Eigenvalue Characterization for a Class of Boundary Value Problems

Chuan Jen Chyan
Department of Mathematics
Tamkang University
Taipei, Taiwan, 251
email: chuanjen@mail.tku.edu.tw

Johnny Henderson
Department of Mathematics
Auburn University
Auburn, Alabama 36849-5310 USA
email: hendej2@mail.auburn.edu

Abstract

We consider the nth order ordinary differential equation $(-1)^{n-k}y^{(n)} = \lambda a(t)f(y)$, $t \in [0,1]$, $n \geq 3$ together with boundary condition $y^{(i)}(0) = 0$, $0 \leq i \leq k-1$, and $y^{(l)}(1) = 0$, $j \leq l \leq j+n-k-1$, for $1 \leq j \leq k-1$ fixed. Values of λ are characterized so that the boundary value problem has a positive solution.

1 Introduction

Let $n \geq 3, 2 \leq k \leq n-1$, and $1 \leq j \leq k-1$ be given. In this paper we shall consider the nth order differential equation

$$(-1)^{n-k}y^{(n)} = \lambda a(t)f(y), \quad t \in [0,1], \tag{1}$$

satisfying the boundary conditions

$$\begin{aligned} &y^{(i)}(0) = 0, \quad 0 \leq i \leq k-1, \\ &y^{(l)}(1) = 0, \quad j \leq l \leq j+n-k-1. \end{aligned} \tag{2}$$

Throughout, we assume the following hypotheses :

(H_1) $a(t)$ is a continuous nonnegative function on $[0,1]$ and is not identically equal to zero on any subinterval of $[0,1]$.

(H_2) $f : R \to [0, \infty)$ is continuous and nonnegative.

(H_3) The limits $f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}$ and $f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}$ exist in $[0, \infty)$.

We shall determine values of λ for which the boundary value problem (1), (2) has a positive solution. By a positive solution y of (1), (2), we mean $y \in C^{(n)}[0, 1]$ satisfies (1) on $[0, 1]$ and fulfills (2), and y is nonnegative and is not identically zero on $[0, 1]$. We let

$$Sp(a) = \{\lambda > 0 \mid (1), (2) \textit{ has a postive solution}\}.$$

The motivation for the present work originates from many recent investigations. In the case $n = 2$ the boundary value problem (1), (2) describes a vast spectrum of scientific phenomena; we refer the reader to [1, 3, 5, 6, 14, 16]. It is noted that only positive solutions are meaningful in those models. Our results complement the work of many authors, see, e.g. [2, 4, 8, 9, 10, 11, 12, 13, 17, 18, 19]. In Section 2, we provide some definitions and background results, and state a fixed point theorem due to Krasnosel'skii [15]. Also, we present some properties of certain Green's function where needed. By defining an appropriate Banach space and cone, in Section 3, we characterize the set $Sp(a)$.

2 Background Notation and Definitions

We first present the definition of a cone in a Banach space and the Krasnosel'skii Fixed Point Theorem. **Definition 2.1.** Let $\mathcal{B}$ be a Banach space over R. A nonempty closed convex set $\mathcal{P} \subset \mathcal{B}$ is said to be a cone provided the following are satisfied:

(a) If $y \in \mathcal{P}$ and $\alpha \geq 0$, then $\alpha y \in \mathcal{P}$;

(b) If $y \in \mathcal{P}$ and $-y \in \mathcal{P}$, then $y = 0$.

Theorem 2.1 *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(i) $\|Tu\| \leq \|u\|, u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|, u \in \mathcal{P} \cap \partial\Omega_2$ *;*

(ii) $\|Tu\| \geq \|u\|, u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|, u \in \mathcal{P} \cap \partial\Omega_2$.

Then T *has a fixed point in* $\mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$.

