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Multiple positive solutions for nonlinear third order general two-point boundary value problems

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Abstract

We consider the existence of positive solutions and multiple positive solutions for the third order nonlinear differential equation subject to the general two-point boundary conditions using different fixed point theorems.

1 Introduction

In this paper we consider the existence of positive solutions to the third order nonlinear differential equation,

$$y'''(t) + f(t,\ y(t)) = 0, \quad t \in [a, b], \tag{1}$$

subject to the general two-point boundary conditions

$$\begin{aligned} \alpha_{11}y(a) - \alpha_{12}y(b) &= 0 \\ \alpha_{21}y'(a) - \alpha_{22}y'(b) &= 0 \\ -\alpha_{31}y''(a) + \alpha_{32}y''(b) &= 0 \end{aligned} \tag{2}$$

where the coefficients α_{11}, α_{12}, α_{21}, α_{22}, α_{31}, α_{32} are positive real constants. The BVPs of this form arise in the modeling of nonlinear diffusions generated by nonlinear sources, in thermal ignition of gases, and in concentration in chemical or biological problems. In these applied settings, only positive solutions are meaningful.

There is much current attention focussed on existence of positive solutions to the boundary value problems for ordinary differential equations, as well as for the finite difference equations; see [5, 6, 8, 9] to name a few. The book by Agarwal, Wong and O'Regan [1] gives a good overview for much of the work

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which has been done and the methods used. Shuhong and Li [22] obtained the existence of single and multiple positive solutions to the nonlinear singular third-order two-point boundary value problem

$$\begin{aligned} y'''(t)+&\lambda a(t)f(y(t)) = 0,\ 0 < t < 1\\ y(0) =&y'(0) = y''(1) = 0 \end{aligned}$$

by using Krasnosel'skii fixed point theorem [16]. In [23], Sun and Wen considered the existence of multiple positive solutions to third order equation,

$$y'''(t) = a(t)f(y(t)),\ 0 < t < 1$$

under the boundary conditions

$$\alpha y'(0) - \beta y''(0) = 0, y(1) = y'(1) = 0.$$

We extend these results to general two-point boundary value problems in the interval $[a, b]$, where $b > a \geq 0$. We use the following notation for simplicity, $\gamma_i = \alpha_{i1} - \alpha_{i2},\quad i = 1, 2, 3$ and $\beta_i = a\alpha_{i1} - b\alpha_{i2},\quad i = 1, 2.$
We assume that throughout the paper:

(A1) $f : [a, b] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, where $\mathbb{R}^+$ is the set of nonnegative real numbers.

(A2) $\gamma_i > 0,\quad i = 1, 2, 3$

(A3) $\frac{\beta_2}{\gamma_2} - \frac{\alpha_{22}\gamma_3}{\alpha_{32}\gamma_2}(b - a) \leq a$ and $\frac{\beta_2}{\gamma_2} + \frac{\alpha_{21}\gamma_3}{\alpha_{31}\gamma_2}(b - a) \leq a$

(A4) $\frac{-\alpha_{11}}{2\gamma_1} + \frac{\alpha_{21}}{\gamma_2} - \frac{\alpha_{31}}{2\gamma_3} < 0, -\frac{\alpha_{12}}{2\gamma_1} + \frac{\alpha_{22}}{\gamma_2} - \frac{\alpha_{32}}{2\gamma_3} < 0.$

We define the nonnegative extended real numbers $f_0,\ \ f^0,\ \ f_\infty$ and f^∞ by

$$\begin{aligned} f_0 = \lim_{y\to 0^+} \min_{t\in[a,b]} \frac{f(t, y)}{y},\ f^0 = \lim_{y\to 0^+} \max_{t\in[a,b]} \frac{f(t, y)}{y},\\ f_\infty = \lim_{y\to \infty} \min_{t\in[a,b]} \frac{f(t, y)}{y},\ \text{ and } f^\infty = \lim_{y\to \infty} \max_{t\in[a,b]} \frac{f(t, y)}{y} \end{aligned}$$

and assume that they will exist. We note that $f^0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f^\infty = 0$ correspond to the sublinear case. By the positive solution of (1)-(2) we mean that $y(t)$ is positive on $[a, b]$ and satisfies the differential equation (1) and the boundary conditions (2).

