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Global boundedness and stabilization in a predator-prey model with cannibalism and prey-evasion

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Abstract. This paper is concerned with a predator-prey model with cannibalism and prey-evasion. The global existence and boundedness of solutions to the system in bounded domains of 1D and 2D are proved for any prey-evasion sensitivity coefficient. It is also shown that prey-evasion driven Turing instability when the prey-evasion coefficient surpasses the critical value. Besides, the existence of Hopf bifurcation, which generates spatiotemporal patterns, is established. And, numerical simulations demonstrate the complex dynamic behavior.

Keywords: predator-prey, cannibalism, prey-evasion, global existence, Turing instability, Hopf bifurcation.

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1 Introduction

Cannibalism, adult preying on juveniles of the same species, has an effective impact on the regulation and equilibration of population density [7,23]. Numerous mathematical modeling and analysis of cannibalism have been developed rapidly over the past few decades [5, 8]. These analyses focused mainly on the stabilizing-destabilizing effect of cannibalism, which seems to strongly depend on the form of the model. For example, Kohlmeier and Ebenhöh [13] found that cannibalism can stabilize population cycles. A high cannibalism rate may cause the internal steady state to change from being unstable to stable due to the interaction between logistic population growth of the prey and a Beddington–DeAngelis functional response. In 1999, Magnússon [18] proposed an age-structured predator-prey model and showed that cannibalism has a destabilizing effect. If the mortality rate of juveniles is high and/or the recruitment rate to the mature population is low, then the equilibrium will be stable for low levels of cannibalism. However, a loss of stability by the Hopf bifurcation will take place as the level of cannibalism increases, and numerical studies indicate that a stable limit cycle exists.

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In 2006, Buonomo and Lacitignola [3] introduced a predator-prey model with age structure and cannibalism in the predator population

$$\begin{cases} \dfrac{dA}{dt} = MJ - d_A A, \\ \dfrac{dJ}{dt} = \eta_1 \delta AP - (1-\eta_c)\sigma AJ - (M + d_J)J, \\ \dfrac{dP}{dt} = r_1 P - r_2 P^2 - \delta AP, \end{cases} \tag{1.1}$$

where $A(t)$ and $J(t)$ represent the densities of individuals of mature and immature predator populations at time t, respectively, and $P(t)$ denotes the number of individuals of prey population. Further, M is the constant maturation rate from juveniles to adults; δ is the inter-specific competition rate; σ is the cannibalism attack rate; η_1 and η_c denote the coefficients in converting preys into new immature predators (juveniles), and juveniles into new juveniles, respectively. r_1 and r_2 are the logistic coefficients, d_A and d_J are the death rates.

By the following non-dimensional variables

$$u = \delta A/d_A, \qquad v = M\delta J/d_A^2, \qquad w = r_2 P/d_A,\ \tau = d_A t,$$

and denoting τ by t again, system (1.1) becomes

$$\begin{cases} \dfrac{du}{dt} = v - u, \\ \dfrac{dv}{dt} = auw - \gamma uv - cv, \\ \dfrac{dw}{dt} = rw - w^2 - uw, \end{cases} \tag{1.2}$$

where $a = \frac{\eta_1 M\delta}{r_2 d_A}$, $\gamma = \frac{\sigma(1-\eta_c)}{\delta}$, $c = \frac{M+d_J}{d_A}$, $r = \frac{r_1}{d_A}$. Obviously, if $ar > c$, then system (1.2) has a unique positive equilibrium point $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}, \tilde{w})$, where

$$\tilde{u} = \frac{ar - c}{a + \gamma}, \qquad \tilde{v} = \frac{ar - c}{a + \gamma}, \qquad \tilde{w} = \frac{\gamma r + c}{a + \gamma}. \tag{1.3}$$

Buonomo and Lacitignola derived that cannibalism is a stabilizing mechanism in the model (1.2). That is, when cannibalism attack rate increases to a level that exceeds the critical value, the coexistence steady state changes from being unstable to stable. Moreover, they provided numerical simulations to demonstrate the mathematical analysis. The same conclusion has been pointed out by Buonomo and coauthors [4]. They also found that the effects of cannibalism and prey growth are opposite. Besides, numerical simulations showed that the higher the uptake of prey by predators, the higher the critical value of cannibalism.

Recently, Jia *et al*. [10] discussed the corresponding pure diffusion system of (1.2) and obtained the result that the effects of prey growth and predator cannibalism rate on the stability of nonnegative constant steady state are opposite. They also proved the nonexistence and existence of nonconstant positive solutions and found that diffusion can cause a periodic solution of spatial inhomogeneity which occurs in unstable area (also the unstable area of ODE). Very recently, in another paper, we investigated the temporal, spatial and spatiotemporal patterns of the corresponding cross-diffusion system of (1.2) in detail. We showed that cannibalism is no longer a stabilizing effect, and cross-diffusion is the decisive factor of destabilizing positive steady state.

