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Global attractivity of a higher order nonlinear difference equation with decreasing terms

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Abstract. In the present paper, we further study the asymptotical behavior of the following higher order nonlinear difference equation

$$x(n+1) = ax(n) + bf(x(n)) + cf(x(n-k)), \qquad n = 0, 1, \dots$$

where a, b and c are constants with $0 < a < 1, 0 \leq b < 1, 0 \leq c < 1$ and $a + b + c = 1$, $f \in C[[0, \infty), [0, \infty)]$ with $f(x) > 0$ for $x > 0$, and k is a positive integer, which has been recently studied in: On global attractivity of a higher order difference equation and its applications [*Electron. J. Qual. Theory Diff. Equ.* **2022**, No. 2, 1–14 pp]. We obtain some new sufficient conditions for the global attractivity of positive solutions of the equation, and show the applications of these results to some population models..

Keywords: higher order nonlinear difference equation, positive equilibrium, global attractivity, population model.

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1 Introduction

Consider the following higher order nonlinear difference equation

$$x(n+1) = ax(n) + bf(x(n)) + cf(x(n-k)), \qquad n = 0, 1, \dots, \tag{1.1}$$

where a, b and c are constants with $0 < a < 1, 0 \leq b < 1, 0 \leq c < 1$ and $a + b + c = 1$, $f \in C[[0, \infty), [0, \infty)]$ with $f(x) > 0$ for $x > 0$ and k is a positive integer. The case when the sum of the main coefficients of a higher order difference equation is equal to one is of a great interest and has been studied a lot see, e.g., [1,2,19–23] and the related references therein. One of the reasons is that such difference equations frequently model some processes in nature or society. Recently, asymptotic behavior of positive solutions of Eq. (1.1) has been studied in [1]. Among other results, the following one was presented therein.

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Theorem A. *Assume that $f(x)$ has a unique positive fixed point $\bar{x}$ and satisfies the negative feedback condition*

$$(x-\bar{x})(f(x)-x) < 0, \qquad x > 0,\ x \neq \bar{x}. \tag{1.2}$$

Suppose also $ax + bf(x)$ is increasing, and $f(x)$ is L-Lipschitz with

$$c\frac{1-a^{k+1}}{c+a^k b}L \leq 1. \tag{1.3}$$

Then every positive solution $\{x(n)\}$ of Eq. (1.1) converges to $\bar{x}$ as $n \to \infty$.

In addition, by using a different approach, a new result on the global attractivity of positive solutions of Eq. (1.1) was obtained in [2] for the special case that f is unimodal, that is, $f(x) = xg(x)$ where $g \in C[[0,\infty),[0,\infty)]$ is decreasing.

In the present paper, we are still interested in the study of global attractivity of positive solutions of Eq. (1.1), but for the case that f is decreasing, and furthermore for the case that f is an S-map, that is, $f : [0,\infty) \to [0,\infty)$ is three times differentiable with $(Sf)(x) < 0$ and $f'(x) < 0$ for $x > 0$ where S is the Schwarzian derivative

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Clearly, if we let

$$x(-k), x(-k+1), \ldots, x(0) \tag{1.4}$$

be $k+1$ given nonnegative numbers with $x(0) > 0$, then Eq. (1.1) has a unique positive solution with initial condition (1.4).

In the next section, we establish two sufficient conditions on the global attractivity of positive solutions of Eq. (1.1) under the conditions that f is a decreasing function and f is an S-map, respectively. Our results can be applied to several difference equations derived from mathematical biology. We show these applications in Section 3.

In the following discussion,we always assume that f is decreasing. In addition, for the sake of convenience, we adopt the notation $\prod_{i=m}^{n} s(i) = 1$ and $\sum_{i=m}^{n} s(i) = 0$ whenever $\{s(n)\}$ is a real sequence and $m > n$.

2 Main results

Since f is decreasing, f has a unique positive fixed point $\bar{x}$ and satisfies the negative feedback condition (1.2). Hence by Lemma 2.1 in [1], every positive solution $\{x(n)\}$ of Eq. (1.1) is bounded and persistent.

In the following, we establish two sufficient conditions for every positive solution of Eq. (1.1) to converge to $\bar{x}$ as $n \to \infty$. By an argument similar to that in the proof of Theorem 2.2 in [1], we know that every nonoscillatory solution of Eq. (1.1) converges to $\bar{x}$. Hence we need to obtain conditions for every oscillatory solution of Eq. (1.1) to converge to $\bar{x}$ also.

The following lemma on the asymptotic behavior of oscillatory solutions of Eq. (1.1) is needed in the proof of our main results.

