Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **52**, 1-9; **http://www.math.u-szeged.hu/ejqtde/**

Periodic Boundary Value Problems for First Order Difference Equations

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Abstract

In this paper, existence criteria for single and multiple positive solutions of periodic boundary value problems for first order difference equations of the form

$$\begin{cases} \triangle x(k) + f(k, x(k+1)) = 0, & k \in [0, T], \\ x(0) = x(T+1), \end{cases}$$

are established by using the fixed point theorem in cones. An example is also given to illustrate the main results.

Keywords: periodic boundary value problem; positive solution; fixed point theorem; difference equation; cone

MSC: 39A10

1 Introduction

Due to the wide application in many fields such as science, economics, neural network, ecology, cybernetics, etc., the theory of nonlinear difference equations has been widely studied since 70's of last century, see, for example, [1, 2, 19, 20]. At the same time, Boundary value problems (BVPs) of difference equations have received much attention from many authors, see [3-12, 14-18, 21, 22, 24-29] and the references therein. However, to the best our knowledge, few papers can be found in the literature for periodic boundary value problems (PBVPs) of difference equations [9, 10, 22, 24, 29].

In this paper, we are concerned with the existence of single and multiple positive solutions of PBVP for first order difference equation

$$\begin{cases} \triangle x(k) + f(k, x(k+1)) = 0, & k \in [0, T], \\ x(0) = x(T+1), \end{cases} \tag{1.1}$$

where T is a fixed positive integer, $\triangle$ denotes the forward difference operator with stepsize 1, and $[a, b] = \{a,\ a+1,\ \cdots,\ b-1,\ b\} \subset Z$ the set of all integers, and $f : [0, T] \times R \to R$ is continuous.

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In [22], by using a fixed point theorem, Sun considered the existence of one positive solution of the PBVP (1.1) when the following condition holds:

(A) There exists a positive number $M > 1$ such that

$$(M-1)x - f(k,\ x) \geq 0 \text{ for } k \in [0,T],\ x \in [0,+\infty)\,.$$

In [24], Wang obtained the existence of multiple positive solutions of PBVP (1.1) by using the Leggett-Williams multiple fixed point theorem and fixed point theorem of cone expansion and compression when condition (A) holds.

Motivated by the results mentioned above, in this paper, we shall obtain existence criteria for single and multiple positive solutions to the PBVP (1.1) by means of a fixed point theorem in cones. It is worth noticing that our hypotheses on nonlinearity f in this paper are weaker than condition (A) of [22, 24]. This paper's ideas come from [23].

Theorem 1.1 ([13])**.** Let X be a Banach space and K is a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let

$$\Phi : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a continuous and completely continuous operator such that

(i) $\|\Phi x\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1$;

(ii) there exists $e \in K\backslash\{0\}$ such that $x \neq \Phi x + \lambda e$ for $x \in K \cap \partial\Omega_2$ and $\lambda > 0$.

Then Φ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Remark 1.1. In Theorem 1.1, if (i) and (ii) are replaced by

(i) $\|\Phi x\| \leq \|x\|$ for $x \in K \cap \partial\Omega_2$;

(ii) there exists $e \in K\backslash\{0\}$ such that $x \neq \Phi x + \lambda e$ for $x \in K \cap \partial\Omega_1$ and $\lambda > 0$, then Φ has also a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2 Preliminaries

Let $C = \{x : [0,T] \to R\}$. For $\sigma \in C$, we first consider the following linear PBVP :

$$\begin{cases} \triangle x(k) + (M-1)x(k+1) = \sigma(k), & k \in [0,T]\,, \\ x(0) = x(T+1), \end{cases} \tag{2.1}$$

where $M > 1$ is a constant.

Let

$$G(k,s) = \begin{cases} \frac{M^{-(k-s)}}{1-M^{-(T+1)}}, & 0 \leq s \leq k-1, \\ \frac{M^{-(T+1+k-s)}}{1-M^{-(T+1)}}, & k \leq s \leq T. \end{cases}$$

Then we have

$$\frac{M^{-(T+1)}}{1-M^{-(T+1)}} \leq G(k,s) \leq \frac{1}{1-M^{-(T+1)}},\ (k,\ s) \in [0,T+1] \times [0,T]. \tag{2.2}$$

It is easy to see that the following lemma holds.

