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On the Fractional Derivatives at Extreme Points

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Abstract

We correct a recent result concerning the fractional derivative at extreme points. We then establish new results for the Caputo and Riemann-Liouville fractional derivatives at extreme points.
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1 Introduction

In recent years several authors have discussed the existence and uniqueness results for wide classes of fractional differential equations [1, 2, 3, 4, 6, 7, 9]. The techniques implemented are mainly fixed point theorems, maximum principle and the method of lower and upper solutions. In this paper we correct a result obtained in [9] and obtain new results concerning the fractional derivatives at extreme points. These results will be of interest for many researchers, especially for those who are working in extending the method of lower and upper solutions to fractional boundary value problems [1, 7]. In the following we present some definitions and main results concerning the Caputo and Riemann-Liouville fractional derivatives.

Definition 1.1. *Let $f \in C[0,1]$, $\delta \geq 0$, and Γ is the Euler gamma function. The left Riemann-Liouville fractional integral is defined by*

$$I^{\delta} f(t) = \begin{cases} \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)ds, & \delta > 0, \\ f(t), & \delta = 0. \end{cases} \tag{1.1}$$

Definition 1.2. *Let $f \in C^n[0,1]$, the left Caputo fractional derivative is defined by*

$$D_C^{\delta} f(t) = I^{n-\delta} \frac{d^n}{dt^n} f(t) = \begin{cases} \frac{1}{\Gamma(n-\delta)} \int_0^t (t-s)^{n-\delta-1} f^{(n)}(s)ds, & n-1 < \delta < n \in Z^+, \\ f^{(n)}(t), & \delta = n \in Z^+. \end{cases}$$

Definition 1.3. *Let $f \in C^n[0,1]$, the left Riemann-Liouville fractional derivative is defined by*

$$D_R^{\delta} f(t) = \frac{d^n}{dt^n} I^{n-\delta} f(t) = \begin{cases} \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\delta-1} f(s)ds, & n-1 < \delta < n \in Z^+, \\ f^{(n)}(t), & \delta = n \in Z^+. \end{cases}$$

It is well-known that if $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$, then $D_C^\delta f(t) = D_R^\delta f(t)$. In general the relation between the Caputo and Riemann-Liouville fractional derivatives is given by [5, 8]

$$D_C^\delta f(t) = D_R^\delta \Big(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \Big), \tag{1.2}$$

where

$$D_R^\delta t^k = \frac{\Gamma(k+1)}{\Gamma(k-\delta+1)} t^{k-\delta}. \tag{1.3}$$

2 Main Results

We first show that the following result claimed in [9] is not correct. The following is claimed as Theorem 2.2 of [9].

• *Let a function* $f \in C^2(0,1) \cap C[0,1]$, *attain its minimum over the interval* $[0,1]$ *at the point* $t_0 \in (0,1]$. *Then* $D_C^\delta f(t_0) \geq 0$, *for all* $1 < \delta \leq 2$.

As a counter example we consider $f(t) = t(t-\frac{1}{2})(t-1),\ 0 \leq t \leq 1$. Direct calculations imply that $f(t)$ has absolute minimum value at $t_0 = \frac{3+\sqrt{3}}{6} < 1$. For $1 < \delta < 2$, we have

$$D_C^\delta t^3 = \frac{\Gamma(4)}{\Gamma(4-\delta)} t^{3-\delta},\ D_C^\delta t^2 = \frac{\Gamma(3)}{\Gamma(3-\delta)} t^{2-\delta} \text{ and } D_C^\delta t = 0.$$

Thus,

$$D_C^{1.1} f(t_0) = \frac{(3+\sqrt{3})^{1.9}}{6^{0.9}\Gamma(2.9)} - \frac{3^{0.1}(3+\sqrt{3})^{0.9}}{2^{0.9}\Gamma(1.9)} = -0.4277\cdots < 0,$$

which contradicts the result in Theorem 2.2 of [9]. We correct the above result by imposing more conditions on f. We have

Theorem 2.1. *Let* $f \in C^2[0,1]$ *attain its minimum at* $t_0 \in (0,1)$, *then*

$$D_C^\delta f(t_0) \geq \frac{t_0^{-\delta}}{\Gamma(2-\delta)} \Big[(\delta-1)(f(0) - f(t_0)) - t_0 f'(0) \Big], \text{ for all } 1 < \delta < 2. \tag{2.1}$$

Proof. We define the auxiliary function $h(t) = f(t) - f(t_0), t \in [0,1]$. Then $h(t)$ satisfies the following in $[0,1]$

$$h(t) \geq 0,\ h(t_0) = h'(t_0) = 0,\ h''(t_0) \geq 0 \text{ and } D_C^\delta h(t) = D_C^\delta f(t).$$

