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Energy Decay of Klein-Gordon-Schrödinger Type with Linear Memory Term*

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Abstract

This paper is concerned with the existence, uniqueness and uniform decay of the solutions of a Klein-Gordon-Schrödinger type system with linear memory term. The existence is proved by means of the Faedo-Galerkin method and the asymptotic behavior is obtained by making use of the multiplier technique combined with integral inequalities.

1 Introduction

This paper aims to prove the global existence and uniform decay for the following system

$$i\psi' + \kappa\Delta\psi + i\alpha\psi = \phi\psi \ , \ \ x \in \Omega \subset \mathbb{R}^n, t > 0, \tag{1.1}$$

$$\phi'' - \Delta\phi + \int_0^t g(t-\tau)\Delta\phi(\tau)d\tau + \phi + \lambda\ \phi' = -Re(F(x)\cdot \nabla\psi), \ \ x \in \Omega \subset \mathbb{R}^n, t > 0, \tag{1.2}$$

satisfying the following initial and boundary conditions

$$\psi(x,0) = \psi_0(x),\ \phi(x,0) = \phi_0(x),\ \phi'(x,0) = \phi_1(x), \qquad x \in \Omega, \tag{1.3}$$

$$\psi(x,t) = \phi(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \leq 2$ with $\kappa, \alpha, \lambda > 0$. The variable ψ stands for the dimensionless low frequency electron field, whereas ϕ denotes the dimensionless low frequency density. This system in one dimension describes the nonlinear interaction between high frequency electron waves and low frequency ion plasma waves in a homogeneous magnetic field, adapted to model the UHH plasma heating scheme. The unusual form of the right side of equation (1.2), as compared to the corresponding Zakharov equation, is a consequence of the different low frequency coupling that was considered, i.e. the polarization drift instead of the ponderomotive force.

*Key Phrases: Klein-Gordon-Schrödinger System; memory term; energy decay.
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Systems of Klein-Gordon-Schrödinger type have been studied for many years. In [4] the authors proved the existence of a strong global attractor in $H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ attracting bounded sets of $H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ for a Klein-Gordon-Schrödinger system with Yukawa coupling. This was extended in [8] where the existence of a strong global attractor in $H^k(\mathbb{R}^N) \times H^k(\mathbb{R}^N), N = 1, 2, 3,$ attracting bounded sets of $H^k(\mathbb{R}^N) \times H^k(\mathbb{R}^N), k \geq 1$ was proved. For a dissipative system of Zakharov type I. Flahaut [3] proved the existence of a weak global attractor in $H_0^1((0, L)) \times H_0^1((0, L)) \bigcap H^2((0, L)) \times H_0^1((0, L)) \bigcap H^3((0, L))$ and obtained upper bounds for its Hausdorff and Fractal dimensions. In [6] the authors studied the one dimensional case of (1.1) - (1.2) and proved the global existence and uniqueness of the solutions and established the necessary conditions for the system to manifest energy decay. Later on the authors in [10] proved the existence of a global attractor in the space $(H_0^1(\Omega) \cap H^2(\Omega))^2 \times H_0^1(\Omega)$ which attracts all bounded sets of $(H_0^1(\Omega) \cap H^2(\Omega))^2 \times H_0^1(\Omega)$ in the norm topology.

The rest of the paper is divided into four sections. In Section 2, the basic notation and assumptions made are stated along with the main results. In Section 3 the existence and uniqueness of the solutions of (1.1) - (1.4) in $(H_0^1(\Omega) \cap H^2(\Omega)^2 \times H_0^1(\Omega)$ are established while in Section 4 the uniform decay of the solutions is proved.

***Notation*:** Let us introduce some notations that will be used throughout this work. Denote by $H^s(\Omega)$ both the standard real and complex Sobolev spaces on (Ω). For simplicity reasons sometimes we use H^s, L^s for $H^s(\Omega)$, $L^s(\Omega)$ and $||.||$, $(.,.)$ for the norm and the inner product of $L^2(\Omega)$ respectively as well as the symbol $\cdot$ denotes the inner product in $\mathbb{R}^n$. Finally, C is a general symbol for any positive constant.

2 Assumptions and main result

Let us consider the Hilbert space $L^2(\Omega)$ of complex valued functions on Ω endowed with the inner product

$$(u, v) = \int_\Omega u(x)\overline{v(x)}dx,$$

and the corresponding norm

$$|u|^2 = (u, u).$$

We consider the Sobolev space $H^1(\Omega)$ endowed with the scalar product

$$(u, v)_{H^1(\Omega)} = (u, v) + (\bigtriangledown u, \bigtriangledown v).$$

We define the subspace of $H^1(\Omega)$, denoted by $H_0^1(\Omega)$, as the closure of $C_0^\infty(\Omega)$ in the strong topology of $H^1(\Omega)$.

Assumption 2.1 *Let the function* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *be a nonnegative and bounded* $\mathbb{C}^2$ *- function such that*

$$l = 1 - \int_0^\infty g(r)dr > 0$$

and for some positive $m_i, i = 0, 1, 2$ *it holds*

$$-m_0 g(t) \le g^{'}(t) \le -m_1 g(t), \ \ \forall\, t \ge 0,$$
$$0 \le g^{''}(t) \le m_2 g(t), \ \ \forall\, t \ge 0.$$

Assumption 2.2 *We assume that* $F(x)$ *is a one dimensional vector function with* $F(x) \in C^1(\Omega)$ *and* $||F(x)||_\infty = M$.

We recall the following inequalities which will be used frequently later:

$$||u||^2 \le c|| \bigtriangledown u||^2, \ \ u \ \in \ H_0^1(\Omega), \tag{2.1}$$

$$||u||_\infty \le c||u||_{H^2}^{n/4} ||u||^{1-(n/4)}, \ \ u \ \in \ H^2(\mathbb{R}^n), \ \ n \le 2, \tag{2.2}$$

and

$$||u||_4 \le || \bigtriangledown u||^{n/4} ||u||^{1-(n/4)} \ \ u \ \in \ H^1(\mathbb{R}^n) \ \ n \le 2. \tag{2.3}$$

We define the energy of (1.1)- (1.2) as

$$E(t) = \frac{1}{2}\left[||\psi||^2 + \kappa|| \bigtriangledown \psi(t)||^2 + ||\phi^{'}||^2 + || \bigtriangledown \phi||^2 + ||\phi||^2 + \frac{1}{2}\int_\Omega \phi|\psi|^2 dx \right]$$

and therefore we have the following main result

Theorem 2.1 *Let* $(\psi_0, \phi_0, \phi_1) \ \in \ (H_0^1(\Omega) \cap H^2(\Omega))^2 \times H_0^1(\Omega)$ *and Assumption 2.1 - 2.2 hold . Then, there exists a unique solution for the system (1.1), (1.4) such that*

$$\begin{aligned}
&\psi \in L^\infty(0, \infty; H_0^1(\Omega) \cap H^2(\Omega)), \ \psi^{'} \ \in L^\infty(0, \infty; L^2(\Omega)), \\
&\phi \in L^\infty(0, \infty; H_0^1(\Omega) \cap H^2(\Omega)), \ \phi^{'} \ \in L^\infty(0, \infty; H_0^1(\Omega)), \\
&\phi^{''} \in L^\infty(0, \infty; L^2(\Omega)), \\
&\psi(x, 0) = \psi_0(x), \ \phi(x, 0) = \phi_0(x), \ \phi^{'}(x, 0) = \phi_1(x), \ x \in \Omega.
\end{aligned}$$

