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Well-posed Dirichlet problems pertaining to the Duffing equation

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Abstract. In this paper we investigate existence and continuous dependence on a functional parameter of Duffing's type equation with Dirichlet boundary value conditions. The method applied relies on variational investigation of auxiliary problems and then in order to prove existence, the Banach fixed point theorem is applied. Uniqueness of solutions is also examined.

Keywords: Dirichlet boundary value problems, nonlinear problems, variational method, Duffing equation.

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1 Introduction

We investigate the classical variational problem for a Duffing type equation. It concerns a nonlinear second order differential equation used to model certain damped and driven oscillators, firstly introduced in [4] by Georg Duffing who was inspired by joint works of O. von Martienssen and J. Biermanns. Variational approach was found successful in proving existence of solution to this problem. The classical variational problem for a Duffing type equation with Dirichlet boundary condition yields whether there exists a function $x \in \mathrm{H}_0^1(0,1)$, such that

$$\frac{d^2}{dt^2}x(t) + r(t)\frac{d}{dt}x(t) + G(t, x(t), u(t)) = 0.$$

Here $r \in \mathrm{C}^1(0,1)$ stands for the friction term, and G is a nonlinear term, satisfying some suitable assumptions. In fact G can correspond to a restoring force for a string in string-damper system. The Duffing's equation was also found applicable for some problems concerning current and flux, thus r and G may as well corresponds to its coefficients. The equation is well known for its chaotical behaviour, well described by Holmes [6, 7, 9, 10, 11] and jointly by Holmes and Moon [18, 8]. Recently in [1, 2, 5, 20] some variational approaches were used

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in order to receive the existence results for both periodic and Dirichlet type boundary conditions. See for example in [1, 2, 12, 20], where variational approaches are applied such as a direct method, mountain geometry, a min-max theorem due to Manashevich. We note also paper [16] where the topological method is used.

Since a Duffing equation serves a mechanical model, it is also important to know whether the solution, once its existence is proved, depends continuously on a functional parameter and also whether this solution is unique. Into the classical variational problem we introduce a control function $u \in \mathrm{H}_0^1(0,1)$ with only function G depending on it. Thus it is of interest to know the conditions which guarantee

a) the existence of solutions,

b) their uniqueness,

c) dependence of solutions on parameters.

This is sometimes known as Hadamard's programme and problems satisfying all three conditions are called well-posed. The question of continuous dependence on parameters has a great impact on future applications of any model since it is desirable to know whether the solution to the small deviation from the model would return, in a continuous way, to the solution of the original model. In our investigations we base somehow on [15] however, we use much simpler approach. As concerns the existence of solutions we use generalization of our earlier result [14].

We consider the problem in the following form,

$$\begin{cases} \frac{d^2}{dt^2}x(t) + r(t)\frac{d}{dt}x(t) + g(t, x(t), u(t)) - f(t, x(t)) = 0, \quad \text{a.e. } t \in (0,1), \\ x(0) = x(1) = 0. \end{cases} \tag{DEq}$$

under the assumptions that $r \in \mathrm{L}^\infty(0,1)$ and some further requirements on f and g. We introduce $u \in \mathrm{L}^q(0,1)$ to be the functional parameter. Solutions to the above problem are investigated in $\mathrm{H}_0^1(0,1)$ and these are understood as the weak solutions. We examine problem (DEq) by a kind of a two step method. It can be described as follows. At first we substitute $h := \frac{dx(t)}{dt}$ and we consider an auxiliary problem of a form

$$\begin{cases} \frac{d^2}{dt^2}x(t) + r(t)h(t) + g(t, x(t), u(t)) - f(t, x(t)) = 0, \quad \text{a.e. } t \in (0,1), \\ x(0) = x(1) = 0. \end{cases} \tag{AuxEq}$$

Once the auxiliary problem is solved, we apply the Banach fixed point theorem using condition (H5) to obtain solutions of (DEq). By the fundamental lemma of calculus of variations any weak solutions x to (DEq) is a classical one i.e.

$$x \in \mathrm{H}_0^1(0,1) \cap \mathrm{W}^{2,1}(0,1).$$

Moreover, the functions $g, G\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f, F\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ will be Carathéodory functions, satisfying the conditions below:

$$F(t,x) = \int_0^x f(t,s)\,ds, \qquad \forall d > 0\ \exists f_d \in \mathrm{L}^1(0,1)\ \forall x \in [-d,d], \quad |f(t,x)| \le f_d(t), \tag{H1}$$

and we assume $p \in (1,2)$, $q \in (1,+\infty)$, $s \in (1,q)$ are such that for all $x, u \in \mathbb{R}$ and a.e. $t \in (0,1)$

$$\begin{aligned} G(t,x,u) &= \int_0^x g(t,s,u)ds, \\ |g(t,x,u)| &\le |x|^{p-1} a(t) |u(t)|^s, \; a \in \mathrm{L}^{\frac{q}{q-s}}(0,1), \\ |G(t,x,u)| &\le \left(|x|^p \frac{a(t)}{p} + b(t)\right) |u(t)|^s, \; b \in \mathrm{L}^{\frac{q}{q-s}}(0,1). \end{aligned} \tag{H2}$$

In case $a, b \in \mathrm{L}^\infty(0,1)$ in the (H2) it is possible to assume that $s \in (1,q]$. We shall consider two versions of additional assumptions that will produce different results.

