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Countably condensing multimaps and fixed points

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Abstract: In this paper we present a generalization of the Daher's fixed point theorem to countably condensing multimaps. We obtain this result as a consequence of a new Mönch type theorem for multimaps having weakly closed graph.
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1 Introduction

Condensing operators have been the object of a wide and deep study in nonlinear functional analysis. This research was started in 1967 by Sadovskii [22]. He introduced the concept of condensing operator for single-valued functions by using the Kuratowski measure of noncompactness and proved that a condensing operator from a closed bounded convex subset of a Banach space into itself has a fixed point, extending the well known Darbo's fixed point theorem [13].
Later, the Sadovskii's result has been improved in different directions: from one hand, Daher [12] showed that it is still true for countably condensing maps (i.e. condensing only on countable subsets); from another, Himmelberg, Porter and Van Vleck [17] ex-

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tended the definition of condensing operator to multimaps and provided a multivalued version of the Sadovskii's theorem.
More recently, Agarwal and O'Regan [1] established a multivalued version of the Daher's theorem for upper semicontinuous multimaps.

The main purpose of the present paper is to state a Daher type theorem (see Theorem 3.2) for multimaps having *weakly closed graph*, which can take some nonconvex values and are *countably condensing*.

By *countably condensing* we mean the following property:

Definition 1.1 *Let D be a nonempty subset of a Banach space X and let β be an abstract measure of noncompactness. A map $F : D \to \mathcal{P}(X)$ is said to be* countably condensing *if*

(I) $F(D)$ is bounded;

(II) $\beta(F(B)) < \beta(B)$ for all countable bounded subsets B of D with $\beta(B) > 0$.

As far as we know, this definition is new and it is an improving of the analogous definition given by Agarwal and O'Regan. In fact, they say that a multimap is countably condensing if it satisfies conditions (I), (II) and the further

(III) F is 1-set contractive, i.e. $\beta(F(B)) \leq \beta(B)$ for all countable bounded subsets B of D

taking β as the Kuratowski measure of noncompactness.
The advantage of our definition is that it is the natural weakening of the classical definition of condensing multimap (see, e.g. [19]) and, moreover, that in the single-valued case it comes down to the Daher's definition of countably condensing function.

To prove our main result, in a preliminary way we provide a new Mönch type theorem for multimaps having weakly closed graph (cf. Theorem 3.1). The Mönch type hypothesis we use has been introduced by O'Regan and Precup [21] in order to extend the Mönch theorems to multimaps. We base our proof on a fixed point theorem for multimaps having weakly closed graph established in [6].

We wish to note that our theorems strictly contain respectively the quoted Agarwal-O'Regan's result [1, Theorem 1.2] and the O'Regan-Precup's Mönch type theorem [21, Theorem 3.1] (see Remark 3.1).

Finally, we remark the relevance of the study of the existence of fixed points due to its applicability in finding the existence of solutions of various kinds of nonlinear differential equations or inclusions (see, e.g. [2], [4], [5], [7], [8], [9], [11], [15], [20]) and nonlinear integral equations or inclusions (see, e.g. [10], [21]).

2 Preliminaries

Let $\mathcal{X}$ be a locally convex Hausdorff topological linear space and $\mathcal{P}(\mathcal{X})$ be the family of all nonempty subsets of $\mathcal{X}$.

We recall (see [16]) that a set $A \subset \mathcal{X}$ is said to be *quasi-convex* if for every balanced convex neighborhood V of zero and for every $\{a_1, \ldots, a_n\} \subset A$ there exists $\{z_1, \ldots, z_n\} \subset A$ such that $z_i - a_i \in V$, $i = 1, \ldots, n$, $co\{z_1, \ldots, z_n\} \subset A$.
Moreover, let D be a nonempty subset of $\mathcal{X}$. As in [6], we say that a map $G : D \to \mathcal{P}(\mathcal{X})$ has *weakly closed graph* in $D \times \mathcal{X}$ if for every net $(x_\delta)_\delta$ in D, $x_\delta \to x$, $x \in D$, and for every net $(y_\delta)_\delta$, $y_\delta \in G(x_\delta)$, $y_\delta \to y$, then $S(x, y) \cap G(x) \neq \emptyset$, where $S(x, y) = \{x + \lambda(y - x) : \lambda \in [0, 1]\}$.
In the sequel we will use the following theorem.