Lemma 2.1. *Assume that $ax + bf(x)$ is increasing and let $\{x(n)\}$ be a positive solution of Eq. (1.1) which oscillates about $\bar{x}$. Then for any nonnegative integer $m \geq 0$, there is a positive integer N_m such that*

$$u(2m) \leq x(n) \leq u(2m+1) \quad \textit{for } n \geq N_m \tag{2.1}$$

where $\{u(n)\}$ *is defined by*

$$\begin{cases} u(n) = c\frac{1-a^{k+1}}{c+a^k b} f(u(n-1)) + \frac{a^k(b+ac)}{c+a^k b}\bar{x}, \; n = 1, 2, \dots, \\ u(0) = \frac{a^k(b+ac)}{c+a^k b}\bar{x}. \end{cases} \tag{2.2}$$

Proof. Let $y(n) = x(n) - \bar{x}$. Then $\{y(n)\}$ satisfies the equation

$$y(n+1) = ay(n) + b(f(y(n) + \bar{x}) - \bar{x}) + c(f(y(n-k) + \bar{x}) - \bar{x}) \tag{2.3}$$

and $\{y(n)\}$ oscillates about zero.

Let $y(i)$ and $y(j)$ be two consecutive members of the solution $\{y(n)\}$ such that

$$y(i) \geq 0, y(j+1) \geq 0 \quad \text{and} \quad y(n) < 0 \quad \text{for } i+1 \leq n \leq j. \tag{2.4}$$

and let

$$y(r) = \min\{y(i+1), y(i+2), \dots, y(j)\}.$$

Then by an argument similar to that in the proof of Theorem 2.2 in [1] (the increasing property of $ax + bf(x)$ is needed in the proof) we may show that

$$r - (i+1) \leq k \tag{2.5}$$

and

$$y(r) \geq \frac{1-a}{c+a^k b} a^r \sum_{n=i}^{r-1} \frac{c}{a^{n+1}} [f(y(n-k) + \bar{x}) - f(\bar{x})]. \tag{2.6}$$

Noting $f(y(n-k) + \bar{x}) \geq 0, f(\bar{x}) = \bar{x}$ and (2.5), we see that

$$y(r) \geq -\bar{x}\frac{1-a}{c+a^k b} a^r \sum_{n=i}^{r-1} \frac{c}{a^{n+1}} = -\bar{x}\frac{1-a}{c+a^k b} c\left(\frac{1-a^{r-i}}{1-a}\right) \geq -c\bar{x}\frac{1-a^{k+1}}{c+a^k b}$$

and so it follows that

$$y(n) \geq -c\bar{x}\frac{1-a^{k+1}}{c+a^k b}, \qquad i \leq n \leq j.$$

Since $y(i)$ and $y(j)$ are two arbitrary members of the solution with property (2.4), we see that there is a positive integer N_0' such that

$$y(n) \geq -c\bar{x}\frac{1-a^{k+1}}{c+a^k b} \overset{\text{def}}{=} z(0), \qquad n \geq N_0'. \tag{2.7}$$

Next, let $y(i)$ and $y(j)$ be two consecutive members of the solution $\{y(n)\}$ with $N_0' + k \leq i < j$ such that

$$y(i) \leq 0, \; y(j+1) \leq 0 \quad \text{and} \quad y(n) > 0 \quad \text{for } i+1 \leq n \leq j \tag{2.8}$$

and

$$y(t) = \max\{y(i+1), y(i+2), \dots, y(j)\}.$$

Then by a similar argument, we may show that

$$t - (i+1) \leq k \tag{2.9}$$

and

$$y(t) \le \frac{1-a}{c+a^k b} a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}} [f(y(n-k)+\bar{x}) - f(\bar{x})]. \tag{2.10}$$

Since

$$z(0)+\bar{x} = \left(1 - c\frac{1-a^{k+1}}{c+a^k b}\right)\bar{x} = \frac{a^k(b+ac)}{c+a^k b}\bar{x} > 0,$$

$f(z(0)+\bar{x})$ is well-defined. Since $z(0) < 0$ (see (2.7)) and f is decreasing, we see that

$$f(y(n-k)+\bar{x}) \le f(z(0)+\bar{x}) \quad \text{for } n \ge N_0' + k.$$

Hence, it follows from (2.9) and (2.10) that

$$y(t) \le \frac{1-a}{c+a^k b} a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}} [f(z(0)+\bar{x}) - f(\bar{x})] \le c\frac{1-a^{k+1}}{c+a^k b}[f(z(0)+\bar{x}) - \bar{x}],$$

which yields

$$y(n) \le c\frac{1-a^{k+1}}{c+a^k b}[f(z(0)+\bar{x}) - \bar{x}], \qquad i \le n \le j.$$

Since $y(i)$ and $y(j)$ are two arbitrary members of the solution with property (2.8), we see that there is a positive integer $N_0 > N_0'$ such that

$$y(n) \le c\frac{1-a^{k+1}}{c+a^k b}[f(z(0)+\bar{x}) - \bar{x}] \stackrel{\text{def}}{=} z(1), \qquad n \ge N_0.$$