1. Convex version: for a.e. $t \in [0,1]$ a function
$$\mathbb{R} \ni x \mapsto F(t,x) \tag{H3}$$
is convex and $f(t,0) \in \mathrm{L}^1(0,1)$.
2. Bounded version: there exist constants $A \in \mathbb{R} \setminus \{0\}, B, C \in \mathbb{R}$ such that
$$F(t,x) \ge A|x|^2 + B|x| + C, \; |A| < \frac{1}{2} \tag{H4}$$
for all $x \in \mathbb{R}$, for a.e. $t \in [0,1]$.

In order to obtain limit solution we assume that for any $u \in \mathrm{L}^q(0,1)$ there exists a constant $L(u) < 1$ such that

$$\begin{aligned} &\int_0^1 \left(g(t,x(t),u(t)) - f(t,x(t)) - g(t,y(t),u(t)) + f(t,y(t))\right)(x(t) - y(t))\, dt \\ &\quad \le L(u) \|x - y\|^2_{\mathrm{H}_0^1(0,1)}, \qquad \frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1 - L(u)} < 1, \end{aligned} \tag{H5}$$

for any $x,y \in \mathrm{H}_0^1(0,1)$, $x \neq y$. To obtain continuous dependence on the functional parameter we consider the following (stronger) version of assumptions. Assume there exists $d^* \in (0,1)$ such

$$\begin{aligned} F(t,x) &= \int_0^x f(t,s)\, ds, \\ \exists \bar{f} \in \mathrm{L}^1(0,1)\, \exists_{M>0} \forall |x| > M, &\quad |f(t,x)| \le \bar{f}(t)\left(1 + |x|^{d*}\right), \\ \exists f_M \in \mathrm{L}^1(0,1)\, \forall x \in [-M,M], &\quad |f(t,x)| \le f_M(t). \end{aligned} \tag{H1c}$$

2 Preliminaries

In this paper we use several well known facts.

Lemma 2.1 ([14]). *Let $1 \le p < q, x \in L^q(0,1), f \in L^{\frac{q}{q-p}}(0,1)$. Then*

$$\int_0^1 |x(t)|^p |f(t)|\, dt \le \|x\|^p_{L^q(0,1)} \cdot \|f\|_{L^{\frac{q}{q-p}}(0,1)}.$$

Lemma 2.2 ([14]). *Let $1 \le p < q$ and $x \in L^q(0,1)$. Then*

$$\|x\|_{L^p(0,1)} \le \|x\|_{L^q(0,1)}.$$

Theorem 2.3 ([19, Thm. 6.18]). *Let x be a μ-measurable function defined on Ω with $\mu(\Omega) < \infty$. If for every $p \in [1,\infty)$, $x \in L^p(\Omega)$ and $\sup\limits_{1\le p<\infty} \|x\|_{L^p(\Omega)} < \infty$ then $x \in L^\infty(\Omega)$ and*

$$\|x\|_{L^\infty(0,1)} = \lim_{p\to\infty} \|x\|_{L^p(\Omega)}.$$

We shall also require Poincaré and Sobolev type inequality in the following form.

Lemma 2.4 (Poincaré inequality [3, Prop. 8.13, p. 218]). *Let $x \in H_0^1(0,1)$. Then*

$$\|x\|_{L^2(0,1)} \le \left\|\frac{dx}{dt}\right\|_{L^2(0,1)}.$$

Lemma 2.5 (Sobolev type inequality [3, Prop. 8.13, p. 218]). *Let $x \in H_0^1(0,1)$. Then*

$$\|x\|_{L^\infty(0,1)} \le \left\|\frac{dx}{dt}\right\|_{L^2(0,1)}.$$

The inequalities are proved in [14].

Since the Poincaré inequality holds we shall use the following norm on $\mathrm{H}_0^1(0,1)$:

$$\|x\|^2_{\mathrm{H}_0^1(0,1)} := \int_0^1 \left(\frac{dx}{dt}(t)\right)^2 dt.$$

Lemma 2.6 (Fundamental lemma of calculus of variations [17, Lemma 1.1, p. 31, sec. 1.3]). *Let $v \in L^2(I,\mathbb{R})$, $I = [0,1]$, $w \in L^1(I,\mathbb{R})$ be such functions that*

$$\int_I v(x)h'(x)\,dx = -\int_I w(x)h(x)\,dx,$$

for any $h \in H_0^1(I)$. Then there exists a constant $c \in \mathbb{R}$ such that

$$v(x) = \int_0^x w(s)\,ds + c,$$

for almost every $x \in I$.

Theorem 2.7 ([17]). *Let E be reflexive Banach space and functional $f\colon E \to \mathbb{R}$ is s.w.l.s.c. and coercive then there exists a function that minimizes f.*

Theorem 2.8 (Krasnoselskii's theorem [13]). *Let $\Omega \subset \mathbb{R}$ be an interval and let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. If for any convergent sequence $(x_n)_{n\in\mathbb{N}} \subset L^2(\Omega)$ there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ and a function $h \in L^p(\Omega)$, $1 \le p < \infty$, such that*

$$|f(t, x_{n_i}(t))| \le h(t),$$

for all $i \in \mathbb{N}$ and $t \in \Omega$ a.e., then the Nemytskii's operator

$$F : L^2(\Omega) \ni (x) \to f(\cdot, x(\cdot)) \in L^p(\Omega)$$

is well-defined and sequentially continuous, that is, if

$$x_n \overset{n\to\infty}{\longrightarrow} x_0 \quad in\ L^2(\Omega)$$

then

$$F(x_n) \overset{n\to\infty}{\longrightarrow} F(x_0) \quad in\ L^p(\Omega).$$

Theorem 2.9 (Duality pairing convergence [3, prop. 3.5. (iv)]). *Let E be a Banach space. If $x_n \rightharpoonup x$ in E and if $f_n \to f$ strongly in E^* then*

$$\langle f_n, x_n \rangle \to \langle f, x \rangle$$

strongly.

