




# Existence and multiplicity of weak quasi-periodic solutions for second order Hamiltonian system with a forcing term

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Received 12 April 2014, appeared 19 December 2014

Communicated by Michal Fečkan

**Abstract.** In this paper, we first obtain three inequalities and two of them, in some sense, generalize Sobolev's inequality and Wirtinger's inequality from periodic case to quasi-periodic case, respectively. Then by using the least action principle and the saddle point theorem, under subquadratic case, we obtain two existence results of weak quasi-periodic solutions for the second order Hamiltonian system:

$$\frac{d[P(t)\dot{u}(t)]}{dt} = \nabla F(t, u(t)) + e(t),$$

which generalize and improve the corresponding results in recent literature [J. Kuang, *Abstr. Appl. Anal.* 2012, Art. ID 271616]. Moreover, when the assumptions  $F(t, x) = F(t, -x)$  and  $e(t) \equiv 0$  are also made, we obtain two results on existence of infinitely many weak quasi-periodic solutions for the second order Hamiltonian system under the subquadratic case.

**Keywords:** second order Hamiltonian system, weak quasi-periodic solution, variational method, subquadratic case.

**2010 Mathematics Subject Classification:** 37J45, 34C25, 70H05.


## 1 Introduction and main results

In this paper, we are concerned with the existence and multiplicity of weak-quasi periodic solutions for the second order Hamiltonian system:

$$\frac{d[P(t)\dot{u}(t)]}{dt} = \nabla F(t, u(t)) + e(t), \quad t \in \mathbb{R} \quad (1.1)$$

where  $u(t) = (u_1(t), \dots, u_N(t))^{\tau}$ ,  $N > 1$  is an integer,  $F \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ ,  $\nabla F(t, x) = (\partial F / \partial x_1, \dots, \partial F / \partial x_N)^{\tau}$ ,  $P(t) = (p_{ij}(t))_{N \times N}$  is a symmetric and continuous  $N \times N$  matrix-value functions on  $\mathbb{R}$ ,  $e: \mathbb{R} \rightarrow \mathbb{R}^N$ ,  $(\cdot)^{\tau}$  stands for the transpose of a vector or a matrix.

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It is well known that the variational method is a very effective tool which investigate the existence and multiplicity of periodic solutions, subharmonic solutions and homoclinic solutions for Hamiltonian systems and in these directions, lots of contributions have been obtained (for example, see [6, 7, 11, 12, 15–28, 30–34] and references therein). However, the results on existence and multiplicity of almost periodic solutions for Hamiltonian systems are not often seen by using variational approach. We refer readers to [1–5, 13, 29]. Especially, when  $P(t) \equiv I_{N \times N}$  and  $e(t) \equiv 0$ , where  $I_{N \times N}$  is the unit matrix, recently, in [13], by using the least action principle and the saddle point theorem, Kuang obtained two existence results of quasi-periodic solutions for system (1.1). Next, we recall two definitions and Kuang's results in [13].

**Definition 1.1** ([8]). A function  $f(t)$  is said to be Bohr almost periodic, if for any  $\varepsilon > 0$ , there is a constant  $l_\varepsilon > 0$ , such that in any interval of length  $l_\varepsilon$ , there exists  $\tau$  such that the inequality  $|f(t + \tau) - f(t)| < \varepsilon$  is satisfied for all  $t \in \mathbb{R}$ .

**Definition 1.2** ([9]). A function  $f \in C^0(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^N)$  is called almost periodic in  $t$  uniformly for  $x \in \mathbb{R}^m$  when, for each compact subset  $K$  in  $\mathbb{R}^m$ , for each  $\varepsilon > 0$ , there exists  $l > 0$ , and for each  $\alpha \in \mathbb{R}$ , there exists  $\tau \in [\alpha, \alpha + l]$  such that

$$\sup_{t \in \mathbb{R}} \sup_{x \in K} \|f(t + \tau, x) - f(t, x)\|_{\mathbb{R}^N} < \varepsilon.$$

Let  $p > 1$  be a positive integer and  $\{T_j\}_{j=1}^p$  be rationally independent positive real constants. Define

$$\Lambda = \cup_{j=1}^p \Lambda_j = \cup_{j=1}^p \left\{ \frac{2m\pi}{T_j} \mid m \in \mathbb{Z} \right\}, \quad (1.2)$$

where  $\Lambda_j = \left\{ \frac{2m\pi}{T_j} \mid m \in \mathbb{Z} \right\}$ .

To be precise, in [13], Kuang obtained the following results.

**Theorem 1.3** ([13, Theorem 2.3]). *Suppose  $F$  satisfies the following conditions:*

- (f<sub>1</sub>)  $F(t, \cdot) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$  and  $F(t, \cdot)$  is almost periodic in  $t$  uniformly for  $x \in \mathbb{R}^N$ ;
- (f<sub>2</sub>)  $\nabla F(t, \cdot)$  is almost periodic in  $t$  uniformly for  $x \in \mathbb{R}^N$ ;
- (f<sub>3</sub>) for any  $\lambda \in \mathbb{R}/\Lambda$ ,  $x \in V$ ,

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \nabla F(t, x) e^{-i\lambda t} dt = 0;$$

- (f<sub>4</sub>) there exists  $g \in L^1_{loc}(\mathbb{R})$ , for a.e.  $t \in \mathbb{R}$  and all  $x \in \mathbb{R}^N$ , such that

$$|\nabla F(t, x)| \leq g(t);$$

- (f<sub>5</sub>)  $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, x) dt \rightarrow +\infty$  as  $|x| \rightarrow \infty$ .

Then (1.1) with  $P(t) \equiv I_{N \times N}$  and  $e(t) \equiv 0$  has at least a quasi periodic solution, where the definition of  $V$  can be seen in Section 2 below.

**Theorem 1.4** ([13, Theorem 2.4]). *Suppose that  $F$  satisfies (f<sub>1</sub>)–(f<sub>4</sub>) and*

$$(f_6) \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, x) dt \rightarrow -\infty \quad \text{as } |x| \rightarrow \infty.$$

Then (1.1) with  $P(t) \equiv I_{N \times N}$  and  $e(t) \equiv 0$  has at least one quasi-periodic solution by saddle point theorem.

Obviously,  $(f_4)$  implies that  $|\nabla F|$  is bounded, which makes lots of functions eliminated. For example, a simple function

$$F(t, x) \equiv \pm |x|^{\frac{3}{2}}, \quad \forall t \in \mathbb{R} \quad (1.3)$$

which does not satisfy  $(f_4)$ . However, in this paper, we obtain that system (1.1) still has quasi-periodic solution for such potential  $F$  like (1.3). To be precise, in this paper, inspired by [10, 13, 15, 24, 28, 32], we obtain the following results.

### (I) Existence of weak quasi-periodic solution

By using the least action principle and the saddle point theorem, we obtain that system (1.1) has at least one weak quasi-periodic solution.