This paper is organized as follows. In Section 2, as a fundamental importance, we estimate the bounds for Green's function corresponding to the BVP

(1)-(2). In Section 3, we present a lemma which is needed in our main result and establish a criterion for the existence of at least one positive solution for the BVP (1)-(2) by using Krasnosel'skii fixed point theorem [16]. In Section 4, some existence criteria for at least three positive solutions to the BVP (1)-(2) are established by using the well-known Leggett-Williams fixed point theorem [18]. And then, for arbitrary positive integer m, existence results for at least $2m-1$ positive solutions are obtained. Finally as an application, we give two examples to demonstrate our results.

2 Green's Function and bounds

In this section, we estimate the bounds of the Green's function for the homogeneous two-point BVP corresponding to (1)- (2).

The Green's function for the homogeneous problem $-y'''=0$, satisfying the boundary conditions (2) can be constructed after computation and is given by

$$G(t,s)=\begin{cases}\frac{-\alpha_{12}\gamma_2\gamma_3(b-s)^2-2\alpha_{22}\gamma_3(-\beta_1+t\gamma_1)(b-s)-\alpha_{32}(A-2t\gamma_1\beta_2+t^2\gamma_1\gamma_2)}{2\gamma_1\gamma_2\gamma_3} & \mathrm{a}\le \mathrm{t}\le \mathrm{s}\le \mathrm{b}\\ \frac{-\alpha_{11}\gamma_2\gamma_3(s-a)^2+2\alpha_{21}\gamma_3(-\beta_1+t\gamma_1)(s-a)-\alpha_{31}(A-2t\gamma_1\beta_2+t^2\gamma_1\gamma_2)}{2\gamma_1\gamma_2\gamma_3} & \mathrm{a}\le \mathrm{s}\le \mathrm{t}\le \mathrm{b}.\end{cases} \tag{3}$$

where $A=2\beta_1\beta_2-\gamma_2(a^2\alpha_{11}-b^2\alpha_{12})$. We now state two Lemmas to minimum and maximum values of Green's function.

Lemma 2.1 *For $t<s$, $G(t,s)$ attains minimum value at*

$$\begin{aligned} t=&\frac{\alpha_{22}^2\gamma_3\beta_1-\alpha_{12}\alpha_{32}\gamma_2\beta_2}{\alpha_{22}^2\gamma_1\gamma_3-\alpha_{12}\alpha_{32}\gamma_2^2}\\ s=&\frac{-\alpha_{22}\alpha_{32}\gamma_1\beta_2+b\alpha_{22}^2\gamma_1\gamma_3+\alpha_{22}\alpha_{32}\gamma_2\beta_1-b\alpha_{12}\alpha_{32}\gamma_2^2}{\alpha_{22}^2\gamma_1\gamma_3-\alpha_{12}\alpha_{32}\gamma_2^2}.\end{aligned}$$

And also, for $s<t$, $G(t,s)$ attains minimum value at

$$\begin{aligned} t=&\frac{\alpha_{11}\alpha_{31}\gamma_2\beta_2-\alpha_{21}^2\gamma_3\beta_1}{\alpha_{11}\alpha_{31}\gamma_2^2-\alpha_{21}^2\gamma_1\gamma_3}\\ s=&\frac{\alpha_{21}\alpha_{31}\gamma_1\beta_2-a\alpha_{21}^2\gamma_1\gamma_3-\alpha_{21}\alpha_{31}\gamma_2\beta_1+a\alpha_{11}\alpha_{31}\gamma_2^2}{\alpha_{11}\alpha_{31}\gamma_2^2-\alpha_{21}^2\gamma_1\gamma_3}.\end{aligned}$$

Lemma 2.2 *Assume that the condition (A4) holds, then $G(s,s)$ has a maximum value at*

$$s=\frac{b\alpha_{12}\gamma_2\gamma_3-b\alpha_{22}\gamma_1\gamma_3-\alpha_{22}\gamma_3\beta_1+\alpha_{32}\gamma_1\beta_2}{\alpha_{12}\gamma_2\gamma_3-2\alpha_{22}\gamma_1\gamma_3+\alpha_{32}\gamma_1\gamma_2}.$$

The above two Lemmas can be proved easily by considering minimum and maximum of function of two variables.

