Electronic Journal of Qualitative Theory of Differential Equations
2014, No. **55**, 1–9; http://www.math.u-szeged.hu/ejqtde/

Oscillatory solutions of nonlinear fourth order differential equations with a middle term

Miroslav Bartušek and **Zuzana Došlá**[✉]

Masaryk University, Faculty of Science, Kotlářská 2, 611 37 Brno, The Czech Republic

Received 1 July 2014, appeared 18 November 2014
Communicated by John R. Graef

Abstract. We study the oscillation of a fourth order nonlinear differential equation with a middle term. Using a certain energy function, we describe the properties of oscillatory solutions. The paper extends oscillation criteria stated for equations with the operator $x^{(4)} + x''$ and completes the results stated for super-linear and sub-linear case. Oscillation results are new also for the linear equation.

Keywords: fourth order nonlinear differential equation, oscillatory solution, oscillation.

2010 Mathematics Subject Classification: Primary 34C10; Secondary 34C15.

1 Introduction

Consider the fourth order nonlinear differential equation

$$x^{(4)}(t) + q(t)x''(t) + r(t)f(x(t)) = 0 \tag{1.1}$$

under the following assumptions:

(i) $q \in C(\mathbb{R}_+)$, $q(t) > 0$ for large t, $r \in C(\mathbb{R}_+)$, $r(t) > 0$ for large t and $\mathbb{R}_+ = [0, \infty)$;

(ii) $f \in C(\mathbb{R})$ satisfies $f(u)u > 0$ for $u \neq 0$ and either

$$|f(u)| \geq |u| \quad \text{for } u \in \mathbb{R} \tag{1.2}$$

or there exists $0 < \lambda < 1$ such that

$$|f(u)| \geq |u|^{\lambda} \quad \text{for } u \in \mathbb{R}, \tag{1.3}$$

where $\mathbb{R} = (-\infty, \infty)$.

A special case of (1.1) is the equation

$$x^{(4)}(t) + q(t)x''(t) + r(t)|x(t)|^{\lambda} \operatorname{sgn} x(t) = 0, \tag{1.4}$$

[✉]Corresponding author. Email: dosla@math.muni.cz

where $\lambda \le 1$.

By a solution of (1.1) we mean a function $x \in C^4[0,\infty)$, which satisfies (1.1) on $[0,\infty)$. A solution is said to be *nonoscillatory* if $x(t) \neq 0$ for large t, otherwise is said to be *oscillatory*. A solution is said to be *proper* if it is nontrivial in any neighbourhood of infinity. Equation (1.1) is *oscillatory* if all its solutions are oscillatory.

The oscillatory behavior of fourth order differential equations enjoys a great deal of interest, see [1–4, 6, 10] and references contained therein. The important role in the investigation of (1.1) is played by the fact whether the associated second order linear equation

$$h''(t) + q(t)h(t) = 0 \tag{1.5}$$

is oscillatory or nonoscillatory. For example, if (1.5) is nonoscillatory, then (1.4) can be written as a two-term equation, see [3], or as a four-dimensional Emden–Fowler differential system, see [10], and oscillation criteria for (1.4) can be obtained by this approach.

If (1.5) is oscillatory and $\lambda \ge 1$, then (1.1) and (1.4) have been investigated in [3]. Here conditions determining that all nonoscillatory solutions are vanishing at infinity have been given, and the oscillation theorem for (1.4) has been proved in the case $\lambda > 1$.

The natural problem is to study oscillation of (1.1) and (1.4) when $\lambda \le 1$. If $\lambda = 1$ and $q(t) \equiv 1$, then (1.4) is the linear equation

$$x^{(4)}(t) + x''(t) + r(t)x(t) = 0 \tag{1.6}$$

and the following well-known result holds, see, e.g., [8, Corollary 1.3].

Theorem A. *Let* (1.2) *hold. If either*

$$\liminf_{t\to\infty} t \int_t^\infty r(s)\,ds > \frac{1}{4} \quad \text{or} \quad \limsup_{t\to\infty} t \int_t^\infty r(s)\,ds > 1,$$

then (1.6) *is oscillatory.*

If $\lambda < 1$ and (1.5) is oscillatory, the following oscillation criterion for (1.4) has been proved in [4, Theorem 2].