Then, by an easy induction, we see that for each $m \ge 0$, there is a positive integer N_m such that

$$z(2m) \le y(n) \le z(2m+1) \quad \text{for } n \ge N_m, \tag{2.11}$$

where $\{z(n)\}$ is defined by

$$\begin{cases} z(n) = c\frac{1-a^{k+1}}{c+a^k b}[f(z(n-1)+\bar{x}) - \bar{x}], & n = 1, 2, \dots, \\ z(0) = -c\bar{x}\frac{1-a^{k+1}}{c+a^k b}. \end{cases} \tag{2.12}$$

Let $u(n) = z(n) + \bar{x}, n = 0, 1, \dots$ Then (2.11) and (2.12) become (2.1) and (2.2), respectively. The proof is complete. $\square$

Theorem 2.2. *Assume that $ax + bf(x)$ is increasing and*

$$\frac{c(1-a^{k+1})}{a^k(b+ac)\bar{x}}(xf(x))' > -1, \qquad x > 0. \tag{2.13}$$

Then every positive solution $\{x(n)\}$ of Eq. (1.1) tends to its positive equilibrium $\bar{x}$ as $n \to \infty$.

Proof. As indicated at the beginning of the section, every nonoscillatory solution of Eq. (1.1) converges to $\bar{x}$. Hence we only need to show that every oscillatory solution converges to $\bar{x}$ also. To this end, let $\{x(n)\}$ be an oscillatory solution of Eq. (1.1). Then by Lemma 2.1, $\{x(n)\}$ satisfies (2.1). Since $u(0) \le u(1)$, from (2.2) and the monotonicity of f it is not difficult to see that $\{u(2m)\}$ is increasing, $\{u(2m+1)\}$ is decreasing and $u(2m) \le \bar{x} \le u(2m+1)$, $m = 0, 1, \dots$ Hence,

$$\lim_{m\to\infty} u(2m) = l \le \bar{x} \quad \text{and} \quad \lim_{m\to\infty} u(2m+1) = L \ge \bar{x}$$

exist, and l and L satisfy the equations

$$\begin{cases} l = c\frac{1-a^{k+1}}{c+a^k b} f(L) + \frac{a^k(b+ac)}{c+a^k b}\bar{x} \\ L = c\frac{1-a^{k+1}}{c+a^k b} f(l) + \frac{a^k(b+ac)}{c+a^k b}\bar{x}. \end{cases} \tag{2.14}$$

We now show that $l = L = \bar{x}$. To this end, let

$$g(x) = c\frac{1-a^{k+1}}{c+a^k b} x f(x) + \frac{a^k(b+ac)}{c+a^k b}\bar{x}x, \qquad x > 0$$

and observe that

$$g'(x) = c\frac{1-a^{k+1}}{c+a^k b} (x f(x))' + \frac{a^k(b+ac)}{c+a^k b}\bar{x}, \qquad x > 0.$$

In view of (2.13), we see that $g'(x) > 0$. However, it follows from (2.14) that $g(l) = g(L) = lL$. Hence $l = L = \bar{x}$ and so $\lim_{n\to\infty} u(n) = \bar{x}$. Then from (2.1) we see that $x(n) \to \bar{x}$ as $n \to \infty$. The proof is complete. □

For the proof of the next theorem, we need the following lemma which is extracted from [12].

Lemma 2.3. *Consider the following difference equation*

$$x(n+1) = h(x(n)), \qquad n = 0, 1, \ldots \tag{2.15}$$

where $h : [0,\infty) \to [0,\infty)$ is an S-map. Assume that $\bar{x}$ is the unique fixed point of h and $|h'(\bar{x})| \le 1$. Then $\bar{x}$ is a global attractor of all solutions of Eq. (2.15).

Theorem 2.4. *Assume that $ax + bf(x)$ is increasing and f is an S-map with*

$$c\frac{1-a^{k+1}}{c+a^k b} f'(\bar{x}) \ge -1. \tag{2.16}$$

Then every positive solution $\{x(n)\}$ of Eq. (1.1) tends to its positive equilibrium $\bar{x}$ as $n \to \infty$.

Proof. We only need to show that every oscillatory solution of Eq. (1.1) converges to $\bar{x}$. Let $\{x(n)\}$ be an oscillatory solution. Then $\{x(n)\}$ satisfies (2.1). Hence, to show that $x(n) \to \bar{x}$ as $n \to \infty$ it suffices to show that $u(n) \to \bar{x}$ as $n \to \infty$. To this end, let

$$h(x) = c\frac{1-a^{k+1}}{c+a^k b} f(x) + \frac{a^k(b+ac)}{c+a^k b}\bar{x}.$$

Clearly, $h : [0,\infty) \to [0,\infty)$, $\bar{x}$ is the unique fixed point of h, $h'(x) = c\frac{1-a^{k+1}}{c+a^k b} f'(x) < 0$ and $(Sh)(x) = (Sf)(x) < 0$ for $x > 0$. Hence, h is an S-map. In addition, (2.16) yields $|h'(\bar{x})| \le 1$. Therefore, all the conditions assumed in Lemma 2.3 are satisfied and so $u(n) \to \bar{x}$ as $n \to \infty$. Then it follows that $x(n) \to \bar{x}$ as $n \to \infty$. The proof is complete. □

Remark 2.5. By comparing Theorems 2.2 and 2.4 with Theorem A, we see that when f is a decreasing function, the condition (2.13) is different from the condition (1.3); while when f is an S-map, the condition (2.16) is better than the condition (1.3).

