u_1(t)}\,dt \le (r_1 - D_1)^M \int_0^\omega e^{u_1(t)}\,dt + D_1^M \int_0^\omega e^{u_2(t)}\,dt,$$

which yields

$$a_1^L \int_0^\omega e^{2u_1(t)}\,dt \le (r_1 - D_1)^M \int_0^\omega e^{u_1(t)}\,dt + D_1^M \int_0^\omega e^{u_2(t)}\,dt. \tag{2.10}$$

Similarly, multiplying the second equation in (2.3) by $e^{u_2(t)}$ and integrating over $[0,\omega]$ gives

$$-\sum_{k=1}^{p} b_{2k} e^{u_1(t_k)} + \int_0^\omega a_2(t) e^{2u_2(t)}\,dt = \int_0^\omega (r_2(t) - D_2(t)) e^{u_2(t)}\,dt + \int_0^\omega D_2(t) e^{u_1(t)}\,dt,$$

which implies

$$a_2^L \int_0^\omega e^{2u_2(t)}\,dt < (r_2 - D_2)^M \int_0^\omega e^{u_2(t)}\,dt + D_2^M \int_0^\omega e^{u_1(t)}\,dt. \tag{2.11}$$

By using the inequalities

$$\left(\int_0^\omega e^{u_i(t)}\,dt\right)^2 \le \omega \int_0^\omega e^{2u_i(t)}\,dt, \qquad i = 1, 2,$$

it follows from (2.10) and (2.11) that

$$a_1^L \left(\int_0^\omega e^{u_1(t)}\,dt\right)^2 < \omega (r_1 - D_1)^M \int_0^\omega e^{u_1(t)}\,dt + D_1^M \omega \int_0^\omega e^{u_2(t)}\,dt, \tag{2.12}$$

$$a_2^L \left(\int_0^\omega e^{u_2(t)}\,dt\right)^2 < \omega (r_2 - D_2)^M \int_0^\omega e^{u_2(t)}\,dt + D_2^M \omega \int_0^\omega e^{u_1(t)}\,dt. \tag{2.13}$$

If $\int_0^\omega e^{u_2(t)}\,dt \le \int_0^\omega e^{u_1(t)}\,dt$, then we derive from (2.12) that

$$a_1^L \left(\int_0^\omega e^{u_1(t)}\,dt\right)^2 < \omega (r_1 - D_1)^M \int_0^\omega e^{u_1(t)}\,dt + D_1^M \omega \int_0^\omega e^{u_1(t)}\,dt,$$

which implies

$$\int_0^\omega e^{u_2(t)}\,dt \le \int_0^\omega e^{u_1(t)}\,dt < \frac{\omega (r_1 - D_1)^M + \omega D_1^M}{a_1^L}. \tag{2.14}$$

If $\int_0^\omega e^{u_1(t)}\,dt \le \int_0^\omega e^{u_2(t)}\,dt$, then we can conclude

$$\int_0^\omega e^{u_1(t)}\,dt \le \int_0^\omega e^{u_2(t)}\,dt < \frac{\omega (r_2 - D_2)^M + \omega D_2^M}{a_2^L}. \tag{2.15}$$

Set

$$A = \max\left\{ \frac{(r_1 - D_1)^M + D_1^M}{a_1^L}, \frac{(r_2 - D_2)^M + D_2^M}{a_2^L} \right\}. \tag{2.16}$$

Then it follows from (2.14)–(2.16) that

$$\int_0^\omega e^{u_i(t)}\,dt < \omega A, \qquad i = 1, 2. \tag{2.17}$$

This, together with (2.7) and (2.8), yields

$$\int_0^\omega |u_1'(t)|\,dt \le 2\omega\left(a_1^M A + \frac{c_0\overline{r}_3}{a_3^L}\right) + 2\omega\overline{D}_1 - \sum_{k=1}^{q}\ln(1+b_{1k}) =: c_1,$$
$$\int_0^\omega |u_2'(t)|\,dt \le 2\omega a_2^M A + 2\omega\overline{D}_2 - \sum_{k=1}^{q}\ln(1+b_{2k}) =: c_2. \tag{2.18}$$

Since $u(t) \in X$, there exist $\xi_i, \eta_i \in [0,\omega]$ $(i = 1,2,3)$ such that

$$u_i(\xi_i) = \min_{t\in[0,\omega]} u_i(t), \qquad u_i(\eta_i) = \max_{t\in[0,\omega]} u_i(t), \qquad i = 1,2,3. \tag{2.19}$$