3 Global Existence

Let us represent by w_n a basis in $H_0^1(\Omega) \cap H^2(\Omega)$ formed by the eigenfunctions of $-\Delta$, also by V_m the subspace of $H_0^1(\Omega) \cap H^2(\Omega)$ generated by the first m vectors and by

$$\psi_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i, \ \ \phi_m(t) = \sum_{i=1}^{m} h_{im}(t) w_i,$$

where $(\psi_m(t), \ \phi_m(t), \phi_m^{'}(t))$ is a solution of the following Cauchy problem

$$i(\psi_m^{'}, u) + \kappa(\Delta\psi_m, u) + i\alpha(\psi_m, u) \ \ = \ \ (\phi_m \psi_m, u) \ \ \forall \ u \ \in V_m, \tag{3.1}$$

$$\begin{aligned}
(\phi_m^{''}, v) - (\Delta\phi_m, v) + \int_0^t g(t-\tau) \ \ (\Delta\phi_m \ \ (\tau), v) d\tau + (\phi_m, v) + \lambda \ (\phi_m^{'}, v) \\
= \ \ -Re(F(x) \cdot \bigtriangledown \psi_m, v), \ \ \forall \ v \ \in V_m,
\end{aligned} \tag{3.2}$$

with initial conditions

$$\begin{aligned}
\psi_m(x, 0) = \psi_{0m} \to \psi^0, \ \ \phi(x, 0) = \phi_{0m} \to \phi^0 \ \in \ H_0^1(\Omega) \cap H^2(\Omega), \\
\phi_m^{'}(0) = \phi_{1m} \to \phi^1 \ \in \ H_0^1(\Omega).
\end{aligned} \tag{3.3}$$

In this section we derive a priori estimates for the solutions of the (3.1)-(3.3) system.

3.1 A Priori Estimate I

Letting $u = \bar{\psi}_m(t)$ and by taking the imaginary part equation of (3.1) and integrating over Ω we obtain

$$\frac{1}{2}\frac{d}{dt}||\psi_m(t)||^2 + \alpha||\psi_m(t)||^2 = 0. \tag{3.4}$$

Applying Gronwall's Lemma produces

$$||\psi_m(t)|| \leq ||\psi_m(0)||e^{-2\alpha t}. \tag{3.5}$$

Therefore

$$||\psi_m(t)|| \leq R \text{ for all } t > 0. \tag{3.6}$$

Next let $u = -\bar{\psi}_m'(t)$, then by taking the real part of (3.1) and integrating over Ω (3.1) becomes

$$\frac{\kappa}{2}\frac{d}{dt}\int_\Omega | \bigtriangledown \psi_m|^2 dx + \alpha Im \int_\Omega \psi_m \bar{\psi}_m' dx = -Re\int_\Omega \phi_m\psi_m\bar{\psi}_m' dx.$$

For the right hand side of the equation above we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \phi_m|\psi_m|^2 dx = \frac{1}{2}\int_\Omega \phi_m'|\psi_m|^2 dx + Re\int_\Omega \phi_m\psi_m\bar{\psi}_m' dx.$$

But from (3.1) we also obtain

$$\alpha Im\int_\Omega \psi_m\bar{\psi}_m' dx = \kappa\alpha\int_\Omega | \bigtriangledown \psi_m|^2 dx + \alpha\int_\Omega \phi_m|\psi_m|^2 dx.$$

Therefore

$$\begin{aligned}\frac{\kappa}{2}\frac{d}{dt}\int_\Omega | \bigtriangledown \psi_m|^2 dx + \kappa\alpha\int_\Omega | \bigtriangledown \psi_m|^2 dx + \alpha\int_\Omega \phi_m|\psi_m|^2 dx \\ = -\frac{1}{2}\frac{d}{dt}\int_\Omega \phi_m|\psi_m|^2 dx + \frac{1}{2}\int_\Omega \phi_m'|\psi_m|^2 dx.\end{aligned} \tag{3.7}$$

Next, substituting $v = \phi_m'(t)$ into (3.2) and then integrating over Ω (3.2) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big[||\phi_m'||^2 + || \bigtriangledown \phi_m||^2 + ||\phi_m||^2\Big] + \lambda||\phi_m'||^2 = \frac{1}{2}\frac{d}{dt}\Big(\int_0^t g(t-\tau)\int_\Omega \bigtriangledown\phi_m(\tau)\bigtriangledown\phi_m(t)dxd\tau\Big) \\ -\int_0^t g'(t-\tau)\int_\Omega \bigtriangledown\phi_m(\tau)\bigtriangledown\phi_m(t)dxd\tau - g(0)|| \bigtriangledown \phi_m||^2 - \int_\Omega (F(x)\cdot\bigtriangledown\psi_m)\phi_m' dx.\end{aligned} \tag{3.8}$$

Hence, by adding (3.4), (3.7) and (3.8) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[||\psi_m||^2 + \kappa|| \bigtriangledown \psi_m||^2 + ||\phi_m'||^2 + || \bigtriangledown \phi_m||^2 + ||\phi_m||^2 + \int_\Omega \phi_m|\psi_m|^2 dx\Big] + \lambda||\phi_m'||^2 \\ &+ \kappa\alpha|| \bigtriangledown \psi_m||^2 + \alpha||\psi_m||^2 + \alpha\int_\Omega \phi_m|\psi_m|^2 dx = \frac{1}{2}\int_\Omega \phi_m'|\psi_m|^2 dx - \int_\Omega (F(x)\cdot\bigtriangledown\psi_m)\phi_m' dx \\ &+ \frac{d}{dt}\Big(\int_0^t g(t-\tau)\int_\Omega \bigtriangledown\phi_m(\tau)\bigtriangledown\phi_m(t)dxd\tau\Big) - \int_0^t g'(t-\tau)\int_\Omega \bigtriangledown\phi_m(\tau)\bigtriangledown\phi_m(t)dxd\tau \\ &- g(0)|| \bigtriangledown \phi_m||^2.\end{aligned} \tag{3.9}$$

Evaluating the integrals of (3.9) by using Assumption 2.2, the compact embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ and Young's inequality, we obtain

$$\left|\frac{1}{2}\int_\Omega \phi_m^{'}|\psi_m|^2dx\right| \leq \frac{\lambda}{4}\int_\Omega |\phi_m^{'}|^2dx + \frac{\kappa\alpha}{2}\int_\Omega |\bigtriangledown \psi_m|^2dx + C,$$
$$\left|\int_\Omega (F(x)\cdot \bigtriangledown\psi_m)\phi_m^{'}dx\right| \leq \frac{\lambda}{2}\int_\Omega |\phi_m^{'}|^2dx + \frac{M^2}{2\lambda}\int_\Omega |\bigtriangledown \psi_m|^2dx.$$

Also, considering Cauchy-Schwarz Inequality, Young's Inequality and Assumption 2.1 we have the following estimate

$$\begin{aligned}
&\left|\int_0^t g^{'}(t-\tau)\int_\Omega \bigtriangledown\phi_m(\tau)\bigtriangledown\phi_m(t)dxd\tau\right| \\
&\leq \int_0^t |g^{'}(t-\tau)|\Big(\int_\Omega |\bigtriangledown\phi_m(\tau)|^2dx\Big)^{1/2}\Big(\int_\Omega |\bigtriangledown\phi_m(t)|^2dx\Big)^{1/2}d\tau \\
&\leq \frac{m_1^2}{2}||\bigtriangledown\phi_m(t)||^2 + \frac{1}{2}\Big(\int_0^t g(t-\tau)||\bigtriangledown\phi_m(\tau)||d\tau\Big)^2 \\
&\leq \frac{m_1^2}{2}||\bigtriangledown\phi_m(t)||^2 + \frac{1}{2}||g||_{L^1}\int_0^t g(t-\tau)||\bigtriangledown\phi_m(\tau)||^2d\tau.
\end{aligned}$$