3 Applications

In this section, we apply our results obtained in the last section to some difference equations derived from mathematical biology.

Consider the following system of difference equations

$$\begin{cases} x(n+1) = (1-\epsilon)f(x(n)) + \epsilon y(n), \\ y(n+1) = (1-\epsilon)y(n) + \epsilon f(x(n)), \qquad n = 0, 1, \dots, \\ x(0) \geq 0,\ y(0) \geq 0,\ x(0) + y(0) > 0, \end{cases} \tag{3.1}$$

where $0 < \epsilon < 1$ is a positive constant and $f \in C[[0, \infty), [0, \infty)]$ with $f(x) > 0$ for $x > 0$. Sys. (3.1) is a population model proposed by Newman et al. [18] which assumes symmetric dispersal between active population $x(n)$ and refuge population $y(n)$. The chaotic behavior of positive solutions of Sys. (3.1) is studied in [18] by numerical simulations, whereas in [3] various properties of solutions of (3.1) are studied and several results on the asymptotic behavior of solutions of (3.1) are obtained. Recently, a sufficient condition on the global stability of positive solutions of (3.1) is obtained in [1].

Notice that Sys. (3.1) can be converted into the second order difference equation

$$x(n+1) = (1-\epsilon)x(n) + (1-\epsilon)f(x(n)) + (2\epsilon - 1)f(x(n-1)), \qquad n = 0, 1, \dots \tag{3.2}$$

When f is decreasing and $\epsilon \geq 1/2$, Eq. (3.2) is in the form of (1.1) and f has a unique positive fixed point $\bar{x}$. Clearly, $\bar{x}$ is the unique positive equilibrium of Eq. (3.2) and $(\bar{x}, \bar{x})$ is the unique positive equilibrium of Sys. (3.1).

By Theorems 2.2 and 2.4, we may have the following result on the global attractivity of positive solutions of Sys. (3.1).

Corollary 3.1. *Assume that $1/2 \leq \epsilon < 1$, f is decreasing and $x + f(x)$ is increasing. Suppose also that either $xf(x)$ is differentiable with*

$$\frac{(2\epsilon - 1)(2-\epsilon)}{2(1-\epsilon)^2 \bar{x}}(xf(x))' > -1, \qquad x > 0 \tag{3.3}$$

or f is an S-map with

$$(2-\epsilon)(2 - 1/\epsilon)f'(\bar{x}) \geq -1. \tag{3.4}$$

Then every positive solution $(x(n), y(n))$ of Sys. (3.1) tends to its positive equilibrium $(\bar{x}, \bar{x})$ as $n \to \infty$.

Proof. As indicated above, Sys. (3.1) can be converted into (3.2) which is in the form of Eq. (1.1) with $a = b = 1 - \epsilon, c = 2\epsilon - 1$ and $k = 1$. By the assumption, $ax + bf(x) = (1-\epsilon)(x + f(x))$ is increasing. In addition, noting

$$\frac{c(1 - a^{k+1})}{a^k(b + ac)} = \frac{(2\epsilon - 1)(2 - \epsilon)}{2(1-\epsilon)^2} \tag{3.5}$$

and

$$c\frac{1 - a^{k+1}}{c + a^k b} = (2-\epsilon)(2 - 1/\epsilon) \tag{3.6}$$

we see that when (3.5) or (3.6) holds, (2.13) or (2.16) holds respectively. Then by Theorems 2.2 and 2.4, every positive solution $\{x(n)\}$ of Eq. (3.2) converges to $\bar{x}$ as $n \to \infty$. Then from (3.1) we see that

$$\epsilon y(n) = x(n+1) - (1-\epsilon)f(x(n)) \to \bar{x} - (1-\epsilon)f(\bar{x}) \quad \text{as } n \to \infty,$$

which yields

$$y(n) \to \bar{x} \quad \text{as } n \to \infty.$$

Hence, it follows that every positive solution $(x(n), y(n))$ of Sys. (3.1) converges to $(\bar{x}, \bar{x})$. The proof is complete. $\square$

Next, consider the following difference equation in the form

$$x(n+1) = \alpha x(n) + \beta g(x(n)) + \gamma g(x(n-k)), \qquad n = 0, 1, \ldots, \tag{3.7}$$

where $0 < \alpha < 1, \beta \geq 0$ and $\gamma \geq 0$ with $\beta + \gamma > 0$ are constants, $g \in C[[0,\infty), [0,\infty)]$ and k is a positive integer, observe that it can be written as

$$x(n+1) = \alpha x(n) + \frac{\beta(1-\alpha)}{\beta+\gamma}\left[\frac{\beta+\gamma}{1-\alpha}g(x(n))\right] + \frac{\gamma(1-\alpha)}{\beta+\gamma}\left[\frac{\beta+\gamma}{1-\alpha}g(x(n-k))\right], \tag{3.8}$$

which is in the form of (1.1) with

$$a = \alpha, \qquad b = \frac{\beta(1-\alpha)}{\beta+\gamma}, \qquad c = \frac{\gamma(1-\alpha)}{\beta+\gamma} \quad \text{and} \quad f(x) = \frac{\beta+\gamma}{1-\alpha}g(x).$$