Theorem 2.3 *Let $G(t,s)$ be the Green's function for the homogeneous BVP corresponding to* (1)-(2), *then*

$$\gamma G(s,s) \leq G(t,s) \leq G(s,s), \quad for\ all\ (t,s) \in [a,b] \times [a,b] \tag{4}$$

where $0 < \gamma = \min\{m_1, m_2\} \leq 1$.

Proof: The Green's function $G(t,s)$ for the homogeneous problem of the BVP (1)-(2) is given in (3). Clearly

$$G(t,s) > 0 \ \text{ on } [a,b] \times [a,b]. \tag{5}$$

First we establish the right side inequality by assuming the conditions given by $(A2)$-$(A3)$. For $t < s$,

$$\begin{aligned} G(t,s) &\leq -\frac{\alpha_{12}}{2\gamma_1}(b-s)^2 - \frac{\alpha_{22}}{\gamma_2}\left(\frac{-\beta_1}{\gamma_1} + s\right)(b-s) - \frac{\alpha_{32}}{2\gamma_3}\left(\frac{A}{\gamma_1\gamma_2} - 2s\frac{\beta_2}{\gamma_2} + s^2\right) \\ &= G(s,s) \end{aligned}$$

and for $s < t$,

$$\begin{aligned} G(t,s) &\leq -\frac{\alpha_{11}}{2\gamma_1}(s-a)^2 + \frac{\alpha_{21}}{\gamma_2}\left(\frac{-\beta_1}{\gamma_1} + s\right)(s-a) - \frac{\alpha_{31}}{2\gamma_3}\left(\frac{A}{\gamma_1\gamma_2} - 2s\frac{\beta_2}{\gamma_2} + s^2\right) \\ &= G(s,s). \end{aligned}$$

Hence,

$$G(t,s) \leq G(s,s).$$

By assuming the conditions given by $(A2)$-$(A4)$, we establish the other inequality.

For $t < s$, from Lemma 2.1 and Lemma 2.2, we have

$$\frac{G(t,s)}{G(s,s)} \geq \frac{\min G(t,s)}{\max G(s,s)} = m_1$$

and for $s < t$, we have

$$\frac{G(t,s)}{G(s,s)} \geq \frac{\min G(t,s)}{\max G(s,s)} = m_2.$$

Therefore,

$$\gamma G(s,s) \leq G(t,s), \quad \text{for all} \quad (t,s) \in [a,b] \times [a,b],$$

where $0 < \gamma = \min\{m_1, m_2\} \leq 1$. □

3 Existence of Positive Solutions

In this section, first we prove a lemma which is needed in our main result and establish criteria for the existence of at least one positive solution of the BVP (1)-(2).

Let $y(t)$ be the solution of the BVP (1)-(2), and be given by

$$y(t) = \int_a^b G(t,s)f(s,y(s))ds, \quad \text{for all } t \in [a,b]. \tag{6}$$

Define

$$X = \left\{ \ y \mid y \in C[a,b] \ \right\},$$

with norm

$$\| \ y \ \| = \max_{t\in[a,b]} \mid y(t) \mid .$$

Then $(X, \| \ . \ \|)$ is a Banach space. Define a set κ by

$$\kappa = \left\{ u \in X : u(t) \geq 0 \ \text{ on } [a,b] \ \text{ and } \ \min_{t\in[a,b]} u(t) \geq \gamma \parallel u \parallel \right\}, \tag{7}$$

then it is easy to see that κ is a positive cone in X.
Define the operator $T : \kappa \to X$ by

$$(Ty)(t) = \int_a^b G(t,s)f(s,y(s))ds, \quad \text{for all } t \in [a,b]. \tag{8}$$

If $y \in \kappa$ is a fixed point of T, then y satisfies (6) and hence y is a positive solution of the BVP (1)-(2). We seek the fixed points of the operator T in the cone κ.