**Theorem 1.5.** *Suppose that  $(f_1)$ – $(f_3)$  hold. If*

(P)  $p_{ij}(t)$ ,  $i, j = 1, 2, \dots, N$  are Bohr almost periodic and there exists  $m > \frac{1}{2}$  such that

$$(P(t)x, x) > m|x|^2, \quad \text{for all } (t, x) \in \mathbb{R} \times \{\mathbb{R}^N \setminus \{0\}\};$$

(E)  $e$  is Bohr almost periodic and

$$\lim_{T \rightarrow \infty} \int_{-T}^T e(t) dt = 0;$$

(W) there exist constants  $c_0 > 0$ ,  $k_1 > 0$ ,  $k_2 > 0$ ,  $\alpha \in [0, 1)$  and a nonnegative function  $w \in C([0, +\infty), [0, +\infty))$  with the properties:

- (i)  $w(s) \leq w(t)$ ,  $\forall s \leq t, s, t \in [0, +\infty)$ ,
- (ii)  $w(s+t) \leq c_0(w(s) + w(t))$ ,  $\forall s, t \in [0, +\infty)$ ,
- (iii)  $0 \leq w(t) \leq k_1 t^\alpha + k_2$ ,  $\forall t \in [0, +\infty)$ ,
- (iv)  $w(t) \rightarrow +\infty$ , as  $t \rightarrow \infty$ ;

$(f_4)'$  there exist  $g, h \in L^1_{loc}(\mathbb{R}, \mathbb{R}^+)$  such that

$$|\nabla F(t, x)| \leq g(t)w(|x|) + h(t), \quad \text{for a.e. } t \in \mathbb{R};$$

$$(f_5)' \quad \frac{1}{w^2(|x|)} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, x) dt > \frac{c_0^2 \sum_{j=1}^p \frac{T_j^2}{12}}{2m} \left( \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g(t) dt \right)^2, \quad \text{as } |x| \rightarrow \infty,$$

then system (1.1) has at least one weak quasi-periodic solution.

**Theorem 1.6.** *Suppose that (P), (E), (W),  $(f_1)$ – $(f_3)$  and  $(f_4)'$  hold. If*

$(f_5)''$

$$\begin{aligned} & \frac{1}{w^2(|x|)} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, x) dt \\ & < - \frac{c_0^2 (\|P\| + 2m) \sum_{j=1}^p \frac{T_j^2}{12}}{2(2m-1)} \left( \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g(t) dt \right)^2 \quad \text{as } |x| \rightarrow \infty, \end{aligned}$$

where

$$\begin{aligned} \|P\| &= \sup_{t \in [0, T]} \max_{|x|=1, x \in \mathbb{R}^N} |P(t)x| \\ &= \sup_{t \in [0, T]} \max \left\{ \sqrt{\lambda(t)} : \lambda(t) \text{ is the eigenvalue of } P^\tau(t)P(t) \right\}, \end{aligned}$$

then system (1.1) has at least one weak quasi-periodic solution.

**Remark 1.7.** Obviously, Theorem 1.5 and Theorem 1.6 generalize and improve Theorem 1.3 and Theorem 1.4, respectively. It is easy to verify that  $F(t, x) \equiv |x|^{3/2}$  and  $F(t, x) \equiv -|x|^{3/2}$  satisfy Theorem 1.5 and Theorem 1.6, respectively, but do not satisfy Theorem 1.3 and Theorem 1.4. Moreover, similar to the argument of Remark 2.5 in [13], when  $P(t) \equiv I_{N \times N}$ ,  $e(t) \equiv 0$ ,  $V$  only contains a frequency  $2\pi/T$  and  $F(t, x)$  is periodic in  $t$  with period  $T$ , in some sense, Theorem 1.5 and Theorem 1.6 improve the corresponding results in [15] because of the presence of  $(\mathcal{W})$  and  $(f_4)'$ .  $(\mathcal{W})$  and  $(f_4)'$  were given by Wang and Zhang in [28], which present some advantages compared to the well known condition: there exist  $g, h \in L^1([0, T]; \mathbb{R}^+)$  and  $\alpha \in [0, 1)$  such that

$$|\nabla F(t, x)| \leq g(t)|x|^\alpha + h(t). \quad (1.4)$$

Finally, one can also compare Theorem 1.5 and Theorem 1.6 with the corresponding results in [32], in which, Zhang and Tang investigated the existence of  $T$ -periodic solution under  $(\mathcal{W})$  and the following condition: there exist  $g \in L^2([0, T]; \mathbb{R}^+)$ ,  $h \in L^1([0, T]; \mathbb{R}^+)$  and  $\alpha \in [0, 1)$  such that

$$|\nabla F(t, x)| \leq g(t)w(|x|) + h(t), \quad (1.5)$$

where  $g \in L^2([0, T]; \mathbb{R}^+)$  is demanded from proofs of their theorems. In our Theorem 1.5 and Theorem 1.6, when  $P(t) \equiv I_{N \times N}$ ,  $e(t) \equiv 0$ ,  $V$  only contains a frequency  $2\pi/T$  and  $F(t, x)$  is periodic in  $t$  with period  $T$ , we only demand that  $g \in L^1([0, T]; \mathbb{R}^+)$ . Hence, our results are different from those in [32].

## (II) Multiplicity of weak quasi-periodic solutions

Moreover, by using a critical point theorem due to Ding in [6], we obtain the following multiplicity results.

**Theorem 1.8.** *Suppose that  $(\mathcal{P})$ ,  $(\mathcal{W})$ ,  $(f_1)$ – $(f_3)$ ,  $(f_4)'$  and  $(f_5)'$  hold. If*

$$(\mathcal{E})' \quad e(t) \equiv 0, \quad \forall t \in \mathbb{R};$$

$$(f_8) \quad F(t, 0) \equiv 0 \quad \text{and} \quad F(t, x) = F(t, -x) \quad \text{for all } (t, x) \in \mathbb{R} \times \mathbb{R}^N;$$

$$(f_9) \quad \lim_{|x| \rightarrow 0} \frac{F(t, x)}{|x|^2} = -\infty \quad \text{uniformly for all } t \in \mathbb{R},$$

then system (1.1) has infinitely many weak quasi-periodic solutions.

**Theorem 1.9.** *Suppose that  $(\mathcal{P})$ ,  $(\mathcal{E})'$ ,  $(\mathcal{W})$ ,  $(f_1)$ – $(f_3)$ ,  $(f_4)'$ ,  $(f_5)''$ ,  $(f_8)$  and  $(f_9)$  hold. Then system (1.1) has infinitely many weak quasi-periodic solutions.*

## 2 Preliminaries

In this section, we need to make some preliminaries. Some knowledge and statements below come from [3,4,8,9,13].

Define

$$AP^0(\mathbb{R}^N) = \{u : \mathbb{R} \rightarrow \mathbb{R}^N \mid u \text{ is Bohr almost periodic}\},$$

endowed with the norm  $\|u\|_\infty = \sup_{t \in \mathbb{R}} |u(t)|$ . Then  $(AP^0(\mathbb{R}^N), \|\cdot\|_\infty)$  is a Banach space.