From biological characteristics, it can be seen that in addition to the random diffusion of predators, the spatial movements between predators and prey can also be pursuit and evasion,that is to say, predators pursuing preys and preys escaping from predators. Such movement is not random but directed, that is predators move toward the gradient direction of prey distribution (called "prey-taxis"), and/or preys move opposite to the gradient of predator distribution (called "prey-evasion" or "predator-taxis") [28]. These processes are well known to be important in biological control and ecological balance such as regulating prey (pest) population or incipient outbreaks of prey or forming large scale aggregation for survival [20,31].

Tsyganov and coauthors [22] proposed a predator-prey model with both prey-taxis and predator-taxis, and found that the taxis terms change the shape of the propagating waves and increase the propagation speed. Since then, there are many mathematical literatures demonstrating and explaining the pursuit-evasion phenomenon. Meanwhile, various reaction-diffusion models with prey-taxis and (or) predator-taxis have been proposed to study global existence, traveling wave, pattern formation, and bifurcation analysis [11,12,14,15,17,19,24,27,30]. Recently, Wu and coauthors [28] considered a reaction-diffusion predator-prey model system with predator-taxis, which is a similar situation occurs when susceptible population avoids the infected ones in epidemic spreading. They proved the global existence and boundedness of solutions in bounded domains of arbitrary spatial dimension and any predator-taxis sensitivity coefficient. It is also shown that a smaller predator-taxis effect can destabilize the positive constant steady state and generate non-constant spatial pattern.

Inspired by the above discussion, the main aim of this paper is to investigate the global existence and dynamical behavior in a predator-prey model with both cannibalism and prey-evasion

$$\begin{cases} u_t - d_1\Delta u = -u + v, & x\in\Omega,\ t>0, \\ v_t - d_2\Delta v = auw - \gamma uv - cv, & x\in\Omega,\ t>0, \\ w_t - d_3\Delta w - \xi\nabla\cdot(w\nabla u) = rw - w^2 - uw, & x\in\Omega,\ t>0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x), & x\in\Omega, \end{cases} \tag{1.4}$$

where $-\xi\nabla\cdot(w\nabla u)$ is prey-evasion, which shows the tendency of prey moving toward the opposite direction of the increasing predator gradient direction. $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. ν is the outer normal directional derivative on $\partial\Omega$. The homogeneous Neumann boundary condition indicates that this system is self-contained with zero population flux across the boundary. The initial values $u_0(x), v_0(x), w_0(x)$ are nonnegative smooth functions which are not identically zero.

Our main results on the global existence and boundedness of solutions of system (1.4) are as follows.

Theorem 1.1. *Let $n=\{1,2\}$ and $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary. For any $(u_0,v_0,w_0)\in[W^{1,p}(\Omega)]^3$ where $p>n$, satisfying $u_0(x)\geq 0, v_0(x)\geq 0, w_0(x)\geq 0$ for $x\in\Omega$, the system (1.4) has a unique nonnegative and bounded global classical solution $(u(x,t),v(x,t),w(x,t))$, and $(u,v,w)\in\left(C([0,\infty);W^{1,p}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))\right)^3$.*

The rest of the paper is organized as follows. In Section 2, we obtain some preliminary results. Section 3 is devoted to prove the global existence and uniform boundedness of the classical solution of (1.4). The dynamical behavior and pattern formation of the prey-evasion system are studied in Section 4. And, numerical simulations are emphasized the theoretical results. The last section is a brief discussion.

2 Preliminaries

2.1 Existence and uniqueness of local solutions

We first give a claim concerning the local-in-time existence of a classical solution to (1.4).

Lemma 2.1. *Assume that the initial data u_0, v_0, and w_0 be nonnegative and satisfy $(u_0, v_0, w_0) \in [W^{1,p}(\Omega)]^3$ for $p > n$. Then the following statements for the model (1.4) hold.*

(1) There exists a positive constant $T_{\max}$ (the maximal existence time) such that the problem (1.4) has a unique local in time classical solution $(u(x,t), v(x,t), w(x,t))$ satisfying

$$(u,v,w) \in \left(C([0,T_{\max}); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max}))\right)^3.$$

Moreover, u, v, and w satisfy the inequalities

$$u > 0, \qquad v > 0, \qquad w > 0 \quad \text{in } \Omega \times (0, T_{max}). \tag{2.1}$$

(2) If for each $T > 0$ there exists a constant $C(T)$ (depending on T and $\|(u_0, v_0, w_0)\|_{W^{1,p}(\Omega)}$ only) such that

$$\|(u(t), v(t), w(t))\|_{L^\infty} \le C(T), \qquad 0 < t < \min\{T, T_{\max}\}, \tag{2.2}$$

then $T_{\max} = +\infty$.