Combining the results above (3.9) can be rewritten as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[||\psi_m||^2 + \kappa||\bigtriangledown\psi_m||^2 + ||\phi_m^{'}||^2 + ||\bigtriangledown\phi_m||^2 + ||\phi_m||^2 + \int_\Omega \phi_m|\psi_m|^2dx\Big] + \frac{\lambda}{4}||\phi_m^{'}||^2 \\
&+ \alpha||\psi_m||^2 + \frac{\kappa\alpha}{2}||\bigtriangledown\psi_m||^2 + \alpha\int_\Omega \phi_m|\psi_m|^2dx \leq C + \frac{1}{2}||g||_{L^1}\int_0^t g(t-\tau)||\bigtriangledown\phi_m(\tau)||^2d\tau \qquad (3.10)\\
&+ \frac{d}{dt}\Big(\int_0^t g(t-\tau)\int_\Omega \bigtriangledown\phi_m(\tau)\bigtriangledown\phi_m(t)dxd\tau\Big) + (\frac{m_1^2}{2} - g(0))||\bigtriangledown\phi_m||^2 + \frac{M^2}{2\lambda}||\bigtriangledown\psi_m||^2.
\end{aligned}$$

Integrating the above expression over $(0,t)$ and considering (3.3) it follows that

$$\begin{aligned}
&\frac{1}{2}\Big[||\psi_m||^2 + \kappa||\bigtriangledown\psi_m||^2 + ||\phi_m^{'}||^2 + ||\bigtriangledown\phi_m||^2 + ||\phi_m||^2 + \frac{1}{2}\int_\Omega \phi_m|\psi_m|^2dx\Big] \\
&+ \int_0^t\Big(\alpha||\psi_m(s)||^2 + \frac{\lambda}{4}||\phi_m^{'}(s)||^2 + \frac{\kappa\alpha}{2}||\bigtriangledown\psi_m(s)||^2 + \alpha\int_\Omega \phi_m|\psi_m|^2dx\Big)ds \\
&\leq C + (\frac{m_1^2}{2} - g(0))\int_0^t ||\bigtriangledown\phi_m(s)||^2ds + \int_0^t g(s-\tau)\int_\Omega \bigtriangledown\phi_m(\tau)\bigtriangledown\phi_m(t)dxd\tau \qquad (3.11)\\
&+ \frac{1}{2}||g||_{L^1}\int_0^t\int_0^s g(s-\tau)||\bigtriangledown\phi_m(\tau)||^2d\tau ds + \frac{M^2}{2\lambda}\int_0^t ||\bigtriangledown\psi_m(s)||^2ds.
\end{aligned}$$

Evaluating the following terms

$$\begin{aligned}
&\left|\int_0^t g(t-\tau)\int_\Omega \nabla\phi_m(\tau)\nabla\phi_m(t)dxd\tau\right| \\
&\le \int_0^t |g(t-\tau)|\Big(\int_\Omega |\nabla\phi_m(\tau)|^2dx\Big)^{1/2}\Big(\int_\Omega |\nabla\phi_m(t)|^2dx\Big)^{1/2}d\tau \\
&\le \frac{1}{2}||\nabla\phi_m(t)||^2+\frac{||g||_{L^1}||g||_{L^\infty}}{2}\int_0^t||\nabla\phi_m(\tau)||^2d\tau,
\end{aligned} \tag{3.12}$$

$$\left|\int \phi_m|\psi_m|^2dx\right|\le ||\phi_m||||\psi_m||_4^2\le C||\phi_m||||\nabla\psi_m||\ ||\psi_m||\le\frac{1}{2}||\nabla\phi_m||^2+\frac{\kappa}{2}||\nabla\psi_m||^2+C.$$

Substituting the results above in (3.11) and applying Gronwall's Lemma we obtain the first estimate

$$\begin{aligned}
&||\psi_m||^2+||\nabla\psi_m||^2+||\phi_m'||^2+||\nabla\phi_m||^2+||\phi_m||^2 \\
&\quad+\int_0^t\Big\{||\psi_m(s)||^2+||\phi_m'(s)||^2+||\nabla\psi_m(s)||^2\Big\}ds\le L_1
\end{aligned} \tag{3.13}$$

where L_1 is a positive constant independent of $m\in\mathbb{N}$.

3.2 A Priori Estimate II

Let $u=\Delta\bar{\psi}_m'(t)+\alpha\Delta\bar{\psi}_m(t)$ in (3.1), then by taking the real part and integrating over Ω we have

$$\frac{1}{2}\frac{d}{dt}\kappa||\Delta\psi_m||^2+\kappa\alpha||\Delta\psi_m||^2=Re\int_\Omega\phi_m\psi_m\Delta\bar{\psi}_m'dx+\alpha Re\int_\Omega\phi_m\psi_m\Delta\bar{\psi}_mdx. \tag{3.14}$$

Next, let $v=-\Delta\phi_m'(t)$ in (3.2). Therefore by integrating we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(||\nabla\phi_m'||^2+||\Delta\phi_m||^2+||\nabla\phi_m||^2\Big)+\lambda||\nabla\phi_m'||^2 \\
&\quad-\int_0^t g(t-\tau)(\Delta\phi_m(\tau),\Delta\phi_m'(t))dx=Re\int_\Omega(F(x)\cdot\nabla\psi_m)\Delta\phi_m'dx.
\end{aligned} \tag{3.15}$$

Noticing that

$$Re\int_\Omega\phi_m\psi_m\Delta\bar{\psi}_m'dx=\frac{d}{dt}Re\int_\Omega\phi_m\psi_m\Delta\bar{\psi}_mdx-Re\int_\Omega\phi_m'\psi_m\Delta\bar{\psi}_mdx-Re\int_\Omega\phi_m\psi_m'\Delta\bar{\psi}_mdx$$

while by $\psi_m'=-i(-\Delta\psi_m-i\alpha\psi_m-\phi_m\psi_m)$, we have the following estimate

$$\begin{aligned}
-Re\int_\Omega\phi_m\psi_m'\Delta\bar{\psi}_mdx&=Re\int_\Omega i\phi_m[-\Delta\psi_m-i\alpha\psi_m-\phi_m\psi_m]\Delta\bar{\psi}_mdx \\
&=\alpha Re\int_\Omega\phi_m\psi_m\Delta\bar{\psi}_mdx+Im\int_\Omega\phi_m^2\psi_m\Delta\bar{\psi}_mdx.
\end{aligned}$$

Substituting the expressions above into (3.14) deduces

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\kappa||\Delta\psi_m||^2-2Re\int_\Omega\phi_m\psi_m\Delta\bar{\psi}_m dx\right)+\kappa\alpha||\Delta\psi_m||^2\\&=2\alpha\int_\Omega\phi_m\psi_m\Delta\bar{\psi}_m dx+Im\int_\Omega\phi_m^2\psi_m\Delta\bar{\psi}_m dx-Re\int_\Omega\phi_m^{'}\psi_m\Delta\bar{\psi}_m dx.\end{aligned}\tag{3.16}$$

Hence, adding (3.15) and (3.16) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\kappa||\Delta\psi_m||^2-2Re\int_\Omega\phi_m\psi_m\Delta\bar{\psi}_m dx+||\bigtriangledown\phi_m^{'}||^2+||\Delta\phi_m||^2+||\bigtriangledown\phi_m||^2\right)\\&+\kappa\alpha||\Delta\psi_m||^2+\lambda||\bigtriangledown\phi_m^{'}||^2-\int_0^t g(t-\tau)(\Delta\phi_m(\tau),\Delta\phi_m^{'}(t))dx=2\alpha\int_\Omega\phi_m\psi_m\Delta\bar{\psi}_m dx\\&+Im\int_\Omega\phi_m^2\psi_m\Delta\bar{\psi}_m dx-Re\int_\Omega\phi_m^{'}\psi_m\Delta\bar{\psi}_m dx+Re\int_\Omega(F(x)\cdot\bigtriangledown\psi_m)\Delta\phi_m^{'}dx.\end{aligned}\tag{3.17}$$