Lemma 3.1 *The operator T defined by* (8) *is a self map on κ.*

Proof: If $y \in \kappa$, then by (4)

$$\begin{aligned}(Ty)(t) &= \int_a^b G(t,s)f(s,y(s))ds \\ &\leq \int_a^b G(s,s)f(s,y(s))ds,\end{aligned}$$

then

$$\parallel Ty \parallel \leq \int_a^b G(s,s)f(s,y(s))ds, \quad t \in [a,b].$$

Moreover, if $y \in \kappa$,

$$
\begin{aligned}
(Ty)(t) &= \int_a^b G(t,s)f(s,y(s))ds \\
&\geq \int_a^b \gamma G(s,s)f(s,y(s))ds \\
&\geq \gamma \int_a^b \max_{t\in[a,b]} G(t,s)f(s,y(s))ds \\
&\geq \gamma \max_{t\in[a,b]} \int_a^b G(t,s)f(s,y(s))ds \\
&= \gamma \parallel Ty \parallel .
\end{aligned}
$$

Therefore,

$$\min_{t\in[a,b]} (Ty)(t) \geq \gamma \parallel Ty \parallel .$$

Also, from the positivity of $G(t,s)$, it clear that for $y \in \kappa$, that $(Ty)(t) \geq 0,\ \ a \leq t \leq b$, and so $Ty \in \kappa$; thus $T : \kappa \to \kappa$. Further arguments yields that T is completely continuous. $\square$ The

existence of at least one positive solution of (1)-(2) is based on an application of the following fixed point theorem [16].

Theorem 3.2 (***Krasnosel'skii***) *Let X be a Banach space, $K \subseteq X$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is completely continuous operator such that either*

(i) $\parallel Tu \parallel \leq \parallel u \parallel,\ u \in K \cap \partial\Omega_1$ *and* $\parallel Tu \parallel \geq \parallel u \parallel,\ u \in K \cap \partial\Omega_2,$ *or*

(ii) $\parallel Tu \parallel \geq \parallel u \parallel,\ u \in K \cap \partial\Omega_1$ *and* $\parallel Tu \parallel \leq \parallel u \parallel,\ u \in K \cap \partial\Omega_2$

holds. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 3.3 *Assume that conditions $(A1) - (A4)$ are satisfied. If $f^0 = 0$ and $f_\infty = \infty$, then the BVP (1)-(2) has at least one positive solution that lies in κ.*

Proof: Let T be the cone preserving, completely continuous operator defined as in (8). Since $f^0 = 0$, we may choose $H_1 > 0$ so that

$$\max_{t\in[a,b]} \frac{f(t,y)}{y} \leq \eta_1, \quad \text{for} \quad 0 < y \leq H_1,$$

where $\eta_1 > 0$ satisfies

$$\eta_1 \int_a^b G(s,s)ds \leq 1.$$

Thus, if $y \in \kappa$ and $\| y \| = H_1$, then we have

$$\begin{aligned}
(Ty)(t) &= \int_a^b G(t,s)f(s,y(s))ds \\
&\leq \int_a^b G(s,s)f(s,y(s))ds \\
&\leq \int_a^b G(s,s)\eta_1 y(s)ds \\
&\leq \eta_1 \int_a^b G(s,s) \| y \| ds \\
&\leq \| y \| .
\end{aligned}$$

Therefore,

$$\| Ty \| \leq \| y \| .$$

Now if we let

$$\Omega_1 = \{y \in X : \| y \| < H_1\},$$

then

$$\| Ty \| \leq \| y \|, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_1. \tag{9}$$