3 Existence result for the auxiliary problem

In order to solve (DEq) we introduce an auxiliary problem

$$\begin{cases} \frac{d^2}{dt^2}x(t) + r(t)h(t) + g(t, x(t), u(t)) - f(t, x(t)) = 0, \quad \text{a.e. } t \in (0,1), \\ x(0) = x(1) = 0. \end{cases} \tag{AuxEq}$$

The above problem is in a variational form. We consider the following functional

$$J_u(x) = \int_0^1 \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - r(t)h(t)x(t) + F(t, x(t)) - G(t, x(t), u(t))\, dt.$$

We prove that critical points to J_u are the weak solutions to (AuxEq). In order to prove that problem (AuxEq) has at least one solution it is sufficient to show that:

1. functional J_u is well defined and differentiable in sense of Gâteaux,
2. functional J_u is coercive and sequentially weakly lower semicontinuous,
3. and critical points of J_u are the solutions of (AuxEq).

In the sequel we shall assume $u \in \mathrm{L}^q(0,1)$ to be a fixed parameter. In order to simplify proofs we introduce the following functionals.

$$J_u^1(x) = \int_0^1 \frac{1}{2}\left(\frac{dx}{dt}\right)^2 dt,$$

$$J_u^2(x) = \int_0^1 r(t)h(t)x(t)\, dt,$$

$$J_u^3(x) = \int_0^1 F(t, x(t))\, dt,$$

$$J_u^4(x) = \int_0^1 G(t, x(t), u(t))\, dt.$$

Then $J_u = J_u^1 - J_u^2 + J_u^3 - J_u^4$.

We start by proving that the functional J_u is well defined and admits a Gâteaux derivative.

Lemma 3.1. *If* (H1) *and* (H2) *hold then the functional* J_u *is well defined for any* $x \in H_0^1(0,1)$.

Proof of this fact is elementary.

Lemma 3.2. *Assume* (H1) *holds. Then*

$$\lim_{\lambda \to 0} \int_0^1 \frac{F(t,x(t)+\lambda v(t)) - F(t,x(t))}{\lambda}\, dt = \int_0^1 \lim_{\lambda \to 0} \frac{F(t,x(t)+\lambda v(t)) - F(t,x(t))}{\lambda}\, dt$$

for every x,$v \in H_0^1(0,1)$.

Lemma 3.3. *Assume* (H2) *holds. Then*

$$\lim_{\lambda \to 0} \int_0^1 \frac{G(t,x(t)+\lambda v(t),u(t)) - G(t,x(t),u(t))}{\lambda}\, dt$$
$$= \int_0^1 \lim_{\lambda \to 0} \frac{G(t,x(t)+\lambda v(t),u(t)) - G(t,x(t),u(t))}{\lambda}\, dt$$

for every $x, v \in H_0^1(0,1)$.

The proof of the above properties follows from Lebesgue's dominated convergence theorem.

Lemma 3.4. *Assume that* (H1) *and* (H2) *hold. Then functional* J_u *is differentiable in sense of Gâteaux and its derivative is equal to*

$$\partial J_u(x;v) = \int_0^1 \frac{dx(t)}{dt}\frac{dv(t)}{dt} + \left[-r(t)h(t) + f(t,x(t)) - g(t,x(t),u(t))\right] v(t)\, dt. \tag{3.1}$$

for all $v \in H_0^1(0,1)$.

Proof for this fact is elementary.

We will now focus on proving that problem of finding critical points of functional J_u is equivalent to solving problem (AuxEq).

Definition 3.5. Every $x \in \mathrm{H}_0^1(0,1)$ which satisfies the equality

$$\forall v \in \mathrm{H}_0^1(0,1) \quad \partial J_u(x;v) = 0, \tag{CPP}$$

shall be called a critical point for J_u.

Definition 3.6. Every $x \in \mathrm{H}_0^1(0,1)$ which satisfies the equality

$$\int_0^1 \left(r(t)h(t) + g(t,x(t),u(t)) - f(t,x(t))\right) v(t) - \frac{dx(t)}{dt}\frac{dv(t)}{dt} dt = 0, \tag{WSP}$$

for all $v \in \mathrm{H}_0^1(0,1)$ shall be called a weak solution to (AuxEq).

Lemma 3.7. *Assume that* (H1) *and* (H2) *hold. Let* $x \in H_0^1(0,1)$. *Then the following conditions are equivalent:*

(1) x is a critical point to J_u (x solves (CPP));

(2) x is a weak solution to (AuxEq) (x solves (WSP)).

Proof follows from Lemma (3.4). We also prove that the solution has better regularity than $H_0^1(0,1)$.