Therefore

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\kappa||\Delta\psi_m||^2-2Re\int_\Omega\phi_m\psi_m\Delta\bar{\psi}_m dx+||\bigtriangledown\phi_m^{'}||^2+||\Delta\phi_m||^2+||\bigtriangledown\phi_m||^2\right)+\kappa\alpha||\Delta\psi_m||^2\\&+\lambda||\bigtriangledown\phi_m^{'}||^2=2\alpha\int_\Omega\phi_m\psi_m\Delta\bar{\psi}_m dx+Im\int_\Omega\phi_m^2\psi_m\Delta\bar{\psi}_m dx-Re\int_\Omega\phi_m^{'}\psi_m\Delta\bar{\psi}_m dx\\&+Re\int_\Omega(F(x)\cdot\bigtriangledown\psi_m)\Delta\phi_m^{'}dx-g(0)||\Delta\phi_m(t)||^2+\frac{d}{dt}\left(\int_0^t g(t-\tau)(\Delta\phi_m(\tau),\Delta\phi_m(t))d\tau\right)\\&-\int_0^t g^{'}(t-\tau)(\Delta\phi_m(\tau),\Delta\phi_m(t))d\tau.\end{aligned}\tag{3.18}$$

Estimating the integrals on the right hand side of (3.18) using the Sobolev embedding theorem and Young's Inequality gives the following results

$$\begin{aligned}\left|Re\int_\Omega\phi_m\psi_m\Delta\bar{\psi}_m dx\right|&\le||\phi_m||_4||\psi_m||_4||\Delta\psi_m||\\&\le\frac{1}{4}||\Delta\psi_m||^2+C||\bigtriangledown\phi_m||^2||\bigtriangledown\psi_m||^2,\\\left|Im\int_\Omega\phi_m^2\psi_m\Delta\bar{\psi}_m dx\right|&\le||\phi_m||_6^2||\psi_m||_6||\Delta\psi_m||\le\frac{1}{4}||\Delta\psi_m||^2+C||\bigtriangledown\phi_m||^4||\bigtriangledown\psi_m||^2,\\\left|-Re\int_\Omega\phi_m^{'}\psi_m\Delta\bar{\psi}_m dx\right|&\le||\phi_m^{'}||_4||\psi_m||_4||\Delta\psi_m||\le\frac{1}{4}||\Delta\psi_m||^2+C||\bigtriangledown\phi_m^{'}||^2||\bigtriangledown\psi_m||^2.\end{aligned}$$

Now evaluating the last term of (3.15)

$$\begin{aligned}\int_\Omega(F(x)\cdot\bigtriangledown\psi_m)\Delta\phi_m^{'}dx=&-\int_\Omega(F(x)\cdot\Delta\psi_m)\bigtriangledown\phi_m^{'}dx-\int_\Omega(\bigtriangledown F(x)\cdot\bigtriangledown\psi_m)\bigtriangledown\phi_m^{'}dx\\&-\int_\Omega(\bigtriangledown\psi_m\times(\bigtriangledown\times F(x)))\bigtriangledown\phi_m^{'}dx\end{aligned}$$

and taking into consideration Assumption 2.2 we evaluate the integrals on the right hand side

$$\left|-\int_\Omega (F(x)\cdot \Delta\psi_m)\bigtriangledown \phi_m^{'} dx\right| \le C||\Delta\psi_m|| ||\bigtriangledown \phi_m^{'}||$$

$$\left|-\int_\Omega (\bigtriangledown F(x)\cdot \bigtriangledown\psi_m)\bigtriangledown \phi_m^{'} dx\right| \le C||\bigtriangledown\psi_m|| ||\bigtriangledown \phi_m^{'}||,$$

$$\left|-\int_\Omega (\bigtriangledown\psi_m \times (\bigtriangledown \times F(x)))\bigtriangledown \phi_m^{'} dx\right| \le C||\bigtriangledown\psi_m|| ||\bigtriangledown \phi_m^{'}||,$$

also we obtain

$$\left|\int_0^t g^{'}(t-\tau)(\Delta\phi_m(\tau),\Delta\phi_m(t))d\tau\right| \le ||\Delta\phi_m(t)||\int_0^t |g^{'}(t-\tau)|||\Delta\phi_m(\tau)||d\tau$$
$$\le \frac{m_1^2}{2}||\Delta\phi_m(t)||^2 + \frac{1}{2}||g||_{L^1(0,\infty)}\int_0^t g(t-\tau)||\Delta\phi_m(\tau)||^2 d\tau.$$

Substituting the expressions above into (3.18) gives the following result

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\kappa||\Delta\psi_m||^2 - 2Re\int_\Omega \phi_m\psi_m\Delta\bar{\psi}_m dx + ||\bigtriangledown\phi_m^{'}||^2 + ||\Delta\phi_m||^2 + ||\bigtriangledown\phi_m||^2\right) + \kappa\alpha||\Delta\psi_m||^2\\ &+\lambda||\bigtriangledown\phi_m^{'}||^2 \le C[||\Delta\psi_m|| ||\bigtriangledown\phi_m^{'}|| + ||\Delta\psi_m||^2 + ||\bigtriangledown\phi_m||^2||\bigtriangledown\psi_m||^2 + ||\bigtriangledown\phi_m||^4||\bigtriangledown\psi_m||^2\\ &+||\bigtriangledown\phi_m^{'}||^2||\bigtriangledown\psi_m||^2 + ||\bigtriangledown\psi_m|| ||\bigtriangledown\phi_m^{'}||] + \frac{d}{dt}\left(\int_0^t g(t-\tau)(\Delta\phi_m(\tau),\Delta\phi_m(t))d\tau\right)\\ &+\frac{m_1^2}{2}||\Delta\phi_m(t)||^2 + \frac{1}{2}||g||_{L^1(0,\infty)}\int_0^t g(t-\tau)||\Delta\phi_m(\tau)||^2 d\tau + C.\end{aligned} \tag{3.19}$$

Integrating (3.19) over $(0,t)$ and considering (3.3) it follows that

$$\begin{aligned}&\frac{1}{2}\left(\kappa||\Delta\psi_m||^2 - 2Re\int_\Omega \phi_m\psi_m\Delta\bar{\psi}_m dx + ||\bigtriangledown\phi_m^{'}||^2 + ||\Delta\phi_m||^2 + ||\bigtriangledown\phi_m||^2\right)\\ &+\int_0^s\left\{\kappa\alpha||\Delta\psi_m(s)||^2 + \lambda||\bigtriangledown\phi_m^{'}(s)||^2\right\}ds \le C\int_0^t\left\{||\Delta\psi_m(s)||^2 + ||\bigtriangledown\phi_m(s)||^2||\bigtriangledown\psi_m(s)||^2\right.\\ &+||\bigtriangledown\phi_m(s)||^4||\bigtriangledown\psi_m(s)||^2 + ||\bigtriangledown\phi_m^{'}(s)||^2||\bigtriangledown\psi_m(s)||^2 + ||\bigtriangledown\psi_m(s)|| ||\bigtriangledown\phi_m^{'}(s)||\\ &\left.+||\Delta\psi_m(s)|| ||\bigtriangledown\phi_m^{'}(s)|| + ||\Delta\phi_m(s)||^2\right\}ds + \int_0^t g(t-\tau)(\Delta\phi_m(\tau),\Delta\phi_m(t))d\tau\\ &+\frac{1}{2}||g||_{L^1(0,\infty)}\int_0^t\int_0^s g(s-\tau)||\Delta\phi_m(\tau)||^2 d\tau ds + C.\end{aligned} \tag{3.20}$$

Using Cauchy Schwarz inequality and Young's inequality imply

$$\left|\int_0^t g(t-\tau)\int_\Omega \Delta\phi_m(\tau)\Delta\phi_m(t)d\tau\right| \le \frac{1}{2}||\Delta\phi_m(t)||^2 + \frac{1}{2}||g||_{L^1(0,\infty)}||g||_{L^\infty(0,\infty)}\int_0^s ||\Delta\phi_m(\tau)||^2 d\tau.$$

Substituting the expression above into (3.20) and applying Gronwall's Lemma we obtain the second estimate

$$||\Delta\psi_m||^2 + ||\bigtriangledown\phi_m^{'}||^2 + ||\Delta\phi_m||^2 + ||\bigtriangledown\phi_m||^2 + \int_0^s\left\{||\Delta\psi_m(s)||^2 + ||\bigtriangledown\phi_m^{'}(s)||^2\right\}ds \le L_2 \tag{3.21}$$

where L_2 is a positive constant independent of $m \in \mathbb{N}$.

