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Three nontrivial solutions for some elliptic equation involving the N-Laplacian

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Abstract. A new variational approach is used in order to establish the existence of at least three nontrivial solutions to some elliptic equation involving the N-Laplacian and whose nonlinearity term enjoys a critical exponential growth. The well known Ambrosetti–Rabinowitz condition is not needed.

Keywords: N-Laplacian, Trudinger–Moser inequality, critical exponential growth, variational method, local minima, Palais–Smale condition.

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1 Introduction

In this paper, we deal with the following problem

$$-\operatorname{div}\left(|\nabla u|^{N-2}\nabla u\right)+V(x)\,|u|^{N-2}u=\lambda\left(\exp\left(a\,|u|^{\frac{N}{N-1}}\right)+f(x,u)\right),\quad \text{in } \mathbb{R}^N, \qquad (P_\lambda)$$

where $N\geq 2$, a is some positive constant and λ is some positive parameter. We assume

(V_1) $V\colon \mathbb{R}^N \to]0,+\infty[$ is a continuous function such that

$$V(x)\geq V_0, \qquad \forall\, x\in\mathbb{R}^N,$$

where V_0 is a positive constant.

(F_1) $f\colon \mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. We assume that $f(x,s)\geq 0$, $\forall\,(x,s)\in \mathbb{R}^N\times[0,+\infty[$ and there exist $C_0>0$, $p>0$, $\alpha\geq 0$ and $\beta\geq 0$ such that

$$|f(x,s)|\leq C_0\left(|s|^\alpha+|s|^\beta\left(\exp\left(p\,|s|^{\frac{N}{N-1}}\right)-S_{N-2}(p,\,s)\right)\right), \qquad \forall\,(x,s)\in\mathbb{R}^N\times\mathbb{R},$$

where $S_{N-2}(p,\,s)=\sum_{k=0}^{N-2}\frac{p^k}{k!}|s|^{\frac{kN}{N-1}}$.

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Elliptic problems similar to (P_λ), i.e. containing the N-Laplacian and a nonlinear term which behaves like $\exp\left(\alpha|s|^{N/(N-1)}\right)$, as $|s| \to +\infty$ have been treated by many authors. We can for example cite [2–5, 8–12, 14–18, 20, 21, 23]. This interest on that type of nonlinear equations is motivated by the Trudinger–Moser inequality (see [1, 13, 16, 19, 22]) which allows a variational analysis of these equations. Our work is a contribution in this direction. Here, we have to highlight the fact that in our paper we do not assume that the famous Ambrosetti–Rabinowitz condition (AR), that is

(AR) there are constants $\theta > N$ and $s_0 > 0$ such that

$$0 < \theta \int_0^s f(x,t)\,dt \leq s f(x,s), \qquad \forall\ |s| \geq s_0,\ \forall\ x \in \mathbb{R}^N,$$

or its weaker form,

(ARR) there exists $s_0 > 0$ and $M > 0$ such that

$$0 < \int_0^s f(x,t)\,dt \leq M\,|f(x,s)|\,, \qquad \forall\ |s| \geq s_0,\ \forall\ x \in \mathbb{R}^N,$$

holds. Knowing the important role of this condition in the establishment of existence and multiplicity results, we see that proving the existence of at least three nontrivial solutions could be considered as interesting. Some works dealing with exponential nonlinearities and where the (AR) condition is dropped were published (see, for example, [10–12]). In [12], the authors treated the case $N = 2$ and they used an appropriate version of the mountain pass theorem introduced by G. Cerami. In order to get the boundedness of some Palais–Smale sequence, they assumed that there exist $C_* \geq 0$, $\theta \geq 1$ such that

$$(H)\ \ H(x,t) \leq \theta H(x,s) + C_* \quad \forall\ 0 < t < s,\ \forall\ x \in \Omega, \quad \text{where} \quad H(x,u) = uf(x,u) - 2F(x,u).$$

This work is extended to N-dimensional space in [11]. In [10, Section 7], the authors assumed that (H) holds true with $\theta = 1$ and $C_* = 0$. In addition, they assumed that there exists $c > 0$ such that for all $(x,s) \in \mathbb{R}^N \times [0,+\infty[$, $F(x,s) \leq c(|s|^N + f(x,s))$.