Define

$$AP^1(\mathbb{R}^N) = \left\{ u \in AP^0(\mathbb{R}^N) \cap C^1(\mathbb{R}, \mathbb{R}^N) \mid u'(t) \in AP^0(\mathbb{R}^N) \right\},$$

endowed with the norm

$$\|u\| = \|u\|_\infty + \|u'\|_\infty.$$

Then  $(AP^1(\mathbb{R}^N), \|\cdot\|)$  is also a Banach space.

Let  $f \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$ , that is  $f$  is locally Lebesgue integrable from  $\mathbb{R}$  to  $\mathbb{R}^N$ . Then the mean value of  $f$  is the limit (when it exists)

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t) dt.$$

A fundamental property of almost periodic functions is that such functions have convergent means, that is, the limit

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T u(t) dt$$

exists.

Let  $p \in \mathbb{Z}^+$ .  $B^p(\mathbb{R}^N)$  is the completion of  $AP^0(\mathbb{R}^N)$  into  $L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$  with respect to the norm

$$\|u\|_p = \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |u(t)|^p dt \right\}^{1/p}.$$

The elements of these spaces  $B^p(\mathbb{R}^N)$  are called Besicovitch almost periodic functions.

For  $u \in B^p(\mathbb{R}^N)$ , if

$$\lim_{r \rightarrow 0} \frac{u(t+r) - u(t)}{r}$$

exists, then define

$$\nabla u = \lim_{r \rightarrow 0} \frac{u(t+r) - u(t)}{r}.$$

For  $u, v \in B^p(\mathbb{R}^N)$ , if  $\|u - v\|_p = 0$ , then we say that  $u, v$  belong to a class of equivalence. We will identify the equivalence class  $u$  with its continuous representant

$$u(t) = \int_0^t \nabla u(t) dt + c.$$

When  $p = 2$ ,  $B^2(\mathbb{R}^N)$  is a Hilbert space with its norm  $\|\cdot\|_2$  and the inner product

$$\langle u, v \rangle_2 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (u(t), v(t)) dt.$$

When  $u \in B^2(\mathbb{R}^N)$ , define

$$a(u, \lambda) := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{-i\lambda t} u(t) dt$$

which are complex vector and are called Fourier–Bohr coefficients of  $u$ . Let  $\Lambda(u) = \{\lambda \in \mathbb{R} \mid a(u, \lambda) \neq 0\}$ .

Define

$$B^{1,2}(\mathbb{R}^N) = \left\{ u \in B^2(\mathbb{R}^N) \mid \nabla u \text{ exists and } \nabla u \in B^2(\mathbb{R}^N) \right\},$$

endowed with the inner product

$$\begin{aligned} \langle u, v \rangle &= \langle u, v \rangle_2 + \langle \nabla u, \nabla v \rangle_2 \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (u(t), v(t)) dt + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (\nabla u(t), \nabla v(t)) dt, \end{aligned} \quad (2.1)$$

and the corresponding norm

$$\|u\| = \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |u(t)|^2 dt + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\nabla u(t)|^2 dt \right\}^{1/2}$$

Define

$$V = \left\{ u \in B^{1,2}(\mathbb{R}^N) \mid \Lambda(u) \subset \Lambda \right\}.$$

Then  $V$  is a linear subspace of  $B^{1,2}(\mathbb{R}^N)$  and  $(V, \langle \cdot, \cdot \rangle)$  is a Hilbert space.

Inspired by [13] and [16], we present the following two lemmas:

**Lemma 2.1.** *If  $u \in V$ , then*

$$u(t) = \sum_{j=1}^p u_j(t) \in AP^0(\mathbb{R}^N),$$

where

$$u_j(t) = \sum_{m=-\infty}^{+\infty} a(u, \lambda_m^{(j)}) e^{i\lambda_m^{(j)} t}, \quad \lambda_m^{(j)} := \frac{2m\pi}{T_j} \in \Lambda_j,$$

and

$$\|u\|_\infty \leq \sqrt{p^2 + \sum_{j=1}^p \frac{T_j^2}{12}} \|u\| \quad (2.2)$$

*Proof.* Since  $V \subseteq B^{1,2}(\mathbb{R}^N) \subseteq B^2(\mathbb{R}^N)$ , then

$$u(t) \sim \sum_{m=-\infty}^{+\infty} a(u, \lambda_m) e^{i\lambda_m t}, \quad \lambda_m \in \Lambda$$

and

$$\nabla u(t) \sim \sum_{m=-\infty}^{+\infty} i\lambda_m a(u, \lambda_m) e^{i\lambda_m t}, \quad \lambda_m \in \Lambda.$$

Combining (1.2), we obtain that

$$u(t) \sim \sum_{j=1}^p u_j(t), \quad \nabla u(t) \sim \sum_{j=1}^p \nabla u_j(t). \quad (2.3)$$

By Parseval's equality, we have

$$\|u\|_2^2 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |u(t)|^2 dt = \sum_{m=-\infty}^{+\infty} |a(u, \lambda_m)|^2, \quad (2.4)$$

$$\|\nabla u\|_2^2 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\nabla u(t)|^2 dt = \sum_{m=-\infty}^{+\infty} \lambda_m^2 |a(u, \lambda_m)|^2. \quad (2.5)$$

Hence

$$\sum_{m=-\infty}^{+\infty} |a(u, \lambda_m)|^2 = \sum_{j=1}^p \sum_{m=-\infty}^{+\infty} |a(u, \lambda_m^{(j)})|^2 < +\infty.$$

Then by [14, Theorem 3.5-2], we have

$$u(t) = \sum_{m=-\infty}^{+\infty} a(u, \lambda_m) e^{i\lambda_m t}, \quad \lambda_m \in \Lambda \quad (2.6)$$

and

$$\nabla u(t) = \sum_{m=-\infty}^{+\infty} i\lambda_m a(u, \lambda_m) e^{i\lambda_m t}, \quad \lambda_m \in \Lambda. \quad (2.7)$$

Since

$$\sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{1}{m^2} = \frac{\pi^2}{3},$$