From (2.17) and (2.19), we see that

$$u_i(\xi_i) < \ln A, \qquad i = 1,2. \tag{2.20}$$

Thus, from (2.18) and (2.20) we have

$$u_1(t) = \begin{cases} u_1(\xi_1) + \int_{\xi_1}^t u_1'(s)\,ds + \sum_{\xi_1<t_k<t}\ln(1+b_{1k}), & t\in(\xi_1,\omega] \\ u_1(\xi_1) + \int_{\xi_1}^t u_1'(s)\,ds - \sum_{t\le t_k\le \xi_1^-}\ln(1+b_{1k}), & t\in[0,\xi_1] \end{cases}$$
$$\le u_1(\xi_1) + \int_0^\omega |u_1'(t)|\,dt - \sum_{k=1}^{q}\ln(1+b_{1k})$$
$$< \ln A + c_1 - \sum_{k=1}^{q}\ln(1+b_{1k}), \tag{2.21}$$

$$u_2(t) \le u_2(\xi_2) + \int_0^\omega |u_2'(t)|\,dt - \sum_{k=1}^{q}\ln(1+b_{2k})$$
$$< \ln A + c_2 - \sum_{k=1}^{q}\ln(1+b_{2k}). \tag{2.22}$$

It follows from (2.4) that

$$(\overline{r_1 - D_1})\omega < \int_0^\omega a_1(t)e^{u_1(t)}\,dt + \int_0^\omega g(t, e^{u_1(t-\tau_1)})e^{u_3(t)-u_1(t)}dt - \sum_{k=1}^{q}\ln(1+b_{1k})$$
$$\le \int_0^\omega a_1(t)e^{u_1(t)}\,dt + c_0\int_0^\omega e^{u_3(t)-u_1(t)}\,dt - \sum_{k=1}^{q}\ln(1+b_{1k})$$
$$\le \int_0^\omega a_1(t)e^{u_1(t)}\,dt + \frac{c_0\overline{r}_3\omega}{a_3^L} - \sum_{k=1}^{q}\ln(1+b_{1k})$$

This, together with (2.19), deduces

$$e^{u_1(\eta_1)}\overline{a}_1\omega \ge \int_0^\omega a_1(t)e^{u_1(t)}\,dt \ge (\overline{r_1 - D_1})\omega - \frac{c_0\overline{r}_3\omega}{a_3^L} + \sum_{k=1}^{q}\ln(1+b_{1k}),$$

which implies

$$u_1(\eta_1) \ge \ln\left(\frac{(\overline{r}_1 - \overline{D}_1)\omega - (c_0\overline{r}_3/a_3^L)\omega + \sum_{k=1}^{q}\ln(1+b_{1k})}{\overline{a}_1\omega}\right) =: d_1.$$

This, together with (2.18), leads to

$$\begin{aligned} u_1(t) &\ge u_1(\eta_1) - \int_0^\omega |u_1'(t)|\,dt + \sum_{k=1}^{q} \ln(1+b_{1k}) \\ &> d_1 - c_1 + \sum_{k=1}^{q} \ln(1+b_{1k}). \end{aligned} \tag{2.23}$$

Let

$$R_1 = \max\left\{ |\ln A| + c_1 - \sum_{k=1}^{q} \ln(1+b_{1k}),\ |d_1| + c_1 - \sum_{k=1}^{q} \ln(1+b_{1k}) \right\},$$

it follows from (2.21) and (2.23) that

$$\max_{t\in[0,\omega]} |u_1(t)| < R_1. \tag{2.24}$$

From (2.5) we have

$$\begin{aligned} \bar{a}_2 e^{u_2(\eta_2)} &\ge \overline{(r_2 - D_2)} + \bar{D}_2 e^{u_1(\xi_1) - u_2(\eta_2)} + \sum_{k=1}^{q} \ln(1+b_{2k})/\omega \\ &> \overline{(r_2 - D_2)} + \bar{D}_2 e^{u_1(\xi_1)} \cdot e^{-u_2(\eta_2)} + \sum_{k=1}^{q} \ln(1+b_{2k})/\omega \\ &\ge \overline{(r_2 - D_2)} + \bar{D}_2 e^{-R_1} \cdot e^{-u_2(\eta_2)} + \sum_{k=1}^{q} \ln(1+b_{2k})/\omega, \end{aligned}$$

which implies

$$e^{u_2(\eta_2)} \ge \frac{d + \sqrt{d^2 + 4\bar{a}_2 \bar{D}_2 e^{-R_1}}}{2\bar{a}_2},$$

where $d = \overline{(r_2 - D_2)} + \sum_{k=1}^{q} \ln(1+b_{2k})/\omega$, so

$$u_2(\eta_2) \ge \ln \frac{d + \sqrt{d^2 + 4\bar{a}_2 \bar{D}_2 e^{-R_1}}}{2\bar{a}_2} =: d_2. \tag{2.25}$$