(3) The total mass of $u(x,t), v(x,t)$ and $w(x,t)$ satisfies

$$\int_\Omega w dx \le m_1 := \max\left\{\int_\Omega w_0 dx, r|\Omega|\right\}, \qquad t \in (0, T_{\max}), \tag{2.3}$$

$$\int_\Omega v dx \le m_2 := \max\left\{\int_\Omega (v_0 + a w_0) dx, \frac{a(r+c)}{c} m_1\right\}, \qquad t \in (0, T_{\max}), \tag{2.4}$$

$$\int_\Omega u dx \le m_3 := \max\left\{\int_\Omega u_0 dx, m_2\right\}, \qquad t \in (0, T_{\max}). \tag{2.5}$$

Proof. We first let $\eta = (u, v, w)^T$, then the system (1.4) can be reformulated as the abstract form

$$\begin{cases} \eta_t - \nabla \cdot (\mathcal{A}(\eta)\nabla \eta) = \mathcal{F}(\eta), & x \in \Omega, t > 0, \\ \frac{\partial \eta}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ \eta(\cdot, 0) = (u_0, v_0, w_0)^T, & x \in \Omega, \end{cases} \tag{2.6}$$

where

$$\mathcal{A}(\eta) = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ \xi w & 0 & d_3 \end{pmatrix}, \qquad \mathcal{F}(\eta) = \begin{pmatrix} -u + v \\ auw - \gamma uv - cv \\ rw - w^2 - uw \end{pmatrix}.$$

System (2.6) is normally parabolic since all the eigenvalues of $\mathcal{A}(\eta)$ are positive. Then from Theorem 7.3 and Corollary 9.3 in Ref. [1] or Theorem 14.4 and 14.6 in Ref. [2], we obtain that there exists a unique classical solution. Next, the estimates (2.1) follow from the maximum principle.

Furthermore, since the system (2.6) is a lower triangular system, then we can invoke Theorem 15.5 of Ref. [2] to conclude that $T_{\max} = \infty$ if (2.2) holds.

Finally, we show that the solution $(u(x,t), v(x,t), w(x,t))$ is bounded in $L^1(\Omega)$. Integrating the third equation in (1.4) over Ω and using the Cauchy–Schwarz inequality we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega w dx &= r\int_\Omega w dx - \int_\Omega w^2 dx - \int_\Omega uw dx \\ &\le r\int_\Omega w dx - \frac{1}{|\Omega|}\left(\int_\Omega w dx\right)^2, \qquad t \in (0, T_{\max}).\end{aligned}$$

By an ODE comparison principle, we derive

$$\int_\Omega w dx \le \max\left\{\int_\Omega w_0 dx, r|\Omega|\right\} =: m_1.$$

Then we have

$$\begin{aligned}\int_\Omega (v_t + aw_t) dx &= \frac{d}{dt}\int_\Omega (v + aw) dx \\ &= \int_\Omega [d_2\Delta v + d_3 a\Delta w + \xi a\nabla\cdot(w\nabla u)] dx + \int_\Omega (raw - aw^2 - \gamma uv - cv) dx \\ &= \int_\Omega [raw + acw - aw^2 - \gamma uv - c(v + aw)] dx \\ &\le \int_\Omega [aw(r + c) - c(v + aw)] dx\end{aligned}$$

since $\int_\Omega w dx \le m_1$, it gets

$$\int_\Omega v dx \le \int_\Omega (v + aw) dx \le \max\left\{\int_\Omega (v_0 + aw_0) dx, \frac{a(r+c)}{c} m_1\right\} =: m_2.$$

Similarly, it can be derived

$$\int_\Omega u dx \le \max\left\{\int_\Omega u_0 dx, m_2\right\} =: m_3.$$

This completes the proof of part (3). □

2.2 Relationship between bounds for u, ∇v and w in the case $n \ge 2$

In this subsection, by using appropriate smoothing estimates for the Neumann heat semigroup to the system (1.4), which have been inspired by Winkler [26], we establish some relationships between the quantities

$$\sup_{s\in(0,t)} \|u(\cdot,s)\|_{L^\infty}, \qquad \sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|_{L^q}, \qquad \sup_{s\in(0,t)} \|w(\cdot,s)\|_{L^p}, \qquad t \in (0, T_{\max}),$$

for suitably wide ranges of the free parameters $p \in (1,\infty]$ and $q \in (1,\infty)$ when $n \ge 2$.