To obtain a solution for (1) and (2), we require a mapping whose kernel $G(t,s)$ is the Green's function of the boundary value problem

$$(-1)^{n-k} y^{(n)} = 0, \tag{3}$$

$$\begin{aligned} y^{(i)}(0) &= 0, \quad 0 \leq i \leq k-1, \\ y^{(l)}(1) &= 0, \quad j \leq l \leq j+n-k-1. \end{aligned}$$

Wong and Agarwal [20] have found that if y satisfies

$$(-1)^{n-p} y^{(n)} \geq 0, \tag{4}$$

$$\begin{aligned} y^{(i)}(0) &= 0, \quad 0 \leq i \leq p-1, \\ y^{(l)}(1) &= 0, \quad 0 \leq l \leq n-p-1, \end{aligned} \tag{5}$$

then, for $\delta \in (0, \frac{1}{2})$ and $t \in [\delta, 1-\delta]$,

$$y(t) \geq \min\{b(p)\min\{c(p),\ c(n-p-1)\},\quad b(p-1)\min\{c(p-1),\ c(n-p)\}\}\|y\| \tag{6}$$

where the functions b and c are defined as

$$b(x) = \frac{(n-1)^{n-1}}{x^x(n-x-1)^{n-x-1}}, \quad c(x) = \delta^x(1-\delta)^{n-x-1}.$$

Aided by this, we have the following lemma.

Lemma 2.2 *Let* $n \geq 3$. *Assume* $u \in C^{(n)}[0,1]$, $(-1)^{n-k}u^{(n)}(t) \geq 0, \quad 0 \leq t \leq 1$ *and* u *satifies* (2). *Then for* $0 \leq t \leq 1$,

$$u^{(j)}(t) \geq 0$$

and for $t \in [\delta, 1-\delta]$

$$u^{(j)}(t) \geq \sigma_1 |u^{(j)}|_\infty$$

where

$$\sigma_1 = \min\{b(k-j)\min\{c(k-j),\ c(n-k-1)\},\quad b(k-j-1)\min\{c(k-j-1),\ c(n-k)\}\}.$$

Proof: First, $u^{(j)} \in C^{(n-j)}[0,1]$. Also $u^{(j)}$ satisfies

$$(-1)^{n-k} y^{(n-j)}(t) \geq 0.$$

Let the boundary condition (2) be partitioned into two parts:

$$\begin{aligned} y^{(i)}(0) &= 0, \quad j \leq i \leq k-1 \\ y^{(l)}(1) &= 0, \quad j \leq l \leq j+n-k-1 \end{aligned} \tag{7}$$

and

$$y^{(i)}(1) = 0, \quad 0 \leq i \leq j-1. \tag{8}$$

Now u satisfies (7), so $u^{(j)}$ satisfies $(k-j, n-k)$ homogeneous conjugate boundary conditions. The conclusion then follows from inequality (6).

Lemma 2.3 *Let $n \geq 3$. Assume $u \in C^{(n)}[0,1]$, $(-1)^{n-k} y^{(n)}(t) \geq 0$, $0 \leq t \leq 1$, and u satifies (2). Then for $0 \leq t \leq 1$,*

$$u(t) \geq 0$$

and for $t \in [\frac{1}{2}, 1-\delta]$,

$$u(t) \geq \sigma_2 |u^{(j)}|_\infty$$

where $\sigma_2 = \frac{\sigma_1(\frac{1}{2}-\delta)^j}{j!}$ and $|u^{(j)}|_\infty = \max_{t\in[0,1]} |u^{(j)}(t)|$.

Proof: Since u satisfies (2), u satisfies (8) as well. Thus for $0 \leq t \leq 1$,

$$u(t) = \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} u^{(j)}(s) ds.$$

Aided by Lemma 2.2

$$\begin{aligned} u(t) &= \int_\delta^t \frac{(t-s)^{j-1}}{(j-1)!} u^{(j)}(s) ds + \int_0^\delta \frac{(t-s)^{j-1}}{(j-1)!} u^{(j)}(s) ds. \\ &\geq \int_\delta^t \frac{(t-s)^{j-1}}{(j-1)!} u^{(j)}(s) ds \\ &\geq \frac{\sigma_1 (t-\delta)^j}{j!} |u^{(j)}|_\infty. \end{aligned}$$

Consequently, for $t \in [\frac{1}{2}, 1-\delta]$,

$$u(t) \geq \frac{\sigma_1 (\frac{1}{2}-\delta)^j}{j!} |u^{(j)}|_\infty.$$

The nonnegativity of u follows.