then

$$\begin{aligned} |u(t)| &\leq \sum_{j=1}^p |u_j(t)| \\ &\leq \sum_{j=1}^p \sum_{m=-\infty}^{+\infty} |a(u, \lambda_m^{(j)})| |e^{i\lambda_m^{(j)} t}| \\ &= \sum_{j=1}^p \sum_{m=-\infty}^{+\infty} |a(u, \lambda_m^{(j)})| \\ &= \sum_{j=1}^p |a(u, \lambda_0^{(j)})| + \sum_{j=1}^p \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} |a(u, \lambda_m^{(j)})| \\ &= \sum_{j=1}^p |a(u, 0)| + \sum_{j=1}^p \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{1}{|\lambda_m^{(j)}|} |\lambda_m^{(j)} a(u, \lambda_m^{(j)})| \\ &= \sum_{j=1}^p |a(u, 0)| + \sum_{j=1}^p \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{T_j}{2\pi|m|} |\lambda_m^{(j)} a(u, \lambda_m^{(j)})| \\ &\leq p \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |u(t)| dt + \sum_{j=1}^p \left( \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{T_j^2}{4\pi^2 m^2} \right)^{1/2} \left( \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} |\lambda_m^{(j)} a(u, \lambda_m^{(j)})|^2 \right)^{1/2} \\ &\leq p \lim_{T \rightarrow \infty} \frac{1}{\sqrt{2T}} \left( \int_{-T}^T |u(t)|^2 dt \right)^{1/2} + \sum_{j=1}^p \sqrt{\frac{T_j^2}{12}} \left( \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} |\lambda_m^{(j)} a(u, \lambda_m^{(j)})|^2 \right)^{1/2} \\ &\leq p \|u\|_2 + \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{1/2} \left( \sum_{j=1}^p \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} |\lambda_m^{(j)} a(u, \lambda_m^{(j)})|^2 \right)^{1/2} \\ &\leq \left( p^2 + \sum_{j=1}^p \frac{T_j^2}{12} \right)^{1/2} (\|u\|_2^2 + \|\nabla u\|_2^2)^{1/2}. \end{aligned} \quad (2.8)$$

Hence (2.2) holds. (2.2) implies that the embedding from  $V$  into  $AP^0(\mathbb{R}^N)$  is continuous. So  $u \in AP^0(\mathbb{R}^N)$ . Thus we complete the proof.  $\square$

**Lemma 2.2.** *If  $u \in V$  and*

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T u(t) dt = 0, \quad (2.9)$$

then

$$\|u\|_\infty \leq \sqrt{\sum_{j=1}^p \frac{T_j^2}{12}} \|\nabla u\|_2 \quad (2.10)$$

and

$$\|u\|_2 \leq \max \left\{ \frac{T_j}{2\pi} \mid j = 1, \dots, p \right\} \|\nabla u\|_2. \quad (2.11)$$

*Proof.* By (2.8) and (2.9), it is obvious that (2.10) holds. Moreover, by (2.5) and (2.9), we have

$$\begin{aligned} \|\nabla u\|_2^2 &= \sum_{m=-\infty}^{+\infty} \lambda_m^2 |a(u, \lambda_m)|^2 \\ &= \sum_{j=1}^p \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} |\lambda_m^{(j)} a(u, \lambda_m^{(j)})|^2 \\ &= \sum_{j=1}^p \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{4m^2 \pi^2}{T_j^2} |a(u, \lambda_m^{(j)})|^2 \\ &\geq \sum_{j=1}^p \frac{4\pi^2}{T_j^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} |a(u, \lambda_m^{(j)})|^2 \\ &\geq \min \left\{ \frac{4\pi^2}{T_j^2} \mid j = 1, \dots, p \right\} \|u\|_2^2. \end{aligned}$$

Hence, (2.11) holds.  $\square$

**Remark 2.3.** A version of Lemma 2.1 and (2.10) has been given in [13] (see [13, Lemma 3.1 and Lemma 3.3]), where the author obtained that there exists a constant  $C > 0$  such that

$$\|u\|_\infty \leq (C+1)\|u\|, \quad \forall u \in V,$$

and when (2.9) holds,

$$\|u\|_\infty \leq C\|\nabla u\|_2.$$

However, the value of  $C$  are not given. Our Lemma 2.1 and Lemma 2.2 present the value of  $C$ , which will play an important role in our main results and their proofs. Moreover, we also present the inequality (2.11). One can compare (2.10) and (2.11) with Sobolev's inequality and Wirtinger's inequality in [16] which investigate periodic functions  $u \in W_T^{1,2}$ . It is easy to see that when  $V$  only contains a frequency  $2\pi/T$ , (2.11) reduces to Wirtinger's inequality.

**Lemma 2.4** ([13, Lemma 3.2]). *For any  $\{u_n\} \subset V$ , if the sequence  $\{u_n\}$  converges weakly to  $u$ , then  $\{u_n\}$  converges uniformly to  $u$  on any compact subset of  $\mathbb{R}$ .*



**Lemma 2.5.** Suppose  $F$  satisfies  $(f_1)$ – $(f_3)$ , then the functional  $\varphi: V \rightarrow \mathbb{R}$ , defined by

$$\varphi(u) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[ \frac{1}{2} (P(t) \nabla u(t), \nabla u(t)) + F(t, u(t)) + (e(t), u(t)) \right] dt \quad (2.12)$$

is continuously differentiable on  $V$ , and  $\varphi'(u)$  is defined by

$$\langle \varphi'(u), v \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[ (P(t) \nabla u(t), \nabla v(t)) + (\nabla F(t, u(t)), v(t)) + (e(t), v(t)) \right] dt \quad (2.13)$$

for  $v \in V$ . Moreover, if  $u$  is a critical point of  $\varphi$  in  $V$ , then

$$\nabla(P(t) \nabla u(t)) = \nabla F(t, u(t)) + e(t). \quad (2.14)$$

*Proof.* The proof with  $P(t) \equiv I_{N \times N}$  and  $e(t) \equiv 0$  can be seen in [13, Theorem 2.1]. With the aid of the conditions  $(\mathcal{P})$  and  $(\mathcal{E})$ , it is easy to see that the proof is the essentially same as Theorem 2.1 of [13]. So we omit the details. we refer readers to Theorem 2.1 and its proof in [13].  $\square$

**Definition 2.6.** When  $u$  satisfies (2.14), we say that  $u$  is a weak solution of system (1.1).

### 3 Existence

In this section, we will use the least action principle (see [16, Theorem 1.1]) to prove Theorem 1.5 and use the saddle point theorem (see [19]) to prove Theorem 1.6.

Define

$$\tilde{V} = \left\{ u \in V \mid \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T u(t) dt = 0 \right\}$$

and

$$\bar{V} = \{ u \mid u \in V \cap \mathbb{R}^N \}.$$

Then  $V = \tilde{V} \oplus \bar{V}$ . For  $u \in V$ ,  $u$  can be written as  $u = \bar{u} + \tilde{u}$ , where

$$\bar{u} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T u(t) dt \in \bar{V}.$$

It is easy to obtain that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \tilde{u}(t) dt = 0.$$

Then  $\tilde{u} \in \tilde{V}$ . For the sake of convenience, we denote

$$\begin{aligned} M_1 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g(t) dt, & M_2 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T h(t) dt, \\ M_3 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |e(t)| dt, & C^* &= \max \left\{ \frac{T_j}{2\pi} \mid j = 1, \dots, p \right\}. \end{aligned}$$

*Proof of Theorem 1.5.* Since  $V$  is a Hilbert space, then  $V$  is reflexive. Note that  $P$  is positive definite. Then  $\frac{1}{2}(P(t) \nabla u(t), \nabla u(t))$  and  $(e(t), u(t))$  are convex and continuous. Then by the proof of [13, Theorem 2.3], we know that  $\varphi$  is weakly lower semi-continuous. Condition  $(f_5)'$  implies that there exists  $a_1 > \frac{1}{m} \sum_{j=1}^p (T_j^2/12)$  such that

$$\lim_{|x| \rightarrow \infty} \left[ \frac{1}{w^2(|x|)} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, x) dt \right] > \frac{a_1 c_0^2 M_1^2}{2}. \quad (3.1)$$

