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Strongly nonlinear problem of infinite order with L^1 data

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Abstract

In this paper, we prove the existence of solutions for the strongly nonlinear equation of the type

$$Au + g(x, u) = f$$

where A is an elliptic operator of infinite order from a functional space of Sobolev type to its dual. $g(x, s)$ is a lower order term satisfying essentially a sign condition on s and the second term f belongs to $L^1(\Omega)$.

Keywords: Strongly nonlinear problem, infinite order, L^1 data, monotonicity condition, sign condition.
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1 Introduction

In a recent paper, Benkirane, Chrif and El Manouni [5] studied a class of anisotropic problems involving operators of finite and infinite higher order in the variational case. They proved the existence of solutions in generalized Sobolev spaces, also called anisotropic Sobolev spaces. The goal of this paper is to study a strongly nonlinear elliptic equation in which the operator of infinite order

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha (A_\alpha(x, D^\gamma u)),\ |\gamma| \le |\alpha|$$

is involved. Such operators includes as a special case Leray–Lions types in the usual sense. Especially, we consider the strongly nonlinear problem associated to the following differential equations

$$Au + g(x, u) = f, \tag{1.1}$$

for $x \in \Omega$, where Ω is a bounded domain in $I\!R^N$ and A is an operator of infinite order defined as above. The functions A_α are assumed to satisfy some growth and coerciveness conditions without supposing a monotonicity condition. So that, we prove the existence results and generalize the isotropic case for the problem (1.1). The nonlinear term g has to fulfil only the sign condition $g(x, s)s \geq 0$, but we do not assume any growth conditions with respect to $|u|$. As regards the second member, we suppose that $f \in L^1(\Omega)$. If A is a Leray–Lions operator, let us recall that several studies have been devoted to the investigation of related problems with L^1 data and a lot of papers have appeared in the classical Sobolev spaces under the additional monotonicity condition (cf. [7, 12, 14, 15]). Let us point out that in the L^1- case, a recent work can be found in [11] where the authors have studied some anisotropic problems of finite order of equations of (1.1) type. In this context of Leray–Lions operators, in the variational case (i.e., where f belongs to the dual space), the problem (1.1) has been extensively studied by many authors, we cite in the case of classical Sobolev spaces the works of Webb [17], Brezis and Browder [8]...etc, and the work of Benkirane and Gossez [3] in Sobolev-Orlicz spaces.

2 Preliminaries

Let Ω be a bounded domain in $I\!R^N$. Further $a_\alpha \geq 0, p_\alpha > 1$ are real numbers for all multi-indices α, and $\|\cdot\|_{p_\alpha}$ is the usual norm in the Lebesgue space $L^{p_\alpha}(\Omega)$. For a positive integer m, we define the following vector of real numbers:

$$\vec{p} = \{p_\alpha, |\alpha| \leq m\},$$

and denote $\underline{\mathrm{p}} = \min\{p_\alpha, |\alpha| \leq m\}$.

Now, let us consider the generalized functional Sobolev space

$$W^{m,\vec{p}}(\Omega) = \{u \in L^{p_0}(\Omega), D^\alpha u \in L^{p_\alpha}(\Omega), 0 < |\alpha| \leq m\}$$

equipped with the norm

$$\|u\| = \sum_{|\alpha|=0}^{m} \|D^\alpha u\|_{p_\alpha}. \tag{2.1}$$

Here $D^\alpha = \dfrac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \dots (\partial x_N)^{\alpha_N}}$. We define the space $W_0^{m,\vec{p}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{m,\vec{p}}(\Omega)$ with respect to the norm (2.1). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{m,\vec{p}}(\Omega)$. Both of $W^{m,\vec{p}}(\Omega)$ and $W_0^{m,\vec{p}}(\Omega)$ are separable if $1 \leq p_\alpha < \infty$, reflexives if $1 < p_\alpha < \infty$ for all $|\alpha| \leq m$ (the proof of this is an adaptation from Adams [1]). $W^{-m,\vec{p}'}(\Omega)$ designs its dual where $\vec{p}'$ is the conjugate of $\vec{p}$, i.e., $p'_\alpha = \frac{p_\alpha}{p_\alpha - 1}$ for all $|\alpha| \leq m$.