Further, since $f_\infty = \infty$, there exists $\overline{H}_2 > 0$ such that

$$\min_{t\in[a,b]} \frac{f(t,y)}{y} \geq \eta_2, \quad \text{for} \quad y \geq \overline{H}_2,$$

where $\eta_2 > 0$ is chosen so that

$$\eta_2\gamma^2 \int_a^b G(s,s)ds \geq 1.$$

Let

$$H_2 = \max\left\{2H_1, \frac{1}{\gamma}\overline{H}_2\right\},$$

and

$$\Omega_2 = \{y \in X : \| y \| < H_2\},$$

then $y \in \kappa$ and $\| y \| = H_2$ implies

$$\min_{t\in[a,b]} y(t) \geq \gamma \| y \| \geq \overline{H}_2,$$

and so

$$
\begin{aligned}
(Ty)(t) &= \int_a^b G(t,s)f(s,y(s))ds \\
&\geq \int_a^b \gamma G(s,s)f(s,y(s))ds \\
&\geq \gamma \int_a^b G(s,s)\eta_2 y(s)ds \\
&\geq \gamma^2\eta_2 \int_a^b G(s,s) \parallel y \parallel ds \\
&\geq \parallel y \parallel .
\end{aligned}
$$

Hence,

$$
\parallel Ty \parallel \geq \parallel y \parallel, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_2. \tag{10}
$$

Therefore, by part (i) of the Theorem 3.2 applied to (9) and (10), T has a fixed point $y(t) \in \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $H_1 \leq \parallel y \parallel \leq H_2$. This fixed point is a positive solution of the BVP (1)-(2). □

Theorem 3.4 *Assume that conditions* $(A1)-(A4)$ *are satisfied. If* $f_0 = \infty$ *and* $f^\infty = 0$*, then the BVP* (1)-(2) *has at least one positive solution that lies in* κ*.*

Proof: Let T be the cone preserving, completely continuous operator defined as in (8). Since $f_0 = \infty$, we choose $J_1 > 0$ such that

$$
\min_{t\in[a,b]} \frac{f(t,y)}{y} \geq \overline{\eta}_1, \quad \text{for} \quad 0 < y \leq J_1,
$$

where $\overline{\eta}_1\gamma^2 \int_a^b G(s,s)ds \geq 1$. Then for $y \in \kappa$ and $\parallel y \parallel = J_1$, we have

$$
\begin{aligned}
(Ty)(t) &= \int_a^b G(t,s)f(s,y(s))ds \\
&\geq \int_a^b \gamma G(s,s)f(s,y(s))ds \\
&\geq \gamma \int_a^b G(s,s)\overline{\eta}_1 y(s)ds \\
&\geq \gamma^2\overline{\eta}_1 \int_a^b G(s,s) \parallel y \parallel ds \\
&\geq \parallel y \parallel .
\end{aligned}
$$

Thus, we may let

$$\Omega_1 = \{y \in X : \parallel y \parallel < J_1\},$$

so that

$$\parallel Ty \parallel \geq \parallel y \parallel, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_1. \tag{11}$$

Now, since $f^{\infty} = 0$, there exists $\overline{J}_2 > 0$ so that

$$\max_{t\in[a,b]} \frac{f(t,y)}{y} \leq \overline{\eta}_2, \quad \text{for} \quad y \geq \overline{J}_2,$$

where $\overline{\eta}_2 > 0$ satisfies

$$\overline{\eta}_2 \int_a^b G(s,s)ds \leq 1.$$

It follows that

$$f(t,y) \leq \overline{\eta}_2 y, \ \text{for } y \geq \overline{J}_2.$$

We consider two cases:
Case(i) f is bounded. Suppose $L > 0$ is such that $f(t,y) \leq L$, for all $0 < y < \infty$. In this case, we may choose

$$J_2 = \max\left\{2J_1, L\int_a^b G(s,s)ds\right\},$$

so that $y \in \kappa$ with $\parallel y \parallel = J_2$, we have

$$\begin{aligned}
(Ty)(t) &= \int_a^b G(t,s)f(s,y(s))ds \\
&\leq \int_a^b G(s,s)f(s,y(s))ds \\
&\leq L\int_a^b G(s,s)ds \\
&\leq J_2 = \parallel y \parallel,
\end{aligned}$$

and therefore

$$\parallel Ty \parallel \leq \parallel y \parallel.$$

Case(ii) f is unbounded. Choose $J_2 > \max\{2J_1, \overline{J}_2\}$ be such that $f(t,y) \leq$