It follows from  $(\mathcal{W})$ ,  $(f_4)'$  and Lemma 2.2 that

$$\begin{aligned}
& \left| \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [F(t, u(t)) - F(t, \bar{u})] dt \right| \\
&= \lim_{T \rightarrow \infty} \left| \frac{1}{2T} \int_{-T}^T [F(t, u(t)) - F(t, \bar{u})] dt \right| \\
&= \lim_{T \rightarrow \infty} \left| \frac{1}{2T} \int_{-T}^T \int_0^1 (\nabla F(t, \bar{u} + s\tilde{u}(t)), \tilde{u}(t)) ds dt \right| \\
&\leq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 |\nabla F(t, \bar{u} + s\tilde{u}(t))| |\tilde{u}(t)| ds dt \\
&\leq \|\tilde{u}\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 |\nabla F(t, \bar{u} + s\tilde{u}(t))| ds dt \\
&\leq \|\tilde{u}\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 [g(t)w(|\bar{u} + s\tilde{u}(t)|) + h(t)] ds dt \\
&\leq \|\tilde{u}\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 [c_0 g(t)w(|\bar{u}|) + c_0 g(t)w(|s\tilde{u}(t)|) + h(t)] ds dt \tag{3.2} \\
&\leq c_0 \|\tilde{u}\|_\infty w(|\bar{u}|) \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g(t) dt \\
&\quad + c_0 \|\tilde{u}\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 g(t) [k_1 |s\tilde{u}(t)|^\alpha + k_2] ds dt + \|\tilde{u}\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T h(t) dt \\
&\leq \frac{\|\tilde{u}\|_\infty^2}{2a_1} + \frac{a_1 c_0^2 M_1^2 w^2(|\bar{u}|)}{2} + \frac{c_0 M_1 k_1}{\alpha + 1} \|\tilde{u}\|_\infty^{\alpha+1} + c_0 k_2 M_1 \|\tilde{u}\|_\infty + M_2 \|\tilde{u}\|_\infty \\
&\leq \frac{1}{2a_1} \left( \sum_{j=1}^p \frac{T_j^2}{12} \right) \|\nabla u\|_2^2 + \frac{a_1 c_0^2 M_1^2 w^2(|\bar{u}|)}{2} + \frac{c_0 M_1 k_1}{\alpha + 1} \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{\alpha+1}{2}} \|\nabla u\|_2^{\alpha+1} \\
&\quad + (c_0 k_2 M_1 + M_2) \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{1}{2}} \|\nabla u\|_2.
\end{aligned}$$

It follows from (3.2) and Lemma 2.2 that

$$\begin{aligned}
\varphi(u) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[ \frac{1}{2} (P(t) \nabla u(t), \nabla u(t)) + F(t, u(t)) - F(t, \bar{u}) + F(t, \bar{u}) + (e(t), u(t)) \right] dt \\
&\geq \frac{m}{2} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\nabla u(t)|^2 dt - \frac{1}{2a_1} \left( \sum_{j=1}^p \frac{T_j^2}{12} \right) \|\nabla u\|_2^2 - \frac{a_1 c_0^2 M_1^2 w^2(|\bar{u}|)}{2} \\
&\quad - \frac{c_0 M_1 k_1}{\alpha + 1} \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{\alpha+1}{2}} \|\nabla u\|_2^{\alpha+1} - (c_0 k_2 M_1 + M_2) \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{1}{2}} \|\nabla u\|_2 \\
&\quad + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, \bar{u}) dt + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (e(t), \bar{u}(t)) dt \\
&\geq \left( \frac{m}{2} - \frac{1}{2a_1} \sum_{j=1}^p \frac{T_j^2}{12} \right) \|\nabla u\|_2^2 - \frac{c_0 M_1 k_1}{\alpha + 1} \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{\alpha+1}{2}} \|\nabla u\|_2^{\alpha+1} \\
&\quad - (c_0 k_2 M_1 + M_2 + M_3) \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{1}{2}} \|\nabla u\|_2 \tag{3.3} \\
&\quad + w^2(|\bar{u}|) \left( w^{-2}(|\bar{u}|) \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, \bar{u}) dt - \frac{a_1 c_0^2 M_1^2}{2} \right).
\end{aligned}$$

Note that  $a_1 > \frac{1}{m} \sum_{j=1}^p (T_j^2/12)$ . Since  $\|u\| \rightarrow \infty$  if and only if

$$\left( |\bar{u}|^2 + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\nabla u(t)|^2 dt \right)^{1/2} \rightarrow \infty,$$

(3.3), (3.1) and (W)(iv) imply that

$$\varphi(u) \rightarrow +\infty, \quad \text{as } \|u\| \rightarrow \infty.$$

Then by the least action principle (see [16, Theorem 1.1]), we know that  $\varphi$  has at least one critical point  $u^*$  which minimizes  $\varphi$ . Thus we complete the proof.  $\square$

*Proof of Theorem 1.6.* It follows from  $(f_5)''$  that there exists  $a_2 > \sum_{j=1}^p (T_j^2/12)$  such that

$$\lim_{|x| \rightarrow \infty} \frac{1}{w^2(|x|)} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, x) dt < - \left( \frac{a_2 \|P\|}{4m-2} + \frac{m\sqrt{a_2}}{2m-1} \sqrt{\sum_{j=1}^p \frac{T_j^2}{12}} \right) c_0^2 M_1^2. \quad (3.4)$$

At first, we prove that  $\varphi$  satisfies (PS) condition. Assume that  $\{u_n\} \subset V$  such that  $\varphi(u_n)$  is bounded and  $\varphi'(u_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Then there exists a constant  $D_0$  such that

$$|\varphi(u_n)| \leq D_0, \quad \|\varphi'(u_n)\| \leq D_0. \quad (3.5)$$

Similar to the argument of (3.2), we have

$$\begin{aligned} & \left| \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 (\nabla F(t, u_n(t)), \tilde{u}_n(t)) ds dt \right| \\ & \leq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 |\nabla F(t, u_n(t))| |\tilde{u}_n(t)| ds dt \\ & \leq \|\tilde{u}_n\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 |\nabla F(t, u_n(t))| ds dt \\ & \leq \|\tilde{u}_n\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 [g(t)w(|\bar{u}_n + \tilde{u}_n(t)|) + h(t)] ds dt \\ & \leq \|\tilde{u}_n\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 [c_0 g(t)w(|\bar{u}_n|) + c_0 g(t)w(|\tilde{u}_n(t)|) + h(t)] ds dt \\ & \leq c_0 \|\tilde{u}_n\|_\infty w(|\bar{u}_n|) \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g(t) dt + c_0 \|\tilde{u}_n\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g(t) [k_1 |\tilde{u}_n(t)|^\alpha + k_2] dt \\ & \quad + \|\tilde{u}_n\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T h(t) dt \\ & \leq \frac{\|\tilde{u}_n\|_\infty^2}{2a_2} + \frac{a_2 c_0^2 M_1^2 w^2(|\bar{u}_n|)}{2} + c_0 M_1 k_1 \|\tilde{u}_n\|_\infty^{\alpha+1} + c_0 k_2 M_1 \|\tilde{u}_n\|_\infty + M_2 \|\tilde{u}_n\|_\infty \\ & \leq \frac{1}{2a_2} \sum_{j=1}^p \frac{T_j^2}{12} \|\nabla u_n\|_2^2 + \frac{a_2 c_0^2 M_1^2 w^2(|\bar{u}_n|)}{2} + c_0 M_1 k_1 \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{\alpha+1}{2}} \|\nabla u_n\|_2^{\alpha+1} \\ & \quad + (c_0 k_2 M_1 + M_2) \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{1}{2}} \|\nabla u_n\|_2. \end{aligned} \quad (3.6)$$