The Sobolev space of infinite order is the functional space defined by

$$W^\infty(a_\alpha, p_\alpha)(\Omega) = \Big\{u \in C^\infty(\Omega) : \, \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_{p_\alpha}^{p_\alpha} \, < \infty \Big\}.$$

We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in Ω with continuous derivatives of arbitrary order.

Since we shall deal with the Dirichlet problem, we shall use the functional space $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ defined by

$$W_0^\infty(a_\alpha, p_\alpha)(\Omega) = \Big\{ u \in C_0^\infty(\Omega) : \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_{p_\alpha}^{p_\alpha} < \infty \Big\}.$$

We say that $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ is a nontrivial space if it contains at least a nonzero function. The dual space of $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ is defined as follows:

$$W^{-\infty}(a_\alpha, p'_\alpha)(\Omega) = \Big\{ h : h = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha h_\alpha,\ \rho^{'}(h) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|h_\alpha\|_{p'_\alpha}^{p'_\alpha} < \infty \Big\},$$

where $h_\alpha \in L^{p'_\alpha}(\Omega)$ and p'_α is the conjugate of p_α, i.e., $p'_\alpha = \frac{p_\alpha}{p_\alpha - 1}$ (for more details about these spaces, see [13]).

We need the anisotropic Sobolev embeddings result.

lemma 2.1 *Let Ω be a bounded open subset of $I\!R^N$.*

If $m \cdot \underline{p} < N$, then $W_0^{m,\vec{p}}(\Omega) \subset L^q(\Omega)$ for all $q \in [\underline{p}, p^*[$ with $\frac{1}{p^*} = \frac{1}{\underline{p}} - \frac{m}{N}$.

If $m \cdot \underline{p} = N$, then $W_0^{m,\vec{p}}(\Omega) \subset L^q(\Omega)$ for all $q \in [\underline{p}, +\infty[$.

If $m \cdot \underline{p} > N$, then $W_0^{m,\vec{p}}(\Omega) \subset L^\infty(\Omega) \cap C^k(\overline{\Omega})$ where $k = E(m - \frac{N}{\underline{p}})$.

Moreover, the embeddings are compacts.

The proof follows immediately from the corresponding embedding theorems in the isotropic case by using the fact that $W^{m,\vec{p}}(\Omega) \subset W^{m,\underline{p}}(\Omega)$.

3 Main results

We denote by λ_α the number of multi-indices γ such that $|\gamma| \leq |\alpha|$. Let A be a nonlinear operator of infinite order from $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ into its dual $W^{-\infty}(a_\alpha, p'_\alpha)(\Omega)$ defined as

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha (A_\alpha(x, D^\gamma u)), \quad |\gamma| \leq |\alpha|$$

where $A_\alpha : \Omega \times I\!R^{\lambda_\alpha} \mapsto I\!R$ is a real function satisfying the following assumptions

(H_1) $A_\alpha(x, \xi_\gamma)$ is a Carathéodory function for all $\alpha, |\gamma| \leq |\alpha|$.

(H_2) For a.e. $x \in \Omega$, all $m \in I\!N^*$, all $\xi_\gamma, \eta_\alpha, |\gamma| \leq |\alpha|$ and some constant $c_0 > 0$, we assume that

$$\Big| \sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma)\eta_\alpha \Big| \leq c_0 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha - 1} |\eta_\alpha|,$$

where $a_\alpha \geq 0, p_\alpha > 1$ are reals numbers for all multi-indices α, and for all bounded sequence $(p_\alpha)_\alpha$.

(H$_3$) There exist constants $c_1 > 0, c_2 \geq 0$ such that for all $m \in I\!N^*$, for all $\xi_\gamma, \xi_\alpha; |\gamma| \leq |\alpha|$, we have

$$\sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma)\xi_\alpha \geq c_1 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha} - c_2.$$

(H$_4$) The space $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ is nontrivial.

As regards to the function g, assume that $g : \Omega \times I\!R \mapsto I\!R$ is a Carathéodory function, that is, measurable with respect to x in Ω for every s in $I\!R$, and continuous with respect to s in $I\!R$ for almost every x in Ω, satisfying the following conditions

(G$_1$) For all $\delta > 0$,

$$\sup_{|u|\leq\delta} |g(x, u)| \leq h_\delta(x) \in L^1(\Omega).$$

(G$_2$) The "sign condition" $g(x, u)u \geq 0$, for a.e. $x \in \Omega$ and all $u \in I\!R$.