Using a new variational approach, we establish the existence of at least three nontrivial solutions to the problem (P_λ). For this purpose, we will adapt some arguments developed in [7]. In fact, we will make use of a new Palais–Smale condition introduced by G. Bonanno in [7] to prove the existence of at least two local minima of the energy functional which corresponds to the problem (P_λ). A third solution is obtained by a suitable version of the mountain pass theorem.

The functional space in which the problem (P_λ) will be studied is

$$E = \left\{ u \in W^{1,N}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} V(x)\,|u|^N\,dx < +\infty \right\},$$

which is a reflexive Banach space equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^N + V(x)\,|u|^N \right) dx \right)^{\frac{1}{N}}.$$

First, we recall the Trudinger–Moser inequality for the whole space $\mathbb{R}^N$, $N \geq 2$. In fact, we have the following result (for $N = 2$, see [8, 19], and for $N \geq 2$, see [1, 15])

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha\,|u|^{\frac{N}{N-1}} \right) - S_{N-2}(\alpha,u) \right] dx < +\infty \quad \text{for } u \in W^{1,N}(\mathbb{R}^N) \text{ and } \alpha > 0,$$

where $S_{N-2}(\alpha,\ u) = \sum_{k=0}^{N-2} \frac{\alpha^k}{k!} |u|^{\frac{kN}{N-1}}$. Moreover, if $|\nabla u|_{L^N(\mathbb{R}^N)} \le 1,\ |u|_{L^N(\mathbb{R}^N)} \le M < +\infty$ and $\alpha < \alpha_N$, then there exists a positive constant $C = C(N, M, \alpha)$, which depends only on N, M and α such that

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha\, |u|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, u)\right] dx \le C, \tag{1.1}$$

where $\alpha_N = N W_{N-1}^{\frac{1}{N-1}}$ and W_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$. Furthermore, using the above results together with Hölder's inequality, if $\alpha > 0$ and $q > 0$ then we have

$$\int_{\mathbb{R}^N} |u|^q \left[\exp\left(\alpha\, |u|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, u)\right] dx < +\infty, \qquad \forall\ u \in W^{1,N}(\mathbb{R}^N).$$

More precisely, if $\|u\|_{W^{1,N}(\mathbb{R}^N)} \le M$ with $\alpha M^{\frac{N}{N-1}} < \alpha_N$, then there exists a positive constant $C = C(\alpha,\ M,\ q,\ N)$ such that

$$\int_{\mathbb{R}^N} |u|^q \left[\exp\left(\alpha\, |u|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, u)\right] dx \le C\, \|u\|^q_{W^{1,N}(\mathbb{R}^N)}, \tag{1.2}$$

where

$$\|u\|_{W^{1,N}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^N + |u|^N\right) dx\right)^{\frac{1}{N}}$$

is the norm in the Sobolev space $W^{1,N}(\mathbb{R}^N)$. Observe that since V is positive and bounded from below, then clearly

$$E \hookrightarrow W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \qquad \forall\ N \le q < +\infty,$$

with continuous embeddings. Thus, there exists a positive constant χ_0 such that

$$\|u\|_{W^{1,N}(\mathbb{R}^N)} \le \chi_0\, \|u\|, \qquad \forall\ u \in E.$$

This last inequality together with (1.2) implies that there exists a constant $C' = C'(\alpha, M, q) > 0$ such that

$$\int_{\mathbb{R}^N} |u|^q \left[\exp\left(\alpha\, |u|^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, u)\right] dx \le C'\, \|u\|^q, \tag{1.3}$$

provided that $\|u\| \le M$ with $M < \frac{1}{\chi_0}(\frac{\alpha_N}{\alpha})^{\frac{N-1}{N}}$.
Assume that

(V_2) the function $(V(x))^{-1}$ belongs to $L^{\frac{1}{N-1}}(\mathbb{R}^N)$.