Hence, by (3.5), (3.6), (P) and Lemma 2.2, we have

$$\begin{aligned}
D_0 \|\tilde{u}_n\| &\geq \langle \varphi'(u_n), \tilde{u}_n \rangle \\
&= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[ (P(t) \nabla u_n(t), \nabla u_n(t)) + (\nabla F(t, u_n(t)), \tilde{u}_n(t)) + (e(t), \tilde{u}_n(t)) \right] dt \\
&\geq \left( m - \frac{1}{2a_2} \sum_{j=1}^p \frac{T_j^2}{12} \right) \|\nabla u_n\|_2^2 - \frac{a_2 c_0^2 M_1^2 w^2(|\tilde{u}_n|)}{2} \\
&\quad - c_0 M_1 k_1 \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{\alpha+1}{2}} \|\nabla u_n\|_2^{\alpha+1} - (c_0 k_2 M_1 + M_2 + M_3) \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{1}{2}} \|\nabla u_n\|_2.
\end{aligned} \tag{3.7}$$

Moreover, by Lemma 2.2,

$$\begin{aligned}
D_0 \|\tilde{u}_n\| &= D_0 \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\tilde{u}_n(t)|^2 dt + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\nabla u_n(t)|^2 dt \right\}^{1/2} \\
&\leq D_0 ((C^*)^2 + 1)^{1/2} \|\nabla u_n\|_2.
\end{aligned} \tag{3.8}$$

Note that  $m > \frac{1}{2}$ . So (3.7) and (3.8) imply that

$$\frac{1}{2m-1} a_2 c_0^2 M_1^2 w^2(|\tilde{u}_n|) \geq \|\nabla u_n\|_2^2 + D_1, \tag{3.9}$$

where

$$\begin{aligned}
D_1 &= \min_{s \in [0, +\infty)} \left\{ \left( \frac{1}{2} - \frac{1}{2a_2} \sum_{j=1}^p \frac{T_j^2}{12} \right) s^2 - c_0 M_1 k_1 \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{\alpha+1}{2}} s^{\alpha+1} \right. \\
&\quad \left. - \left[ (c_0 k_2 M_1 + M_2 + M_3) \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{1}{2}} - D_0 ((C^*)^2 + 1)^{1/2} \right] s \right\}.
\end{aligned}$$

Since  $a_2 > \sum_{j=1}^p (T_j^2/12)$ , then  $0 > D_1 > -\infty$ . Hence, there exists a positive constant  $D_2$  such that

$$\|\nabla u_n\|_2 \leq \sqrt{\frac{a_2}{2m-1}} c_0 M_1 w(|\tilde{u}_n|) + D_2. \tag{3.10}$$

Similar to (3.2), we have

$$\begin{aligned}
&\left| \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [F(t, u_n(t)) - F(t, \tilde{u}_n)] dt \right| \\
&\leq c_0 \|\tilde{u}_n\|_\infty w(|\tilde{u}_n|) \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g(t) dt + c_0 \|\tilde{u}_n\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 g(t) [k_1 |s \tilde{u}_n(t)|^\alpha + k_2] ds dt \\
&\quad + \|\tilde{u}_n\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T h(t) dt \\
&\leq \frac{\|\tilde{u}_n\|_\infty^2}{2\sqrt{a_2} \sqrt{\sum_{j=1}^p \frac{T_j^2}{12}}} + \frac{\sqrt{a_2} \sqrt{\sum_{j=1}^p \frac{T_j^2}{12}} c_0^2 M_1^2 w^2(|\tilde{u}_n|)}{2}
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
& + \frac{c_0 M_1 k_1}{\alpha + 1} \|\tilde{u}_n\|_\infty^{\alpha+1} + c_0 k_2 M_1 \|\tilde{u}_n\|_\infty + M_2 \|\tilde{u}_n\|_\infty \\
& \leq \frac{1}{2\sqrt{a_2}} \sqrt{\sum_{j=1}^p \frac{T_j^2}{12}} \|\nabla u_n\|_2^2 + \frac{\sqrt{a_2} \sqrt{\sum_{j=1}^p \frac{T_j^2}{12} c_0^2 M_1^2 w^2(|\bar{u}_n|)}}{2} + \frac{c_0 M_1 k_1}{\alpha + 1} \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{\alpha+1}{2}} \|\nabla u_n\|_2^{\alpha+1} \\
& \quad + (c_0 k_2 M_1 + M_2) \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{1}{2}} \|\nabla u_n\|_2.
\end{aligned}$$

It follows from (3.5), (3.9), (3.10) and (3.11) that

$$\begin{aligned}
& -D_0 \leq \varphi(u_n) \\
& = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[ \frac{1}{2} (P(t) \nabla u_n(t), \nabla u_n(t)) + F(t, u_n(t)) - F(t, \bar{u}_n) + F(t, \bar{u}_n) + (e(t), u_n(t)) \right] dt \\
& \leq \frac{\|P\|}{2} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\nabla u_n(t)|^2 dt + \frac{1}{2\sqrt{a_2}} \sqrt{\sum_{j=1}^p \frac{T_j^2}{12}} \|\nabla u_n\|_2^2 + \frac{\sqrt{a_2} \sqrt{\sum_{j=1}^p \frac{T_j^2}{12} c_0^2 M_1^2 w^2(|\bar{u}_n|)}}{2} \\
& \quad + \frac{c_0 M_1 k_1}{\alpha + 1} \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{\alpha+1}{2}} \|\nabla u_n\|_2^{\alpha+1} + (c_0 k_2 M_1 + M_2) \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{1}{2}} \|\nabla u_n\|_2 \\
& \quad + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, \bar{u}_n) dt + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (e(t), \tilde{u}_n(t)) dt \\
& \leq \left( \frac{\|P\|}{2} + \frac{1}{2\sqrt{a_2}} \sqrt{\sum_{j=1}^p \frac{T_j^2}{12}} \right) \left[ \frac{1}{2m-1} a_2 c_0^2 M_1^2 w^2(|\bar{u}_n|) - D_1 \right] \\
& \quad + \frac{c_0 M_1 k_1}{\alpha + 1} \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{\alpha+1}{2}} \left[ \sqrt{\frac{a_2}{2m-1}} c_0 M_1 w(|\bar{u}_n|) + D_2 \right]^{\alpha+1} \\
& \quad + (c_0 k_2 M_1 + M_2 + M_3) \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{1}{2}} \left[ \sqrt{\frac{a_2}{2m-1}} c_0 M_1 w(|\bar{u}_n|) + D_2 \right] \\
& \quad + \frac{\sqrt{a_2} \sqrt{\sum_{j=1}^p \frac{T_j^2}{12} c_0^2 M_1^2}}{2} w^2(|\bar{u}_n|) + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, \bar{u}_n) dt \\
& \leq w^2(|\bar{u}_n|) \left[ w^{-2}(|\bar{u}_n|) \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, \bar{u}_n) dt + \left( \frac{a_2 \|P\|}{4m-2} + \frac{m\sqrt{a_2}}{2m-1} \sqrt{\sum_{j=1}^p \frac{T_j^2}{12}} \right) c_0^2 M_1^2 \right] \\
& \quad + D_3 w^{\alpha+1}(|\bar{u}_n|) + D_4 w(|\bar{u}_n|) + D_5, \tag{3.12}
\end{aligned}$$