Lemma 2.2. *Assume that $n \ge 2$ and $q > \max\{1, \frac{n}{3}\}$. Then for any $\varepsilon > 0$, there exists $C(\varepsilon, q) > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C(\varepsilon,q) + C(\varepsilon,q)\cdot\left\{\sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|_{L^q(\Omega)}\right\}^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon}, \qquad t \in (0, T_{\max}). \quad (2.7)$$

Proof. Since $q > \frac{n}{3}$, without loss of generality we may assume that ε satisfies $(n+1-\frac{n}{q})\varepsilon < 2$ and $(n+1-\frac{n}{q})q\varepsilon < 3q-n$. Here the former property ensures that

$$r \equiv r(\varepsilon, q) := \frac{n}{2-(n+1-\frac{n}{q})\varepsilon}$$

is a positive number satisfying $r > \frac{n}{2} \geq 1$ as well as

$$\frac{(n-q)r}{n} = \frac{n-q}{2-(n+1-\frac{n}{q})\varepsilon} < \frac{n-q}{2-\frac{3q-n}{q}} = q.$$

Hence, the Gagliardo–Nirenberg inequality gives $c_1 = c_1(\varepsilon, q) > 0$ such that with $a := a(\varepsilon, q) := \frac{n-\frac{n}{r}}{n+1-\frac{n}{q}} \in (0,1)$ we have

$$\|\phi\|_{L^r(\Omega)} \leq c_1 \|\nabla\phi\|^a_{L^q(\Omega)} \|\phi\|^{1-a}_{L^1(\Omega)} + c_1\|\phi\|_{L^1}, \qquad \phi \in W^{1,q}(\Omega), \tag{2.8}$$

and moreover we can employ smoothing estimates for the Neumann heat semi-group $(e^{t\Delta})_{t\leq 0}$ [25] to find $c_2 = c_2(\varepsilon, q) > 0$ fulfilling

$$\|e^{t\Delta}\phi\|_{L^\infty(\Omega)} \leq c_2(1+t^{-\frac{n}{2r}})\|\phi\|_{L^r(\Omega)}, \qquad t > 0,\ \phi \in L^r(\Omega). \tag{2.9}$$

As Lemma 2.1 provides that with some $m_2 > 0$ we have $\|v(\cdot,t)\|_{L^1(\Omega)} \leq m_2$ for all $t \in (0, T_{\max})$, based on a variation-of-constants representation we can combine (2.8) with (2.9) to see that due to the maximum principle,

$$\begin{aligned}
&\|u(\cdot,t)\|_{L^\infty(\Omega)} \\
&= \|e^{t(d_1\Delta-1)}u_0 + \int_0^t e^{(t-s)(d_1\Delta-1)}v(\cdot,s)ds\|_{L^\infty(\Omega)} \\
&\leq e^{-t}\|u_0\|_{L^\infty(\Omega)} + c_2\int_0^t (1+(t-s)^{-\frac{n}{2r}})e^{-(t-s)}\|v(\cdot,s)\|_{L^r(\Omega)}ds \\
&\leq \|u_0\|_{L^\infty(\Omega)} + c_1c_2\int_0^t (1+(t-s)^{-\frac{n}{2r}})e^{-(t-s)}\|\nabla v(\cdot,s)\|^a_{L^q(\Omega)}\|v(\cdot,s)\|^{1-a}_{L^1(\Omega)}ds \\
&\quad + c_1c_2\int_0^t (1+(t-s)^{-\frac{n}{2r}})e^{-(t-s)}\|v(\cdot,s)\|_{L^1}ds \\
&\leq \|u_0\|_{L^\infty(\Omega)} + \{c_1c_2m_2^{1-a}\|\nabla v\|^a_{L^\infty((0,t);L^q(\Omega))} + c_1c_2m_2\}\cdot\int_0^t (1+(t-s)^{-\frac{n}{2r}})e^{-(t-s)}ds \\
&\leq \|u_0\|_{L^\infty(\Omega)} + \{c_1c_2m_2^{1-a}\|\nabla v\|^a_{L^\infty((0,t);L^q(\Omega))} + c_1c_2m_2\}\cdot\left(1+\Gamma\left(1-\frac{n}{2r}\right)\right)
\end{aligned}$$

for all $t \in (0, T_{\max})$. Here $\Gamma(1-\frac{n}{2r})$ is the Gamma function which is positive and real-valued according to $r > \frac{n}{2}$, this already entails (2.7) due to the fact that

$$a = \frac{n-(2-(n+1-\frac{n}{q})\varepsilon)}{n+1-\frac{n}{q}} = \frac{n-2}{n+1-\frac{n}{q}} + \varepsilon$$

by definition of a and r. □

A similar argument shows that the regularity of ∇v depends on L^p bounds for w and L^∞ bounds for u.