Finally, we assume that

$$f \in L^1(\Omega), \tag{3.1}$$

and we shall prove the existence result without assuming any monotonicity condition.

Example 3.1 *Let us consider the operator*

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha (a_\alpha |D^\alpha u|^{p_\alpha - 2} D^\alpha u),$$

where $a_\alpha \geq 0, p_\alpha > 1$ are real numbers such that the space $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ is nontrivial. We can easily show that the conditions (H$_1$), (H$_2$) and (H$_3$) are satisfied.
As second example, let us consider the differential operator of infinite order

$$[cosD]u(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} D^{2n} u(x) \quad x \in I,$$

where I is a bounded open interval in $I\!R$. The corresponding Sobolev space of infinite order is $W_0^\infty(\frac{1}{(2n)!}, 2)(I)$ which is nontrivial (see [13]) and the conditions above are obviously verified.
Some more explicit examples of such an operator of infinite order and of a function g that satisfies the conditions (H$_1$), (H$_2$) and (H$_3$) can be found in [5].

Theorem 3.1 *Assume that the assumptions* $(\mathrm{H}_1) - (\mathrm{H}_4)$, $(\mathrm{G}_1), (\mathrm{G}_2)$ *and (3.1) hold, then there exists at least one solution $u \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ of (1.1) in the following sense*

$$\begin{cases} g(x, u) \in L^1(\Omega), \ g(x, u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x, u)v\, dx = \langle f, v\rangle, \quad for\ all\ v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega). \end{cases}$$

Proof. In order to prove our result, we proceed by steps.

Step (1) The approximate problem.

Consider $\varphi \in C_0^\infty(I\!R^N)$ such that $0 < \varphi(x) < 1$ and $\varphi(x) = 1$ for x close to 0. Let f_n be a sequence of regular functions defined by

$$f_n(x) = \varphi\Big(\frac{x}{n}\Big) T_n f(x),$$

where T_n is the usual truncation given by

$$T_n\xi = \begin{cases} \xi & \text{if } |\xi| < n \\ \frac{n\xi}{|\xi|} & \text{if } |\xi| \geq n. \end{cases}$$

It is clear that $|f_n| \leq n$ for a.e. $x \in \Omega$. Thus, it follows that $f_n \in L^\infty(\Omega)$. Using Lebesgue's dominated convergence theorem, since $f_n \longrightarrow f$ a.e. $x \in \Omega$ and $|f_n| \leq |f| \in L^1(\Omega)$, we conclude that f_n strongly converges to f in $L^1(\Omega)$. Define the operator of order $2n+2$ by

$$A_{2n+2}(u) = \sum_{|\alpha|=n+1} (-1)^{n+1} c_\alpha D^{2\alpha} u + \sum_{|\alpha|=0}^{n} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \quad |\gamma| \leq |\alpha|,$$

and consider the following approximate problem with boundary Dirichlet conditions

$$A_{2n+2}(u_n) + g(x, u_n) = f_n. \tag{P_n}$$

Note that c_α are constants small enough such that they fulfil the conditions of the following lemma introduced in [13].

lemma 3.1 (cf. [13]) . *For all nontrivial space $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$, there exists a nontrivial space $W_0^\infty(c_\alpha, 2)(\Omega)$ such that $W_0^\infty(a_\alpha, p_\alpha)(\Omega) \subset W_0^\infty(c_\alpha, 2)(\Omega)$.*

As in [5], the operator A_{2n+2} is clearly monotone since the term of higher order of derivation is linear and satisfies the monotonicity condition. Moreover from assumptions (H_1), (H_2) and (H_3), we deduce that A_{2n+2} satisfies the growth, the coerciveness and the monotonicity conditions. Thanks to [[5], Theorem 3.2], there exists at least one solution $u_n \in W_0^{n+1,\vec{p}}(\Omega)$ of the problem (P_n) in the following sense

$$\begin{cases} g(x, u_n) \in L^1(\Omega), g(x, u_n)u_n \in L^1(\Omega) \\ \langle A_{2n+2}(u_n), v\rangle + \displaystyle\int_\Omega g(x, u_n) v \, dx = \langle f_n, v\rangle \quad v \in W_0^{n+1,\vec{p}}(\Omega) \end{cases}$$

Step (2) A priori estimates.