It is noted that Eloe [7] proved that $G^{(j)}(t,s) = \frac{\partial^j}{\partial t^j}G(t,s)$ is the Green's function of $y^{(n-j)} = 0$ subject to the boundary conditions

$$\begin{aligned} y^{(i)}(0) &= 0, \quad 0 \le i \le k-j-1, \\ y^{(l)}(1) &= 0, \quad 0 \le l \le n-k-1. \end{aligned} \tag{9}$$

The proof follows from the four properties of the Green's function. Consequently we have the following result, whose conclusion follows from Lemma 2.2.

Lemma 2.4 *For each* $s \in (0,1)$, *and* $t \in [\delta, 1-\delta]$

$$(-1)^{n-k}G^{(j)}(t,s) \ge \sigma_1 |G^{(j)}(\cdot, s)|_\infty$$

where $|G^{(j)}(\cdot,s)|_\infty = \max_{0\le t\le 1}|G^{(j)}(t,s)|$.

3 Main Results

We are now in a position to give some chacterization of $Sp(a)$. Define a Banach space, $\mathcal{B}$, by

$$\mathcal{B} = \{u \in C^{(j)}[0,1] | u \ satisfies \ (8)\}$$

with norm $\|u\| = max_{0\le t\le 1}|u^{(j)}(t)|$.

Let $\sigma = \sigma_2 = \frac{\sigma_1(\frac{1}{2}-\delta)^j}{j!}$. Define a cone, $\mathcal{P}_\sigma \subset \mathcal{B}$, by

$$\mathcal{P}_\sigma = \{u \in \mathcal{B} |\, u^{(j)}(t) \ge 0 \ \ on [0,1], \quad and \ \min_{t\in[\delta,1-\delta]} u(t) \ge \sigma \|u\|\}.$$

Let

$$Tu(t) = (-1)^{n-k}\int_0^1 G(t,s)a(s)f(u(s))\,ds, \; 0 \le t \le 1, \quad u \in \mathcal{B}.$$

To obtain a solution of (1), (2), we shall seek a fixed point of the operator λT in the cone $\mathcal{P}_\sigma$. In order to apply the Krasnosel'skii Fixed Point Theorem, for $\lambda > 0$, we need the following.

Lemma 3.1 *For* $\lambda > 0, \lambda T : \mathcal{P}_\sigma \to \mathcal{P}_\sigma$ *and is a completely continuous operator.*

Proof: Let $u \in \mathcal{P}_\sigma$. It suffices to verify this lemma when $\lambda = 1$. By properties of $(-1)^{n-k} G^{(j)}(t, s)$, it is clear that $(Tu)^{(j)}(t) \geq 0$ and $(Tu)^{(j)}(t)$ is continuous on $[0, 1]$.

Furthermore, for any $0 \leq \tau \leq 1$

$$\begin{aligned}
\min_{t\in[\delta,1-\delta]}(Tu)^{(j)}(t) &\geq \int_0^1 \min_{t\in[\delta,1-\delta]}(-1)^{n-k}G^{(j)}(t,s)a(s)f(u(s))ds \\
&\geq \sigma\int_0^1 (-1)^{n-k}G^{(j)}(\tau,s)a(s)f(u(s))ds \\
&\geq \sigma\int_0^1 |G^{(j)}(\cdot,s)|_\infty a(s)f(u(s))ds \\
&\geq \sigma\|Tu\|.
\end{aligned}$$

Also, the standard arguments yield that λT is completely continuous.