Theorem B. *Let $\lambda < 1$ and* (1.5) *be oscillatory. Assume that*

$$q(t) \ge q_0 > 0, \quad q'(t) \le 0, \quad q''(t) \ge 0 \quad \text{for large } t, \tag{1.7}$$

and

$$\lim_{t\to\infty} t^{2(\lambda-1)} r(t) = \infty. \tag{1.8}$$

Then (1.4) *is oscillatory.*

Motivated by these results, we study oscillation of (1.1), and properties of zeros of oscillatory solutions. We allow that the function q can tend to zero or to infinity as $t \to \infty$ and both cases that the corresponding second order equation (1.5) is nonoscillatory/oscillatory are considered. Our approach is based on a suitable energy function for (1.1) and a comparison method for (1.1) and (1.4). Our results are applicable to the equation

$$x^{(4)}(t) + kx''(t) + r(t)f(x(t)) = 0\,, \quad (k > 0)\,, \tag{1.9}$$

studied in [7]. If f is a locally Lipschitz function, then this equation is known as *the Swift–Hohenberg equation*.

2 Classification of solutions

We start with the possible types of nonoscillatory solutions of (1.1). Due to the sign-condition on f, we can focus on eventually positive solutions of (1.1).

To this aim, a function g, defined in a neighborhood of infinity, is said to change its sign, if there exists a sequence $\{t_k\} \to \infty$ such that $g(t_k)g(t_{k+1}) < 0$.

Lemma 2.1. *Every eventually positive solution x of* (1.1) *is one of the following type:*

Type (a): $x(t) > 0$, $x'(t) > 0$, $x''(t) \le 0$ *for large* t,

Type (b): $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$, $x'''(t) > 0$ *for large* t,

Type (c): x'' *changes sign.*

Moreover, if (1.5) *is nonoscillatory, then x is of Type* (a) *or* (b)*, and if* (1.5) *is oscillatory, then x is of Type* (a) *or* (c)*.*

Proof. From Theorem 2 and Theorem 2′ in [3] it follows that if (1.5) is nonoscillatory, then every eventually positive solution x satisfies $x'(t) > 0$ and x'' is of one sign for large t, whereby if (1.5) is oscillatory, then every eventually positive solution x satisfies either $x''(t) \le 0$ or x'' changes sign.

Assume that $x(t) > 0$ and $x''(t) \le 0$ for large t. If $x'(t) \le 0$, then x is nonincreasing and concave, which is a contradiction with the positivity of x.

Assume that $x(t) > 0$, $x'(t) > 0$ and $x''(t) > 0$ for large t. Then $x^{(4)}(t) < 0$ and so x''' is of one sign for large t. If $x'''(t) \le 0$, then x'' is positive nonincreasing and concave function, which is a contradiction with the positivity of x''.

Finally, if (1.5) is oscillatory, then the last conclusion follows from Theorem 2, part (b) in [3]. □

In the sequel, we consider equation (1.4) with $\lambda \le 1$.

Lemma 2.2. *Let* (1.5) *be nonoscillatory. If there exists $\lambda \le 1$ such that*

$$\int_0^\infty t^{2\lambda} r(t)\, dt = \infty\,, \tag{2.1}$$

then (1.4) *has no solution of Type* (b)*.*

Proof. Let (1.5) be nonoscillatory and (2.1) hold for $\lambda \le 1$. Assume that (1.4) has a solution x of Type (b), i.e., there exists $t_0 \ge 0$ such that $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$ and $x'''(t) > 0$ for $t \ge t_0$. Then from (1.4), $x^{(4)}(t) < 0$ for $t \ge t_0$. Thus there exists $t_1 \ge t_0$ such that x''' is positive and decreasing for $t \ge t_1$ and there exist $C > 0$ and $t_2 \ge t_1$ such that $x''(t) \ge C$ and $x(t) \ge Ct^2$ for $t \ge t_2$. From here, integrating (1.4) from t_2 to t, we get

$$\begin{aligned} x'''(t_2) - x'''(t) &\ge -\int_{t_2}^t x^{(4)}(s)\, ds = \int_{t_2}^t \Big(q(s)x''(s) + r(s)x^\lambda(s) \Big)\, ds \\ &\ge C^\lambda \int_{t_2}^t r(s)s^{2\lambda}\, ds\,. \end{aligned}$$

Letting $t \to \infty$, we get a contradiction to the boundedness of x'''. □

3 Oscillation theorems

In this section we state two oscillation theorems for (1.1).