Theorem 2.1 *[6, Teorema I] Let $\mathcal{X}$ be a locally convex Hausdorff topological linear space, K be a nonempty compact subset of $\mathcal{X}$ and $G : K \to \mathcal{P}(K)$ be a map taking nonempty closed values and with the properties*

(i) there exists $A \subset K$, A quasi-convex, $\overline{A} = K$ such that $G(x)$ is convex for every $x \in A$;

(ii) G has weakly closed graph.

Under these conditions, there exists $x \in K$ such that $x \in G(x)$.

From now on, X will be a Banach space endowed with the norm $\|\cdot\|$. We will use the following notations: $\mathcal{P}_b(X) = \{H \subset X : H \neq \emptyset, H \text{ bounded}\}$; $\mathcal{P}_k(X) = \{H \subset X : H \neq \emptyset, H \text{ compact}\}$.

We recall (see, e.g. [3], [19]) that a function $\beta : \mathcal{P}_b(X) \to \mathbb{R}_0^+$ is said to be a *measure of noncompactness* (MNC, for short) if

$$\beta(\overline{co}(\Omega)) = \beta(\Omega) \ , \text{ for every } \Omega \in \mathcal{P}_b(X) \ . \tag{1}$$

In the sequel, we consider a measure of noncompactness β verifying the following properties:

(β_1) *regularity:* $\beta(\Omega) = 0$ if and only if $\overline{\Omega}$ is compact;

(β_2) *monotonicity:* $\Omega_1 \subset \Omega_2$ implies $\beta(\Omega_1) \leq \beta(\Omega_2)$;

(β_3) *semiadditivity:* $\beta(\Omega_1 \cup \Omega_2) = \max\{\beta(\Omega_1), \beta(\Omega_2)\}$;

where $\Omega, \Omega_1, \Omega_2 \in \mathcal{P}_b(X)$.
As examples of measures of noncompactness which satisfy all the previous properties, we recall the Hausdorff and the Kuratowski measures of noncompactness.

Remark 2.1 *We note that if β is a regular MNC, property (II) of Definition 1.1 can be equivalently formulated as*

(II)' for every countable bounded subset B of D the relation $\beta(B) \leq \beta(F(B))$ implies that $\overline{B}$ is compact.

In the single-valued case, this is the definition of countably condensing map sometimes adopted in the literature.

3 The fixed point theorems

In this section, we will provide a Daher type theorem for multimaps. In a preliminary way, we give a fixed point theorem for multimaps by using a Mönch type assumption which has been introduced by O'Regan and Precup [21] in order to extend the classical Mönch theorem to set-valued maps.

Theorem 3.1 *Let D be a closed convex subset of a Banach space X and $F : D \to \mathcal{P}_k(D)$ be a map such that*

(i) there exists $A \subset D$, A quasi-convex, $\overline{A} = D$ such that $F(x)$ is convex for every $x \in A$;

(ii) F has weakly closed graph;

(iii) F maps compact sets into relatively compact sets;

(M) there exists $x_0 \in D$ such that

$$\left.\begin{array}{l} M \subset D,\ M = co(\{x_0\} \cup F(M)) \\ \text{and } \overline{M} = \overline{C} \text{ with } C \subset M \text{ countable} \end{array}\right\} \quad \Rightarrow \quad \overline{M} \text{ is compact.}$$

Then there exists $x \in D$ such that $x \in F(x)$.

Proof. Let $x_0 \in D$ be as by hypothesis *(M)*. We consider the iterative sequence $(M_n)_{n\in I\!N}$ of sets:

$$M_0 = \{x_0\} \ ; \quad M_n = co(\{x_0\} \cup F(M_{n-1})) \ , \ n \in I\!N^+ \ .$$