where  $D_3$ ,  $D_4$  and  $D_5$  are positive constants. Then (3.12), (3.4) and (W)(iv) imply that  $\{w(|\bar{u}_n|)\}$  and  $\{\bar{u}_n\}$  are bounded. Furthermore, by (3.10), we obtain that  $\{\|\nabla u_n\|_2\}$  is bounded. Hence  $\{u_n\}$  is bounded in  $V$ , that is, there is a constant  $D_6 > 0$  such that  $\|u_n\| \leq D_6$ . Then there is a subsequence, still denoted by  $\{u_n\}$ , such that  $\{u_n\}$  weakly converges to  $u^* \in V$ . Next we prove that  $u_n \rightarrow u^*$  in  $V$ , as  $n \rightarrow \infty$ . The proof is similar to [13]. By Lemma 2.4, we know that  $\{u_n\}$  converges uniformly to  $u^*$  on any compact subset of  $\mathbb{R}$ . Then for any  $T > 0$ ,

we have

$$\max_{t \in [-T, T]} |u_n(t) - u^*(t)| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.13)$$

Since  $\nabla F(t, x)$  is almost periodic in  $t$  uniformly for  $x \in \mathbb{R}^N$  and continuously differentiable in  $x$ , then  $\{\nabla F(t, u_n(t))\}$  is bounded in  $\mathbb{R} \times [-\sqrt{p^2 + T^2/12} D_6, \sqrt{p^2 + T^2/12} D_6]$  and  $\{\nabla F(t, u^*(t))\}$  is bounded in  $\mathbb{R} \times [-\|u^*\|_\infty, \|u^*\|_\infty]$ . Hence, by (3.13), there exists  $D_7 > 0$  such that

$$\begin{aligned} & \left| \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (\nabla F(t, u_n(t)) - \nabla F(t, u^*(t)), u_n(t) - u^*(t)) dt \right| \\ & \leq \lim_{T \rightarrow \infty} \frac{\max_{t \in [-T, T]} |u_n(t) - u^*(t)|}{2T} \int_{-T}^T |\nabla F(t, u_n(t)) - \nabla F(t, u^*(t))| dt \\ & \leq D_7 \lim_{T \rightarrow \infty} \max_{t \in [-T, T]} |u_n(t) - u^*(t)| \\ & \rightarrow 0, \quad \text{as } n \rightarrow \infty \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} & \left| \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (e(t), u_n(t) - u^*(t)) dt \right| \\ & \leq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |e(t)| |u_n(t) - u^*(t)| dt \\ & \leq \lim_{T \rightarrow \infty} \frac{\max_{t \in [-T, T]} |u_n(t) - u^*(t)|}{2T} \int_{-T}^T |e(t)| dt \\ & = M_3 \lim_{T \rightarrow \infty} \frac{\max_{t \in [-T, T]} |u_n(t) - u^*(t)|}{2T} \\ & \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.15)$$

Since  $u_n$  weakly converges to  $u^*$  and  $\varphi'(u_n) \rightarrow 0$  as  $n \rightarrow \infty$ , the boundedness of  $\{u_n\}$  implies that

$$\langle \varphi'(u_n) - \varphi'(u^*), u_n - u^* \rangle \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.16)$$

Note that

$$\begin{aligned} & \langle \varphi'(u_n) - \varphi'(u^*), u_n - u^* \rangle \\ & = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[ (P(t) \nabla u_n(t) - P(t) \nabla u^*(t), \nabla u_n(t) - \nabla u^*(t)) \right. \\ & \quad + (\nabla F(t, u_n(t)) - \nabla F(t, u^*(t)), u_n(t) - u^*(t)) \\ & \quad \left. + (e(t), u_n(t) - u^*(t)) \right] dt. \end{aligned} \quad (3.17)$$

Then (3.14), (3.15), (3.16) and (3.17) imply that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (P(t) \nabla u_n(t) - P(t) \nabla u^*(t), \nabla u_n(t) - \nabla u^*(t)) dt \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (3.18)$$

which, combining with  $(\mathcal{P})$ , implies that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\nabla u_n(t) - \nabla u^*(t)|^2 dt \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.19)$$

By (3.13), it is easy to obtain that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |u_n(t) - u^*(t)|^2 dt \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.20)$$

Hence, (3.19) and (3.20) imply that  $\|u_n - u^*\| \rightarrow 0$  in  $V$ . Thus we prove that  $\varphi$  satisfies the (PS) condition.

Next, we prove that

$$\varphi(u) \rightarrow +\infty, \quad \text{as } u \in \tilde{V} \text{ and } \|u\| \rightarrow \infty, \quad (3.21)$$

$$\varphi(u) \rightarrow -\infty, \quad \text{as } u \in \tilde{V} \text{ and } |u| \rightarrow \infty. \quad (3.22)$$

In fact, when  $u \in \tilde{V}$ , it follows from  $(f_4)'$  and Lemma 2.2 that

$$\begin{aligned} & \left| \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [F(t, u(t)) - F(t, 0)] dt \right| \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \left| \int_{-T}^T [F(t, u(t)) - F(t, 0)] dt \right| \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \left| \int_{-T}^T \int_0^1 (\nabla F(t, su(t)), u(t)) ds dt \right| \\ &\leq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 |\nabla F(t, su(t))| |u(t)| ds dt \\ &\leq \|u\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 [g(t)w(|su(t)|) + h(t)] ds dt \\ &\leq \|u\|_\infty \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_0^1 [g(t)k_1 s^\alpha |u(t)|^\alpha + k_2 g(t) + h(t)] ds dt \\ &\leq \frac{M_1 k_1}{\alpha + 1} \|u\|_\infty^{\alpha+1} + (k_2 M_1 + M_2) \|u\|_\infty \end{aligned} \quad (3.23)$$

$$\leq \frac{M_1 k_1}{\alpha + 1} \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{\alpha+1}{2}} \|\nabla u\|_2^{\alpha+1} + (k_2 M_1 + M_2) \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{1}{2}} \|\nabla u\|_2. \quad (3.24)$$