Theorem 3.1. *Let* (1.2) *hold. Assume that*

$$\lim_{t\to\infty} \frac{r(t)}{q(t)} = \infty\,, \tag{3.1}$$

$$q^2(t) \le 4r(t) \quad \textit{for large } t\,, \tag{3.2}$$

and, in addition if (1.5) *is nonoscillatory, that*

$$\int_0^\infty t^2 r(t)\,dt = \infty\,. \tag{3.3}$$

Then (1.1) *is oscillatory.*

To prove this result, we introduce the following energy function used for (1.4) in [4].

Definition 3.2. Let x be a solution (possibly oscillatory or nonoscillatory) of (1.1). Define the function F as

$$F(t) = -x'''(t)\,x(t) + x'(t)\,x''(t), \quad t \in \mathbb{R}_+\,.$$

Lemma 3.3. *Let* (1.2) *hold and x be a proper solution of* (1.1)*. If* (3.2) *holds, then the function F is nondecreasing for large t, and* (1.1) *has no solutions of Type* (c)*.*

Proof. Let x be a proper solution of (3.6). We have

$$F'(t) = r(t)x(t)f(x(t)) + q(t)x''(t)x(t) + \big(x''(t)\big)^2\,. \tag{3.4}$$

If $x(t) \ne 0$, then by (1.2) and (3.2)

$$F'(t) = \left(\sqrt{r(t)}\sqrt{f(x(t))x(t)}\,\operatorname{sgn} x(t) + \frac{q(t)}{2\sqrt{r(t)}}x''(t)\sqrt{x(t)/f(x(t))}\right)^2 + \big(x''(t)\big)^2\left(1 - \frac{q^2(t)}{4r(t)}\frac{x(t)}{f(x(t))}\right) \ge 0\,.$$

If $x(\bar t) = 0$ at some $\bar t > 0$, then $F'(t) \ge 0$ in a neighbourhood of $\bar t$. By (3.4), F' is continuous for $t > 0$ and thus $F'(t) \ge 0$ for large t and we get the monotonicity of F for large t.

Let $x(t) > 0$ for $t \ge T_1 \ge 0$ and by contradiction, suppose that x is of Type (c), i.e., x'' changes sign. Let $\{t_k\}_{k=1}^\infty$ and $\{\tau_k\}_{k=1}^\infty$, $T_1 \le t_k < \tau_k < t_{k+1}$, $k = 1, 2, \ldots$ be sequences of zeros of x'' tending to ∞ such that

$$x''(t) > 0 \quad \text{on } (t_k, \tau_k),\ k = 1, 2, \ldots \tag{3.5}$$

Then (1.4) implies $x^{(4)}(t) < 0$ on $[t_k, \tau_k]$ and, hence, x''' is decreasing. According to (3.5) and the fact that $x''(t_k) = x''(\tau_k) = 0$, numbers $\xi_k \in (t_k, \tau_k)$ exist such that $x'''(\xi_k) = 0$, $k = 1, 2, \ldots$ From this and from the fact that x''' is decreasing, we have

$$x'''(t_k) > 0 \quad \text{and} \quad x'''(\tau_k) < 0\,, \quad k = 1, 2, \ldots$$

Hence,

$$F(t_k) = -x'''(t_k)\,x(t_k) < 0, \qquad F(\tau_k) = -x'''(\tau_k)\,x(\tau_k) > 0, \quad k = 1, 2, \ldots$$

In view of the monotonicity of F, we get a contradiction. Thus x'' does not change sign and this proves the lemma. □

Proof of Theorem 3.1. Step 1. We prove first the statement for the linear equation

$$x^{(4)}(t) + q(t)x''(t) + r(t)x(t) = 0\,. \tag{3.6}$$

Let $T > 0$ be such that (3.2) holds for $t \geq T$. Without loss of generality, consider a solution x of (3.6) such that $x(t) > 0$ for $t \geq T$. Using Lemma 3.3, the function F is nondecreasing for large t, and in view of Lemmas 2.1, 2.2 and 3.3, x is of Type (a), i.e., $x'(t) > 0$, $x''(t) \leq 0$. Then either x''' oscillates or $x'''(t) > 0$ for large t; observe that the case $x'''(t) < 0$ for large t is impossible as x' would change sign. Consider a sequence $\{t_k\}$ such that $t_1 \geq T$, $\lim_{t\to\infty} t_k = \infty$ and $x'''(t_k) = 0$ in case x''' oscillates; otherwise it can be arbitrary. In both cases we have $F(t_k) < 0$ for $k = 1, 2, \dots$. According to Lemma 3.3, F is nondecreasing, so $F(t) < 0$ for $t \geq t_1$. Define the function