Then, it is not difficult to show that $E \hookrightarrow L^q(\mathbb{R}^N),\ \forall\ 1 \le q < +\infty$, with compact embedding. Let $u \in E$. We have

$$\begin{aligned}\int_{\mathbb{R}^N} \left(\int_0^{u(x)} \exp\left(a\, |s|^{\frac{N}{N-1}}\right) ds\right) dx &\le \int_{\mathbb{R}^N} \exp\left(a\, |u|^{\frac{N}{N-1}}\right) |u|\, dx \\ &= \int_{\mathbb{R}^N} \left(\exp\left(a\, |u|^{\frac{N}{N-1}}\right) - S_{N-2}(a, u)\right) |u|\, dx \\ &\quad + \int_{\mathbb{R}^N} S_{N-2}(a, u)\, |u|\, dx.\end{aligned}$$

Thus, using (1.3), one can easily find a positive constant $C_a > 0$ such that

$$\int_{\mathbb{R}^N} \left(\int_0^{u(x)} \exp\left(a\, |s|^{\frac{N}{N-1}}\right) ds\right) dx \le C_a\, \|u\|, \qquad u \in E,\ \|u\| \le \inf\left(1,\ \frac{1}{2\chi_0}\left(\frac{\alpha_N}{a}\right)^{\frac{N-1}{N}}\right). \tag{1.4}$$

On the other hand, since $\alpha+1 \geq 1$, then the continuous (and also compact) embedding $E \hookrightarrow L^{\alpha+1}(\mathbb{R}^N)$ holds and by consequence there exists a positive constant $C_\alpha > 0$ such that

$$\int_{\mathbb{R}^N} |u|^{\alpha+1}\, dx \leq C_\alpha \|u\|^{\alpha+1}, \qquad \forall\, u \in E. \tag{1.5}$$

Next, for $u \in E$ with $\|u\| \leq \frac{1}{2\chi_0}\left(\frac{\alpha_N}{p}\right)^{\frac{N-1}{N}}$, by (1.3) there exists a constant $C_{\beta,p} > 0$ such that

$$\int_{\mathbb{R}^N} |u|^{\beta+1}\left(\exp\left(p\,|u|^{\frac{N}{N-1}}\right) - S_{N-2}(p,u)\right) dx \leq C_{\beta,p} \|u\|^{\beta+1}. \tag{1.6}$$

Definition 1.1. A point $u \in E$ is said to be a weak solution of the problem (P_λ) if it satisfies

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u|^{N-2} \nabla u \nabla v dx + \int_{\mathbb{R}^N} V(x)\, |u|^{N-2} uv\, dx \\ &\quad = \int_{\mathbb{R}^N} \exp\left(a\,|u|^{\frac{N}{N-1}}\right) v\, dx + \int_{\mathbb{R}^N} f(x,u) v\, dx, \qquad \forall\, v \in E. \end{aligned}$$

Now, we are ready to state our main results in the present paper. It consists of the following theorem.

Theorem 1.2. *Assume that (V_1), (V_2), and (F_1) hold true. If there exists $R > 0$ such that*

$$\begin{aligned} &\left[W_{N-1}\frac{(R+1)^N - R^N}{N} + \int_{|x|<R} V(x)\, dx + \int_{R\leq|x|\leq R+1} V(x)\,(R+1-|x|)^N\, dx\right] \\ &\quad < N\left[\frac{W_{N-1}R^N}{2N(4N)^{\frac{1}{N}}\left(C_a + C_0\left(C_\alpha + C_{\beta,p}\right)\right)}\right]^N, \end{aligned} \tag{1.7}$$

then there exist $0 < \lambda_* < \lambda^* < +\infty$ such that (P_λ) admits at least three nontrivial weak solutions provided that $\lambda_* < \lambda < \lambda^*$.

Example 1.3. We can take $V(x) = 1 + \sigma\,|x|^\alpha$ with $N(N-1) < \alpha$ and σ small enough. In this case, (1.7) holds for R chosen large enough.

2 Proof of Theorem 1.2

For $u \in E$ and $\lambda > 0$, define

$$\begin{aligned} \Phi(u) &= \int_{\mathbb{R}^N} \frac{|\nabla u|^N + V(x)\,|u|^N}{N}\, dx = \frac{\|u\|^N}{N} \\ \Psi(u) &= \int_{\mathbb{R}^N} \left(\int_0^{u(x)} \exp\left(a\,|s|^{\frac{N}{N-1}}\right) ds + F(x,u)\right) dx, \\ I_\lambda(u) &= \Phi(u) - \lambda\Psi(u), \end{aligned}$$

where $F(x,s) = \int_0^s f(x,t)\, dt$, $(x,s) \in \mathbb{R}^N \times \mathbb{R}$. Clearly, the functional I_λ is well defined on E and by classical arguments (see [6]) it is of class C^1 and the critical points of I_λ are nontrivial weak solutions of the problem (P_λ).