Integration by parts of

$$D_C^\delta h(t_0) = \frac{1}{\Gamma(2-\delta)} \int_0^{t_0} (t_0 - s)^{1-\delta} h''(s) ds,$$

yields

$$\Gamma(2-\delta) D_C^\delta h(t_0) = (t_0 - s)^{1-\delta} h'(s)|_0^{t_0} - (\delta - 1) \int_0^{t_0} (t_0 - s)^{-\delta} h'(s) ds. \tag{2.2}$$

Since $h'(t_0) = 0$ and $h''(t_0)$ is bounded, there exists $\mu_1(t) \in C[0,1]$ such that $h'(t) = (t_0 - t)\mu_1(t)$. We have for $1 < \delta < 2$

$$\lim_{t \to t_0} \frac{h'(t)}{(t_0 - t)^{\delta-1}} = \lim_{t \to t_0} \frac{(t_0 - t)\mu_1(t)}{(t_0 - t)^{\delta-1}} = \lim_{t \to t_0} (t_0 - t)^{2-\delta} \mu_1(t) = 0.$$

Hence

$$\Gamma(2-\delta)D_C^\delta h(t_0) = -t_0^{1-\delta}h'(0) - (\delta-1)\int_0^{t_0}(t_0-s)^{-\delta}h'(s)ds. \tag{2.3}$$

Since $h(t_0) = h'(t_0) = 0$ and $h''(t_0)$ is bounded, there exists $\mu_2(t) \in C[0,1]$ such that $h(t) = (t_0-t)^2\mu_2(t)$. Thus

$$\int_0^{t_0}(t_0-s)^{-\delta-1}h(s)ds = \int_0^{t_0}(t_0-s)^{-\delta+1}\mu_2(s)ds,$$

is bounded and

$$\lim_{t\to t_0}\frac{h(t)}{(t_0-t)^\delta} = \lim_{t\to t_0}\frac{(t_0-t)^2\mu_2(t)}{(t_0-t)^\delta} = \lim_{t\to t_0}(t_0-t)^{2-\delta}\mu_2(t) = 0.$$

Integrating Eq. (2.3) by parts and using the above result together with $h(t) \geq 0$ on $[0,1]$ yields

$$\begin{aligned}
\Gamma(2-\delta)D_C^\delta h(t_0) &= -t_0^{1-\delta}h'(0) - (\delta-1)\left[(t_0-s)^{-\delta}h(s)|_0^{t_0} - \delta\int_0^{t_0}(t_0-s)^{-\delta-1}h(s)ds\right], \\
&= -t_0^{1-\delta}h'(0) - (\delta-1)\left[-t_0^{-\delta}h(0) - \delta\int_0^{t_0}(t_0-s)^{-\delta-1}h(s)ds\right] \\
&= -t_0^{1-\delta}h'(0) + (\delta-1)t_0^{-\delta}h(0) + \delta(\delta-1)\int_0^{t_0}(t_0-s)^{-\delta-1}h(s)ds \\
&\geq -t_0^{1-\delta}h'(0) + (\delta-1)t_0^{-\delta}h(0) = -t_0^{1-\delta}f'(0) + (\delta-1)t_0^{-\delta}(f(0)-f(t_0))
\end{aligned}$$

and the result is obtained. □

Corollary 2.1. *Let $f \in C^2[0,1]$ attain its minimum at $t_0 \in (0,1)$, and $f'(0) \leq 0$. Then $D_C^\delta f(t_0) \geq 0$, for all $1 < \delta < 2$.*

Proof. By Theorem 2.1 there holds $D_C^\delta f(t_0) \geq \frac{1}{\Gamma(2-\delta)}\left[(\delta-1)t_0^{-\delta}(f(0)-f(t_0)) - t_0^{1-\delta}f'(0)\right]$. Since $f(t_0) \leq f(0)$, $t_0 > 0$ and $f'(0) \leq 0$, we obtain $D_C^\delta f(t_0) \geq 0$. □

The following result is obtained as Theorem 1 of [7].

- *Let a function $f \in W_t^1((0,T)) \cap C([0,T])$ attain its maximum over the interval $[0,T]$ at the point $\tau = t_0, t_0 \in (0,T]$. Then*

$$D_C^\delta f(t_0) \geq 0, \quad 0 < \delta < 1,$$

where $W_t^1((0,T))$ denotes the space of functions $f \in C^1((0,T])$ such that $f' \in L((0,T))$ and $L((0,T))$ being the set of functions Lebesgue integrable on $(0,T)$.

By substituting $g = -f$, we have the following result.

- *Let a function $g \in W_t^1((0,T)) \cap C([0,T])$ attain its minimum over the interval $[0,T]$ at the point $\tau = t_0, t_0 \in (0,T]$. Then $D_C^\delta g(t_0) \leq 0, \quad 0 < \delta < 1$.*

The following result is a simple generalization to the above one for $t \in (0,1)$.