Lemma 2.3. *Let $n \geq 2$. Assume that $p \in (1, \infty]$ and $q > \frac{n}{n-1}$ be such that $(n-p)q < np$. Then for each $\varepsilon > 0$ there exists $C(\varepsilon, p, q) > 0$ such that*

$$\begin{aligned}
&\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \\
&\quad \leq C(\varepsilon,p,q) + C(\varepsilon,p,q)\cdot\left\{1+\sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^p(\Omega)}\right\}^{\frac{n-1-\frac{n}{q}}{n(1-\frac{1}{p})}+\varepsilon}\cdot\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^\infty(\Omega)} \\
&\qquad + C(\varepsilon,p,q)\cdot\left\{1+\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}\right\}^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon}\cdot\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^\infty(\Omega)} \\
&\qquad + C(\varepsilon,p,q)\cdot\left\{1+\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}\right\}^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon}, \qquad t\in(0,T_{\max}). \qquad (2.10)
\end{aligned}$$

Proof. Since $(n-p)q < np$ and thus $\frac{1}{q} + \frac{1}{n} - \frac{1}{p} > 0$, we assume that apart from $(1-\frac{1}{p})\varepsilon < \frac{1}{n}$ the inequality $(1-\frac{1}{p})\varepsilon < \frac{1}{q} + \frac{1}{n} - \frac{1}{p}$ holds about ε, so that

$$\lambda \equiv \lambda(\varepsilon,p,q) := \frac{1}{\frac{1}{q}+\frac{1}{n}-(1-\frac{1}{p})\varepsilon}$$

is a positive number satisfying $\lambda < q$. Moreover

$$\lambda > \frac{1}{\frac{1}{q}+\frac{1}{n}} > 1 \qquad (2.11)$$

thanks to the condition $q > \frac{n}{n-1}$.

By applying Duhamel representation and smoothing properties of the Neumann heat semigroup, for all $t \in (0, T_{\max})$ one can estimate

$$\begin{aligned}
&\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \\
&\quad = \|\nabla e^{t(d_2\Delta-1)}v_0 + a\int_0^t \nabla e^{(t-s)(d_2\Delta-1)}u(\cdot,s)w(\cdot,s)ds - \gamma\int_0^t \nabla e^{(t-s)(d_2\Delta-1)}u(\cdot,s)v(\cdot,s)ds \\
&\qquad + (1-c)\nabla e^{(t-s)(d_2\Delta-1)}v(\cdot,s)ds\|_{L^q(\Omega)} \\
&\quad \leq c_1e^{-t}\|v_0\|_{L^q(\Omega)} + c_2a\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-(t-s)}\|u(\cdot,s)w(\cdot,s)\|_{L^\lambda(\Omega)}ds \\
&\qquad + c_2\gamma\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-(t-s)}\|u(\cdot,s)v(\cdot,s)\|_{L^\lambda(\Omega)}ds \\
&\qquad + c_2|1-c|\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-(t-s)}\|v(\cdot,s)\|_{L^\lambda(\Omega)}ds. \qquad (2.12)
\end{aligned}$$

Furthermore, by the Hölder inequality, since $\lambda < p$ we have

$$\begin{aligned}
\|u(\cdot,s)w(\cdot,s)\|_{L^\lambda(\Omega)} &\leq \|w(\cdot,s)\|_{L^p(\Omega)}^{a_1}\|w(\cdot,s)\|_{L^1}^{1-a_1}\|u(\cdot,s)\|_{L^\infty(\Omega)} \\
&\leq m_1^{1-a_1}\|w(\cdot,s)\|_{L^p(\Omega)}^{a_1}\|u(\cdot,s)\|_{L^\infty(\Omega)}, \qquad s\in(0,T_{\max})
\end{aligned}$$

with $a_1 = a_1(\varepsilon, p, q) := \frac{1-\frac{1}{\lambda}}{1-\frac{1}{p}} \in (0,1)$, and with $m_1 := \sup_{t\in(0,T_{\max})} \|w(\cdot,t)\|_{L^1(\Omega)}$ being finite according to Lemma 2.1. And the Gagliardo–Nirenberg inequality yields

$$\begin{aligned}\|v(\cdot,s)\|_{L^\lambda(\Omega)} &\le \|v(\cdot,s)\|_{L^r(\Omega)}\\ &\le c_3\|\nabla v(\cdot,s)\|^{a_2}_{L^q(\Omega)}\|v(\cdot,s)\|^{1-a_2}_{L^1(\Omega)} + c_3\|v(\cdot,s)\|_{L^1(\Omega)}\\ &\le c_3 m_2^{1-a_2}\|\nabla v(\cdot,s)\|^{a_2}_{L^q(\Omega)} + c_3 m_2\end{aligned}$$

with $a_2 \equiv a_2(\varepsilon, p, q) := \frac{n-\frac{n}{\lambda}}{n+1-\frac{n}{q}} \in (0,1)$, and $\lambda < r$ which is given in Lemma 2.7.