$$Z(t) = -x''(t)\,x(t) + \big(x'(t)\big)^2$$

for $t \geq t_1 \geq T$. Then $Z'(t) = F(t) < 0$ and taking into account that $x''(t) \leq 0$, we have $Z(t) \geq 0$. Thus,

$$0 \leq -x''(t)\,x(t) \leq Z(t_1)\,, \quad x(t) \geq K\,,$$

for $t \geq t_1$ and $K = x(t_1)$. Hence, there exists a constant $M > 0$ such that $|x''(t)| \leq M$ for $t \geq t_1$. From this and (3.6),

$$x^{(4)}(t) = -q(t)x''(t) - r(t)x(t) \leq Mq(t) - Kr(t)$$

for $t \geq t_1$ and (3.1) implies the existence of $\tau \geq t_1$ such that

$$x^{(4)}(t) \leq -Cr(t) < 0 \quad \text{for} \quad t \geq \tau \quad \text{and} \quad C = K^{\lambda}/2\,. \tag{3.7}$$

Since x''' is decreasing for $t \geq \tau$, there exists $\tau_1 \geq \tau$ such that $x'''(t) > 0$ for $t \geq \tau_1$. From this and the fact that $x'(t) > 0$ and $x''(t) \leq 0$, we have $\lim_{t\to\infty} x^{(j)}(t) = 0$ for $j = 2, 3$. Therefore,

$$\big|x^{(j)}(t)\big| = \int_t^{\infty} \big|x^{(j+1)}(s)\big|\,ds\,, \quad j = 2, 3\,,$$

and using (3.7), for $t \geq \tau_1$ we have

$$x'''(t) = \int_t^{\infty} \big|x^{(4)}(s)\big|\,ds \geq C\int_t^{\infty} r(s)\,ds\,,$$

so $r \in L^1(\mathbb{R}_+)$. Proceeding in the same way, $|x''(t)| = \int_t^{\infty} |x'''(s)|\,ds$, thus

$$x'(t) - x'(\tau_1) = \int_{\tau_1}^{t} \big|x''(s)\big|\,ds \geq C\int_{\tau_1}^{t} s^2 r(s)\,ds\,.$$

Since x' is bounded, letting $t \to \infty$ we get a contradiction to (3.3). Thus, a solution of Type (a) does not exist and equation (3.6) is oscillatory.

Step 2. Consider nonlinear equation (1.1) and assume, by contradiction, that (1.1) has a solution $x(t) > 0$ for $t \geq T$. Then $y = x$ is the solution of the linear equation

$$y^{(4)} + q(t)y'' + R(t)y = 0\,, \tag{3.8}$$

where

$$R(t) = \frac{r(t)f(x(t))}{x(t)}\,.$$

According to (1.2), we have $R(t) \geq r(t)$ for $t \geq T$. Thus, using (3.1), (3.2) and (3.3), we get

$$4R(t) \geq q^2(t), \qquad \lim_{t\to\infty} \frac{R(t)}{q(t)} = \infty, \qquad \int_0^\infty t^2 R(t)\, dt = \infty.$$

According to the first part of the proof, equation (3.8) is oscillatory. This is a contradiction to the fact that x is a nonoscillatory solution. □

Our next result extends Theorem A to (1.1).

Theorem 3.4. *Let (1.3) hold. If (1.7) and (1.8) hold, then (1.1) is oscillatory.*

Proof. Assume, by contradiction, that (1.1) has a solution $x(t) > 0$ for $t \geq T$. Since (1.7) holds, (1.5) is oscillatory, and by Lemma 2.1, x is of Type (a) or (c). Moreover, $y = x$ is a solution of the equation

$$y^{(4)} + q(t)y'' + R(t)|y(t)|^\lambda \operatorname{sgn} y(t) = 0 \tag{3.9}$$

for $t \geq T$, where

$$R(t) = \frac{r(t) f(x(t))}{x^\lambda(t)} \geq r(t).$$

From here and (1.8) we have

$$\lim_{t\to\infty} t^{2(\lambda-1)} R(t) = \infty.$$

Applying Theorem A to (3.9), the oscillation of (3.9) follows. This is a contradiction to the fact that x is a nonoscillatory solution. □

The following examples illustrate our results.