3.3 A Priori Estimate III

Differentiating with respect to time equations (3.1) and (3.2) and substituting $u = -\bar{\psi}_m'(t)$ in (3.1) , taking the imaginary part and substituting $v = \phi_m''(t)$ in (3.2) produces

$$\frac{1}{2}\frac{d}{dt}||\psi_m'||^2 + \alpha||\psi_m'||^2 = -Re\int_\Omega \phi_m'\psi_m\bar{\psi}_m' dx - Re\int_\Omega \phi_m\psi_m'\bar{\psi}_m' dx \tag{3.22}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(||\phi_m''||^2 + ||\nabla\phi_m'||^2 + ||\phi_m'||^2\Big) - g(0)(\nabla\phi_m(t), \nabla\phi_m''(t))\\ &-\int_0^t g'(t-\tau)(\nabla\phi_m(\tau), \nabla\phi_m''(t))d\tau + \lambda||\phi_m''||^2 = -Re\int_\Omega (F(x)\cdot\nabla\psi_m')\phi_m'' dx.\end{aligned} \tag{3.23}$$

Now adding (3.22) and (3.23) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(||\psi_m'||^2 + ||\phi_m''||^2 + ||\nabla\phi_m'||^2 + ||\phi_m'||^2\Big) + \alpha||\psi_m'||^2 + g(0)||\nabla\phi_m'||^2 + \lambda||\phi_m''||^2\\ &+ Re\int_\Omega \phi_m'\bar{\psi}_m\psi_m' + \int_\Omega \phi_m|\psi_m'|^2 dx = -g'(0)(\nabla\phi_m(t), \nabla\phi_m'(t)) - Re\int_\Omega (F(x)\cdot\nabla\psi_m')\phi_m'' dx\\ &-\int_0^t g''(t-\tau)(\nabla\phi_m(\tau), \nabla\phi_m'(t)) + \frac{d}{dt}\Big(\int_0^t g'(t-\tau)(\nabla\phi_m(\tau), \nabla\phi_m'(t))\Big)\\ &+ g(0)\frac{d}{dt}(\nabla\phi_m(t), \nabla\phi_m'(t)).\end{aligned} \tag{3.24}$$

But, by using (3.2) we have the following estimate

$$\begin{aligned}-Re\int_\Omega (F(x)\cdot\nabla\psi_m')\phi_m'' dx &= \int_\Omega (F(x)\cdot\nabla\psi_m')\Delta\phi_m + \int_\Omega (F(x)\cdot\nabla\psi_m)(F(x)\cdot\nabla\psi_m')dx\\ &+ \int_\Omega (F(x)\cdot\nabla\psi_m')\phi_m + \lambda\int_\Omega (F(x)\cdot\nabla\psi_m')\phi_m' dx\\ &- \int_0^t g(t-\tau)(F(x)\cdot\nabla\psi_m'(\tau), \Delta\phi_m(t))dxd\tau\end{aligned} \tag{3.25}$$

where

$$\frac{d}{dt}\Big(\int_\Omega (F(x)\cdot\nabla\psi_m)\phi_m' dx\Big) = \int_\Omega (F(x)\cdot\nabla\psi_m')\phi_m' dx + \int_\Omega (F(x)\cdot\nabla\psi_m)\phi_m'' dx.$$

Analyzing the terms on the right hand side gives

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega (F(x)\cdot\nabla\psi_m)\Delta\phi_m dx\Big) &= \int_\Omega (F(x)\cdot\nabla\psi_m')\Delta\phi_m dx + \int_\Omega (F(x)\cdot\nabla\psi_m)\Delta\phi_m' dx\\ &= \int_\Omega (F(x)\cdot\nabla\psi_m')\Delta\phi_m dx - \int_\Omega \nabla(F(x)\cdot\nabla\psi_m)\nabla\phi_m' dx\\ &= -\int_\Omega (F(x)\cdot\Delta\psi_m)\nabla\phi_m' dx - \int_\Omega (\nabla F(x)\cdot\nabla\psi_m)\nabla\phi_m' dx\\ &- \int_\Omega (\nabla\psi_m\times(\nabla\times F(x)))\nabla\phi_m' dx + \int_\Omega (F(x)\cdot\nabla\psi_m')\Delta\phi_m dx.\end{aligned} \tag{3.26}$$

Similarly we have

$$\frac{d}{dt}\bigg(\int_\Omega (F(x)\cdot\nabla\psi_m)\phi_m dx\bigg) = \int_\Omega (F(x)\cdot\nabla\psi_m^{'})\phi_m dx + \int_\Omega (F(x)\cdot\nabla\psi_m)\phi_m^{'} dx, \tag{3.27}$$

with

$$\frac{d}{dt}\bigg(\int_\Omega (F(x)\cdot\nabla\psi_m)\phi_m^{'} dx\bigg) = \int_\Omega (F(x)\cdot\nabla\psi_m^{'})\phi_m^{'} dx + \int_\Omega (F(x)\cdot\nabla\psi_m)\phi_m^{''} dx, \tag{3.28}$$

and

$$\begin{aligned}\frac{d}{dt}\bigg(\int_0^t g(t-\tau)(F(x)\cdot\nabla\psi_m(\tau),\Delta\phi_m(t))dxd\tau\bigg) &= g(0)\int_\Omega (F(x)\cdot\nabla\psi_m(t))\Delta\phi_m(t)\\ &+\int_0^t g^{'}(t-\tau)\int_\Omega (F(x)\cdot\nabla\psi_m(\tau))\Delta\phi_m(t)dxd\tau\\ &+\int_0^t g(t-\tau)\int_\Omega (F(x)\cdot\nabla\psi_m^{'}(t))\Delta\phi_m(\tau)dxd\tau.\end{aligned} \tag{3.29}$$

Substituting (3.25), (3.26), (3.27), (3.28) and (3.29) into (3.24) produces

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\bigg(||\psi_m^{'}||^2+||\phi_m^{''}||^2+||\nabla\phi_m^{'}||^2+||\phi_m^{'}||^2\bigg)+\alpha||\psi_m^{'}||^2+g(0)||\nabla\phi_m^{'}||^2+\lambda||\phi_m^{''}||^2\\ &+Re\int_\Omega \phi_m^{'}\bar{\psi}_m\psi_m^{'}dx+\int_\Omega \phi_m|\psi_m^{'}|^2dx \le \frac{d}{dt}\bigg(\int_0^t g^{'}(t-\tau)(\nabla\phi_m(\tau),\nabla\phi_m^{'}(t))d\tau\bigg)\\ &+\frac{d}{dt}\bigg(\int_\Omega (F(x)\cdot\nabla\psi_m)\Delta\phi_m dx\bigg)+\frac{d}{dt}\bigg(\int_\Omega (F(x)\cdot\nabla\psi_m)\phi_m dx\bigg)-\lambda Re\int_\Omega (F(x)\cdot\nabla\psi_m)\phi_m^{''}dx\\ &+\lambda\frac{d}{dt}\bigg(\int_\Omega (F(x)\cdot\nabla\psi_m)\phi_m^{'}dx\bigg)+\frac{d}{dt}\bigg(\int_0^t g(t-\tau)(F(x)\cdot\nabla\psi_m(\tau),\Delta\phi_m(t))d\tau\bigg)\\ &-g^{'}(0)(\nabla\phi_m(t),\nabla\phi_m^{'}(t))+g(0)\frac{d}{dt}(\nabla\phi_m(t),\nabla\phi_m^{'}(t))+\frac{d}{dt}\bigg(M\int_\Omega |(F(x)\cdot\nabla\psi_m)|^2dx\bigg)\\ &-g(0)\int_\Omega (F(x)\cdot\nabla\psi_m(t))\Delta\phi_m(t)dx+\int_0^t g^{'}(t-\tau)\int_\Omega (F(x)\cdot\nabla\psi_m(\tau))\Delta\phi_m(t)dxd\tau\\ &-\int_0^t g^{''}(t-\tau)(\nabla\phi_m(\tau),\nabla\phi_m^{'}(t))d\tau-\int_\Omega (F(x)\cdot\nabla\psi_m)\phi_m^{''}dx-\int_\Omega (F(x)\cdot\Delta\psi_m)\nabla\phi_m^{'}dx\\ &-\int_\Omega (\nabla F(x)\cdot\nabla\psi_m)\nabla\phi_m^{'}dx-\int_\Omega (\nabla\psi_m\times(\nabla\times F(x)))\nabla\phi_m^{'}dx-\int_\Omega (F(x)\cdot\nabla\psi_m)\phi_m^{'}dx.\end{aligned} \tag{3.30}$$