In order to prove our multiplicity results, we make use of a recent critical points results established by G. Bonanno in [7] by using a new Palais–Smale condition.

Definition 2.1. Let Φ_0 and Ψ_0 be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r_1, r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$; we say that the functional $I_0 = \Phi_0 - \Psi_0$ verifies the Palais–Smale condition cut off lower at r_1 and upper at r_2 (in short $^{[r_1]}(PS)^{[r_2]}$) if any sequence $(u_n) \subset X$ such that

(i) $(I_0(u_n))$ is bounded,

(ii) $I_0'(u_n) \to 0$ in X^* (where X^* denotes the topological dual of X,)

(iii) $r_1 < \Phi_0(u_n) < r_2$, $\forall\, n \in \mathbb{N}$,

has a convergent subsequence. Clearly, if $r_1 = -\infty$ and $r_2 = +\infty$ it coincides with the classical (PS) condition. Moreover, if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ it is denoted by $(PS)^{[r_2]}$.

The main tool to prove the existence of the two first weak solutions of (P_λ) is the following theorem.

Theorem 2.2 ([7, Theorem 5.1]). *Let X be a real Banach space and let $\Phi, \Psi \colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Assume that there are $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, such that*

$$\varsigma(r_1, r_2) = \inf_{r_1 < \Phi(v) < r_2} \frac{\sup_{r_1 < \Phi(u) < r_2} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)} < \rho(r_1, r_2) = \sup_{r_1 < \Phi(v) < r_2} \frac{\Psi(v) - \sup_{\Phi(u) \le r_1} \Psi(u)}{\Phi(v) - r_1},$$

and for each $\lambda \in]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\varsigma(r_1,r_2)}[$ the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies $^{[r_1]}(PS)^{[r_2]}$ condition. Then, for each $\lambda \in]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\varsigma(r_1,r_2)}[$ there is $u_\lambda \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_\lambda) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I_\lambda'(u_\lambda) = 0$.

Remark 2.3. Obviously, the critical point u_λ of I_λ given by Theorem 2.2 is a local minimum of the functional I_λ.

Lemma 2.4. *Assume that the hypotheses of Theorem 1.2 hold true. For each*

$$0 < r < \frac{1}{N} \inf\left(1, \left(\frac{1}{2\chi_0}\right)^N \left(\frac{\alpha_N}{N'p}\right)^{N-1}\right) \quad \text{with} \quad N' = \frac{N}{N-1}$$

and $\lambda > 0$, the functional I_λ satisfies $(PS)^{[r]}$.

Proof. Let $(u_n) \subset E$ be such that $(I_\lambda(u_n))$ is bounded, $I_\lambda'(u_n) \to 0$ and $\Phi(u_n) < r$, $\forall\, n \in \mathbb{N}$. Since $\|u_n\| < (Nr)^{\frac{1}{N}}$, $\forall\, n \in \mathbb{N}$, then there exists $u \in E$ such that $u_n \rightharpoonup u$ weakly in E. By (F_1) we have

$$\begin{aligned}\int_{\mathbb{R}^N} &|f(x, u_n)|^{N'}\,dx \\ &\le c_1 \left(\int_{\mathbb{R}^N} |u_n|^{N'\alpha}\,dx + \int_{\mathbb{R}^N} |u_n|^{N'\beta} \left(\exp\left(N'p\,|u|^{\frac{N}{N-1}}\right) - S_{N-2}(N'p, u)\right) dx \right).\end{aligned}$$

Since $\|u_n\| \le (Nr)^{\frac{1}{N}} < \frac{1}{2\chi_0}\left(\frac{\alpha_N}{N'p}\right)^{\frac{N-1}{N}}$ and by (1.3) we deduce that

$$\sup_{n \in \mathbb{N}} \left(\int_{\mathbb{R}^N} |f(x, u_n)|^{N'}\,dx \right) < +\infty.$$