Example 3.5. Consider the equation

$$x^{(4)}(t) + \frac{c}{t^2} x''(t) + \frac{1}{t^{2-\varepsilon}} f(x(t)) = 0 \quad (t \geq 1), \tag{3.10}$$

where $c > 0$, $\varepsilon > 0$, and

$$f(u) = \begin{cases} \frac{4}{\pi} \arctan u & \text{for } |u| \leq 1, \\ u & \text{for } |u| > 1. \end{cases}$$

By Theorem 3.1, (3.10) is oscillatory.

Example 3.6. Consider the equation

$$x^{(4)}(t) + \left(1 + \frac{1}{t}\right) x''(t) + t \ln(t+1) f(x(t)) = 0, \quad (t \geq 1), \tag{3.11}$$

where

$$f(u) = \begin{cases} \sqrt{u} & \text{for } |u| \leq 1, \\ u & \text{for } |u| > 1. \end{cases}$$

By Theorem 3.4, (3.11) is oscillatory.

4 Existence and zeros of oscillatory solutions

We start with the existence of oscillatory solutions for (1.4).

Proposition 4.1. *Assume* (1.2) *and*

$$\limsup_{u\to\infty} \frac{f(u)}{u} < \infty . \tag{4.1}$$

If (1.5) *is oscillatory and*

$$q^2(t) \le 4r(t) \quad \text{for } t \in \mathbb{R}_+ , \tag{4.2}$$

then (1.1) *has proper oscillatory solutions.*

Proof. According to [8, Theorem 11.5], all solutions of (1.1) are defined on $\mathbb{R}_+$. By Lemmas 2.1 and 3.3, we have that any solution of (1.4) is either proper oscillatory, or trivial in a neighbourhood of infinity, or of Type (a).

Consider the function F from Definition 3.2. If x is of type Type (a), then $F(t) < 0$ for large t, and by Lemma 3.3, $F(t) < 0$ for $t \in \mathbb{R}_+$. If $x(t) \equiv 0$ for large t, then $F(t) \equiv 0$ for large t. Hence, any solution of (1.1) with the initial condition $F(0) > 0$ is proper oscillatory. □

In the sequel, we describe zeros of proper oscillatory solutions x of (1.1) and of their derivatives. As a motivation, consider equation (1.1) with $q(t) \equiv 0$. Then any oscillatory solution has the following properties in the neighbourhood of infinity: any zero of x and x' is simple (i.e. is not double or triple), and zeros of x and x' separate each other, i.e., between two zeros of x [x'] there exists exactly one zero of x' [x]. Here we prove that the same properties remain to hold for (1.1).

Theorem 4.2. *Assume* (1.2) *and* (3.2). *Then for any proper oscillatory solution x of* (1.1) *there exists $T > 0$ such that all zeros of x and x' are simple, and between two zeros of x [x'] there exists exactly one zero of x' [x] on $[T, \infty)$.*

Proof. Let x be a proper solution of (1.1) such that $x(t_k) = 0$, where $\{t_k\}_{k=1}^{\infty}$ tends to infinity. By Lemma 3.3, the function F is nondecreasing for $t \ge T$.

If $F(t) \equiv 0$ for large t, then $Z(t) \equiv 0$ for $t \ge T_1 > T$ and from the definition of Z we have $x''(t)x(t) \ge 0$ and

$$0 \equiv F'(t) = r(t)x(t)f(x(t)) + q(t)x''(t)x(t) + \left(x''(t)\right)^2 \ge r(t)x(t)f(x(t)) \ge 0 .$$

Since $r(t) > 0$ and $f(u)u > 0$ for $u \ne 0$, we get $x(t) \equiv 0$ for large t, which is a contradiction to the fact that x is proper.

Define the function

$$Z(t) = -x''(t)\,x(t) + \left(x'(t)\right)^2$$

for $t \ge t_1 \ge T$. Then $Z'(t) = F(t)$ and $Z(t_k) \ge 0$. If $F(t) > 0$ ($F(t) < 0$) for large t, then Z is increasing (decreasing) and taking into account that $Z(t_k) \ge 0$, we have

$$Z(t) > 0 \qquad \text{for } t \ge T_1 > T . \tag{4.3}$$

If $\tau \ge T_1$ is such that $x'(\tau) = 0$, then, from (4.3), $x''(\tau)x(\tau) < 0$, and so τ is a simple zero of x'.