By choosing $v = u_n$ as test function in (P_n), we have

$$\langle A_{2n+2}(u_n), u_n\rangle + \int_\Omega g(x, u_n) u_n \, dx = \langle f_n, u_n\rangle,$$

which implies that

$$\sum_{|\alpha|=n+1} c_\alpha \int_\Omega |D^\alpha u_n|^2 \, dx + \sum_{|\alpha|=0}^{n} \int_\Omega A_\alpha(x, D^\gamma u_n) D^\alpha u_n \, dx \leq \int_\Omega f_n u_n \, dx.$$

Hence, in view of (H_3), (G_2), the Hölder inequality and by using the fact that $|f_n| \le |f|$, we get the estimates

$$\sum_{|\alpha|=n+1} c_\alpha \|D^\alpha u_n\|_2^2 + \sum_{|\alpha|=0}^{n} a_\alpha \|D^\alpha u_n\|_{p_\alpha}^{p_\alpha} \le K \tag{3.2}$$

$$\int_\Omega g(x, u_n) u_n \, dx \le K \tag{3.3}$$

for some constant $K = K(f) > 0$. The estimate (3.2) is equivalent to

$$\sum_{|\alpha|=0}^{n+1} a_\alpha \|D^\alpha u_n\|_{p_\alpha}^{p_\alpha} \le K \tag{3.4}$$

with $a_\alpha = c_\alpha$ and $p_\alpha = 2$ for $|\alpha| = n+1$. Consequently,

$$\|u_n\|_{W_0^{n+1,\vec{p}}(\Omega)} \le K. \tag{3.5}$$

By using the same approach as in [5] and via a diagonalization process, there exists a subsequence still, denoted by u_n, which converges uniformly to an element $u \in C_0^\infty(\Omega)$, also for all derivatives there holds $D^\alpha u_n \to D^\alpha u$ (for more details we refer to [13]). So that,

$$u \in W_0^\infty(a_\alpha, p_\alpha)(\Omega),$$

since $(p_\alpha)_\alpha$ is a bounded sequence.

Step (3) Convergence of the approximate problem (P_n).

There exists a solution u_n of problem (P_n), $n = 1, 2, \ldots$. Then by passing to the limit, we have

$$\lim_{n \to +\infty} \langle A_{2n+2}(u_n), v \rangle + \lim_{n \to +\infty} \int_\Omega g(x, u_n) v \, dx = \lim_{n \to +\infty} \langle f_n, v \rangle,$$

for $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$. Since $f_n \longrightarrow f$ strongly in $L^1(\Omega)$, it is clear that

$$\lim_{n \to +\infty} \langle f_n, v \rangle = \langle f, v \rangle$$

for all $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$.

Now, we shall prove that

$$\lim_{n \to +\infty} \langle A_{2n+2}(u_n), v \rangle = \langle Au, v \rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega).$$

In fact, let n_0 be a fix number sufficiently large ($n > n_0$) and let $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$. Set $\langle A(u) - A_{2n+2}(u_n), v \rangle = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= \sum_{|\alpha|=0}^{n_0} \langle A_\alpha(x, D^\gamma u) - A_\alpha(x, D^\gamma u_n), D^\alpha v \rangle \\ I_2 &= \sum_{|\alpha|=n_0+1}^{\infty} \langle A_\alpha(x, D^\gamma u), D^\alpha v \rangle \\ I_3 &= -\sum_{|\alpha|=n_0+1}^{n+1} \langle A_\alpha(x, D^\gamma u_n), D^\alpha v \rangle \end{aligned}$$

or in another form,

$$I_3 = -\sum_{|\alpha|=n_0+1}^{n} \langle A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle - \sum_{|\alpha|=n+1} c_\alpha \langle D^\alpha u_n, D^\alpha v\rangle$$

with $A_\alpha(x, \xi_\gamma) = c_\alpha \xi_\alpha$ and $c_\alpha \geq 0$ for $|\alpha| = n+1$.