This fact together with the compactness of the embedding $E \hookrightarrow L^N(\mathbb{R}^N)$ implies

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} f(x,u_n)(u_n-u)\,dx = 0. \tag{2.1}$$

On the other hand, we have

$$\int_{\mathbb{R}^N} \exp\left(a\,|u_n|^{\frac{N}{N-1}}\right)(u_n-u)\,dx = \int_{\mathbb{R}^N} \exp\left(a\,|u_n|^{\frac{N}{N-1}} - S_{N-2}(a,u_n)\right)(u_n-u)\,dx + \int_{\mathbb{R}^N} S_{N-2}(a,u_n)(u_n-u)\,dx.$$

Using the compact embeddings $E \hookrightarrow L^1(\mathbb{R}^N)$ and $E \hookrightarrow L^2(\mathbb{R}^N)$ together with (1.1), it is not difficult to prove that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \exp\left(a\,|u_n|^{\frac{N}{N-1}}\right)(u_n-u)\,dx = 0. \tag{2.2}$$

Combining (2.1) and (2.2) with the fact

$$\langle I'_\lambda(u_n), u_n-u\rangle \to 0 \quad \text{as } n\to+\infty,$$

we conclude that (u_n) is strongly convergent to u in E. This ends the proof of Lemma 2.4. □

Fix a positive real number r such that

$$r \le \frac{1}{4N} \inf\left(1, \left(\frac{1}{\chi_0}\right)^N \left(\frac{\alpha_N}{N'p}\right)^{N-1}, \left(\frac{1}{\chi_0}\right)^N \left(\frac{\alpha_N}{a}\right)^{N-1}\right).$$

Lemma 2.5. *Assume that the hypotheses of Theorem 1.2 hold true. Then, there is $\lambda^* > 0$ such that: if $0 < \lambda < \lambda^*$, then the functional I_λ admits a nontrivial critical point u_λ satisfying*

$$0 < \Phi(u_\lambda) < r \quad \text{and} \quad I_\lambda(u_\lambda) \le I_\lambda(w) \quad \text{for all } w \in \Phi^{-1}(]0,r[).$$

Proof. For $\lambda > 0$ and $R > 0$ as in (1.7), define the function

$$\vartheta_\lambda = \begin{cases} \delta_\lambda & \text{if } |x| < R, \\ \delta_\lambda\,(R+1-|x|) & \text{if } R \le |x| \le R+1, \\ 0 & \text{if } |x| > R+1, \end{cases}$$

with δ_λ is a real number satisfying

$$0 < \delta_\lambda < \inf\left(\left(\frac{\lambda B}{A}\right)^{\frac{1}{N-1}}, \left(\frac{r}{A}\right)^{\frac{1}{N}}\right), \tag{2.3}$$

where

$$A = \frac{1}{N}\left[W_{N-1}\frac{(R+1)^N - R^N}{N} + \int_{|x|<R} V(x)\,dx + \int_{R\le|x|\le R+1} V(x)\,(R+1-|x|)^N\,dx\right], \tag{2.4}$$

and

$$B = W_{N-1}\frac{R^N}{N}. \tag{2.5}$$

It is clear that $\vartheta_\lambda \in E$ and we have

$$\int_{\mathbb{R}^N} |\nabla \vartheta_\lambda|^N \, dx = \int_{R \le |x| \le R+1} \delta_\lambda^N \, dx = \delta_\lambda^N W_{N-1} \frac{(R+1)^N - R^N}{N}.$$

On the other hand, we have

$$\int_{\mathbb{R}^N} V(x) \vartheta_\lambda^N \, dx = \delta_\lambda^N \left[\int_{|x|<R} V(x) \, dx + \int_{R \le |x| \le R+1} V(x) \left(R+1-|x|\right)^N dx \right].$$

Thus,

$$\Phi(\vartheta_\lambda) = A\delta_\lambda^N. \tag{2.6}$$

Next, since $F(x, \vartheta_\lambda) \ge 0$, and $\exp\left(a|s|^{N/N-1}\right) \ge 1$, we get

$$\Psi(\vartheta_\lambda) \ge \int_{|x|<R} \left(\int_0^{\vartheta_\lambda(x)} \exp\left(a\,|s|^{\frac{N}{N-1}}\right) ds \right) dx \ge \int_{|x|<R} \vartheta_\lambda(x) \, dx \ge B\delta_\lambda. \tag{2.7}$$