Evaluating some of the integrals above by taking into consideration Young's inequality and the following embedding $H^1(\Omega)\hookrightarrow L^q(\Omega)$, with $q\in[1,6]$ and inequality (2.2) we obtain

$$\begin{aligned}&\bigg|\int_\Omega \phi_m|\psi_m^{'}|^2dx\bigg| \le ||\phi_m||_\infty||\psi_m^{'}||^2 \le C||\Delta\phi_m||||\psi_m^{'}||^2,\\ &\bigg|\int_\Omega \phi_m^{'}\psi_m\psi_m^{'}dx\bigg| \le ||\phi_m^{'}||\ ||\psi_m^{'}||||\psi_m||_\infty \le \epsilon||\nabla\phi_m^{'}||^2+C(\epsilon)||\psi_m^{'}||^2||\Delta\psi_m||^2.\end{aligned} \tag{3.31}$$

with

$$\left|\int_0^t g^{''}(t-\tau)\int_\Omega \bigtriangledown \phi_m(\tau)\bigtriangledown \phi_m^{'}(t)dxd\tau\right| \\ \leq \frac{1}{2}||\bigtriangledown \phi_m^{'}(t)||^2+\frac{m_2^2}{2}||g||_{L^1(0,\infty)}\int_0^t g(t-\tau)||\bigtriangledown \phi_m(\tau)||^2d\tau. \tag{3.32}$$

and

$$|g^{'}(0)(\bigtriangledown\phi_m(t),\bigtriangledown\phi_m^{'}(t))|\leq \frac{(g^{'}(0))^2}{2}||\bigtriangledown \phi_m(t)||^2+\frac{1}{2}||\bigtriangledown \phi_m^{'}(t)||^2. \tag{3.33}$$

Also

$$\begin{aligned} &\left|Re\int_\Omega (F(x)\cdot\bigtriangledown\psi_m)\phi_m^{''}dx\right|\leq M||\bigtriangledown \psi_m||\;||\phi_m^{''}||\leq\frac{1}{4}||\bigtriangledown \psi_m||^2+C||\phi_m^{''}||^2,\\ &\left|\int_\Omega (F(x)\cdot\bigtriangledown\psi_m)\phi_m^{'}dx\right|\leq M||\bigtriangledown \psi_m||\;||\phi_m^{'}||\leq\frac{1}{4}||\bigtriangledown \psi_m||^2+C||\phi_m^{'}||^2,\\ &\left|g(0)\int_\Omega (F(x)\cdot\bigtriangledown\psi_m)\Delta\phi_m dx\right|\leq Mg(0)||\bigtriangledown \psi_m||\;||\Delta\phi_m||\leq\frac{1}{4}||\bigtriangledown \psi_m||^2+C||\Delta\phi_m||^2. \end{aligned} \tag{3.34}$$

Substituting (3.31), (3.32) and (3.33), (3.34) into (3.30) and integrating over $(0,t)$ we obtain

$$\begin{aligned} &\frac{1}{2}\Big(||\psi_m^{'}||^2+||\phi_m^{''}||^2+||\bigtriangledown \phi_m^{'}||^2+||\phi_m^{'}||^2\Big)+\alpha\int_0^t||\psi_m^{'}(s)||^2ds+g(0)\int_0^t||\bigtriangledown \phi_m^{'}(s)||^2ds\\ &+\lambda\int_0^t||\phi_m^{''}(s)||^2ds\leq C+\int_0^t g^{'}(t-\tau)(\bigtriangledown\phi_m(\tau),\bigtriangledown\phi_m^{'}(t))d\tau+g(0)(\bigtriangledown\phi_m(t),\bigtriangledown\phi_m^{'}(t))\\ &+C[||\bigtriangledown \phi_m^{'}||^2+||\phi_m^{''}||^2+||\bigtriangledown \psi_m||^2+||\phi_m^{'}||^2+||\bigtriangledown \phi_m||^2+||\Delta\phi_m||^2+||\Delta\psi_m||^2]\\ &+\frac{m_2^2}{2}||g||_{L^1(0,\infty)}\int_0^t\int_0^s g(s-\tau)||\bigtriangledown \phi_m(\tau)||^2d\tau ds+\int_0^t g(t-\tau)(F(x)\cdot\bigtriangledown\psi_m(\tau),\Delta\phi_m(t))d\tau. \end{aligned} \tag{3.35}$$

Furthermore

$$\int_0^t g^{'}(t-\tau)(\bigtriangledown\phi_m(\tau),\bigtriangledown\phi_m^{'}(t))\leq\frac{m_1^2}{4\eta}||g||_{L^1(0,\infty)}||g||_{L^\infty(0,\infty)}\int_0^t||\bigtriangledown \phi_m(\tau)||^2d\tau+\eta||\bigtriangledown \phi_m^{'}(t)||^2$$

with

$$g(0)(\bigtriangledown\phi_m(t),\bigtriangledown\phi_m^{'}(t))\leq\frac{(g(0))^2}{4\eta}||\bigtriangledown \phi_m||^2+\eta||\bigtriangledown \phi_m^{'}||^2$$

and

$$\int_0^t g(t-\tau)(F(x)\cdot\bigtriangledown\psi_m(\tau),\Delta\phi_m(t))d\tau\leq M||\Delta\phi_m(t)||||g||^{1/2}_{L^1(0,\infty)}\Big(\int_0^t g(t-\tau)||\bigtriangledown \psi_m||^2\Big)^{1/2} \\ \leq\frac{1}{8}||\Delta_m(t)||^2+2M^2||g||_{L^1(0,\infty)}||g||_{L^\infty(0,\infty)}\int_0^t||\bigtriangledown \psi_m||^2.$$

Next, we are going to estimate the $L^2(\Omega)$ norm of $\psi_m^{'}(0)$ and $\phi_m^{''}(0)$. Letting $u = \psi_m^{'}(0)$ and $v = \phi_m^{''}(0)$ in (3.1) and (3.2) produces

$$||\psi_m^{'}(0)|| \leq \kappa||\Delta\psi_m(0)|| + \alpha||\psi_m(0)|| + ||\phi_m(0)||_4||\psi_m(0)||_4 \tag{3.36}$$

and

$$||\phi_m^{''}(0)|| \leq ||\Delta\phi_m(0)|| + ||\phi_m(0)|| + \lambda||\phi_m^{'}(0)|| + M|| \bigtriangledown \psi_m(0)||. \tag{3.37}$$

From which using Sobolev embeddings it may be concluded that

$$||\psi_m^{'}(0)|| \leq C \ \text{ and } \ ||\phi_m^{''}(0)|| \leq C \ \ \forall m \ \in \mathbb{N}.$$

Combining the above inequalities and employing Gronwall's Lemma in (3.35) we obtain the third estimate

$$||\psi_m^{'}||^2 + ||\phi_m^{''}||^2 + || \bigtriangledown \phi_m^{'}||^2 + ||\phi_m^{'}||^2 + \int_0^t [||\psi_m^{'}(s)||^2 + || \bigtriangledown \phi_m^{'}(s)||^2 + ||\phi_m^{''}(s)||^2]ds \leq L_3. \tag{3.38}$$