Therefore, for all $t \in (0, T_{\max})$, (2.12) can be simplified as follows

$$\begin{aligned}\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le{}& c_1\|v_0\|_{W^{1,\infty}(\Omega)} + ac_2 m_3^{1-a_1}\cdot\left\{\sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^p(\Omega)}\right\}^{a_1}\\ &\cdot\left\{\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\right\}\cdot\int_0^t (1+(t-s)^{-\frac12-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-(t-s)}ds\\ &+(c_2\gamma + c_2|1-c|)\left(c_3 m_2^{1-a_2}\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|^{a_2}_{L^q(\Omega)} + c_3 m_2\right)\\ &\cdot\left\{\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\right\}\cdot\int_0^t (1+(t-s)^{-\frac12-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-(t-s)}ds.\end{aligned}$$

Noting that for all $t > 0$ we have

$$\begin{aligned}\int_0^t (1+(t-s)^{-\frac12-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-(t-s)}ds \le c_4(\varepsilon,p,q) &:= \int_0^t (1+\sigma^{-\frac12-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-\sigma}d\sigma\\ &= \Gamma\left(\frac12 - \frac{n}{2}\left(\frac{1}{\lambda}-\frac{1}{q}\right)\right),\end{aligned}$$

that $c_4 < \infty$ thanks to the inequality $\frac{1}{\lambda} < \frac{1}{q} + \frac{1}{n}$ contained in (2.11), and then

$$a_1 = \frac{1-\left\{\frac{1}{q}+\frac{1}{n}-\left(1-\frac{1}{p}\right)\varepsilon\right\}}{1-\frac{1}{p}} = \frac{n-1-\frac{n}{q}}{n\left(1-\frac{1}{p}\right)} + \varepsilon,$$

we conclude as intended. □

Combining the previous two lemmata allows us to eliminate the dependence on u in (2.10) as follows.

Lemma 2.4. *Let $2 \le n < 5$. Assume that $p \in (1,\infty]$ and that $q > \frac{n}{n-1}$ satisfy $q > \frac{n}{5-n}$ and $(n-p)q < np$. Then for all $\varepsilon > 0$ there exists $C(\varepsilon,p,q) > 0$ with the property that*

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le C(\varepsilon,p,q)\cdot\left\{1+\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^p(\Omega)}\right\}^{\frac{(n-1-\frac{n}{q})(n+1-\frac{n}{q})}{n(1-\frac{1}{p})(5-n-\frac{n}{q})}+\varepsilon}, \qquad t\in(0,T_{\max}). \tag{2.13}$$

Proof. We note that $n+1-\frac{n}{q} > 2(n-2)$ since the assumption that $q > \frac{n}{5-n}$, and that $n-1-\frac{n}{q} > 0$ due to $q > \frac{n}{n-1}$. Then, there exists $\tilde{\varepsilon} = \tilde{\varepsilon}(p,q) > 0$ such that

$$\theta(\varepsilon_1) := \left\{\frac{n-1-\frac{n}{q}}{n(1-\frac{1}{p})} + \varepsilon\right\}\cdot\frac{n+1-\frac{n}{q}}{(n+1-\frac{n}{q})(1-2\varepsilon_1)-2(n-2)}$$

is well-defined for all $\varepsilon_1 \in (0, \tilde{\varepsilon})$, with

$$\theta(\varepsilon_1) \to \theta_0 := \frac{(n-1-\frac{n}{q})(n+1-\frac{n}{q})}{n(1-\frac{1}{p})(5-n-\frac{n}{q})} \quad \text{as } \varepsilon_1 \searrow 0.$$