By (2.6) and (2.7), it yields

$$\frac{\Psi(\vartheta_\lambda)}{\Phi(\vartheta_\lambda)} \ge \frac{B}{A\delta_\lambda^{N-1}}.$$

This inequality with (2.3) leads to

$$\frac{\Psi(\vartheta_\lambda)}{\Phi(\vartheta_\lambda)} > \frac{1}{\lambda}. \tag{2.8}$$

Now, let $u \in E$ be such that $\Phi(u) < r$. Clearly, $\|u\| < (Nr)^{\frac{1}{N}}$. Since

$$(Nr)^{\frac{1}{N}} < \frac{1}{2\chi_0} \left(\frac{\alpha_N}{N'p} \right)^{\frac{N-1}{N}} \le \frac{1}{2\chi_0} \left(\frac{\alpha_N}{p} \right)^{\frac{N-1}{N}},$$

then by (F_1) we have

$$\int_{\mathbb{R}^N} F(x,u) \, dx \le C_0 \left(C_\alpha \|u\|^{\alpha+1} + C_{\beta,p} \|u\|^{\beta+1} \right). \tag{2.9}$$

On the other hand, having in mind that $\|u\| < (Nr)^{\frac{1}{N}} \le \inf\left(1, \frac{1}{2\chi_0}\left(\frac{\alpha_N}{a}\right)^{\frac{N-1}{N}}\right)$, and using (1.4) it yields

$$\int_{\mathbb{R}^N} \left(\int_0^{u(x)} \exp\left(a\,|s|^{\frac{N}{N-1}}\right) ds \right) dx \le C_a \|u\|. \tag{2.10}$$

By (2.9) and (2.10), we deduce

$$\Psi(u) \le C_a \|u\| + C_0 \left(C_\alpha \|u\|^{\alpha+1} + C_{\beta,p} \|u\|^{\beta+1} \right).$$

Since $\inf(\alpha+1, \beta+1) \ge 1$ and $\|u\| < 1$, it follows that

$$\Psi(u) \le \left(C_a + C_0(C_\alpha + C_{\beta,p})\right) \|u\| \le \left(C_a + C_0(C_\alpha + C_{\beta,p})\right) (Nr)^{\frac{1}{N}}. \tag{2.11}$$

Set

$$\lambda^* = \frac{r^{\frac{N-1}{N}}}{(4N)^{\frac{1}{N}} \left(C_a + C_0(C_\alpha + C_{\beta,p})\right)}.$$

By (2.11), we infer

$$\frac{\sup_{\Phi(u)<r} \Psi(u)}{r} \le \frac{1}{\lambda^*}. \tag{2.12}$$

Thus,

$$\frac{\sup_{\Phi(u)<r}\Psi(u)}{r}<\frac{1}{\lambda},\quad \text{provided that } \lambda<\lambda^*. \tag{2.13}$$

From (2.13) and (2.8), we obtain

$$\frac{\sup_{\Phi(u)<r}\Psi(u)}{r}<\frac{1}{\lambda}<\frac{\Psi(\vartheta_\lambda)}{\Phi(\vartheta_\lambda)}.$$

Keeping in mind that $0<\Phi(\vartheta_\lambda)<r$, we easily deduce that

$$\varsigma(0,r)<\rho(0,r),\quad \text{and}\quad \left]\frac{\Phi(\vartheta_\lambda)}{\Psi(\vartheta_\lambda)},\frac{r}{\sup_{\Phi(u)<r}\Psi(u)}\right[\subset\left]\frac{1}{\rho(0,r)},\frac{1}{\varsigma(0,r)}\right[.$$

Finally, for $0<\lambda<\lambda^*$, Theorem 2.2 guarantees the existence of a critical point u_λ of I_λ such that $0<\Phi(u_\lambda)<r$ and $I_\lambda(u_\lambda)\le I_\lambda(w)$, $\forall\, w\in\Phi^{-1}(]0,r[)$. This ends the proof of Lemma 2.5. □

Now, we will try to prove the existence of another critical point of I_λ as a second local minimum. This is made in the following lemma.