From (3.13), (3.21) and (3.38) we get

$$\begin{aligned} &\{\psi_m\} \text{ is bounded in } L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)),\\ &\{\phi_m\} \text{ is bounded in } L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)),\\ &\{\psi_m^{'}\} \text{ is bounded in } L^\infty(0,T; L^2(\Omega)),\\ &\{\phi_m^{'}\} \text{ is bounded in } L^\infty(0,T; H_0^1(\Omega)),\\ &\{\phi_m^{''}\} \text{ is bounded in } L^\infty(0,T; L^2(\Omega)). \end{aligned} \tag{3.39}$$

Therefore we can extract weekly * convergent subsequences denoted again as (ψ_m, ϕ_m) such that

$$\psi_m \to^{w*} \psi, \quad \phi_m \to^{w*} \phi,$$

The above convergences are sufficient to pass to the limit in (3.1) and (3.2) and it results thanks to the elliptic regularity that

$$\psi \ \in \ L^\infty(0,\infty; H_0^1(\Omega) \cap H^2(\Omega)).$$

Following similar procedure as in Theorem 2.1 of [11] we prove the uniqueness of the solutions. Therefore the proof of Theorem 2.1 is completed.

4 Energy Decay

Due to the previous results the corresponding energy functional for the system (1.1) and (1.2) is

$$E(t) = \frac{1}{2}\left[||\psi||^2 + \kappa|| \bigtriangledown \psi||^2 + ||\phi^{'}||^2 + || \bigtriangledown \phi||^2 + ||\phi||^2 + \frac{1}{2}\int_\Omega \phi|\psi|^2 dx\right].$$

The integral cannot affect the asymptotic value of the energy which remains positive as seeing below using (3.12)

$$E(t) \geq \frac{1}{2}\left[||\psi||^2 + \frac{\kappa}{2}|| \bigtriangledown \psi||^2 + ||\phi^{'}||^2 + \frac{1}{2}|| \bigtriangledown \phi||^2 + ||\phi||^2 + C\right],$$

and

$$E(t) \leq \frac{1}{2}\left[||\psi||^2 + \frac{3\kappa}{2}||\bigtriangledown\psi||^2 + ||\phi^{'}||^2 + \frac{3}{2}||\bigtriangledown\phi||^2 + ||\phi||^2 + C\right],$$

Let $u = -(\bar{\psi}^{'}(t) + \alpha\bar{\psi}(t)), v = \phi^{'}(t)$ in (3.1) and (3.2) respectively and then by integrating and adding them up we obtain

$$\frac{1}{2}\frac{d}{dt}\left(||\psi||^2 + \kappa||\bigtriangledown\psi||^2 + ||\phi^{'}||^2 + ||\bigtriangledown\phi||^2 + ||\phi||^2 + \frac{1}{2}\int_\Omega \phi|\psi|^2dx\right) + \alpha||\psi||^2 + \kappa\alpha||\bigtriangledown\psi||^2$$
$$+\lambda||\phi^{'}||^2 + \alpha\int_\Omega \phi|\psi|^2dx = \int_0^t g(t-\tau)(\bigtriangledown\phi(\tau), \bigtriangledown\phi^{'}(t))d\tau + \frac{1}{2}\int_\Omega \phi^{'}|\psi|^2dx - Re\int_\Omega (F(x)\cdot\bigtriangledown\psi)\phi^{'}dx$$

hence

$$\begin{aligned} E^{'}(t) \leq & -\alpha||\psi||^2 - \lambda||\phi^{'}||^2 - \kappa\alpha||\bigtriangledown\psi||^2 + \int_0^t g(t-\tau)(\bigtriangledown\phi, \bigtriangledown\phi^{'})(\bigtriangledown\phi(\tau), \bigtriangledown\phi^{'}(t))d\tau \\ & -\alpha\int_\Omega \phi|\psi|^2dx + \frac{1}{2}\int_\Omega \phi^{'}|\psi|^2dx - Re\int_\Omega (F(x)\cdot\bigtriangledown\psi)\phi^{'}dx. \end{aligned} \tag{4.1}$$

Define the modified energy as

$$\begin{aligned} e(t) =& \frac{1}{2}\Bigg[||\psi||^2 + \kappa||\bigtriangledown\psi||^2 + ||\phi^{'}||^2 + ||\bigtriangledown\phi||^2 + ||\phi||^2 + \frac{1}{2}\int_\Omega \phi|\psi|^2dx \\ & + \left(1 - \int_0^t g(s)ds\right)||\bigtriangledown\phi||^2 + \int_0^t g(t-\tau)||\bigtriangledown\phi(t) - \bigtriangledown\phi(\tau)||^2d\tau\Bigg] \end{aligned}$$

and taking into consideration that

$$\int_0^t g(t-\tau)(\bigtriangledown\phi(\tau), \bigtriangledown\phi^{'}(t))d\tau = \frac{1}{2}\int_0^t g^{'}(t-\tau)||\bigtriangledown\phi(t) - \bigtriangledown\phi(\tau)||^2d\tau - \frac{1}{2}g(t)||\bigtriangledown\phi||^2$$
$$+\frac{d}{dt}\left(\frac{1}{2}\left(\int_0^t g(s)ds\right)||\bigtriangledown\phi(t)||^2\right) - \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t-\tau)||\phi(t)-\phi(\tau)||^2d\tau\right)$$

we obtain

$$\begin{aligned} e^{'}(t) =& -\alpha||\psi||^2 - \lambda||\phi^{'}||^2 - \kappa\alpha||\bigtriangledown\psi||^2 - \frac{1}{2}g(t)||\bigtriangledown\phi||^2 - \frac{m_1}{2}\int_0^t g(t-\tau)||\phi(t)-\phi(\tau)||^2d\tau \\ & -\alpha\int_\Omega \phi|\psi|^2dx + \frac{1}{2}\int_\Omega \phi^{'}|\psi|^2dx - Re\int_\Omega (F(x)\cdot\bigtriangledown\psi)\phi^{'}dx \end{aligned}$$

evaluating the integrals we have

$$\left|\frac{1}{2}\int_\Omega \phi^{'}|\psi|^2dx\right| \leq \frac{\epsilon_1}{4}\int_\Omega |\phi^{'}|^2dx + \frac{C^2}{4\epsilon_1}\int_\Omega |\bigtriangledown\psi|^2dx,$$
$$\left|\int_\Omega (F(x)\cdot\bigtriangledown\psi)\phi^{'}dx\right| \leq \frac{\epsilon_1}{2}\int_\Omega |\phi^{'}|^2dx + \frac{M^2}{2\epsilon_1}\int_\Omega |\bigtriangledown\psi|^2dx,$$
$$\left|\alpha\int_\Omega \phi|\psi|^2dx\right| \leq \frac{\epsilon}{2\mu}\int_\Omega |\phi|^2dx + \frac{\alpha^2\epsilon_0^2C^2\mu}{2\epsilon}\int_\Omega |\bigtriangledown\psi|^2dx.$$

Therefore

$$
\begin{aligned}
e^{'}(t) \leq & -\alpha||\psi||^2 - (\lambda - \frac{3\epsilon_1}{4})||\phi^{'}||^2 - (\kappa\alpha - \frac{M^2}{2\epsilon_1} - \frac{C^2}{4\epsilon_1} - \frac{\alpha^2\epsilon_0^2 C^2\mu}{2\epsilon})||\bigtriangledown\psi||^2 \\
& + \frac{\epsilon}{2\mu}||\phi||^2 - \frac{1}{2}g(t)||\bigtriangledown\phi||^2 - \frac{m_1}{2}\int_0^t g(t-\tau)||\phi(t)-\phi(\tau)||^2 d\tau.
\end{aligned} \tag{4.2}
$$

Following [5] , for $\epsilon > 0$ we introduce the perturbed energy

$$
e_{pert}(t) = e(t) + \epsilon p(t), \tag{4.3}
$$

where $p(t) = ||\psi||^2 + (\phi^{'}, \phi)$. We have the following results

Proposition 4.1 *There exists* $C_1 > 0$ *such that*

$$
|e_{pert}(t) - e(t)| \leq \epsilon C_1 e(t)
$$

for all $\epsilon > 0$ *and* $t \geq 0$.