Lemma 2.1. Suppose $M > 1$. Then for any $\sigma \in C$, PBVP (2.1) has a unique solution:

$$x(k) = \sum_{s=0}^{T} G(k,s)\sigma(s),\ \ k \in [0, T+1].$$

In addition, if we choose $\sigma(k) \equiv 1$, then we know

$$\sum_{s=0}^{T} G(k,s) = \frac{1}{M-1}.$$

Let $E = \{x : [0, T+1] \to R\}$ be equipped with the norm $\|x\| = \max_{k \in [0,T+1]} |x(k)|$, then E is a Banach space.

Let

$$K = \{x \in E : x(k) \geq 0,\ \ \min_{0 \leq k \leq T+1} x(k) \geq \delta \|x\|\},$$

where $\delta = M^{-(T+1)} < 1$, one may readily verify that K is a cone in E.

Now for $u \in K$, we consider the following PBVP:

$$\begin{cases} \triangle x(k) + (M-1)x(k+1) = (M-1)u(k+1) - f(k, u(k+1)), & k \in [0, T], \\ x(0) = x(T+1). \end{cases} \tag{2.3}$$

It follows from Lemma 2.1 that PBVP (2.3) has a unique solution:

$$x(k) = \sum_{s=0}^{T} G(k,s)[(M-1)u(s+1) - f(s, u(s+1))],\ \ k \in [0, T+1].$$

Define an operator $\Phi : K \to E$:

$$(\Phi x)(k) = \sum_{s=0}^{T} G(k,s)[(M-1)x(s+1) - f(s, x(s+1))],\ \ k \in [0, T+1].$$

It is obviously that fixed points of Φ are solutions of PBVP (1.1) and $\Phi : K \to E$ is continuous and completely continuous.

3 Main results

In this section, by defining an appropriate cones, we impose the conditions on f which allow us to apply the fixed point theorem in cones to establish the existence criteria for single and multiple positive solutions of the PBVP (1.1).

Theorem 3.1. Suppose that there exist a positive number $M > 1$ and $0 < \alpha < \beta$ such that

$$(M-1)x - f(k,\ x) \geq 0 \text{ for } k \in [0, T],\ \ x \in [\delta\alpha, \beta].$$

Then the PBVP (1.1) has at least one positive solution if one of the following two conditions holds

(i)

$$\begin{aligned} f(k,x) &\leq 0 \text{ for } k\in[0,T],\ x\in[\delta\alpha,\alpha]\,, \\ f(k,x) &\geq 0 \text{ for } k\in[0,T],\ x\in[\delta\beta,\beta]\,; \end{aligned}$$

(ii)

$$\begin{aligned} f(k,x) &\geq 0 \text{ for } k\in[0,T],\ x\in[\delta\alpha,\alpha]\,, \\ f(k,x) &\leq 0 \text{ for } k\in[0,T],\ x\in[\delta\beta,\beta]\,. \end{aligned}$$

Proof. Define the open sets

$$\begin{aligned} \Omega_1 &= \{x\in E : \|x\|<\alpha\}, \\ \Omega_2 &= \{x\in E : \|x\|<\beta\}. \end{aligned}$$

Firstly, we claim that $\Phi : K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$.

In fact, for any $x\in K\cap(\overline{\Omega}_2\setminus\Omega_1)$, we have $\delta\alpha\leq x\leq\beta$, by (2.2)

$$\|\Phi x\|\leq\frac{1}{1-M^{-(T+1)}}\sum_{s=0}^{T}[(M-1)x(s+1)-f(s,x(s+1))]$$

and

$$\begin{aligned} (\Phi x)(k) &= \sum_{s=0}^{T}G(k,s)[(M-1)x(s+1)-f(s,x(s+1))] \\ &\geq \frac{M^{-(T+1)}}{1-M^{-(T+1)}}\sum_{s=0}^{T}[(M-1)x(s+1)-f(s,x(s+1))]. \end{aligned}$$

So

$$(\Phi x)(k)\geq M^{-(T+1)}\|\Phi x\|=\delta\|\Phi x\|\,,\ \text{i.e., } \Phi x\in K.$$

Therefore, $\Phi : K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$.

Secondly, we prove the result provided conditions (i) holds.