It follows from (2.18) and (2.25) leads to

$$\begin{aligned} u_2(t) &\ge u_2(\eta_2) - \int_0^\omega |u_2'(t)|\,dt + \sum_{k=1}^{q} \ln(1+b_{2k}) \\ &\ge d_2 - c_2 + \sum_{k=1}^{q} \ln(1+b_{2k}). \end{aligned} \tag{2.26}$$

This, together with (2.22), leads to

$$\max_{t\in[0,\omega]} |u_2(t)| < \max\left\{ |\ln A| + c_2 - \sum_{k=1}^{q} \ln(1+b_{2k}), |d_2| + c_2 - \sum_{k=1}^{q} \ln(1+b_{2k}) \right\} =: R_2.$$

From (2.6) and (2.19), we have

$$e^{u_3(\eta_3) - u_1(\xi_1)} \bar{a}_3 \omega \ge \int_0^\omega a_3(t) e^{u_3(t-\tau_2) - u_1(t-\tau_2)}\,dt = \bar{r}_3 \omega + \sum_{k=1}^{q} \ln(1+b_{3k}),$$

$$e^{u_3(\xi_3)-u_1(\eta_1)}\bar{a}_3\omega \le \int_0^\omega a_3(t)e^{u_3(t-\tau_2)-u_1(t-\tau_2)}\,dt = \bar{r}_3\omega + \sum_{k=1}^{q}\ln(1+b_{3k}) \le \bar{r}_3\omega,$$

which imply

$$u_3(\eta_3) \ge u_1(\xi_1) + \ln\left(\frac{\bar{r}_3\omega + \sum_{k=1}^{q}\ln(1+b_{3k})}{\bar{a}_3\omega}\right),$$

and

$$u_3(\xi_3) \le u_1(\eta_1) + \ln\left(\frac{\bar{r}_3}{\bar{a}_3}\right),$$

These together with (2.9) and (2.24), yield that

$$\begin{aligned} u_3(t) &\le u_3(\xi_3) + \int_0^\omega |u_3'(t)|\,dt - \sum_{k=1}^{q}\ln(1+b_{3k}) \\ &< u_1(\eta_1) + \ln\left(\frac{\bar{r}_3}{\bar{a}_3}\right) + 2\bar{r}_3\omega - \sum_{k=1}^{q}\ln(1+b_{3k}) \\ &< R_1 + \ln\left(\frac{\bar{r}_3}{\bar{a}_3}\right) + 2\bar{r}_3\omega - \sum_{k=1}^{q}\ln(1+b_{3k}) \end{aligned} \tag{2.27}$$

$$\begin{aligned} u_3(t) &\ge u_3(\eta_3) - \int_0^\omega |u_3'(t)|\,dt + \sum_{k=1}^{q}\ln(1+b_{3k}) \\ &\ge u_1(\xi_1) + \ln\left(\frac{\bar{r}_3\omega + \sum_{k=1}^{q}\ln(1+b_{3k})}{\bar{a}_3\omega}\right) - 2\bar{r}_3\omega + \sum_{k=1}^{q}\ln(1+b_{3k}) \\ &\ge -R_1 + \ln\left(\frac{\bar{r}_3\omega + \sum_{k=1}^{q}\ln(1+b_{3k})}{\bar{a}_3\omega}\right) - 2\bar{r}_3\omega + \sum_{k=1}^{q}\ln(1+b_{3k}). \end{aligned} \tag{2.28}$$

This, together with (2.27), leads to

$$\max_{t\in[0,\omega]} |u_3(t)| < R_3,$$

here

$$R_3 = \max\left\{\left|\ln\frac{\bar{r}_3}{\bar{a}_3}\right|, \left|\frac{\bar{r}_3\omega + \sum_{k=1}^{q}\ln(1+b_{3k})}{\bar{a}_3\omega}\ln\right|\right\} + R_1 + 2\bar{r}_3\omega - \sum_{k=1}^{q}\ln(1+b_{3k}).$$

Clearly, R_1, R_2 and R_3 are independent of λ. Similarly to the proof of Theorem 2.1 of [4], we can find a sufficiently large $M > 0$, denote the set

$$\Omega = \left\{u(t) = (u_1(t), u_2(t), u_3(t))^T \in X : \|u\| < M,\ u(t_k^+) \in \Omega,\ k = 1, 2, \ldots, q\right\},$$

it follows that for each $u \in \operatorname{Ker} L \cap \partial\Omega$, $QNu \ne 0$ and

$$\deg\{JQNu, \Omega \cap \operatorname{Ker} L, 0\} = -1 \ne 0.$$

By now we have proved that Ω verifies all the requirements in Lemma 2.1. Hence (2.2) has at least one ω-periodic solution. Accordingly, system (1.3) has at least one positive ω-periodic solution. The proof is complete. □

Remark 2.3. If $b_{ik} \equiv 0$, $(i = 1, 2, 3)$, $k = 1, 2\ldots$, then (1.3) is translated to (1.2). In this case, the condition (C_6) is the same as (H_3) of Theorem 1.1, but we see that (H_2) is not needed here. Hence our result improves and generalizes the corresponding result of [4].