Lemma 2.6. *Assume that the hypotheses of Theorem 1.2 hold true. Then, there exists $\lambda_*\in]0,\lambda^*[$ such that: if $\lambda_*<\lambda<\lambda^*$, then the functional I_λ admits a critical point $\widetilde{u_\lambda}$ which satisfies*

$$r<\Phi(\widetilde{u_\lambda})<2r,\quad \textit{and}\quad I_\lambda(\widetilde{u_\lambda})\le I_\lambda(w),\qquad \forall\, w\in\Phi^{-1}(]r,2r[).$$

Proof. First set

$$\lambda_*=\left(\frac{r}{A}\right)^{\frac{N-1}{N}}\frac{2A}{B},$$

where A and B are given by (2.4) and (2.5). By (1.7), we get

$$\lambda_*<\lambda^*.$$

For $\lambda_*<\lambda<\lambda^*$, we keep using the function

$$\vartheta_\lambda=\begin{cases}\delta_\lambda & \text{if } |x|<R,\\ \delta_\lambda\,(R+1-|x|) & \text{if } R\le|x|\le R+1,\\ 0 & \text{if } |x|>R+1,\end{cases}$$

with different conditions on δ_λ. In fact, here we choose δ_λ such that

$$\left(\frac{r}{A}\right)^{\frac{1}{N}}<\delta_\lambda<\inf\left(\left(\frac{2r}{A}\right)^{\frac{1}{N}},\left(\frac{B\lambda}{2A}\right)^{\frac{1}{N-1}}\right). \tag{2.14}$$

By (2.6) and (2.7), we get

$$\frac{\Psi(\vartheta_\lambda)}{2\Phi(\vartheta_\lambda)}\ge\frac{B}{2A\delta_\lambda^{N-1}}.$$

Taking (2.14) into account, we infer

$$\frac{\Psi(\vartheta_\lambda)}{2\Phi(\vartheta_\lambda)}>\frac{1}{\lambda}. \tag{2.15}$$

Now, replacing r by $(2r)$ in (2.11), it yields

$$\begin{aligned}\frac{\sup_{\Phi(u)<2r}\Psi(u)}{2r} &\leq N^{\frac{1}{N}}\left(C_a + C_0(C_\alpha + C_{\beta,p})\right)(2r)^{-\frac{N-1}{N}} \\ &\leq N^{\frac{1}{N}}\left(C_a + C_0(C_\alpha + C_{\beta,p})\right) r^{-\frac{N-1}{N}} \\ &\leq \frac{1}{\lambda^*} < \frac{1}{\lambda}.\end{aligned} \tag{2.16}$$

By (2.15) and (2.16), we obtain

$$\frac{\Psi(\vartheta_\lambda)}{2\Phi(\vartheta_\lambda)} > \frac{1}{\lambda} > \frac{\sup_{\Phi(u)<2r}\Psi(u)}{2r}, \quad \text{for each } \lambda_* < \lambda < \lambda^*. \tag{2.17}$$

On the other hand, since $\frac{1}{\lambda} > \frac{1}{\lambda^*}$ and $\delta_\lambda < (\frac{B\lambda}{2A})^{\frac{1}{N-1}} < (\frac{B\lambda}{A})^{\frac{1}{N-1}}$, we infer

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} < \frac{1}{\lambda} < \frac{\Psi(\vartheta_\lambda)}{\Phi(\vartheta_\lambda)}. \tag{2.18}$$

One can easily show that (2.17) implies

$$\frac{1}{\lambda} > \frac{\sup_{r<\Phi(u)<2r}\Psi(u) - \Psi(\vartheta_\lambda)}{2r - \Phi(\vartheta_\lambda)}. \tag{2.19}$$

Similarly, inequality (2.18) implies

$$\frac{1}{\lambda} < \frac{\Psi(\vartheta_\lambda) - \sup_{\Phi(u)\leq r}\Psi(u)}{\Phi(\vartheta_\lambda) - r}. \tag{2.20}$$

Next, by (2.14) and (2.6) it yields $r < \Phi(\vartheta_\lambda) < 2r$. Consequently,

$$\varsigma(r,2r) \leq \frac{\sup_{r<\Phi(u)<2r}\Psi(u) - \Psi(\vartheta_\lambda)}{2r - \Phi(\vartheta_\lambda)}.$$