By the first inequality of (i), we have

$$(M-1)x-f(k,\ x)\geq(M-1)x,\ \ k\in[0,T],\ \ x\in[\delta\alpha,\alpha]\,.$$

Let $e\equiv 1$, then $e\in K$. We assert that

$$x\neq\Phi x+\lambda e \text{ for } x\in K\cap\partial\Omega_1 \text{ and } \lambda>0. \tag{3.1}$$

If not, there would exist $x_0\in K\cap\partial\Omega_1$ and $\lambda_0>0$ such that $x_0=\Phi x_0+\lambda_0 e$.

Since $x_0 \in K \cap \partial\Omega_1$, then $\delta\alpha = \delta \|x_0\| \le x_0(k) \le \alpha$. Let $\mu = \min_{0\le k\le T+1} x_0(k)$, then for any $k \in [0, T+1]$, we have

$$\begin{aligned} x_0(k) &= (\Phi x_0)(k) + \lambda_0 \\ &= \sum_{s=0}^{T} G(k,s)[(M-1)x_0(s+1) - f(s, x_0(s+1))] + \lambda_0 \\ &\ge \sum_{s=0}^{T} G(k,s)(M-1)x_0(s+1) + \lambda_0 \\ &\ge \mu \sum_{s=0}^{T} G(k,s)(M-1) + \lambda_0 \\ &= \mu + \lambda_0. \end{aligned}$$

This implies that $\mu \ge \mu + \lambda_0$, and this is a contradiction. Therefore (3.1) holds.

On the other hand, by using the second inequality of (i), we have

$$(M-1)x - f(k,\ x) \le (M-1)x,\ k \in [0,T],\ x \in [\delta\beta, \beta].$$

We assert that

$$\|\Phi x\| \le \|x\| \ \text{ for } x \in K \cap \partial\Omega_2. \tag{3.2}$$

In fact, for any $x \in K \cap \partial\Omega_2$, then $\delta\beta = \delta \|x\| \le x(k) \le \beta$, we have

$$\begin{aligned} (\Phi x)(k) &= \sum_{s=0}^{T} G(k,s)[(M-1)x(s+1) - f(s, x(s+1))] \\ &\le \sum_{s=0}^{T} G(k,s)(M-1)x(s+1) \\ &\le \sum_{s=0}^{T} G(k,s)(M-1)\|x\| \\ &= \|x\|. \end{aligned}$$

Therefore, $\|\Phi x\| \le \|x\|$.

It follows from Remark 1.1, (3.1) and (3.2) that Φ has a fixed point $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In a similar way, we can prove the result by Theorem 1.1 if condition (ii) holds.

Theorem 3.2. Suppose that there exist a positive number $M > 1$ and $0 < \alpha < \rho < \beta$ such that

$$(M-1)x - f(k,\ x) \ge 0 \text{ for } k \in [0,T],\ x \in [\delta\alpha, \beta].$$

Then the PBVP (1.1) has at least two positive solutions if one of the following two conditions holds

(i)

$$\begin{aligned} f(k,x) &\le 0 \text{ for } k\in[0,T],\ x\in[\delta\alpha,\alpha]\,,\\ f(k,x) &> 0 \text{ for } k\in[0,T],\ x\in[\delta\rho,\rho]\,,\\ f(k,x) &\le 0 \text{ for } k\in[0,T],\ x\in[\delta\beta,\beta]\,; \end{aligned}$$

(ii)

$$\begin{aligned} f(k,x) &\ge 0 \text{ for } k\in[0,T],\ x\in[\delta\alpha,\alpha]\,,\\ f(k,x) &< 0 \text{ for } k\in[0,T],\ x\in[\delta\rho,\rho]\,,\\ f(k,x) &\ge 0 \text{ for } k\in[0,T],\ x\in[\delta\beta,\beta]\,. \end{aligned}$$

Proof. We only prove the result when condition (i) holds. In a similar way we can obtain the result if condition (ii) holds.