$f(t, J_2)$, for $0 < y \leq J_2$. Then for $y \in \kappa$ and $\| y \| = J_2$, we have

$$
\begin{aligned}
(Ty)(t) &= \int_a^b G(t,s) f(s, y(s)) ds \\
&\leq \int_a^b G(s,s) f(s, y(s)) ds \\
&\leq \int_a^b G(s,s) f(s, J_2) ds \\
&\leq \int_a^b G(s,s) \overline{\eta}_2 J_2 ds \\
&\leq \overline{\eta}_2 \int_a^b G(s,s) J_2 ds \\
&\leq J_2 = \| y \| .
\end{aligned}
$$

Therefore, in either case we put

$$\Omega_2 = \{ y \in X : \| y \| < J_2 \},$$

we have

$$\| Ty \| \leq \| y \|, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_2. \tag{12}$$

Therefore, by the part (ii) of Theorem 3.2 applied to (11) and (12), T has a fixed point $y(t) \in \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $J_1 \leq \| y \| \leq J_2$. This fixed point is a positive solution of the BVP (1)-(2). □

4 Existence of Multiple Positive Solutions

In this section, we establish the existence of at least three positive solutions to the BVP (1)-(2). And then, for an arbitrary positive integer m, existence of at least $2m-1$ positive solutions are obtained.

Let E be a real Banach space with cone P. A map $S : P \to [0, \infty)$ is said to be a nonnegative continuous concave functional on P, if S is continuous and

$$S(\lambda x + (1-\lambda) y) \geq \lambda S(x) + (1-\lambda) S(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$. Let α and β be two real numbers such that $0 < \alpha < \beta$ and S be a nonnegative continuous concave functional on P. We define the following convex sets

$$P_\alpha = \{ y \in P : \| y \| < \alpha \},$$

and

$$P(S,\alpha,\beta)=\{y\in P:\alpha\leq S(y),\parallel y\parallel\leq\beta\}.$$

We now state the famous Leggett-Williams fixed point theorem [18].

Theorem 4.1 *Let $T:\overline{P_{a_3}}\rightarrow\overline{P_{a_3}}$ be completely continuous and S be a non-negative continuous concave functional on P such that $S(y)\leq\parallel y\parallel$ for all $y\in\overline{P_{a_3}}$. Suppose that there exist $0<d<a_1<a_2\leq a_3$ such that the following conditions hold.*

(i) $\{y\in P(S,a_1,a_2):S(y)>a_1\}\neq\emptyset$ and $S(Ty)>a_1$ for all $y\in P(S,a_1,a_2)$;

(ii) $\parallel Ty\parallel<d$ for all $y\in\overline{P_d}$;

(iii) $S(Ty)>a_1$ for $y\in P(S,a_1,a_3)$ with $\parallel Ty\parallel>a_2$.

Then, T has at least three fixed points y_1,y_2,y_3 in $\overline{P_{a_3}}$ satisfying

$$\parallel y_1\parallel<d,a_1<S(y_2),\parallel y_3\parallel>d,S(y_3)<a_1.$$

For convenience, we let

$$D=\max_{t\in[a,b]}\int_a^b G(t,s)ds;\quad C=\min_{t\in[a,b]}\int_a^b G(t,s)ds.$$

Theorem 4.2 *Assume that the conditions $(A1)-(A4)$ are satisfied and also there exist real numbers d_0, d_1 and c with $0<d_0<d_1<\frac{d_1}{\gamma}<c$ such that*

$$f(t,y(t))<\frac{d_0}{D},\ for\ y\in[0,d_0], \tag{13}$$

$$f(t,y(t))>\frac{d_1}{C},\ for\ y\in[d_1,\frac{d_1}{\gamma}], \tag{14}$$

$$f(t,y(t))<\frac{c}{D},\ for\ y\in[0,c]. \tag{15}$$

Then the BVP (1)-(2) has at least three positive solutions.