This inequality together with (2.19) gives

$$\varsigma(r,2r) < \frac{1}{\lambda}. \tag{2.21}$$

Similarly, we get

$$\frac{\Psi(\vartheta_\lambda) - \sup_{\Phi(u)\leq r}\Psi(u)}{\Phi(\vartheta_\lambda) - r} \leq \rho(r,2r).$$

Taking (2.20) into account, it follows that

$$\frac{1}{\lambda} < \rho(r,2r). \tag{2.22}$$

Combining (2.21) and (2.22), we deduce that

$$\frac{1}{\rho(r,2r)} < \lambda < \frac{1}{\varsigma(r,2r)}, \qquad \forall\, \lambda_* < \lambda < \lambda^*.$$

Since

$$0 < 2r < \frac{1}{N}\inf\left(1, \left(\frac{1}{\chi_0^2}\right)^N \left(\frac{\alpha_N}{N'p}\right)^{N-1}\right),$$

then by Lemma 2.4, the functional I_λ satisfies ${}^{[r]}(PS)^{[2r]}$. Hence, all the conditions of Theorem 2.2 are fulfilled. We conclude that, for each $\lambda_* < \lambda < \lambda^*$, the functional I_λ admits a critical point $\widetilde{u_\lambda}$ satisfying

$$r < \Phi(\widetilde{u_\lambda}) < 2r, \quad \text{and} \quad I_\lambda(\widetilde{u_\lambda}) \leq I_\lambda(w), \qquad \forall\, w \in \Phi^{-1}(]r, 2r[).$$

Since $\Phi(u_\lambda) < r$, then $u_\lambda \neq \widetilde{u_\lambda}$. □

In order to prove the existence of a third critical point of I_λ, the following result is needed.

Theorem 2.7 ([7, Theorem 6.2])**.** *Let X be a real Banach space and let $\Phi_0, \Psi_0 \colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ_0 convex. Put $I_0 = \Phi_0 - \Psi_0$ and assume that $x_0, x_1 \in X$ are two local minima of I_0. Put $m_0 = \min_{t\in[0,1]} \Psi_0(tx_1 + (1-t)x_0)$ and assume that there are $r_0 > \max\{\Phi_0(x_0), \Phi_0(x_1)\}$ and $s_0 \geq 0$ such that*

$$\sup_{\Phi_0(x) < r_0 + s_0} \Psi_0(x) < s_0 + m_0,$$

and I_0 satisfies the $(PS)^{[r_0+s_0]}$ condition. Then, I_0 admits at least a third critical point x_3 such that $\Phi_0(x_3) < r_0 + s_0$.

For $\lambda_* < \lambda < \lambda^*$, take $x_1 = u_\lambda$, $x_2 = \widetilde{u_\lambda}$, $r_0 = 3r$ and $s_0 = r$. Since

$$r_0 + s_0 = 4r < \frac{1}{N} \inf\left(1, \left(\frac{1}{2\chi_0}\right)^N \left(\frac{\alpha_N}{N'p}\right)^{N-1}\right),$$

then I_λ satisfies $(PS)^{[r_0+s_0]}$. Arguing as in (2.11), we can easily obtain

$$\sup_{\Phi(u)<4r} \Psi(u) \leq (4N)^{\frac{1}{N}} \left(C_a + C_0\left(C_\alpha + C_{\beta,p}\right)\right) r^{\frac{1}{N}}.$$

Thus,

$$\frac{\sup_{\Phi(u)<4r} \Psi(u)}{r} \leq (4N)^{\frac{1}{N}} \left(C_a + C_0\left(C_\alpha + C_{\beta,p}\right)\right) r^{-\frac{N-1}{N}} = \frac{1}{\lambda^*}.$$

Since $\lambda < \lambda^*$, then

$$\sup_{\Phi(u)<4r} \Psi(u) < \frac{r}{\lambda}.$$

By the virtue of Theorem 2.7, the functional I_λ admits at least a third critical point $\widetilde{\widetilde{u_\lambda}}$ such that $\Phi(\widetilde{\widetilde{u_\lambda}}) < r_0 + s_0 = 4r$. This ends the proof of Theorem 1.2.

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