Then

$$\begin{aligned} \varphi(u) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[ \frac{1}{2} (P(t) \nabla u(t), \nabla u(t)) + F(t, u(t)) - F(t, 0) + F(t, 0) + (e(t), u(t)) \right] dt \\ &\geq m \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\nabla u(t)|^2 dt - \frac{M_1 k_1}{\alpha + 1} \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{\alpha+1}{2}} \|\nabla u\|_2^{\alpha+1} \\ &\quad - (k_2 M_1 + M_2 + M_3) \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{1}{2}} \|\nabla u\|_2, \quad \text{for all } u \in \tilde{V}. \end{aligned} \quad (3.25)$$

By Lemma 2.2, it is easy to see that  $\|\nabla u\|_2$  is equivalent to  $\|u\|$  in  $\tilde{V}$ . So (3.25) implies that (3.21) holds.

Moreover, by  $(f_5)''$  and  $(\mathcal{W})(iv)$ , it is easy to see that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, x) dt \rightarrow -\infty, \quad \text{as } |x| \rightarrow \infty.$$

Then when  $u \in \tilde{V}$ , by  $(\mathcal{E})$ , we have

$$\varphi(u) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t, u) dt \rightarrow -\infty, \quad \text{as } |u| \rightarrow \infty.$$

(3.22) holds. Thus, by using the saddle point theorem, we complete the proof of Theorem 1.6.  $\square$

## 4 Multiplicity

In this section, we will use the following critical point theorem due to Ding [6] to prove Theorem 1.8.

**Lemma 4.1** ([6, Lemma 2.4]). *Let  $E$  be an infinite dimensional Banach space and let  $f \in C^1(E, \mathbb{R})$  be even, satisfy (PS), and  $f(0) = 0$ . If  $E = E_1 \oplus E_2$ , where  $E_1$  is finite dimensional, and  $f$  satisfies*

- (i)  $f$  is bounded from above on  $E_2$ ,
- (ii) for each finite dimensional subspace  $\tilde{E} \subset E$ , there are positive constants  $\rho = \rho(\tilde{E})$  and  $\sigma = \sigma(\tilde{E})$  such that  $f \geq 0$  on  $B_\rho \cap \tilde{E}$  and  $f|_{\partial B_\rho \cap \tilde{E}} \geq \sigma$  where  $B_\rho = \{x \in E; \|x\| \leq \rho\}$ ,

then  $f$  possesses infinitely many nontrivial critical points.

*Proof of Theorem 1.8.* Let  $-\varphi = f$ . Obviously, the critical points of  $-\varphi$  are still the solutions of system (1.1). Then  $(f_8)$  implies that  $-\varphi$  is even and  $-\varphi(0) = 0$ . Let  $E = V$ ,  $E_1 = \tilde{V}$  and  $E_2 = \tilde{V}$ . By  $(f_4)'$  and Lemma 2.2, we know that (3.25) holds. Then

$$\begin{aligned} -\varphi(u) &\leq -m \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\nabla u(t)|^2 dt + \frac{M_1 k_1}{\alpha + 1} \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{\alpha+1}{2}} \|\nabla u\|_2^{\alpha+1} \\ &\quad + (k_2 M_1 + M_2 + M_3) \left( \sum_{j=1}^p \frac{T_j^2}{12} \right)^{\frac{1}{2}} \|\nabla u\|_2, \quad \text{for all } u \in \tilde{V}, \end{aligned}$$

which implies that

$$-\varphi(u) \rightarrow -\infty, \quad \text{as } u \in \tilde{V} \text{ and } \|u\| \rightarrow \infty.$$

So  $-\varphi$  is bounded from above on  $E_2$ . Condition (i) of Lemma 4.1 holds. Next we prove that  $-\varphi$  also satisfies (ii) of Lemma 4.1. For each finite dimensional subspace  $\tilde{E} \subset V$ , all norms are equivalent on  $\tilde{E}$ . Then there exist positive constants  $d_1 := d_1(\tilde{E})$  and  $d_2 := d_2(\tilde{E})$  such that

$$d_1 \|u\| \leq \|u\|_2 \leq d_2 \|u\|.$$

It follows from  $(f_9)$  that there exist  $M > \frac{\|P\|}{2d_1^2}$  and  $r > 0$  such that

$$F(t, x) \leq -M|x|^2, \quad \forall |x| \leq r.$$

Note that  $e(t) \equiv 0$ . Let

$$\rho = \frac{r}{\sqrt{p^2 + \sum_{j=1}^p \frac{T_j^2}{12}}}.$$



Then for all  $u \in B_\rho \cap \tilde{V}$ , we have

$$\begin{aligned} -\varphi(u) &= -\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[ \frac{1}{2} (P(t) \nabla u(t), \nabla u(t)) + F(t, u(t)) \right] dt \\ &\geq -\frac{\|P\|}{2} \|\nabla u\|_2^2 + M \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |u(t)|^2 dt \\ &\geq -\frac{\|P\|}{2} \|u\|^2 + Md_1^2 \|u\|^2 \\ &\geq 0. \end{aligned}$$

Let  $\sigma = (Md_1^2 - \frac{\|P\|}{2})\rho$ . Then (ii) of Lemma 4.1 holds.

By (3.3), we know that

$$-\varphi(u) \rightarrow -\infty, \quad \text{as } u \in V \text{ and } \|u\| \rightarrow \infty. \quad (4.1)$$

Then for any sequence  $\{u_n\} \subset V$  such that  $-\varphi(u_n)$  is bounded and  $-\varphi'(u_n) \rightarrow 0$  as  $n \rightarrow \infty$ , (4.1) implies that  $\{u_n\}$  is bounded. Similar to the argument of Theorem 1.6,  $\{u_n\}$  has a convergent subsequence in  $V$ . Hence,  $-\varphi$  satisfies (PS) condition. Thus by Lemma 4.1, we obtain that  $-\varphi$  has infinitely many nontrivial critical points. The proof is complete.  $\square$

*Proof of Theorem 1.9.* By the proof of Theorem 1.6, it is easy to see that  $-\varphi$  also satisfies (PS) condition. Moreover, by the proof of Theorem 1.8, we know that  $(f_4)'$  and Lemma 2.2 imply that (i) of Lemma 4.1 and  $(f_9)$  implies that (ii) of Lemma 4.1 holds. Thus by Lemma 4.1, we obtain that  $-\varphi$  has infinitely many nontrivial critical points. The proof is complete.  $\square$

## Acknowledgements

This work is supported by the National Natural Science Foundation of China (No: 11301235), Tianyuan Fund for Mathematics of the National Natural Science Foundation of China (No: 11226135) and the Fund for Fostering Talents in Kunming University of Science and Technology (No: KKSJ201207032).

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