For $\varepsilon > 0$, we can find $\varepsilon_1 = \varepsilon_1(\varepsilon, p, q) \in (0, \tilde{\varepsilon})$ such that

$$\theta(\varepsilon_1) \le \theta_0 + \varepsilon, \tag{2.14}$$

and then from Lemma 2.2 and Lemma 2.3 provide $c_1 = c_1(\varepsilon, q) > 0$ and $c_2 = c_2(\varepsilon, p, q) > 0$ such that

$$L(t) := 1 + \sup_{s\in(0,t)} \|w(\cdot, s)\|_{L^p(\Omega)}, \qquad t \in (0, T_{\max}),$$

and

$$M(t) := \sup_{s\in(0,t)} \|\nabla v(\cdot, s)\|_{L^q(\Omega)}, \qquad t \in (0, T_{\max}),$$

as well as

$$N(t) := \sup_{s\in(0,t)} \|u(\cdot, s)\|_{L^\infty}(\Omega), \qquad t \in (0, T_{\max}),$$

satisfy

$$N(t) \le c_1 + c_1 M^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon_1}(t), \qquad t \in (0, T_{\max}) \tag{2.15}$$

and

$$M(t) \le c_2 + c_2 L^{\frac{n-1-\frac{n}{q}}{n(1-\frac{1}{p})}+\varepsilon_1}(t) M(t) + c_2 M^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon_1} N(t) + c_2 M^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon_1}, \qquad t \in (0, T_{\max}). \tag{2.16}$$

In the case of $t \in (0, T_{\max})$ and $M(t) \ge 1$, from (2.15) we obtain that

$$N(t) \le 2c_1 M^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon_1}(t)$$

and by (2.16),

$$\begin{aligned} M(t) &\le c_2 + 2c_1c_2 L^{\frac{n-1-\frac{n}{q}}{n(1-\frac{1}{p})}+\varepsilon_1}(t) M^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon_1}(t) + 2c_1c_2 M^{\frac{2(n-2)}{n+1-\frac{n}{q}}+2\varepsilon_1}(t) + c_2 M^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon_1}(t) \\ &\le (2c_2 + 4c_1c_2) L^{\frac{n-1-\frac{n}{q}}{n(1-\frac{1}{p})}+\varepsilon_1}(t) M^{\frac{2(n-2)}{n+1-\frac{n}{q}}+2\varepsilon_1}(t), \end{aligned}$$

because $L(t) \ge 1$ by definition. Since for any such t we therefore have

$$M^{1-2\varepsilon_1-\frac{2(n-2)}{n+1-\frac{n}{q}}}(t) \le (2c_2 + 4c_1c_2) L^{\frac{n-1-\frac{n}{q}}{n(1-\frac{1}{p})}+\varepsilon_1}(t),$$

and since

$$1 - 2\varepsilon_1 - \frac{2(n-2)}{n+1-\frac{n}{q}} = \frac{(n+1-\frac{n}{q})(1-2\varepsilon_1) - 2(n-2)}{n+1-\frac{n}{q}} > 0$$

by positivity of $\theta(\varepsilon_1)$, from this we can infer that actually for arbitrary $t \in (0, T_{\max})$, regardless of the sign of $M(t) - 1$,

$$M(t) \le c_3 L^{\theta(\varepsilon_1)}(t)$$

with $c_3 \equiv c_3(\varepsilon, p, q) := \max\left\{1, (2c_2 + 4c_1c_2)^{\frac{n+1-\frac{n}{q}}{(n+1-\frac{n}{q})(1-2\varepsilon_1)-2(n-2)}}\right\} > 0$. Once again since $L(t) \ge 0$ for all $t \in (0, T_{\max})$, in view of (2.14) this establishes (2.13). □

Lemma 2.5. *Let $n = 2$. Then whenever $p \in (\frac{n}{n-1}, \infty]$ and $q > n$, for all $\varepsilon > 0$ there exists $C(\varepsilon, p, q) > 0$ such that*

$$\|w(\cdot,t)\|_{L^p(\Omega)} \leq C(\varepsilon,p,q) + C(\varepsilon,p,q) \cdot \left\{\sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|_{L^q(\Omega)}\right\}^{\frac{1-\frac{1}{p}}{\frac{2}{n}-\frac{1}{q}}+\varepsilon} \tag{2.17}$$

for all $t \in (0, T_{\max})$.