Assume that $\bar{x}$ is the unique positive fixed point of $f(x)$, that is, $\bar{x}$ is the only positive number satisfying

$$g(\bar{x}) = \frac{1-\alpha}{\beta+\gamma}\bar{x}.$$

Clearly $\bar{x}$ is the unique positive equilibrium of Eq. (3.7). Observing that

$$\frac{c(1-a^{k+1})}{a^k(b+ac)\bar{x}} = \frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}}$$

and

$$c\frac{1-a^{k+1}}{c+a^k b}f'(\bar{x}) = \frac{\gamma(1-\alpha^{k+1})(\beta+\gamma)}{(1-\alpha)(\gamma+\alpha^k\beta)}g'(\bar{x}),$$

we see that the following corollary on the global attractivity of $\bar{x}$ is a direct consequence of Theorems 2.2 and 2.4.

Corollary 3.2. *Assume that g is decreasing and $\alpha x + \beta g(x)$ is increasing. Let $\bar{x}$ be the unique positive equilibrium of Eq. (3.7) and suppose that either $xg(x)$ is differentiable with*

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}}(xg(x))' > -1 \tag{3.9}$$

or g is an S-map with

$$\frac{\gamma(1-\alpha^{k+1})(\beta+\gamma)}{(1-\alpha)(\gamma+\alpha^k\beta)}g'(\bar{x}) \geq -1. \tag{3.10}$$

Then every positive solution of Eq. (3.7) tends to $\bar{x}$ as $n \to \infty$.

When $\gamma = 0$, Eq. (3.7) reduces to

$$x(n+1) = \alpha x(n) + \beta g(x(n)), \qquad n = 0, 1, \ldots \tag{3.11}$$

Clearly, (3.9) is automatically satisfied since the left side is 0. From Corollary 3.2 we know that when g is decreasing and $\alpha x + \beta g(x)$ is increasing, every positive solution $\{x(n)\}$ of Eq. (3.11) tends to its positive equilibrium $\bar{x}$ as $n \to \infty$ where $\bar{x}$ is the unique positive number satisfying $\bar{x} = \frac{\beta}{1-\alpha} g(\bar{x})$.

When $\beta = 0$, Eq. (3.7) reduces to

$$x(n+1) = \alpha x(n) + \gamma g(x(n-k)), \qquad n = 0, 1, \ldots, \tag{3.12}$$

which includes several discrete models derived from mathematical biology. For instance, when $g(x) = \frac{1}{1+x^p}$ where p is a positive constant, Eq. (3.12) is a discrete analogue of a model that has been used to study blood cells production [13]; when $g(x) = e^{-qx}$ where q is a positive constant, Eq. (3.12) is a discrete version of a model of the survival of red blood cells in an animal [25]. Due to its theoretical interest and applications, asymptotic behavior of positive solutions of Eq. (3.12) and some related forms have been studied by numerous authors, see, for example, [1, 2, 4–11, 13–25] and the references cited therein. As a special case of Eq. (3.7), our results can be applied to Eq. (3.12) also.

In the following, we discuss the global attractivity of positive solutions of Eq. (3.7) when $g(x) = \frac{1}{1+x^p}$ and $g(x) = e^{-qx}$ where p and q are positive constants, respectively. When $g(x) = \frac{1}{1+x^p}$, Eq. (3.7) becomes

$$x(n+1) = \alpha x(n) + \frac{\beta}{1+x^p} + \frac{\gamma}{1+x^p(n-k)}, \qquad n = 0, 1, \ldots \tag{3.13}$$

Clearly, g is decreasing and has a unique positive number $\bar{x}$ satisfying $g(\bar{x}) = \frac{1-\alpha}{\beta+\gamma}\bar{x}$ which is the only positive equilibrium of Eq. (3.13). When $\beta = 0$, $\alpha x + \beta g(x) = \alpha x$ is increasing; when $\beta > 0$ and $p \geq 1$, noting

$$g'(x) = \frac{-px^{p-1}}{(1+x^p)^2}$$

and

$$g''(x) = \frac{-px^{p-2}((p-1)-(p+1)x^p)}{(1+x^p)^3}$$

we see that $g'(x)$ takes minimum at $x^* = (\frac{p-1}{p+1})^{1/p}$ and

$$g'(x^*) = -\frac{1}{4p}(p-1)^{1-1/p}(1+p)^{1+1/p}.$$

Hence, if $p \geq 1$ and

$$\frac{\beta}{4p}(p-1)^{1-1/p}(1+p)^{1+1/p} \leq \alpha, \tag{3.14}$$

then

$$(\alpha x + \beta g(x))' \geq \alpha + \beta g'(x^*) = \alpha - \frac{\beta}{4p}(p-1)^{1-1/p}(1+p)^{1+1/p} \geq 0$$

and so $\alpha x + \beta g(x)$ is increasing.