3 Applications

In this section, we will list some applications of our above results.

Example 3.1. Consider the following delayed Holling–Tanner predator–prey system with diffusion and impulse

$$\begin{aligned} x_1'(t) &= x_1(t)(r_1(t) - a_1(t)x_1(t)) - \frac{c(t)x_3(t)x_1(t-\tau_1)}{m(t)+x_1(t-\tau_1)} + D_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) &= x_2(t)(r_2(t) - a_2(t)x_2(t)) + D_2(t)(x_1(t) - x_2(t)), \\ x_3'(t) &= x_3(t)\left(r_3(t) - a_3(t)\frac{x_3(t-\tau_2)}{x_1(t-\tau_2)}\right), \\ \Delta x_i(t_k) &= b_{ik}x_i(t_k), \qquad i = 1,2,3,\ k = 1,2,\dots, \end{aligned} \tag{3.1}$$

where $D_1(t)$, $D_2(t)$, $r_i(t)$, $a_i(t)$ $(i = 1,2,3)$, $c(t)$ and $m(t)$ are positive continuous ω-periodic functions; τ_1 and τ_2 are positive constants, b_{ik} $(i = 1,2,3,\ k = 1,2,\dots,)$ satisfy the condition (C_4). The system (3.1) without impulse has been considered in [17].

From Theorem 2.2 one obtains the following.

Theorem 3.2. *Suppose that* $\bar{r}_3\omega + \sum_{k=1}^{q} \ln(1 + b_{3k}) > 0$ *and*

$$a_3^L(\overline{r_1 - D_1})\omega + a_3^L \sum_{i=1}^{q} \ln(1 + b_{1k}) > c^M \bar{r}_3\omega,$$

hold, then (3.1) *has at least one* ω*-periodic solution with strictly positive components.*

Example 3.3. Consider the following delayed semi-ratio-dependent predator–prey diffusion system with Ivlev functional response and impulse:

$$\begin{aligned} x_1'(t) &= x_1(t)(r_1(t) - a_1(t)x_1(t)) - c(t)x_3(t)\big(1 - e^{-m(t)x_1(t-\tau_1)}\big) + D_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) &= x_2(t)(r_2(t) - a_2(t)x_2(t)) + D_2(t)(x_1(t) - x_2(t)), \\ x_3'(t) &= x_3(t)\left(r_3(t) - a_3(t)\frac{x_3(t-\tau_2)}{x_1(t-\tau_2)}\right), \\ \Delta x_i(t_k) &= b_{ik}x_i(t_k), \qquad i = 1,2,3,\ k = 1,2,\dots, \end{aligned} \tag{3.2}$$

where all functions are defined as above. The system (3.2) without impulse has been considered in [7].

From Theorem 2.2 one obatins the following.

Theorem 3.4. *Suppose that* $\bar{r}_3\omega + \sum_{k=1}^{q} \ln(1 + b_{3k}) > 0$ *and*

$$a_3^L(\overline{r_1 - D_1})\omega + a_3^L \sum_{i=1}^{q} \ln(1 + b_{1k}) > c^M \bar{r}_3\omega,$$

hold, then (3.2) *has at least one* ω*-periodic solution with strictly positive components.*

Example 3.5. Consider the following delayed semi-ratio-dependent predator–prey diffusion system with Monod–Haldane functional response and impulse

$$\begin{aligned} x_1'(t) &= x_1(t)(r_1(t) - a_1(t)x_1(t)) - \frac{c(t)x_3(t)x_1(t-\tau_1)}{m^2(t) + x_1^2(t-\tau_1)} + D_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) &= x_2(t)(r_2(t) - a_2(t)x_2(t)) + D_2(t)(x_1(t) - x_2(t)), \\ x_3'(t) &= x_3(t)\left(r_3(t) - a_3(t)\frac{x_3(t-\tau_2)}{x_1(t-\tau_2)}\right), \\ \Delta x_i(t_k) &= b_{ik}x_i(t_k), \qquad i = 1,2,3,\ k = 1,2,\dots, \end{aligned} \tag{3.3}$$

where all functions are defined as above. The system (3.3) without impulse has been considered in [15].

From Theorem 2.2 we get the following.

Theorem 3.6. *Suppose that* $\bar{r}_3\omega + \sum_{k=1}^{q} \ln(1 + b_{3k}) > 0$ *and*

$$2a_3^L(\overline{r_1 - D_1})\omega + 2a_3^L \sum_{i=1}^{q} \ln(1 + b_{1k}) > \left(\frac{c}{m}\right)^M \bar{r}_3\omega,$$

hold, then (3.3) *has at least one* ω*-periodic solution with strictly positive components.*

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