Lemma 3.8. *Let x be a solution to (WSP). If (H1) and (H2) are both satisfied, then this solution is classical solution to (AuxEq).*

The proof follows from the fundamental lemma of calculus of variations.
Finally we prove the existence of critical point.

Lemma 3.9. *Assume (H1) and (H2) holds. Then the functional J_u is sequentially weakly lower semi-continuous.*

Proof. It is obvious that

$$x \mapsto \int_0^1 \frac{1}{2}\left(\frac{d}{dt}x(t)\right)^2 - r(t)h(t)x(t)\,dt,$$

is s.w.l.s.c. It is easy to show that $-J_u^4$ is s.w.l.s.c. using Lebesgue's domininated convergence theorem. We prove that J_u^3 is s.w.l.s.c. Assume $x_n \rightharpoonup x$ in $H_0^1(0,1)$. We will prove that

$$\liminf J_u^3(x_n) \geq J_u^3(x).$$

We reason by contradiction. Suppose there exists such a subsequence that

$$\lim J_u^3(x_{k_n}) < J_u^3(x).$$

By the Arzelà–Ascoli theorem this subsequence admits a subsubsequence (x_{l_n}) convergent strongly in $C(0,1)$. Thus it is bounded in $C(0,1)$ norm. As (H1) holds, we may reason by using Lebesgue's dominated convergence theorem to obtain the inequality

$$J_u^3(x) > \lim J_u^3(x_{l_n}) = J_u^3(x).$$

Thus it contradicts the supposition. Finally J_u is s.w.l.s.c. □

Lemma 3.10. *Assume (H1) and (H2) hold. If additionally either (H3) or (H4) holds then J_u is coercive.*

Proof. We will prove that functional J_u is bounded from below by a coercive function depending on $\|x\|_{H_0^1(0,1)}$. Let $x \in H_0^1(0,1)$ be taken arbitrarily. We see that $J_u^1(x) = \frac{1}{2}\|x\|^2_{H_0^1(0,1)}$. By the Cauchy–Schwartz inequality one can prove that

$$-J_u^2(x) = \int_0^1 -r(t)h(t)x(t)\,dt \geq -\|r\|_{L^\infty(0,1)}\|h\|_{L^2(0,1)}\|x\|_{H_0^1(0,1)}.$$

We can easily calculate that

$$\begin{aligned}
\int_0^1 -G(t,x(t),u(t))\,dt &\geq \int_0^1 -\left(|x(t)|^p\frac{a(t)}{p} + b(t)\right)|u(t)|^s\,dt \\
&\geq -\frac{1}{p}\|x\|^p_{H_0^1(0,1)}\int_0^1 a(t)\,|u(t)|^s\,dt - \int_0^1 b(t)\,|u(t)|^s\,dt \\
&\geq -\frac{1}{p}\|x\|^p_{H_0^1(0,1)}\|a\|_{L^{\frac{q}{q-s}}(0,1)}\|u\|^s_{L^q(0,1)} - \|b\|_{L^{\frac{q}{q-s}}(0,1)}\|u\|^s_{L^q(0,1)}.
\end{aligned}$$

Assume (H3) holds. Then

$$F(t,x(t)) \geq F(t,0) + f(t,0)x(t), \quad \text{a.e. } t \in (0,1).$$

Then as we integrate by sides, we obtain

$$\int_0^1 F(t,x(t))dt \geq -\|F(\cdot,0)\|_{L^1(0,1)} - \|f(\cdot,0)\|_{L^1(0,1)} \|x\|_{H_0^1(0,1)}.$$

Thus if we assume (H3), functional J_u is obviously bounded from below by coercive function. Now we assume (H4). Then

$$\int_0^1 F(t,x(t))\,dt \geq A\,\|x\|^2_{H_0^1(0,1)} - |B|\,\|x\|_{H_0^1(0,1)} - |C|,$$

and thus functional is obviously coercive since $|A| < \frac{1}{2}$. □

We present the following result.

Theorem 3.11. *Assume* (H1) *and* (H2) *and either* (H3) *or* (H4) *hold. Then there exists at least one solution to problem* (AuxEq)*.*

Proof. By Lemmas 3.9 and 3.10, and reflexivity of $H_0^1(0,1)$, we see that assumptions of Theorem 2.7 are satisfied. Then there exists a critical point. By Lemma 3.8 this critical point is a classical solution to (AuxEq). □

4 Iterative scheme

In this section we shall prove that using equation (AuxEq) we may produce the solution of (DEq). Since we proved that (AuxEq) for each $h \in L^2(0,1)$ admits a classical solution it somehow define a solution operator Λ. Up to the section end we will assume $u \in L^q(0,1)$ to be a fixed parameter.

Theorem 4.1. *Assume that* (H1)*,* (H2) *and* (H5) *are satisfied and either* (H3) *or* (H4) *holds then problem* (DEq) *has exactly one solution.*

Proof. By Theorem (3.11) we know that for any function $h \in L^2(0,1)$ there exists a solution to problem (AuxEq). This means that for any function $v \in H_0^1(0,1)$ there exists a solution x_v to the following problem,

$$\begin{cases} \frac{d^2}{dt^2}x(t) + r(t)\frac{d}{dt}v(t) + g(t,x(t),u(t)) - f(t,x(t)) = 0, & \text{a.e. } t \in (0,1), \\ x(0) = x(1) = 0. \end{cases} \tag{4.1}$$

Let $\Psi\colon H_0^1(0,1) \mapsto 2^{H_0^1(0,1)}$ be a multivalued operator which to any $v \in H_0^1(0,1)$ assigns a set of solutions of (4.1) corresponding to this parameter. Let $\Lambda\colon H_0^1(0,1) \mapsto H_0^1(0,1)$ be an arbitrarily chosen single valued selection of operator Ψ, i.e. for any $v \in H_0^1(0,1)$ $\Lambda v \in \Psi v$. We prove that Λ is a contraction mapping.