Next, observe that

$$(xg(x))' = \frac{1+(1-p)x^p}{(1+x^p)^2}$$

and

$$(xg(x))'' = \frac{px^{p-1}((p-1)x^p-(p+1))}{(1+x^p)^3}.$$

We see that when $p \le 1$, $(xg(x))' > 0$ and so (3.9) is true; when $p > 1$, $(xg(x))'$ has minimum at $x^* = \left(\frac{p+1}{p-1}\right)^{1/p}$. Hence in this case,

$$(xg(x))' \ge (xg(x))'|_{x=x^*} = -\frac{(p-1)^2}{4p}. \tag{3.15}$$

Clearly, if

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}}\left(-\frac{(p-1)^2}{4p}\right) > -1,$$

that is,

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}}\frac{(p-1)^2}{4p} < 1, \tag{3.16}$$

then by noting (3.15) we know that (3.9) is satisfied. Furthermore, by a simple calculation, we find that for $p > 1$,

$$(Sg)(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2 = \frac{1}{2}(1-p)(1+p)x^{-2} < 0, \qquad x > 0,$$

that is, g is an S-map. In addition, by noting

$$g'(\bar{x}) = -\frac{p\bar{x}^{p-1}}{(1+\bar{x}^p)^2} = -p\bar{x}^{p-1}g^2(\bar{x}) = -p\bar{x}^{p-1}\left(\frac{1-\alpha}{\beta+\gamma}\bar{x}\right)^2 = -p\left(\frac{1-\alpha}{\beta+\gamma}\right)^2\bar{x}^{p+1}$$

we see that if

$$\frac{\gamma(1-\alpha^{k+1})(\beta+\gamma)}{(1-\alpha)(\gamma+\alpha^k\beta)}\left(-p\left(\frac{1-\alpha}{\beta+\gamma}\right)^2\bar{x}^{p+1}\right) \ge -1,$$

that is,

$$\frac{\gamma(1-\alpha)(1-\alpha^{k+1})}{(\beta+\gamma)(\gamma+\alpha^k\beta)}p\bar{x}^{p+1} \le 1, \tag{3.17}$$

then (3.10) is satisfied. Hence, by Corollary 3.2, we have the following conclusion: every positive solution of Eq. (3.13) tends to its positive equilibrium $\bar{x}$ as $n \to \infty$ if one of the following holds

(i) $p \le 1$ and $\beta = 0$;

(ii) $p \ge 1$, (3.14) and (3.16) hold;

(iii) $p > 1$, (3.14) and (3.17) hold.

When $f(x) = \frac{1}{1+x^p}$, Sys. (3.1) becomes

$$\begin{cases} x(n+1) = \frac{1-\epsilon}{1+x^p(n)} + \epsilon y(n), \\ y(n+1) = (1-\epsilon)y(n) + \frac{\epsilon}{1+x^p(n)}, \\ x(0) \ge 0,\ y(0) \ge 0,\ x(0)+y(0) > 0, \end{cases} \qquad n = 0, 1, \ldots, \tag{3.18}$$

and it can be converted into Eq. (3.13) with $\alpha = \beta = 1-\epsilon, \gamma = 2\epsilon - 1$ and $k = 1$. Since $\alpha = \beta$, (3.14) reduces to

$$\frac{1}{4p}(p-1)^{1-1/p}(1+p)^{1+1/p} \le 1. \tag{3.19}$$

Hence, when $p \ge 1$ and (3.19) holds, $x + f(x)$ is increasing. Note that $f(x) = g(x)$. From (3.15), (3.17) and the above discussion, we know that when $p \ge 1$,

$$(xf(x))' \ge -(p-1)^2/(4p) \tag{3.20}$$

and

$$f(\bar{x}) = -p\left(\frac{1-\alpha}{\beta+\gamma}\right)^2 \bar{x}^{p+1} = -p\bar{x}^{p+1}. \tag{3.21}$$

Clearly, (3.20) implies that if

$$\frac{(2\epsilon-1)(2-\epsilon)}{2(1-\epsilon)^2\bar{x}}\left(-\frac{(p-1)^2}{4p}\right) > -1,$$

that is,

$$\frac{(2\epsilon-1)(2-\epsilon)}{2(1-\epsilon)^2\bar{x}}\frac{(p-1)^2}{4p} < 1, \tag{3.22}$$

then (3.5) is satisfied, and (3.21) implies that if

$$(2-\epsilon)(2-1/\epsilon)(-p\bar{x}^{p+1}) \ge -1,$$

that is,

$$(2-\epsilon)(2-1/\epsilon)p\bar{x}^{p+1} \le 1, \tag{3.23}$$

then (3.6) is satisfied. In addition, from the above discussion, we know that when $p > 1$, f is an S-map. Hence, by Corollary 3.1, we have the following conclusion: when $1/2 \le \epsilon < 1$, every positive solution of Sys. (3.18) converges to its positive equilibrium $(\bar{x}, \bar{x})$ as $n \to \infty$ if either $p \ge 1$, (3.19) and (3.22) hold, or $p > 1$, (3.19) and (3.23) hold.