Theorem 3.2 *Assume* $(H_1), (H_2)$, *and* (H_3) *with* $f_0 < f_\infty < \infty$. *Assume there exists a value of* λ *such that*

$$\lambda f_0 \int_0^1 \|G(\cdot, s)\| a(s) ds < 1, \tag{10}$$

and

$$\lambda \sigma^2 f_\infty \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds > 1. \tag{11}$$

Then the BVP (1),(2) has a positive solution in the cone $\mathcal{P}_\sigma$.

Proof: For each $\lambda > 0$ satisfying both of the conditions (10) and (11), let $\epsilon(\lambda) > 0$ be sufficiently small such that

$$\lambda(f_0 + \epsilon) \int_0^1 \|G(\cdot, s)\| a(s) ds \leq 1, \tag{12}$$

and

$$\lambda \sigma^2 (f_\infty - \epsilon) \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds \geq 1. \tag{13}$$

Consider f_0 first. There exists $H_1(\epsilon) > 0$ such that $f(u) \leq (f_0 + \epsilon)u$, for all $0 < u \leq H_1$. Let

$$\Omega_1 = \{u \in \mathcal{B} | \|u\| < H_1\}.$$

For all $u \in \partial\Omega_1 \cap \mathcal{P}_\sigma, \quad 0 \leq u(s) \leq \|u\|$, and

$$\begin{aligned}
\|\lambda T u\| &\leq \lambda \int_0^1 \|G(\cdot,s)\| a(s) f(u(s)) ds \\
&\leq \lambda \int_0^1 \|G(\cdot,s)\| a(s)(f_0+\epsilon) u(s) ds \\
&\leq \lambda (f_0+\epsilon) \int_0^1 \|G(\cdot,s)\| a(s) ds \cdot \|u\|.
\end{aligned}$$

Hence, (12) implies that

$$\|\lambda T u\| \leq \|u\|.$$

On the other hand, consider f_∞. There exists $\bar{H}_2(\epsilon) > 0$ such that $f(u) \geq (f_\infty - \epsilon)u$, for all $u \geq \bar{H}_2$. Let

$$H_2 = max\{2H_1, \ \ \frac{1}{\sigma}\bar{H}_2\},$$

$$\Omega_2 = \{u \in \mathcal{B} \mid \|u\| < H_2\}.$$

For all $u \in \partial\Omega_2 \cap \mathcal{P}_\sigma, \quad u(s) \geq \sigma\|u\|, \frac{1}{2} \leq s \leq 1-\delta$, and

$$\begin{aligned}
\|\lambda T u\| &\geq \min_{t\in[\delta,1-\delta]} \lambda T u(t) \\
&\geq \int_0^1 min_{t\in[\delta,1-\delta]} (-1)^{n-k} G(t,s) a(s) f(u(s)) ds \\
&\geq \lambda \int_0^1 \sigma \|G(\cdot,s)\| a(s) f(u(s)) ds \\
&\geq \lambda\sigma \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot,s)\| a(s) f(u(s)) ds \\
&\geq \lambda\sigma \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot,s)\| a(s)(f_\infty - \epsilon) u(s) ds \\
&\geq \lambda\sigma \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot,s)\| a(s)(f_\infty - \epsilon)\sigma \|u\| ds \\
&\geq \lambda\sigma^2 (f_\infty - \epsilon) \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot,s)\| a(s) ds \|u\|.
\end{aligned}$$

Hence, (13) implies that

$$\|\lambda T u\| \geq \|u\|.$$

Finally, we apply part (i) of Krasnosel'skii's Fixed Point Theorem and obtain a fixed point u_1 of λT in $\mathcal{P}_\sigma \cap (\overline{\Omega_2} \backslash \Omega_1)$. Note that for $\frac{1}{2} \leq t \leq 1-\delta$,

$$u_1(t) \geq \sigma\|u_1\| \geq \sigma H_1 > 0.$$

Hence, u_1 is a nontrivial solution of (1),(2). Successive applications of Rolle's theorem imply that u_1 does not vanish on $(0,1)$ and so u_1 is a positive solution.