Let $h, v \in \mathrm{H}_0^1(0,1)$. Denote $x_v := \Lambda v$, $x_h := \Lambda h$. Assume $x_v \neq x_h$. Otherwise condition for contraction mappings holds. Equations (4.1) for h and v are multiplied by $(x_v - x_h)$ and then integrated with respect to $t \in [0,1]$.

$$-\int_0^1 \frac{d^2 x_h}{dt^2}(x_h(t) - x_v(t))\,dt = \int_0^1 \left(r(t)\frac{dh(t)}{dt} + g(t, x_h(t), u(t) - f(t, x_h(t))\right)(x_h - x_v)\,dt,$$
$$-\int_0^1 \frac{d^2 x_v}{dt^2}(x_h(t) - x_v(t))\,dt = \int_0^1 \left(r(t)\frac{dv(t)}{dt} + g(t, x_v(t), u(t) - f(t, x_v(t))\right)(x_h - x_v)\,dt.$$

After subtracting the sides and integrating by parts we get

$$\|x_h - x_v\|^2_{\mathrm{H}_0^1(0,1)} = \int_0^1 \left(r(t)\frac{dh(t)}{dt} + g(t, x_h(t), u(t)) - f(t, x_h(t))\right)(x_h(t) - x_v(t))\,dt$$
$$- \int_0^1 \left(r(t)\frac{dv(t)}{dt} + g(t, x_v(t), u(t)) - f(t, x_v(t))\right)(x_h(t) - x_v(t))\,dt.$$

By $x_h \neq x_v$ relation, (H5) and by Theorem 2.5

$$\|x_h - x_v\|^2_{\mathrm{H}_0^1(0,1)} \leq \left(\|r\|_{\mathrm{L}^\infty(0,1)} \|h - v\|_{\mathrm{H}_0^1(0,1)} + L\,\|x_h - x_v\|_{\mathrm{H}_0^1(0,1)}\right)\|x_h - x_v\|_{\mathrm{H}_0^1(0,1)}\,.$$

Thus

$$\|x_h - x_v\|_{\mathrm{H}_0^1(0,1)} \leq \|r\|_{\mathrm{L}^\infty(0,1)} \|h - v\|_{\mathrm{H}_0^1(0,1)} + L\,\|x_h - x_v\|_{\mathrm{H}_0^1(0,1)}\,.$$

Finally we get

$$\|\Lambda h - \Lambda v\|_{\mathrm{H}_0^1(0,1)} = \|x_h - x_v\|_{\mathrm{H}_0^1(0,1)} \leq \frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1 - L}\|h - v\|_{\mathrm{H}_0^1(0,1)}\,.$$

Thus Λ is a contraction mapping. Then the assumptions of Banach's fixed point theorem are satisfied and thus Λ admits a single fixed point in $\mathrm{H}_0^1(0,1)$, which is a solution of (DEq).

However, since Λ was chosen arbitrarily we cannot be sure that there exists a unique solution. We reason by contradiction. Assume that x and y are two distinct solutions of (DEq). We multiply (DEq) by $(x(t) - y(t))$ and integrate over $[0,1]$ interval.

$$-\int_0^1 \frac{d^2 x}{dt^2}(x(t) - y(t))\,dt = \int_0^1 \left(r(t)\frac{dx(t)}{dt} + g(t, x(t), u(t) - f(t, x(t))\right)(x(t) - y(t))\,dt,$$
$$-\int_0^1 \frac{d^2 y}{dt^2}(x(t) - y(t))\,dt = \int_0^1 \left(r(t)\frac{dy(t)}{dt} + g(t, y(t), u(t) - f(t, y(t))\right)(x(t) - y(t))\,dt.$$

Similarly we obtain

$$\|x - y\|_{\mathrm{H}_0^1(0,1)} \leq \|r\|_{\mathrm{L}^\infty(0,1)} \|x - y\|_{\mathrm{H}_0^1(0,1)} + L\,\|x - y\|_{\mathrm{H}_0^1(0,1)} < \|x - y\|_{\mathrm{L}^\infty(0,1)}\,.$$

Thus it contradicts the assumption that there are two distinct solutions. □

We note that however we obtain the uniqueness of the weak solution, the classical one is also unique since Lemma 3.8 holds in this case as well. We can also prove similar property in the limit case with $p = 2$. In fact one can similarly prove that in case (H5) holds, the operator Λ is actually single-valued.

Lemma 4.2. *If $1 - \|u\|^s_{L^q(0,1)} \|a\|_{L^{\frac{q}{q-s}}(0,1)} > 0$, (H1), (H2), (H3) and (H5) are satisfied then problem (DEq) has at least one solution.*

Lemma 4.3. *If $1 - |A| - \|u\|^s_{L^q(0,1)} \|a\|_{L^{\frac{q}{q-s}}(0,1)} > 0$, (H1), (H2), (H4) and (H5) are satisfied then problem (DEq) has at least one solution.*

Proofs follow the steps from proof of Theorem 4.1. In the next section we will investigate the impact of functional parameter, which until now was considered as fixed.