Example 3.3. Consider the equation

$$x(n+1) = (1/2)x(n) + (3/4)\frac{1}{1+x^2(n)} + (1/4)\frac{1}{1+x^2(n-3)}, \qquad n = 0, 1, \dots, \tag{3.24}$$

which is in the form of Eq. (3.7) with $\alpha = 1/2, \beta = 3/4, \gamma = 1/4, k = 3$ and $g(x) = 1/(1+x^2)$. Note that $\bar{x} = 1$ is the unique positive equilibrium of Eq. (3.24). Since $p = 2$,

$$\frac{\beta}{4p}(p-1)^{1-1/p}(1+p)^{1+1/p} = (3/4)(1/8)3^{3/2} < 1/2 = \alpha,$$

that is, (3.14) is satisfied. In addition, observing that

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}}\frac{(p-1)^2}{4p} = \frac{(1/4)(1-(1/2)^4)}{(1/2)^3((3/4)+(1/2)(1/4))}\cdot\frac{1}{8} = \frac{15}{56} < 1$$

we see that (3.16) is satisfied. Hence, from the above discussion, we know that every positive solution of Eq. (3.24) tends to its positive equilibrium $\bar{x} = 1$ as $n \to \infty$.

Example 3.4. Consider the system

$$\begin{cases} x(n+1) = \frac{7/15}{1+x^3(n)} + (8/15)y(n), \\ y(n+1) = (7/15)y(n) + \frac{8/15}{1+x^3(n)}, \qquad\qquad n = 0,1,\ldots, \\ x(0) \ge 0,\ y(0) \ge 0,\ x(0)+y(0) > 0, \end{cases} \tag{3.25}$$

which is in the form of Sys. (3.18) with $\epsilon = 8/15$ and $f(x) = 1/(1+x^3)$. Since $p = 3$,

$$\frac{1}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} = (1/12)2^{2/3}4^{4/3} < 1,$$

that is, (3.19) is satisfied. In addition, we know that f is an S-map. Sys. (3.25) has the unique positive equilibrium $(\bar{x}, \bar{x})$ where $\bar{x}$ is the unique positive fixed point of f. Observing $\bar{x}(1+\bar{x}^3) = 1$, we see that $\bar{x} < 1$. Then it follows that

$$(2-\epsilon)(2-1/\epsilon)p\bar{x}^{p+1} \le (2-8/15)(2-15/8)3 = 11/20 < 1$$

and so (3.22) is satisfied. Hence, from the above discussion, we know that every positive solution of Sys. (3.25) tends to its positive equilibrium $(\bar{x}, \bar{x})$ as $n \to \infty$.

When $g(x) = e^{-qx}$, Eq. (3.7) becomes

$$x(n+1) = \alpha x(n) + \beta e^{-qx(n)} + \gamma e^{-qx(n-k)}, \qquad n = 0,1,\ldots. \tag{3.26}$$

Since g is decreasing, there is a unique positive number $\bar{x}$ satisfying $g(\bar{x}) = \frac{1-\alpha}{\beta+\gamma}\bar{x}$. Clearly $\bar{x}$ is the only positive equilibrium of Eq. (3.26). Noting

$$(\alpha x + \beta g(x))' = (\alpha x + \beta e^{-qx})' = \alpha - q\beta e^{-qx}$$

we see that $\alpha x + \beta e^{-qx}$ is increasing when

$$\alpha \ge q\beta. \tag{3.27}$$

In addition, observing that

$$(xg(x))' = (1-qx)e^{-qx} \text{ and } (xg(x))'' = q(qx-2)e^{-qx}$$

we find that $(xg(x))'$ takes minimum when $x = 2/q$ and so

$$(xg(x))' \ge (xg(x))'|_{x=q/2} = -e^{-2}. \tag{3.28}$$

Hence, if

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}}(-e^{-2}) > -1,$$

that is,

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}} < e^2, \tag{3.29}$$

then (3.9) is satisfied. Furthermore, by a simple calculation, we find that

$$(Sg)(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2 = -(1/2)q^2 < 0, \qquad x > 0,$$

that is, g is an S-map. In addition, by noting

$$g'(\bar{x}) = -qe^{-q\bar{x}} = -q\frac{1-\alpha}{\beta+\gamma}\bar{x} \tag{3.30}$$

we see that if

$$\frac{\gamma(1-\alpha^{k+1})(\beta+\gamma)}{(1-\alpha)(\gamma+\alpha^k\beta)}\left(-q\frac{1-\alpha}{\beta+\gamma}\bar{x}\right) \geq -1$$

that is,

$$\frac{\gamma(1-\alpha^{k+1})}{\gamma+\alpha^k\beta}q\bar{x} \leq 1, \tag{3.31}$$

then (3.10) is satisfied. Hence, by Corollary 3.2, we have the following conclusion: if (3.27) holds and either (3.29) or (3.31) holds also, then every positive solution of Eq. (3.26) tends to its positive equilibrium as $n \to \infty$.