Clearly,

$$\overline{M_n} \subset D \ , \quad n \in I\!N \ . \tag{2}$$

Let us prove by induction that M_n, $n \in I\!N^+$, is relatively compact.
First, the Mazur Theorem (see, e.g. [18, Theorem A.3.68]) implies that $\overline{co}(\{x_0\} \cup F(M_0))$ is compact. So, $M_1 = co(\{x_0\} \cup F(M_0))$ is relatively compact.
Now, suppose that M_{n-1} is relatively compact, $n \geq 2$. Of course $\overline{M_n} \subset \overline{co}(\{x_0\} \cup \overline{F(\overline{M_{n-1}})})$ (see (2)). By *(iii)* and the Mazur's theorem, we can say that also M_n is relatively compact.
By induction again, we can say that

$$M_{n-1} \subset M_n \ , \quad n \in I\!N^+ \ . \tag{3}$$

Now, for every $n \in I\!N$, let us consider the space (M_n, d), where d is the metric induced on M_n by the metric generated by $\|\cdot\|$. The compactness of $\overline{M_n}$ implies that (M_n, d) is a separable space (cf., e.g. Corollary 1.4.29 and Corollary 1.4.12 in [14]). Hence, there exists a countable set $C_n \subset M_n$ such that

$$\overline{C_n}^{(M_n,d)} = M_n \ . \tag{4}$$

Let us consider the subset of D defined as

$$M = \cup_{n\in I\!N} M_n \tag{5}$$

and its countable subset

$$C = \cup_{n\in I\!N} C_n \ . \tag{6}$$

First of all, it is easy to show that

$$\overline{C} = \overline{\cup_{n\in I\!N} \overline{C_n}} \ . \tag{7}$$

Now, being C_n a subset of M_n, we have

$$\overline{C_n} = \overline{C_n}^{(M_n,d)}$$

and so, from (7), we deduce

$$\overline{C} = \overline{\cup_{n\in I\!N} \overline{C_n}^{(M_n,d)}} \ . \tag{8}$$

Therefore, by (5), (4) and (8), we can say that

$$\overline{M} = \overline{C} \ . \tag{9}$$

Further, some easy considerations, which make use of (3) and (5), lead us to write that

$$M = co(\{x_0\} \cup F(M)) \ . \tag{10}$$

Since M is a set verifying the property of hypothesis *(M)*, we can claim that the set

$$\overline{M} \text{ is compact } . \tag{11}$$

Now, we consider the map $G : \overline{M} \to \mathcal{P}(\overline{M})$ defined by

$$G(x) = F(x) \cap \overline{M} \ , \ x \in \overline{M} \ .$$

As a matter of fact, the multimap G has nonempty values. In fact, fixed $x \in \overline{M}$, there exists a sequence $(x_n)_{n\in I\!N}$ in M such that $x_n \to x$. Let us consider a sequence $(y_n)_{n\in I\!N}$, $y_n \in F(x_n)$, $n \in I\!N$. By (10), $\{y_n\}_{n\in I\!N}$ is included in the compact set $\overline{M}$ (see (11)). Therefore, w.l.o.g., we can say that $y_n \to y \in \overline{M}$. So, by applying hypothesis *(ii)*, $S(x, y) \cap F(x) \neq \emptyset$; by the convexity of $\overline{M}$, we have $S(x, y) \subset \overline{M}$. Therefore $G(x) = F(x) \cap \overline{M} \neq \emptyset$.
Now, let us prove that G has weakly closed graph in $\overline{M} \times X$. We fix a sequence $(x_n)_{n\in I\!N}$ in $\overline{M}$ converging to a point $\bar{x}$ and a sequence $(y_n)_{n\in I\!N}$, $y_n \in G(x_n)$, $n \in I\!N$, converging to a point $\bar{y}$. By hypothesis *(ii)* and by the fact that $S(\bar{x}, \bar{y}) \subset \overline{M}$, we can conclude that $\emptyset \neq S(\bar{x}, \bar{y}) \cap F(\bar{x}) = S(\bar{x}, \bar{y}) \cap F(\bar{x}) \cap \overline{M} = S(\bar{x}, \bar{y}) \cap G(\bar{x})$.
Finally, F takes closed values and satisfies hypothesis *(i)*, so we can conclude that the map G verifies all the assumptions of Theorem 2.1. Therefore, there exists $x \in D$ such that $x \in G(x) \subset F(x)$. □

Now, we can provide our fixed point theorem for Daher type multimaps.

Theorem 3.2 *Let D be a closed, convex subset of a Banach space X and $F : D \to \mathcal{P}_k(D)$ be a map verifying hypotheses (i), (ii), (iii) of Theorem 3.1 and the following*

(c) F is countably condensing.