5 Continuous dependence on functional parameter

We will prove that sequence of solutions corresponding to sequence of parameters is bounded.

Theorem 5.1. *Let $(u_k)_{k\in\mathbb{N}} \subset L^q(0,1)$ be a bounded sequence of functional parameters. Assume (H1c), (H2), (H5) are satisfied and either (H3) or (H4) holds. Then there exists a sequence $(x_k)_{k\in\mathbb{N}}$ of solutions to (DEq), such that each x_k corresponds to a parameter u_k and that sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in $H^1_0(0,1)$.*

Proof. Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence of functional parameters. By Theorem 4.1 for any u_k there exists $x_k \in \mathrm{H}^1_0(0,1) \cap \mathrm{W}^{2,1}(0,1)$ being a solution of (DEq). We may equivalently consider the following problem. For all $k \in \mathbb{N}$, and for all $v \in \mathrm{H}^1_0(0,1)$, we have that:

$$\int_0^1 \frac{d^2x_k(t)}{dt^2} h(t) + \left(r(t)\frac{dx_k}{dt}(t) + g(t, x_k(t), u_k(t)) - f(t, x_k(t)) \right) v(t)\, dt = 0.$$

We shall test against $v := x_k$ function. Then we have that

$$\int_0^1 \frac{d^2x_k(t)}{dt^2} x_k(t) + \left(r(t)\frac{dx_k}{dt}(t) + g(t, x_k(t), u_k(t)) - f(t, x_k(t)) \right) x_k(t)\, dt = 0.$$

We integrate by parts

$$\int_0^1 \left(\frac{dx_k(t)}{dt}\right)^2 dt = \int_0^1 \left(r(t)\frac{dx_k(t)}{dt} + g(t, x_k(t), u_k(t)) - f(t, x_k(t)) \right) x_k(t)\, dt.$$

By Lemma 2.5 we obtain that

$$\|x_k\|^2_{\mathrm{H}^1_0(0,1)} \le \left(\int_0^1 \left| r(t)\frac{dx_k(t)}{dt} + g(t, x_k(t), u_k(t)) - f(t, x_k(t)) \right| dt \right) \|x_k\|_{\mathrm{H}^1_0(0,1)}.$$

If $\|x_k\|_{\mathrm{H}^1_0(0,1)} = 0$, the assertion is trivial. We may assume that $\|x_k\|_{\mathrm{H}^1_0(0,1)} > 0$. Then:

$$\|x_k\|_{\mathrm{H}^1_0(0,1)} \le \int_0^1 \left| r(t)\frac{dx_k(t)}{dt} \right| dt + \int_0^1 |g(t, x_k(t), u_k(t)) - f(t, x_k(t))|\, dt.$$

Suppose the sequence x_k is unbounded in $\mathrm{H}_0^1(0,1)$. By (H1c) and (H2) for sufficiently large k we obtain

$$\|x_k\|_{\mathrm{H}_0^1(0,1)}\left(1-\|r\|_{\mathrm{L}^\infty(0,1)}\right)\le \|x_k\|^{p-1}_{\mathrm{H}_0^1(0,1)}\|u\|^s_{\mathrm{L}^q(0,1)}\|a\|_{\mathrm{L}^{\frac{q}{q-s}}(0,1)}+\|\bar f\|_{\mathrm{L}^1(0,1)}\left(1+\|x_k\|^{d*}_{\mathrm{H}_0^1(0,1)}\right).$$

The above is equivalent to

$$\begin{aligned}\|x_k\|_{\mathrm{H}_0^1(0,1)}\left(1-\|r\|_{\mathrm{L}^\infty(0,1)}\right)-\|x_k\|^{p-1}_{\mathrm{H}_0^1(0,1)}\|u\|^s_{\mathrm{L}^q(0,1)}\|a\|_{\mathrm{L}^{\frac{q}{q-s}}(0,1)}-\|\bar f\|_{\mathrm{L}^1(0,1)}\|x_k\|^{d*}_{\mathrm{H}_0^1(0,1)}\\ \le\|\bar f\|_{\mathrm{L}^1(0,1)}.\end{aligned}\tag{5.1}$$

Since the left-hand side is a coercive functional, its values would go up to infinity as $k\to\infty$. This contradicts (5.1). □

Now we focus on dependence on functional parameter.

Theorem 5.2. *Let $(u_k)\subset L^q(0,1)$, $k\in\mathbb{N}$ be a bounded sequence of functional parameters. Assume (H1c), (H2), (H5) are satisfied and either (H3) holds or (H4) does. Then there exists a sequence of solutions x_k of (DEq) corresponding to u_k. Moreover,*

- *If $u_k\to\overline{u}$ strongly in $L^q(0,1)$ then $(x_k)\rightharpoonup\bar x$ in $H_0^1(0,1)$ and $\bar x$ is a solution to (DEq) corresponding to $\overline{u}$.*
- *If $g(t,x,u)=\overline{g}(t,x)u$, and if assumption (H2) takes a form*

$$|\overline{g}(t,x)|\le|x|^{p-1}a(t),\quad a\in L^{\frac{q}{q-s}}(0,1),\tag{H2c}$$

then for any sequence of parameters $u_k\rightharpoonup\overline{u}$ converging weakly in $L^q(0,1)$ there exists a sequence of solutions to (DEq) such $x_k\rightharpoonup\bar x$ converging weakly in $H_0^1(0,1)$ and $\bar x$ is a solution to (DEq) corresponding to $\overline{u}$.