When $f(x) = e^{-qx}$, Sys. (3.1) is

$$\begin{cases} x(n+1) = (1-\epsilon)e^{-qx(n)} + \epsilon y(n), \\ y(n+1) = (1-\epsilon)y(n) + \epsilon e^{-qx(n)}, \qquad n = 0, 1, \ldots, \\ x(0) \geq 0,\ y(0) \geq 0,\ x(0) + y(0) > 0, \end{cases} \tag{3.32}$$

and it can be converted into Eq. (3.26) with $\alpha = \beta = 1-\epsilon, \gamma = 2\epsilon - 1$ and $k = 1$. Since $\alpha = \beta$, (3.27) reduces to $q \leq 1$. Noting $f(x) = g(x)$, from (3.28), (3.30) and the above discussion, we know that

$$(xf(x))' \geq -e^{-2} \tag{3.33}$$

and

$$f'(\bar{x}) = -qe^{-q\bar{x}} = -q\frac{1-\alpha}{\beta+\gamma}\bar{x} = -q\bar{x}. \tag{3.34}$$

Clearly, (3.33) implies that if

$$\frac{(2\epsilon-1)(2-\epsilon)}{2(1-\epsilon)^2\bar{x}}(-e^{-2}) > -1,$$

that is,

$$\frac{(2\epsilon-1)(2-\epsilon)}{2(1-\epsilon)^2\bar{x}} < e^2, \tag{3.35}$$

then (3.5) is satisfied, and if

$$(2-\epsilon)(2-1/\epsilon)(-q\bar{x}) \geq -1,$$

that is,

$$(2-\epsilon)(2-1/\epsilon)q\bar{x} \leq 1, \tag{3.36}$$

then (3.6) is satisfied. Hence, by Corollary 3.1, we have the following conclusion on the global attractivity of positive solutions of Sys. (3.32): if $q \leq 1$ and either (3.35) or (3.36) holds, then every positive solution of Sys. (3.32) tends to its positive equilibrium $(\bar{x}, \bar{x})$ as $n \to \infty$.

Example 3.5. Consider the equation

$$x(n+1) = (2/3)x(n) + (1/3)e^{-2x(n)} + (1/4)e^{-2x(n-3)}, \qquad n = 0, 1, \ldots \tag{3.37}$$

which is in the form of Eq. (3.26) with $\alpha = 2/3, \beta = 1/3, \gamma = 1/4, k = 3$ and $g(x) = e^{-2x}$. Noting $q = 2$, we see that (3.27) is satisfied. Let $\bar{x}$ be the unique positive equilibrium of Eq. (3.37). Then $\bar{x}$ satisfies $\bar{x}e^{2\bar{x}} = 7/4$. By noting $(1/2)e^{2(1/2)} < 7/4$, we see that $\bar{x} > 1/2$ and so it follows that

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}} < \frac{(1/4)(1-(2/3)^4)}{(2/3)^3((1/3)+(2/3)(1/4))(1/2)} = \frac{65}{24} < e^2,$$

that is, (3.29) holds. Hence, from the above discussion, we know that every positive solution of Eq. (3.37) converges to its positive equilibrium $\bar{x}$ as $n \to \infty$.

Example 3.6. Consider the system

$$\begin{cases} x(n+1) = (2/5)e^{-(1/2)x(n)} + (3/5)y(n), \\ y(n+1) = (2/5)y(n) + (3/5)e^{-(1/2)x(n)}, \qquad n = 0, 1, \dots, \\ x(0) \geq 0,\ y(0) \geq 0,\ x(0) + y(0) > 0, \end{cases} \tag{3.38}$$

which is in the form of Sys. (3.32) with $\epsilon = 3/5$ and $f(x) = e^{-(1/2)x}$. Note that $q = 1/2 < 1$. Sys. (3.38) has the unique positive equilibrium $(\bar{x}, \bar{x})$ where $\bar{x}$ is the unique positive fixed point of f. Noting $\bar{x}e^{(1/2)\bar{x}} = 1$, we see that $\bar{x} < 1$. Then it follows that

$$(2-\epsilon)(2-1/\epsilon)q\bar{x} \leq (2-3/5)(2-5/3)(1/2) = 7/30 < 1$$

and so (3.36) is satisfied. Hence, from the above discussion, we know that every positive solution of Sys. (3.38) tends to its positive equilibrium $(\bar{x}, \bar{x})$ as $n \to \infty$.

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