Then there exists $x \in D$ with $x \in F(x)$.

Proof. We prove that the multimap F satisfies hypothesis *(M)* of Theorem 3.1.

Let us fix $x_0 \in D$ and let M be a subset of D such that $M = co(\{x_0\} \cup F(M))$ and

$$\overline{M} = \overline{C} \ , \tag{12}$$

with C countable subset of M.

Being $C \subset co(\{x_0\} \cup F(M))$, every point of C can be written as a finite combination of points belonging to the set $\{x_0\} \cup F(M)$. Therefore, there exists a countable set $\mathcal{M} \subset M$ such that

$$C \subset co(\{x_0\} \cup F(\mathcal{M})) \ . \tag{13}$$

By hypothesis *(c)*, $F(D)$ is bounded, then also sets M, C and $\mathcal{M}$ are bounded.

Let us prove that $\beta(C) = 0$. First of all, by using (13), (1), (β_2) and (β_3), we have

$$\beta(C) \ \le \ \beta(co(\{x_0\} \cup F(\mathcal{M}))) \ = \ \beta(\{x_0\} \cup F(\mathcal{M})) \ = \ \beta(F(\mathcal{M})) \ . \tag{14}$$

Now, suppose that $\beta(\mathcal{M}) \neq 0$. Then, hypothesis *(c)* yields

$$\beta(F(\mathcal{M})) < \beta(\mathcal{M}) \ . \tag{15}$$

Combining (14) with (15), by (β_2), (1) and (12), we obtain

$$\beta(C) \ < \ \beta(\mathcal{M}) \ \le \ \beta(M) \ = \ \beta(\overline{M}) \ = \ \beta(\overline{C}) \ = \ \beta(C) \ , \tag{16}$$

that is a contradiction.

Hence, it must be $\beta(\mathcal{M}) = 0$. So, by the regularity of β, $\overline{\mathcal{M}}$ is compact. Thus, assumption *(iii)* provides that

$$\beta(\overline{F(\overline{\mathcal{M}})}) = 0 \ .$$

Then, since $F(\mathcal{M}) \subset \overline{F(\overline{\mathcal{M}})}$ and by using (β_2), we have

$$\beta(F(\mathcal{M})) = 0 \ . \tag{17}$$

Therefore, by (14) and (17), we can conclude

$$\beta(C) \ = \ 0 \ .$$

Now, thanks to the definition of β and by (12), we can write

$$\beta(M) \ = \ \beta(\overline{M}) \ = \ \beta(\overline{C}) \ = \ \beta(C) \ = \ 0$$

and so $\beta(M) = 0$ too. Therefore, the set $\overline{M}$ is compact. Hence, hypothesis *(M)* of Theorem 3.1 is verified.

Consequently, by applying Theorem 3.1, we can conclude that there exists a fixed point for F in D. $\square$

Remark 3.1 *We note that our Theorem 3.1 and Theorem 3.2 strictly contain respectively Theorem 3.1 in [21] and Theorem 1.2 in [1]. In fact, let us consider the map*

$$F : [0,2] \to \mathcal{P}([0,2]) \text{ defined by } F(x) = \begin{cases} \{1\} & , \ x \in [0,2[\\ \{2\} & , \ x = 2 \ . \end{cases}$$

It is easy to check that F satisfies either all the assumptions of Theorems 3.1 and 3.2, but it has not closed graph. Therefore, it is not possible to apply to F neither Theorem 3.1 in [21] nor Theorem 1.2 in [1].

Remark 3.2 *We wish to observe that if we strengthen of Definition 1.1 by the following*

Definition 3.1 *Let D be a nonempty subset of X. For $k \in [0,1[$, we say that a map $F : D \to \mathcal{P}(X)$ is* countably k-condensing *if $F(D)$ is bounded and $\beta(F(B)) \leq k\beta(B)$ for all countable bounded subsets B of D .*

then an immediate consequence of Theorem 3.2 is the next Darbo type result:

Corollary 3.1 *Let D be a closed, convex subset of a Banach space X and $F : D \to \mathcal{P}_k(D)$ be a map verifying hypotheses (i), (ii), (iii) of Theorem 3.1 and the following*

(c)' F is countably k-condensing, for $k \in [0,1[$.

Then there exists $x \in D$ with $x \in F(x)$.