Proof. By Theorem 4.1 there exists a sequence of solutions x_k of (DEq) corresponding to each u_k. By Theorem 5.1 this sequence is bounded in $\mathrm{H}_0^1(0,1)$. By the Rellich–Kondrachov theorem this sequence admits a subsequence x_{n_k} convergent strongly in $\mathrm{L}^2(0,1)$ and in $\mathrm{C}([0,1])$ and also weakly in $\mathrm{H}_0^1(0,1)$. Let $\overline{x}$ denote the element such that $x_{n_k}\rightharpoonup\overline{x}$ in $\mathrm{H}_0^1(0,1)$. By the fundamental lemma of calculus of variations we can use equivalently the weak formulation. Let $v\in\mathrm{H}_0^1(0,1)$. Note that

$$-\int_0^1\frac{dx_{n_k}(t)}{dt}\frac{dv(t)}{dt}\,dt+\int_0^1\left(r(t)\frac{dx_{n_k}(t)}{dt}+g(t,x_{n_k}(t),u_{n_k}(t))-f(t,x_{n_k}(t))\right)v(t)\,dt=0.\tag{5.2}$$

Since (x_{n_k}) converges weakly in $\mathrm{H}_0^1(0,1)$ then by definition

$$-\int_0^1\frac{dx_{n_k}(t)}{dt}\frac{dv(t)}{dt}\,dt\to-\int_0^1\frac{d\overline{x}(t)}{dt}\frac{dv(t)}{dt}\,dt.$$

Similarly

$$\int_0^1 r(t)\frac{dx_{n_k}(t)}{dt}v(t)\,dt\to\int_0^1 r(t)\frac{d\overline{x}(t)}{dt}v(t)\,dt.$$

We use Krasnoselskii's theorem in order to obtain the convergence of

$$-\int_0^1 f(t,x_{n_k}(t))v(t)dt \to -\int_0^1 f(t,\overline{x}(t))v(t)dt.$$

Since (x_{n_k}) is bounded then by (H1c) there exist a number $d>0$ and a function $f_d \in \mathrm{L}^1(0,1)$ such that $\|x_{n_k}\| \le d$ and that

$$|f(t,x_{n_k}(t))| \le f_d(t).$$

By Krasnoselskii's theorem 2.8 we obtain

$$-\int_0^1 f(t,x_{n_k}(t))v(t)\,dt \to -\int_0^1 f(t,\overline{x}(t))v(t)\,dt.$$

Thus we see that the continuous dependence on functional parameter u_k is expressed only by function g. Assume $u_k \to \overline{u}$ strongly in $\mathrm{L}^q(0,1)$. Thus it is bounded in $\mathrm{L}^q(0,1)$. By Lebesgue's dominated convergence theorem and since g is Carathéodory function, we have

$$\int_0^1 g(t,x_{n_k}(t),u_{n_k}(t))v(t)\,dt \to \int_0^1 g(t,\overline{x}(t),\overline{u}(t))v(t)\,dt.$$

By uniqueness in Theorem 4.1 and by the fundamental lemma $\overline{x}$ is a solution corresponding to (DEq) to $\overline{u}$. Therefore a convergent subsequence is obtained.

We note the following. Since $u_k \to \overline{u}$ strongly in $\mathrm{L}^q(0,1)$, then any subsequence of u_k is convergent to the same limit. Let $(x_{s_n})_{n\in\mathbb{N}}$ be an arbitrary subsequence of $(x_n)_{n\in\mathbb{N}}$. We apply the above reasoning to x_{s_n} which is bounded since (x_n) was. Thus x_{s_n} admits a subsequence $x_{k_{s_n}}$ convergent to a solution of (DEq) with parameter $\lim u_{s_n} = \overline{u}$. By Theorem 4.1 for fixed u solution is unique. This means that for arbitrary subsequence x_{s_n}, there exists a convergent subsequence, and each of those subsequences share the same limit. Thus (x_n) is convergent strongly in $\mathrm{H}_0^1(0,1)$.

We now consider the second case. Instead of (H2) we assume that $g(t,x,u) = \overline{g}(t,x)u$ and

$$|\overline{g}(t,x)| \le |x|^{p-1} a(t), \quad a \in \mathrm{L}^{\frac{q}{q-s}}(0,1). \tag{H2c}$$

We can assume that $u_n \rightharpoonup \overline{u}$ in $\mathrm{L}^q(0,1)$. By Theorem 5.1 for each u_n there exists a solution x_n to (DEq). Moreover the sequence of solutions is bounded in $\mathrm{H}_0^1(0,1)$. Thus it has a convergent subsequence, weakly in $\mathrm{H}_0^1(0,1)$, strongly both in $\mathrm{L}^2(0,1)$ and $\mathrm{C}([0,1])$. Let $(x_{n_k})_{n\in\mathbb{N}}$ be a selected subsequence convergent to $\overline{x}$. We proceed as in the previous part of the proof, except for the convergence of the term

$$\int_0^1 \overline{g}(t,x_{n_k}(t))u_{n_k}(t)v(t)\,dt \to \int_0^1 \overline{g}(t,\overline{x}(t))\overline{u}(t)v(t)\,dt. \tag{5.3}$$