Proof From the definition of $p(t)$ and (2.1) we obtain

$$
|p(t)| \leq ||\psi(t)||^2 + \frac{1}{2}||\phi^{'}(t)||^2 + \frac{c^*}{2}||\bigtriangledown\phi(t)||^2 \leq (2+c^*)e(t).
$$

From the last inequality we conclude the proof with $C_1 = 2 + c^*$.

Proposition 4.2 *Let* $16\kappa\lambda\alpha > 6M^2 + 3C^2$ *and Assumptions 2.1 , 2.2 hold. Then there exists a* $\tilde{\epsilon}_1 > 0$ *and* $C_2 > 0$ *such that*

$$
e^{'}_{pert}(t) \leq -\epsilon C_2 e(t)
$$

for all $t \geq 0$ *and* $\epsilon \in (0, \tilde{\epsilon}_1]$.

Proof Getting the derivative of $p(t)$ we have

$$
p^{'}(t) = 2Re(\psi^{'}, \psi) + (\phi^{''}, \phi) + ||\phi^{'}||^2 \tag{4.4}
$$

and replacing $\psi^{'}$ and $\phi^{''}$ by using (1.1) and (1.2) we obtain

$$
\begin{aligned}
p^{'}(t) = & -2\alpha||\psi||^2 - ||\bigtriangledown\phi||^2 - ||\phi||^2 + ||\phi^{'}||^2 - \lambda\int_\Omega \phi^{'}\phi dx \\
& - \int_\Omega (F(x)\cdot\bigtriangledown\psi)\phi dx + \int_0^t g(t-\tau)(\bigtriangledown\phi(\tau), \bigtriangledown\phi(t))d\tau.
\end{aligned} \tag{4.5}
$$

Adding and subtracting several terms and also postulating $N = min\{4\alpha, 1\}$, we have

$$
\begin{aligned}
p^{'}(t) \leq & -NE(t) + \frac{\kappa}{2}||\bigtriangledown\psi||^2 + \frac{3}{2}||\phi^{'}||^2 - \frac{1}{2}||\phi||^2 - \frac{1}{2}||\bigtriangledown\phi||^2 - \lambda\int_\Omega \phi^{'}\phi dx \\
& + \frac{1}{2}\int_\Omega \phi|\psi|^2 dx - \int_\Omega (F(x)\cdot\bigtriangledown\psi)\phi dx + \int_0^t g(t-\tau)(\bigtriangledown\phi(\tau), \bigtriangledown\phi(t))d\tau.
\end{aligned} \tag{4.6}
$$

Evaluating the integrals above

$$\begin{aligned}
&\left|\lambda \int_{\Omega} \phi^{'} \phi dx\right| \leq \frac{\lambda^2 \epsilon_2}{2} \int_{\Omega} |\phi^{'}|^2 dx + \frac{1}{2\epsilon_2} \int_{\Omega} |\phi|^2 dx, \\
&\left|\int_{\Omega} (F(x) \cdot \bigtriangledown \psi) \phi dx\right| \leq \frac{1}{2\epsilon_2} \int_{\Omega} |\phi|^2 dx + \frac{M^2 \epsilon_2}{2} \int_{\Omega} |\bigtriangledown \psi|^2 dx, \\
&\left|\frac{1}{2} \int_{\Omega} \phi |\psi|^2 dx\right| \leq \frac{R\epsilon_1}{4} \int_{\Omega} |\phi|^2 dx + \frac{R}{4\epsilon_1} \int_{\Omega} |\bigtriangledown \psi|^2 dx
\end{aligned}$$

and

$$\int_0^t g(t-\tau)(\bigtriangledown \phi(\tau), \bigtriangledown \phi(t)) d\tau \leq \frac{1}{2} \int_0^t g(t-\tau) ||\phi(t) - \phi(\tau)||^2 d\tau + \frac{3}{2} || \bigtriangledown \phi(t)||^2 \int_0^t g(s) ds$$

equation (4.6) becomes

$$\begin{aligned}
p^{'}(t) \leq &- NE(t) + (\frac{\kappa}{2} + \frac{R}{4\epsilon_1} + \frac{M^2 \epsilon_2}{2}) || \bigtriangledown \psi||^2 + (\frac{3}{2} + \frac{\lambda^2 \epsilon_2}{2}) ||\phi^{'}||^2 + (\frac{R\epsilon_1}{4} + \frac{1}{\epsilon_2} - \frac{1}{2}) ||\phi||^2 \\
&+ \frac{1}{2} \int_0^t g(t-\tau) ||\phi(t) - \phi(\tau)||^2 d\tau + (\frac{3}{2} \int_0^t g(s) ds - \frac{1}{2}) || \bigtriangledown \phi(t)||^2.
\end{aligned} \tag{4.7}$$

Now, differentiating (4.3) with respect to t and using equations (4.2) and (4.7) gives

$$\begin{aligned}
e_{pert}^{'}(t) \leq &- \epsilon NE(t) + \left(\epsilon(\frac{\kappa}{2} + \frac{R}{4\epsilon_1} + \frac{M^2 \epsilon_2}{2}) - (\kappa\alpha - \frac{M^2}{2\epsilon_1} - \frac{C^2}{4\epsilon_1} - \frac{\alpha^2 \epsilon_0^2 C^2 \mu}{2\epsilon})\right) || \bigtriangledown \psi||^2 \\
&+ \left(\epsilon(\frac{3}{2} + \frac{\lambda^2 \epsilon_2}{2}) - (\lambda - \frac{3\epsilon_1}{4})\right) ||\phi^{'}||^2 + \epsilon(\frac{R\epsilon_1}{4} - \frac{1}{2} + \frac{1}{\epsilon_2} + \frac{1}{2\mu}) ||\phi||^2 \\
&+ (\frac{1}{2} - \frac{m_1}{2}) \int_0^t g(t-\tau) ||\phi(t) - \phi(\tau)||^2 d\tau + (\frac{3}{2} \int_0^t g(s) ds - \frac{1}{2} - \frac{1}{2} g(t)) || \bigtriangledown \phi(t)||^2.
\end{aligned} \tag{4.8}$$

Now, let all the expressions within the brackets be simultaneously non positive or zero. To achieve this, we introduce the auxiliary constant $\nu > 0$. Then choosing the constants to be $\epsilon_1 = \dfrac{\nu}{R}$, $\epsilon_2 = \dfrac{2}{\nu}$ and setting the third expression equal to zero, we determine the value of ν as $\nu = \dfrac{2(\mu - 1)}{3\mu}$. Next by requiring the first two expressions to be non positive we reach to the conclusion that for $16\kappa\lambda\alpha > 6M^2 + 3C^2$ we have

$$e_{pert}^{'} \leq -\epsilon NE(t).$$

for all $t \geq 0$ and $\epsilon \in (0, \tilde{\epsilon}_1]$.

Let $\tilde{\epsilon}_0 = min\{\tilde{\epsilon}_1, \frac{1}{2C_1}\}$, where C_1 is given in Proposition 4.1. Consider $\epsilon \in (0, \tilde{\epsilon}_0]$. From Proposition 4.1 we obtain

$$\frac{1}{2} e(t) \leq e_{pert}(t) \leq \frac{3}{2} e(t) \leq 2e(t) \quad \text{for all } t \geq 0.$$

Therefore we get

$$e'_{pert}(t) \leq -\frac{\epsilon C_2}{2} e_{pert}(t),$$

for all $t \geq 0$ and $\epsilon \in (0, \tilde{\epsilon}_0]$ which allows us to conclude that

$$e_{pert}(t) \leq 2e(0) exp(-\frac{\epsilon C_2}{2} t).$$

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