We will pass to the limit as $n \longrightarrow +\infty$ to prove that I_1, I_2 and I_3 tend to 0. Starting by I_1, we have $I_1 \to 0$ since $A_\alpha(x, \xi_\gamma)$ is of Carathéodory type. The term I_2 is the remainder of a convergent series, hence $I_2 \to 0$. For what concerns I_3, for all $\varepsilon > 0$, there holds $k(\varepsilon) > 0$ (see [9]) such that

$$\begin{aligned}
\Big|\sum_{|\alpha|=n_0+1}^{n+1} \langle A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle\Big| &\leq \sum_{|\alpha|=n_0+1}^{n+1} \int_\Omega |A_\alpha(x, D^\gamma u_n) D^\alpha v|\, dx \\
&\leq c_0 \sum_{|\alpha|=n_0+1}^{n+1} a_\alpha \int_\Omega |D^\alpha u_n|^{p_\alpha - 1} |D^\alpha v|\, dx \\
&\leq c_0 \sum_{|\alpha|=n_0+1}^{n+1} a_\alpha \|D^\alpha u_n\|_{p_\alpha}^{p_\alpha - 1} \|D^\alpha v\|_{p_\alpha} \\
&\leq \varepsilon c_0 \sum_{|\alpha|=n_0+1}^{n+1} a_\alpha \|D^\alpha u_n\|_{p_\alpha}^{p_\alpha} \\
&+ k(\varepsilon) c_0 \sum_{|\alpha|=n_0+1}^{n+1} a_\alpha \|D^\alpha v\|_{p_\alpha}^{p_\alpha} \\
&\leq \varepsilon c_0 K + k(\varepsilon) c_0 \sum_{|\alpha|=n_0+1}^{\infty} a_\alpha \|D^\alpha v\|_{p_\alpha}^{p_\alpha},
\end{aligned}$$

where K is the constant given in the estimate (3.2). Since the sequence (p_α) is bounded, this implies that $\displaystyle\sum_{|\alpha|=n_0+1}^{\infty} a_\alpha \|D^\alpha v\|_{p_\alpha}^{p_\alpha}$ is the remainder of a convergent series, therefore $I_3 \to 0$ holds. Consequently, we have

$$\langle A_{2n+2}(u_n), v\rangle \to \langle A(u), v\rangle \quad \text{as } n \to +\infty$$

for all $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$.

It remains to show, for our purposes, that

$$\lim_{n\to+\infty} \int_\Omega g(x, u_n) v\, dx = \int_\Omega g(x, u) v\, dx.$$

for $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$. Indeed, we have $u_n \to u$ uniformly in Ω, hence $g(x, u_n) \to g(x, u)$ for a.e. $x \in \Omega$. In view of (3.3), we deduce by Fatou's lemma as $n \to \infty$ that

$$\int_\Omega g(x, u) u\, dx \leq \lim_{n\to+\infty} \int_\Omega g(x, u_n) u_n\, dx \leq K,$$

this implies $g(x, u)u \in L^1(\Omega)$.
On the other hand, for all $\delta > 0$ we have

$$\begin{aligned}
|g(x, u_n)| &\leq \sup_{|t|\leq\delta} |g(x, t)| + \delta^{-1} |g(x, u_n).u_n| \\
&\leq h_\delta(x) + \delta^{-1} |g(x, u_n) u_n|
\end{aligned}$$

On the other hand, we have

$$\int_E |g(x,u_n)|\,dx \le \int_E h_\delta(x)\,dx + \delta^{-1}K,$$

for some measurable subset E of Ω and for some $\varepsilon > 0$. Here, K is the constant of (3.3) which is independent of n. For $|E|$ sufficiently small and $\delta = \frac{2K}{\varepsilon}$, we obtain $\int_E |g(x,u_n)|\,dx < \varepsilon$. Then, we get by using Vitali's theorem

$$g(x,u_n) \to g(x,u) \quad \text{in } L^1(\Omega).$$

Hence it follows that $g(x,u) \in L^1(\Omega)$.