Theorem 2.2. *Let $f \in C^1[0,1]$ attain its minimum at $t_0 \in (0,1)$, then*

$$D_C^\delta f(t_0) \leq \frac{t_0^{-\delta}}{\Gamma(1-\delta)}[f(t_0)-f(0)] \leq 0, \ \textit{for all} \ 0 < \delta < 1. \tag{2.4}$$

Proof. We define the auxiliary function $h(t) = f(t) - f(t_0), t \in [0,1]$. Then $h(t) \geq 0$, on $[0,1]$, $h(t_0) = h'(t_0) = 0$ and $h(t) = (t_0 - t)\mu_3(t)$ for some $\mu_3(t) \in C[0,1]$. Integration by parts of

$$D_C^\delta h(t_0) = \frac{1}{\Gamma(1-\delta)} \int_0^{t_0} (t_0 - s)^{-\delta} h'(s) ds,$$

yields

$$\Gamma(1-\delta) D_C^\delta h(t_0) = (t_0 - s)^{-\delta} h(s)|_0^{t_0} - \delta \int_0^{t_0} (t_0 - s)^{-\delta-1} h(s) ds. \tag{2.5}$$

For $0 < \delta < 1$, we have $\int_0^{t_0} (t_0 - s)^{-\delta-1} h(s) ds = \int_0^{t_0} (t_0 - s)^{-\delta} \mu_3(s) ds$ is bounded and

$$\lim_{t \to t_0} \frac{h(t)}{(t_0 - t)^\delta} = \lim_{t \to t_0} (t_0 - t)^{1-\delta} \mu_3(t) = 0.$$

Thus

$$\Gamma(1-\delta) D_C^\delta h(t_0) = -t_0^{-\delta} h(0) - \delta \int_0^{t_0} (t_0 - s)^{-\delta-1} h(s) ds \leq -t_0^{-\delta} h(0) = -t_0^{-\delta} (f(0) - f(t_0)),$$

and the result is obtained.
In the following we present analogous results concerning the Riemann-Liouville fractional derivative.

Theorem 2.3. *Let $f \in C^2[0,1]$ attain its minimum at $t_0 \in (0,1)$, then*

$$D_R^\delta f(t_0) \geq \frac{t_0^{-\delta}}{\Gamma(2-\delta)} (\delta - 1) f(t_0) \text{ for all } 1 < \delta < 2. \tag{2.6}$$

Moreover, if $f(t) \geq 0$ in $[0,1]$, then $D_R^\delta f(t_0) \geq 0$.

Proof. From Eq.'s (1.2)-(1.3) we have for $1 < \delta < 2$

$$D_R^\delta f(t) = \frac{t^{-\delta}}{\Gamma(2-\delta)} \Big[(1-\delta) f(0) + t f'(0) \Big] + D_C^\delta f(t).$$

Applying the result in Eq. (2.1) yields

$$\begin{aligned} D_R^\delta f(t_0) &\geq \frac{t_0^{-\delta}}{\Gamma(2-\delta)} \Big[(1-\delta) f(0) + t f'(0) \Big] + \frac{t_0^{-\delta}}{\Gamma(2-\delta)} \Big[(\delta - 1)(f(0) - f(t_0)) - t_0 f'(0) \Big] \\ &= \frac{t_0^{-\delta}}{\Gamma(2-\delta)} [(\delta - 1) f(t_0)]. \end{aligned}$$

If $f(t) \geq 0$ then $f(t_0) \geq 0$ and finally $D_R^\delta f(t_0) \geq 0$. □

□

Theorem 2.4. *Let $f \in C^1[0,1]$ attain its minimum at $t_0 \in (0,1)$, then*

$$D_R^\delta f(t_0) \leq \frac{t_0^{-\delta}}{\Gamma(1-\delta)} f(t_0), \text{ for all } 0 < \delta < 1. \tag{2.7}$$

Moreover, if $f(t_0) \leq 0$, then $D_R^\delta f(t_0) \leq 0$.

Proof. From Eq.'s (1.2)-(1.3) we have for $0 < \delta < 1$

$$D_R^\delta f(t) = \frac{t^{-\delta}}{\Gamma(1-\delta)} f(0) + D_C^\delta f(t).$$

Using the result in Eq. (2.4) we obtain

$$D_R^\delta f(t_0) \leq \frac{t_0^{-\delta}}{\Gamma(1-\delta)} f(0) - \frac{t_0^{-\delta}}{\Gamma(1-\delta)} (f(0) - f(t_0)) = \frac{t_0^{-\delta}}{\Gamma(1-\delta)} f(t_0),$$

and $D_R^\delta f(t_0) \leq 0$ provided $f(t_0) \leq 0$. □

Remark 2.1. *Analogous results for the fractional derivatives at absolute maximum points are obtained by applying the above results on $-f(t)$.*

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