Proof: Let the Banach space $E=C[a,b]$ be equipped with the norm

$$\parallel y\parallel=\max_{t\in[a,b]}\mid y(t)\mid.$$

We denote

$$P=\left\{y\in E:y(t)\geq 0,\ t\in[a,b]\right\}.$$

Then, it is obvious that P is a cone in E. For $y \in P$, we define

$$S(y) = \min_{t\in[a,b]} \mid y(t) \mid,$$

and

$$(Ty)(t) = \int_a^b G(t,s)f(s,y(s))ds, \quad t \in [a,b].$$

It is easy to check that S is a nonnegative continuous concave functional on P with $S(y) \leq\| y \|$ forall $y \in P$. Further the operator $T : P \to P$ is a completely continuous by an application of the Ascoli-Arzela theorem [12] and the fixed points of T are the solutions of the BVP (1)-(2).

First, we prove that if there exists a positive number r such that $f(t, y(t)) < \frac{r}{D}$ for $y \in [0, r]$, then $T : \overline{P_r} \to P_r$. Indeed, if $y \in \overline{P_r}$, then for $t \in [a, b]$,

$$\begin{aligned}
(Ty)(t) &= \int_a^b G(t,s)f(s,y(s))ds \\
&< \frac{r}{D}\int_a^b G(t,s)ds \\
&\leq \frac{r}{D}\max_{t\in[a,b]}\int_a^b G(t,s)ds = r.
\end{aligned}$$

Thus, $\| Ty \|< r$, that is, $Ty \in P_r$. Hence, we have shown that if (13) and (15) hold, then T maps $\overline{P_{d_0}}$ into P_{d_0} and $\overline{P_c}$ into P_c.

Next, we show that $\{y \in P(S, d_1, \frac{d_1}{\gamma}) : S(y) > d_1\} \neq \emptyset$ and $S(Ty) > d_1$ for all $y \in P(S, d_1, \frac{d_1}{\gamma})$. In fact, the constant function

$$\frac{d_1 + \frac{d_1}{\gamma}}{2} \in \left\{ y \in P(S, d_1, \frac{d_1}{\gamma}) : S(y) > d_1 \right\},$$

hence it is nonempty. Moreover, for $y \in P(S, d_1, \frac{d_1}{\gamma})$, we have

$$\frac{d_1}{\gamma} \geq\| y \|\geq y(t) \geq \min_{t\in[a,b]} y(t) = S(y) \geq d_1,$$

for all $t \in [a, b]$. Thus, in view of (14) we see that

$$\begin{aligned}
S(Ty) &= \min_{t\in[a,b]}\int_a^b G(t,s)f(s,y(s))ds \\
&> \frac{d_1}{C}\min_{t\in[a,b]}\int_a^b G(t,s)ds \\
&= d_1,
\end{aligned}$$

as required.

Finally, we show that if $y \in P(S, d_1, c)$ with $\| Ty \| > \frac{d_1}{\gamma}$, then $S(Ty) > d_1$. To see this, we suppose that $y \in P(S, d_1, c)$ and $\| Ty \| > \frac{d_1}{\gamma}$, then, by Theorem 2.3, we have

$$\begin{aligned} S(Ty) &= \min_{t \in [a,b]} \int_a^b G(t,s) f(s, y(s)) ds \\ &\geq \gamma \int_a^b G(s,s) f(s, y(s)) ds \\ &\geq \gamma \int_a^b G(t,s) f(s, y(s)) ds, \end{aligned}$$

for all $t \in [a, b]$. Thus

$$\begin{aligned} S(Ty) &\geq \gamma \max_{t \in [a,b]} \int_a^b G(t,s) f(s, y(s)) ds \\ &= \gamma \| Ty \| \\ &> \gamma \frac{d_1}{\gamma} \\ &= d_1. \end{aligned}$$

Hence the hypotheses of the Leggett Williams theorem 4.1 are satisfied, and therefore T has at least three fixed points, that is, the BVP (1)-(2) has at least three positive solutions u, v and w such that

$$\| u \| < d_0, \quad d_1 < \min_{t \in [a,b]} v(t), \quad \| w \| > d_0, \quad \min_{t \in [a,b]} w(t) < d_1.$$