If $\tau_1 \ge T_1$ is such that $x(\tau_1) = 0$, then again from (4.3) we have $x'(\tau_1) \ne 0$ and τ_1 is a simple zero of x.

Let τ_2, τ_3, where $T_1 \le \tau_2 < \tau_3$ be two successive zeros of x' such that $x'(t) > 0$ on (τ_2, τ_3). Then, from (4.3), we have

$$x''(\tau_2)x(\tau_2) < 0 \quad \text{and} \quad x''(\tau_3)x(\tau_3) < 0\,.$$

Since $x''(\tau_2) > 0$ and $x''(\tau_3) < 0$, we get $x(\tau_2) < 0$ and $x(\tau_3) > 0$, and x has a zero on (τ_2, τ_3). Since x is increasing on (τ_2, τ_3), x has a simple zero. From above we get that between two successive zeros of x' there exists exactly one zero of x.

Let τ_4, τ_5, where $T_1 \le \tau_4 < \tau_5$ be two successive zeros of x such that $x(t) > 0$ on (τ_4, τ_5). According to Rolle's theorem, x' has a zero τ_6 in (τ_4, τ_5). The fact that τ_6 is the only zero of x' in (τ_4, τ_5) follows from the fact that between two zeros of x' there exists exactly one zero of x. □

Remark 4.3. If (4.2) holds, then Theorem 4.2 is valid with $T = 0$, i.e., for all zeros of a proper oscillatory solution. For instance, equations (3.10) with $c = 1$ and (3.11) have by Proposition 4.1 and Theorem 4.2 proper oscillatory solutions x such that zeros of x and x' are simple and separate each other.

Example 4.4. Consider equation (1.9) where f satisfies (1.2) and (4.1), and $r(t) \ge k^2/4$ for $t \in \mathbb{R}_+$. By Proposition 4.1 and Theorem 4.2, (1.9) has proper oscillatory solutions and zeros of x and x' are simple and separate each other.

We conclude this paper with the following open question: *Is it possible to relax the assumptions* (1.7) *and* (1.8) *of Theorem* 3.4 *in the sub-linear case, i.e., f satisfies* (1.3)*?*

Acknowledgements.

Research supported by the Grant GAP 201/11/0768 of the Czech Grant Agency.

References

[1] R. P. Agarwal, D. O'Regan, *Infinite interval problems for differential, difference and integral equations*, Kluwer Academic Publishers, Dordrecht, 2001. MR1845855

[2] I. V. Astashova, On the asymptotic behavior at infinity of solutions to quasi-linear ordinary differential equations, *Math. Bohem.* **135**(2010), 373–382. MR2681011

[3] M. Bartušek, Z. Došlá, Asymptotic problems for fourth-order nonlinear differential equations, *Bound. Value Probl.* **2013**, No. 89, 15 pp. MR3070574; url

[4] M. Bartušek, Z. Došlá, Oscillation of fourth order sub-linear differential equations, *Appl. Math. Lett.* **36**(2014), 36–39. MR3215487; url

[5] M. Bartušek, M. Cecchi, Z. Došlá, M. Marini, Asymptotics for higher order differential equations with a middle term, *J. Math. Anal. Appl.* **388**(2012), 1130–1140. MR2869812; url

[6] M. Bartušek, M. Cecchi, Z. Došlá, M. Marini, Fourth-order differential equation with deviating argument, *Abstr. Appl. Anal.* **2012**, Art. ID 185242, 17 pp. MR2898056

[7] E. Berchio, A. Ferrero, F. Gazzola, P. Karageorgis, Qualitative behavior of global solutions to some nonlinear fourth order differential equations, *J. Differential Equations* **251**(2011), 2696–2727. MR2831710

[8] I. Kiguradze, An oscillation criterion for a class of ordinary differential equations, *Differ. Uravn.* **28**(1992), 201–214. MR1184921

[9] I. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Acad. Publ. G., Dordrecht, 1993. MR1220223

[10] T. Kusano, M. Naito, F. Wu, On the oscillation of solutions of 4-dimensional Emden–Fowler differential systems, *Adv. Math. Sci. Appl.* **11**(2001), 685–719. MR1907463