By Krasnoselskii's theorem 2.8 we know that

$$\overline{g}(t,x_{n_k}(t))v(t) \to \overline{g}(t,\overline{x}(t))v(t)$$

in $\mathrm{L}^{\frac{q}{q-1}}(0,1)$. By Theorem 2.9 we got (5.3). $\square$

6 Example

Example 6.1. The above schema can be applied for the following equation

$$\frac{d^2x}{dt^2}(t) + 0.25 \cdot e^{-\frac{t^2}{2}} \frac{dx}{dt}(t) + \frac{1}{4}\frac{x(t)}{1+x(t)^2} \arcsin t \cdot u_n(t) - \frac{1}{2}e^{-t}x(t) = t+1, \tag{6.1}$$

where

$$u_n(t) = \begin{cases} 1, & t \in [0, \frac{1}{n}] \\ 0, & t \in (\frac{1}{n}, 1] \end{cases}$$

is a control function.

Indeed. (H1) is confirmed since

$$F(\cdot, x) := \frac{1}{2}e^{-(\cdot)}x^2 \in \mathrm{L}^1(0,1),$$

and for any $d > 0$ and $x \in [-d, d]$ we have that

$$f(t,x) = \frac{1}{2}e^{-t}x \leq \frac{1}{2}e^{-t}d \in \mathrm{L}^1(0,1),$$

(H2) is satisfied since

$$G(t,x,u) := \frac{1}{4}\frac{x}{1+x^2} \arcsin t \cdot u \leq \left|\frac{x}{1}\right| \left(\frac{1}{4} \arcsin t \cdot u\right)$$

and $\frac{1}{4} \arcsin t \cdot u(t) \in \mathrm{L}^\infty(0,1)$.

Also (H3) is satisfied since $F(\cdot, x) := \frac{1}{2}e^{-(\cdot)}x^2$ is convex with respect to its second variable.

We can observe for f that

$$|f(t,x) - f(t,y)| = \left|\frac{1}{2}e^{-t}(x-y)\right|.$$

After integrating both sides with respect to $t \in [0,1]$, and knowing that

$$(x(t) - y(t)) \leq \|x-y\|_{\mathrm{L}^\infty(0,1)} \leq \|x-y\|_{\mathrm{H}_0^1(0,1)},$$

we obtain:

$$\int_0^1 |f(t,x) - f(t,y)|\, dt \leq \|x-y\|_{\mathrm{H}_0^1(0,1)} \int_0^1 \frac{1}{2}e^{-t}\, dt = \frac{e-1}{2e} \|x-y\|_{\mathrm{H}_0^1(0,1)}.$$

Then for g we see that

$$\begin{aligned} |g(t,x,u) - g(t,y,u)| &= \left|\frac{1}{4}\frac{x}{1+x^2} \arcsin t \cdot u - \frac{1}{4}\frac{y}{1+y^2} \arcsin t \cdot u\right| \\ &= \frac{1}{4} |\arcsin t \cdot u| \left|\frac{x}{1+x^2} - \frac{y}{1+y^2}\right| \\ &\leq \frac{1}{4} |\arcsin t \cdot u|\, |x-y| \frac{|1-xy|}{(1+x^2)\cdot(1+y^2)} \\ &\leq \frac{1}{4} \cdot |\arcsin t \cdot u| |x-y| \leq \frac{\pi}{8}|x-y|. \end{aligned}$$

Similarly we obtain

$$\int_0^1 |g(t,x,u) - g(t,y,u)|dt \le \frac{\pi}{8} \|x-y\|_{H_0^1(0,1)}$$

which jointly implies that

$$\begin{aligned}
&\int_0^1 \left(g(t,x,u) - f(t,x) - g(t,y) + f(t,y)\right)(x-y)\,dt \\
&\quad \le \|x-y\|_{H_0^1(0,1)} \left(\int_0^1 |g(t,x,u) - g(t,y,u)|\,dt + \int_0^1 |f(t,x) - f(t,y)|\,dt \right) \\
&\quad \le \frac{\pi}{8} \|x-y\|^2_{H_0^1(0,1)} + \frac{e-1}{2e} \|x-y\|^2_{H_0^1(0,1)} \le L \|x-y\|^2_{H_0^1(0,1)}
\end{aligned}$$

with $L = 0.71 < 1$. Since $\|r\|_{L^\infty(0,1)} = \|0.25 \cdot e^{-t^2/2}\|_{L^\infty(0,1)} = 0.25$ and $\frac{\|r\|_{L^\infty(0,1)}}{1-L} = \frac{0.25}{1-0.71} < 0.87 < 1$ then by Proposition 4.2 we conclude that problem (6.1) has at least one solution to each u_n. Then the solution to $\overline{u}$ such that $u_n \to \overline{u}$ is $x_n \rightharpoonup \overline{x}$. Thus $\overline{x}$ is a solution to

$$\frac{d^2x}{dt^2}(t) + 0.25 \cdot e^{-\frac{t^2}{2}} \frac{dx}{dt}(t) - \frac{1}{2} e^{-t} x(t) = t + 1.$$

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