$\square$

Theorem 4.3 *Let m be an arbitrary positive integer. Assume that there exist numbers $d_i (1 \leq i \leq m)$ and $a_j (1 \leq j \leq m-1)$ with $0 < d_1 < a_1 < \frac{a_1}{\gamma} < d_2 < a_2 < \frac{a_2}{\gamma} < \ldots < d_{m-1} < a_{m-1} < \frac{a_{m-1}}{\gamma} < d_m$ such that*

$$f(t, y(t)) < \frac{d_i}{D}, \ y \in [0, d_i], 1 \leq i \leq m \tag{16}$$

$$f(t, y(t)) > \frac{a_j}{C}, \ y \in [a_j, \frac{a_j}{\gamma}], 1 \leq j \leq m-1 \tag{17}$$

Then, the BVP (1)-(2) has at least $2m - 1$ positive solutions in $\overline{P_{d_m}}$.

Proof: We use induction on m. First, for $m = 1$, we know from (16) that $T : \overline{P_{d_1}} \to P_{d_1}$, then, it follows from Schauder fixed point theorem that the BVP (1)-(2) has at least one positive solution in $\overline{P_{d_1}}$.

Next, we assume that this conclusion holds for $m = k$. In order to prove that this conclusion holds for $m = k+1$, we suppose that there exist numbers $d_i(1 \leq i \leq k+1)$ and $a_j(1 \leq j \leq k)$ with $0 < d_1 < a_1 < \frac{a_1}{\gamma} < d_2 < a_2 < \frac{a_2}{\gamma} < \ldots < d_k < a_k < \frac{a_k}{\gamma} < d_{k+1}$ such that

$$f(t, y(t)) < \frac{d_i}{D},\ y \in [0, d_i], 1 \leq i \leq k+1, \tag{18}$$

$$f(t, y(t)) > \frac{a_j}{C},\ y \in [a_j, \frac{a_j}{\gamma}], 1 \leq j \leq k \tag{19}$$

By assumption, the BVP (1)-(2) has at least $2k-1$ positive solutions $y_i(i = 1, 2, ..., 2k-1)$ in $\overline{P_{d_k}}$. At the same time, it follows from Theorem 4.2, (18) and (19) that the BVP (1)-(2) has at least three positive solutions u, v and w in $\overline{P_{d_{k+1}}}$ such that

$$\| u \| < d_k,\ a_k < \min_{t\in[a,b]} v(t),\quad \| w \| > d_k,\ \min_{t\in[a,b]} w(t) < a_k.$$

Obviously, v and w are different from $y_i(i = 1, 2, ..., 2k-1)$. Therefore, the BVP(1)-(2) has at least $2k+1$ positive solutions in $\overline{P_{d_{k+1}}}$, which shows that this conclusion also holds for $m = k+1$. $\square$

5 Examples

Now, we give some examples to illustrate the main results.

Example 1

Consider the following boundary value problem

$$\begin{aligned} y''' + y^2(1 + 9e^{-8y}) &= 0 \\ 6y(0) - \frac{11}{2}y(1) &= 0 \\ 3y'(0) - 2y'(1) &= 0 \\ -y''(0) + 3y''(1) &= 0. \end{aligned} \tag{20}$$

It is easy to see that all the conditions of Theorem 3.3 hold. It follows from Theorem 3.3, the BVP (20) has at least one positive solution.

Example 2

Consider the following boundary value problem

$$
\begin{aligned}
y''' + \frac{100(y+1)}{16(y^2+1)} &= 0 \\
5y(0) - \frac{9}{2}y(1) &= 0 \\
3y'(0) - 2y'(1) &= 0 \\
-y''(0) + 2y''(1) &= 0.
\end{aligned} \tag{21}
$$

A simple calculation shows that $\gamma = 0.1757$. If we choose $d_0 = \frac{1}{2}$, $d_1 = 19$, then the conditions (13)-(15) are satisfied. Therefore, it follows from Theorem 4.2 that the BVP (21) has at least three positive solutions.

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