References

[1] R.P. Agarwal, D. O'Regan, A generalization of the Krasnoselskii-Petryshyn compression and expansion theorem: an essential map approach, J. Korean Math. Soc. 38 (2001), no. 3, 669-681.

[2] M. Arias, J. Campos, C. Marcelli, Fastness and continuous dependence in front propagation in Fisher-KPP equations, Discr. Cont. Dyn. Syst., Ser. B, 11 (2009), no. 1, 11-30.

[3] J.M. Ayerbe Toledano, T. Dominguez Benavides, G. López Acedo, Measures of Noncompactness in Metric Fixed Point Theory, Operator Theory: Advances and Applications, 99. Birkhauser Verlag, Basel, 1997.

[4] I. Benedetti, L. Malaguti, V. Taddei, Two-point b.v.p. for multivalued equations with weakly regular r.h.s., Nonlinear Anal. T.M.A. 74 (2011), no. 11, 3657-3670.

[5] I. Benedetti, L. Malaguti, V. Taddei, Semilinear differential inclusions via weak topologies, J. Math. Anal. Appl. 368 (2010), no. 1, 90-102.

[6] T. Cardinali, F. Papalini, Sull'esistenza di punti fissi per multifunzioni a grafo debolmente chiuso, Riv. Mat. Univ. Parma 17 (1991), no. 4, 59-67.

[7] T. Cardinali, F. Portigiani, P. Rubbioni, Nonlocal Cauchy problems and their controllability for semilinear differential inclusions with lower Scorza-Dragoni nonlinearities, Czech. Math. Jour. 61 (2011), no. 1, 225-245.

[8] T. Cardinali, P. Rubbioni, Impulsive mild solutions for semilinear differential inclusions with nonlocal conditions in Banach spaces, Nonlinear Anal. T.M.A. 75 (2012), no. 2, 871-879.

[9] T. Cardinali, P. Rubbioni, On the existence of mild solutions of semilinear evolution differential inclusions, J. Math. Anal. Appl. 308 (2005), no. 2, 620-635.

[10] Y. Chen, S. Hong, Solvability of a class of integral inclusions, Abstr. Appl. Anal. 2012, Article ID 213270 (2012), 12 pages.

[11] G. Cupini, C. Marcelli, F. Papalini, On the solvability of a boundary value problem on the real line, Bound. Value Probl., 2001:26 (2011), 1-17.

[12] S.J. Daher, On a fixed point principle of Sadovskii, Nonlinear Anal. T.M.A. 2 (1978), no. 5, 643-645.

[13] G. Darbo, Punti uniti in trasformazioni a codominio non compatto (Italian), Rend. Sem. Mat. Univ. Padova 24 (1955), 84-92.

[14] Z. Denkowski, S. Migorski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Theory, Kluwer Acad. Publ. Boston/Dordrecht/London, 2003.

[15] Z. Došlá, M. Marini, S. Matucci, A boundary value problem on a half-line for differential equations with indefinite weight, Communications in Applied Analysis 15 (2011), (2,3,4) 341-352.

[16] C.J. Himmelberg, Fixed points of compact multifunctions, J. Math. Anal. Appl. 38 (1972), 205-207.

[17] C.J. Himmelberg, J.R. Porter, F.S. Van Vleck, Fixed point theorems for condensing multifunctions, Proc. Amer. Math. Soc. 23 (1969), 635-641.

[18] S. Hu, N.S. Papageorgiou, Handbook of Multivalued Analysis, vol. I: Theory, Kluwer Acad. Publ., Dordrecht - Boston - London, 1997.

[19] M. Kamenskii, V.V. Obukhovskii, P. Zecca, Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces, De Gruyter Ser. Nonlinear Anal. Appl. 7, Walter de Gruyter, Berlin - New York, 2001.

[20] S. Matucci, P. Řehák, Nonoscillation of half-linear dynamic equations, Comput. Math. Appl., 60 (2010), no. 5, 1421-1429.

[21] D. O'Regan, R. Precup, Fixed point theorems for set-valued maps and existence principles for integral inclusions, J. Math. Anal. Appl. 245 (2000), no. 2, 594-612.

[22] B.N. Sadovskii, A fixed point principle, Functional Anal. Appl. 1 (1967), no. 2, 151-153.

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