This completes the proof.

Corollary 3.3 *Assume all the conditions for Theorem 3.2 hold. Then*

(*i*) *For* $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear),* $Sp(a) = (0, \infty)$.

(*ii*) *For* $f_0 = 0$ *and* $f_\infty < \infty$, $((\sigma^2 f_\infty \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\|a(s)ds)^{-1}, \infty) \subseteq Sp(a)$.

(*iii*) *For* $f_0 > 0$ *and* $f_\infty = \infty$, $(0, (f_0 \int_0^1 \|G(\cdot, s)\|a(s)ds)^{-1}) \subseteq Sp(a)$.

(*iv*) *For* $0 < f_0 < f_\infty < \infty$,
$((\sigma^2 f_\infty \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\|a(s)ds)^{-1}, (f_0 \int_0^1 \|G(\cdot, s)\|a(s)ds)^{-1}) \subseteq Sp(a)$.

Theorem 3.4 *Assume* $(H_1), (H_2)$ *,and* (H_3) *with* $f_\infty < f_0 < \infty$. *Assume there exists a value of* λ *such that*

$$\lambda\sigma^2 f_0 \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\|a(s)ds > 1. \tag{14}$$

In addition, if f *is not bounded, assume also that*

$$\lambda f_\infty \int_0^1 \|G(\cdot, s)\|a(s)ds < 1. \tag{15}$$

Then the BVP (1),(2) has a positive solution in the cone $\mathcal{P}_\sigma$.

Proof: For each $\lambda > 0$ satisfying the condition (14), let $\epsilon(\lambda) > 0$ be sufficiently small such that

$$\lambda\sigma^2(f_0 - \epsilon) \int_{\frac{1}{2}}^{1-\delta} 1\|G(\cdot, s)\|a(s)ds \geq 1. \tag{16}$$

Consider $f_0 \in \mathcal{R}^+$ first. There exists $H_1(\epsilon) > 0$ such that $f(u) \geq (f_0 - \epsilon)u$, for $0 < u \leq H_1$. Let

$$\Omega_1 = \{u \in \mathcal{B} \mid \|u\| < H_1\}.$$

For all $u \in \partial\Omega_1 \cap \mathcal{P}_\sigma$, $u(s) \geq \sigma\|u\|$, $\frac{1}{2} \leq s \leq 1-\delta$, and so

$$\begin{aligned}
\|\lambda T u\| &\geq \min_{t\in[\delta,1-\delta]} \lambda T u(t) \\
&\geq \lambda \int_0^1 \min_{t\in[\delta,1-\delta]} (-1)^{n-k} G(t,s) a(s) f(u(s)) ds \\
&\geq \lambda \int_0^1 \sigma \|G(\cdot,s)\| a(s) f(u(s)) ds \\
&\geq \lambda\sigma \int_0^1 \|G(\cdot,s)\| a(s)(f_0-\epsilon) u(s) ds \\
&\geq \lambda\sigma(f_0-\epsilon) \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot,s)\| a(s) u(s) ds \\
&\geq \lambda\sigma(f_0-\epsilon) \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot,s)\| a(s) \sigma\|u\| ds \\
&\geq \lambda\sigma^2(f_0-\epsilon) \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot,s)\| a(s) ds \|u\|.
\end{aligned}$$

Hence, (16) implies that

$$\|\lambda T u\| \geq \|u\|.$$

On the other hand, consider $f_\infty \in \mathcal{R}^+$. Given $f_0 > f_\infty$, there are two subcases for us to consider:

Case 1: *f is bounded.* Let $\lambda > 0$ satisfying condition (14) be given throughout this case. Let $N > 0$ be large enough so that

$$f(u) \leq N, \quad for \quad all \quad u \geq 0,$$

and

$$\lambda N \int_0^1 \|G(\cdot,s)\| a(s) ds > H_1.$$

Let

$$H_2 = \lambda N \int_0^1 \|G(\cdot,s)\| a(s) ds,$$

and

$$\Omega_2 = \{u \in \mathcal{B} \mid \|u\| < H_2\}.$$

Then, for all $u \in \partial\Omega_2 \cap \mathcal{P}_\sigma$,

$$\begin{aligned}
\|\lambda T u\| &\leq \lambda \int_0^1 \|G(\cdot,s)\| a(s) f(u(s)) ds \\
&\leq \lambda N \int_0^1 \|G(\cdot,s)\| a(s) ds \\
&= \|u\|.
\end{aligned}$$

Coupled with condition (14), we apply part (ii) of Krasnosel'skii's Fixed Point Theorem and obtain a fixed point of λT in $\mathcal{P}_\sigma \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Case 2: *f is not bounded.* Assume now that $\lambda > 0$ also satisfies the condition (15). Without loss of generality, we let the preceding ϵ also satisfy

$$\lambda(f_\infty + \epsilon) \int_0^1 \|G(\cdot, s)\| a(s) ds \leq 1. \tag{17}$$

There exists $\bar{H}_2 > 0$ such that for all $u \geq \bar{H}_2$, $f(u) \leq (f_\infty + \epsilon)u$. Since f is continuous at $u = 0$, it is unbounded on $(0, \infty)$ as u approaches $+\infty$. Let

$$H_2 > max\{2H_1, \bar{H}_2\}$$

be such that

$$f(u) \leq f(H_2)$$

for all $0 \leq u \leq H_2$. Let

$$\Omega_2 = \{u \in \mathcal{B} \mid \|u\| < H_2\}.$$

For all $u \in \partial\Omega_2 \cap \mathcal{P}_\sigma$, $0 \leq s \leq 1$,

$$\begin{aligned} f(u(s)) &\leq f(H_2) \\ &\leq (f_\infty + \epsilon)H_2, \end{aligned}$$

and so,

$$\begin{aligned} \|\lambda T u\| &\leq \lambda \int_0^1 \|G(\cdot, s)\| a(s) f(u(s)) ds \\ &\leq \lambda \int_0^1 \|G(\cdot, s)\| a(s)(f_\infty + \epsilon) H_2 ds \\ &\leq \lambda(f_\infty + \epsilon) \int_0^1 \|G(\cdot, s)\| a(s) ds \cdot \|u\|. \end{aligned}$$

Hence,(17) implies that

$$\|\lambda T u\| \leq \|u\|.$$

Finally, we apply part (ii) of Krasnosel'skii's Fixed Point Theorem and obtain a fixed point u_1 of λT in $\mathcal{P}_\sigma \cap \overline{\Omega_2} \backslash \Omega_1$.

By an argument similar to that in the proof of Theorem 3.2 there is a positive solution, u_1, of (1), (2).

Corollary 3.5 *(Case 1) Assume all the conditions for Theorem 3.4 hold and in addition that f is bounded. Then*

(i) For $f_0 = 0$, $Sp(a) = (0, \infty)$.

(ii) For $f_0 < \infty$, $((\sigma^2 f_0 \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds)^{-1}, \infty) \subseteq Sp(a)$.

Corollary 3.6 *(Case 2) Assume all the conditions for Theorem 3.4 hold. Then*

(i) For $f_0 = \infty$ and $f_\infty = 0$ (Sublinear), $Sp(a) = (0, \infty)$.

(ii) For $f_0 = \infty$ and $f_\infty > 0, (0, (f_\infty \int_0^1 \|G(\cdot, s)\| a(s) ds)^{-1}) \subseteq Sp(a)$.

(iii) For $0 < f_0 < \infty$ and $f_\infty = 0, ((\sigma^2 f_0 \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds)^{-1}, \infty) \subseteq Sp(a)$.

(iv) For $0 < f_\infty < f_0 < \infty$,
$((\sigma^2 f_0 \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds)^{-1}, (f_\infty \int_0^1 \|G(\cdot, s)\| a(s) ds)^{-1}) \subseteq Sp(a)$.

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