By passing to the limit, we obtain

$$\langle Au, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha,p_\alpha)(\Omega).$$

Consequently,

$$\begin{cases} g(x,u) \in L^1(\Omega), g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \displaystyle\int_\Omega g(x,u)v\,dx = \langle f, v\rangle \quad \text{for all } v \in W_0^\infty(a_\alpha,p_\alpha)(\Omega) \end{cases}$$

This completes the proof. □

Remarks.

1. Note that the existence result is given without assuming a monotonicity condition on the operator. Also, no restrictions to the parameters p_α are imposed.
2. In general, some methods of approximation need at least the segment property of the domain, here we assume no regularity condition of Ω.

4 Application

Consider the following class of strongly nonlinear Dirichlet problem

$$B(u) + u|u|^{r+1}h(x) = f \quad \text{in } \Omega, \tag{4.1}$$

where $r > 0$ is a real number, $h \in L^1(\Omega)$ with $h(x) \ge 0$ a.e $x \in \Omega$ and the operator B is defined as

$$B(u) = (-\sqrt{I+\Delta})u.$$

Our technique here consists to exploit certain result in the setting of functional spaces of infinite order. Thus as in [13], we can write the operator B as follows

$$B(u) = (-\sqrt{I+\Delta})u = \sum_{k=0}^{\infty} a_k(-\Delta)^k u, \tag{4.2}$$

where $a_k > 0, k = 0, 1, ...$ are real numbers which guarantee the nontriviality of the corresponding functional space, (for more details see [13]). The characteristic function of our operator is defined by

$$\varphi(x) = \sum_{k}^{\infty} a_k \xi^{2k} \quad \text{with} \quad \xi^2 = \xi_1^2 + ... + \xi_N^2,$$

and the corresponding space of the present strongly nonlinear Dirichlet problem is

$$W_0^\infty(a_k, 2)(\Omega) = \{u \in C_0^\infty(\Omega) \ : \ \rho(u) = \sum_{k=0}^{\infty} a_k \, \|\nabla^k u\|_2^2 < \infty\}.$$

Definition 4.1 *A function $u \in W_0^\infty(a_k, 2)(\Omega)$ is a solution of the Dirichlet problem* (4.1)*, if for any $v \in W_0^\infty(a_k, 2)(\Omega)$ the equality*

$$\sum_{k=0}^{\infty} a_k \int_\Omega \nabla^k u \, \nabla^k v \, dx + \int_\Omega u|u|^{r+1} h(x) v \, dx = \langle f, v \rangle$$

is valid.

The explicit form of the operator B given in (4.2), shows simply that it satisfies the assumptions (H_1), (H_2) and (H_3). Now, taking into account that B is monotone and the term $u|u|^{r+1}h(x)$ fulfils the sign condition, then by the same argument as in the proof of Theorem 3.1 we prove the following existence result.

Theorem 4.1 *For all $f \in L^1(\Omega)$, there exists at least one nontrivial solution $u \in W_0^\infty(a_k, 2)(\Omega)$ such that*

$$\langle Bu, v \rangle + \int_\Omega u|u|^{r+1} h(x) v \, dx = \langle f, v \rangle \qquad \textit{for all } v \in W_0^\infty(a_\alpha, 2)(\Omega).$$

Remark 4.1 *In his book, Dubinskii [13] considered the Sobolev spaces of infinite order corresponding to boundary value problem for differential equations of infinite order and obtained the solvability of these problems in the case where the coefficients of the equation grow polynomially with respect to the derivatives.*
Dyk Van extends the results of Dubinskii to include the case of operators with rapidly or slowly increasing coefficients (see Dyk Van[10]). In their works, Tran Duk Van et al.[16] introduced Sobolev-Orlicz spaces of infinite order and investigated their principal properties. They also established the existence and uniqueness of solutions of some Dirichlet problems for nonlinear differential equations of infinite order.
In particular, let Ω a bounded domain in $I\!R^N$, $N \geq 1$, with boundary $\partial\Omega$. Consider the Dirichlet problem defined by

$$(Pb) \begin{cases} Au(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x, ..., D^\alpha u) = f(x) & x \in \Omega, \\ D^\omega u(x) = 0, \ x \in \partial\Omega, \ |\omega| = 0, 1, ... \end{cases}$$