Define Ω_1, Ω_2 as in Theorem 3.1 and define

$$\Omega_3 = \{x \in E : \|x\| < \rho\}.$$

Similar to the proof of Theorem 3.1, we can prove that

$$x \neq \Phi x + \lambda e \text{ for } x \in K \cap \partial\Omega_1 \text{ and } \lambda > 0, \tag{3.3}$$

$$x \neq \Phi x + \lambda e \text{ for } x \in K \cap \partial\Omega_2 \text{ and } \lambda > 0, \tag{3.4}$$

where $e \equiv 1 \in K$, and

$$\|\Phi x\| < \|x\| \text{ for } x \in K \cap \partial\Omega_3. \tag{3.5}$$

Thus we can obtain the existence of two positive solutions x_1 and x_2 by using Theorem 1.1 and Remark 1.1, respectively. It is easy to see that $\alpha \le \|x_1\| < \rho < \|x_2\| \le \beta$.

Theorem 3.3. Suppose that there exist a positive number $M > 1$ and $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n$ such that

$$(M-1)x - f(k,\ x) \ge 0 \text{ for } k \in [0,T],\ x \in [\delta\alpha_1, \beta_n].$$

Then the PBVP (1.1) has at least n multiple positive solutions x_i $(1 \le i \le n)$ satisfying $\alpha_i \le \|x_i\| < \beta_i$, $1 \le i \le n$, if one of the following two conditions holds

(i)

$$\begin{aligned} f(k,x) &\le 0 \text{ for } k\in[0,T],\ x\in[\delta\alpha_i,\alpha_i]\,,\ 1\le i\le n,\\ f(k,x) &\ge 0 \text{ for } k\in[0,T],\ x\in[\delta\beta_i,\beta_i]\,,\ 1\le i\le n; \end{aligned}$$

(ii)

$$\begin{aligned} f(k,x) &\ge 0 \text{ for } k\in[0,T],\ x\in[\delta\alpha_i,\alpha_i]\,,\ 1\le i\le n,\\ f(k,x) &\le 0 \text{ for } k\in[0,T],\ x\in[\delta\beta_i,\beta_i]\,,\ 1\le i\le n. \end{aligned}$$

Remark 3.1. In theorem 3.3, if (i) and (ii) are replaced by
(iii)

$$\begin{aligned} f(k,x) &< 0 \text{ for } k \in [0,T],\ x \in [\delta\alpha_i, \alpha_i],\ 1 \le i \le n, \\ f(k,x) &> 0 \text{ for } k \in [0,T],\ x \in [\delta\beta_i, \beta_i],\ 1 \le i \le n; \end{aligned}$$

(iv)

$$\begin{aligned} f(k,x) &> 0 \text{ for } k \in [0,T],\ x \in [\delta\alpha_i, \alpha_i],\ 1 \le i \le n, \\ f(k,x) &< 0 \text{ for } k \in [0,T],\ x \in [\delta\beta_i, \beta_i],\ 1 \le i \le n. \end{aligned}$$

Then the PBVP (1.1) has at least $2n-1$ multiple positive solutions.

4 Examples

Example 4.1. Consider the following PBVP:

$$\begin{cases} \triangle x(k) + f(k, x(k+1)) = 0, \quad k \in [0,3], \\ x(0) = x(4), \end{cases} \tag{4.1}$$

where $T = 3$ and $f(k,x) = x - x^{\frac{1}{2}} + \frac{7}{64}$.

Then the PBVP (4.1) has at least three nonnegative solutions.

Proof. Choose $M = 2$, then $\delta = M^{-(T+1)} = 2^{-4} = \frac{1}{16}$. Let $\alpha = \frac{1}{4}$, $\beta = 32$, then it is not difficult to show that

$$(M-1)x - f(k,\ x) = x^{\frac{1}{2}} - \frac{7}{64} \ge \frac{1}{8} - \frac{7}{64} = \frac{1}{64} > 0,\ x \in [\frac{1}{64}, 32] = [\delta\alpha, \beta];$$

$$f(k,\ x) = x - x^{\frac{1}{2}} + \frac{7}{64} \le \frac{1}{64} - \frac{1}{8} + \frac{7}{64} = 0, x \in [\frac{1}{64}, \frac{1}{4}] = [\delta\alpha, \alpha];$$

$$f(k,x) = x - x^{\frac{1}{2}} + \frac{7}{64} > 0,\ x \in [2,32] = [\delta\beta, \beta].$$

By Theorem 3.1, the PBVP (4.1) has at least one positive solutions.

Acknowledgment: The authors thankful to the anonymous referee for his/her helpful suggestions for the improvement of this article.

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(Received May 4, 2012)