Here $A_\alpha : \Omega \times I\!R^{\lambda_\alpha} \mapsto I\!R$ is a real function, λ_α denotes the number of multi-indices γ such that $|\gamma| \leq |\alpha|$, satisfying the following assumptions:

(H'$_1$) There exist an N-function ϕ_α, a function $a_\alpha \in L\{\overline{\phi}_\alpha, \Omega\}$, a continuous bounded function c^1_α, ($1 \le c^1_\alpha(|t|) \le const$) and a constant $b > 0$ such that

$$|A_\alpha(x,\xi)| \le a_\alpha(x) + b\overline{\phi}_\alpha^{-1}\overline{\phi}_\alpha(c^1_\alpha(|\xi_\alpha|)\xi_\alpha)$$

where

$$\sum_{|\alpha|=0}^{\infty} \|a_\alpha\|_{\phi_\alpha} < +\infty.$$

(H'$_2$) There exist functions $b_m \in L_1(\Omega)$, $g_\alpha \in E\{\phi_\alpha, \Omega\}$, a continuous bounded function c^2_α, ($c^2_\alpha(|t|) \ge c^1_\alpha(|t|)$) and a constant $d > 0$ such that

$$\sum_{|\alpha|=m} (A_\alpha(x,\xi) - g_\alpha(x))\xi_\alpha \ge d \sum_{|\alpha|=m} \phi_\alpha(c^2_\alpha(|\xi_\alpha|)\xi_\alpha) - b_m(x),$$

where

$$\sum_{|\alpha|=0}^{\infty} \|g_\alpha\|_{\phi_\alpha} < +\infty \quad \text{and} \quad \sum_{|\alpha|=0}^{\infty} \int_\Omega |b_m(x)|\,dx < +\infty.$$

(H'$_3$) The N-functions ϕ_α are such that the Sobolev-Orlicz space $LW^\infty_0(\phi_\alpha, \Omega)$ is nontrivial.

(H'$_4$) For all $\xi = (\xi_0, ..., \xi_\alpha)$ and $\xi' = (\xi'_0, ..., \xi'_\alpha)$ such that $\xi \ne \xi'$ we have the inequality

$$\sum_{|\alpha|=0}^{m} (A_\alpha(x,\xi) - A_\alpha(x,\xi'))(\xi_\alpha - \xi'_\alpha) \ge 0.$$

The corresponding functional setting is the Sobolev-Orlicz of infinite order given by

$$LW^\infty_0(\phi_\alpha, \Omega) = \{u \in C^\infty_0(\Omega) \,:\, \|u\|_\infty < +\infty\},$$

where

$$\|u\|_\infty = \inf\{k > 0 \,:\, \sum_{|\alpha|=0}^{\infty} \int_\Omega \phi_\alpha(\frac{D^\alpha u}{k})\,dx \le 1\}.$$

The dual space of $LW^\infty_0(\phi_\alpha, \Omega)$ is defined by

$$EW^{-\infty}(\phi_\alpha, \Omega) = \{f \,:\, h(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha f_\alpha(x)\},$$

where

$$f_\alpha \in E_{\phi_\alpha}(\Omega) \quad \text{for all multi-indice } \alpha,$$

and

$$\rho'(f) = \sum_{|\alpha|=0}^{\infty} \|f_\alpha\|_{\phi_\alpha} < +\infty.$$

The duality of the spaces $LW^\infty_0(\phi_\alpha, \Omega)$ and $EW^{-\infty}(\phi_\alpha, \Omega)$ is determined by the expression

$$\langle f, v\rangle = \sum_{|\alpha|=0}^{\infty} \int_\Omega f_\alpha(x) D^\alpha v(x)\,dx,$$

which is obviously correct. (For more details about the definition of N-function and Sobolev-Orlicz spaces of infinite order we refer to Tran Duk Van et al.[16]). When the data f belongs to the dual, under assumptions (H'_1)-(H'_4) the authors in Tran Duk Van et al.[16] proved the existence and uniqueness of the solution of the nonlinear problem (Pb).
In our case, with a suitable choice of the N-functions ϕ_α, we can deal with the boundary value problem (1.1) in the more general class of Sobolev-Orlicz spaces of infinite order using operators satisfying $(H'_1) - (H'_4)$.

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