Proof. Firstly, we observe that $\frac{1}{q} < \frac{1}{n} < \frac{1}{n} + \frac{1}{p} < 1$ thanks to the assumption that $p > \frac{n}{n-1}$ and $q > n$. Then the interval $J_1 := (\frac{1}{q}, \frac{1}{n} + \frac{1}{p}]$ is not empty and

$$\psi_1(\zeta) := \frac{1}{\zeta - \frac{1}{q}}, \qquad \zeta \in J_1,$$

defines a positive function ψ_1 on J_1 which satisfies

$$\frac{\psi_1(\frac{1}{n} + \frac{1}{p})}{p} = \frac{\frac{1}{p}}{\frac{1}{n} + \frac{1}{p} - \frac{1}{q}} < \frac{\frac{1}{p}}{\frac{1}{q} + \frac{1}{p} - \frac{1}{q}} = 1. \tag{2.18}$$

Next, since $q > n$ together with the inequality $p \geq 1$ infer that $\frac{1}{p} + \frac{1}{q} < \frac{1}{n} + \frac{1}{p}$, similarly, it follows that $J_2 := (\frac{1}{p} + \frac{1}{q}, \frac{1}{n} + \frac{1}{p}] \neq \emptyset$, and

$$\psi_2(\zeta) := \frac{1 - \frac{1}{p}}{\zeta - \frac{1}{p} - \frac{1}{q}}, \qquad \zeta \in J_2,$$

is well-defined and nonnegative with

$$\psi_2\left(\frac{1}{n} + \frac{1}{p}\right) = \frac{1 - \frac{1}{p}}{\frac{1}{n} - \frac{1}{q}}. \tag{2.19}$$

According to (2.18), (2.19) and continuity of ψ_1 and ψ_2, we thereby see that for any $\varepsilon > 0$ it is possible to pick $\zeta = \zeta(\varepsilon, p, q) \in J_1 \cap J_2 = J_2$ such that $\zeta < \frac{1}{n} + \frac{1}{p}$ and that $\psi_1(\zeta) < p$ as well as $\psi_2(\zeta) \leq \frac{1-\frac{1}{p}}{\frac{2}{n}-\frac{1}{q}} + \varepsilon$, where we can clearly moreover achieve that $\zeta > \frac{1}{p}$.

Setting $\mu \equiv \mu(\varepsilon, p, q) := \frac{1}{\zeta}$, we can find a positive number μ simultaneously fulfilling

$$\mu < p, \qquad \mu < q, \qquad \frac{1}{\mu} > \frac{1}{p} + \frac{1}{q}, \qquad \frac{1}{\mu} < \frac{1}{n} + \frac{1}{q}, \quad \text{and} \quad \frac{1}{\mu} < \frac{1}{n} + \frac{1}{p} \tag{2.20}$$

as well as

$$\frac{q\mu}{q - \mu} < p \tag{2.21}$$

and

$$\frac{1 - \frac{1}{p}}{\frac{1}{\mu} - \frac{1}{p} - \frac{1}{q}} \leq \frac{1 - \frac{1}{p}}{\frac{2}{n} - \frac{1}{q}} + \varepsilon. \tag{2.22}$$

Furthermore, $\mu > 1$ since $p > \frac{n}{n-1}$ and the rightmost property in (2.20).

Keeping this parameter μ fixed henceforth, using a Duhamel representation, for all $t \in (0, T_{\max})$, we can estimate

$$\begin{aligned}
&\|\nabla u(\cdot,t)\|_{L^q(\Omega)} \\
&= \|\nabla e^{t(d_1\Delta-1)}u_0 + \int_0^t \nabla e^{(t-s)(d_1\Delta-1)}v(\cdot,s)ds\|_{L^q(\Omega)} \\
&\le c_2 e^{-t}\|u_0\|_{L^q(\Omega)} + c_3\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{q})})e^{-(t-s)}\|v(\cdot,s)\|_{L^\mu(\Omega)}ds \\
&\le c_4 + \left\{c_4 \sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|^{a_0}_{L^q(\Omega)}\|v(\cdot,s)\|^{1-a_0}_{L^1(\Omega)} + c_4\|v(\cdot,s)\|_{L^1(\Omega)}\right\} \\
&\quad \cdot \left(1+\Gamma\left(\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\mu}-\frac{1}{q}\right)\right)\right) \\
&\le c_4 + \left(c_4 m_2^{1-a_0}\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|^{a_0}_{L^q(\Omega)} + c_4 m_2\right)\left(1+\Gamma\left(\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\mu}-\frac{1}{q}\right)\right)\right) \\
&\le c_5\left(1+\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|^{a_0}_{L^q(\Omega)}\right) \\
&\le c_6\left(1+\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}\right)^{a_0} \\
&\le c_6 + c_6\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}
\end{aligned}$$

where $a_0 = \frac{n-\frac{n}{\mu}}{n+1-\frac{n}{q}} \in (0,1)$ since $q > n$, and $\Gamma(\frac{1}{2}-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{q})) < \infty$ due to $\frac{1}{\mu} < \frac{1}{n}+\frac{1}{q}$. Apart from that, by the first inequality in (2.20) and regularization features of the Neumann heat semigroup ([25, Lemma 1.3], [29, Lemma 3.3]) one can pick $c_1 = c_1(\varepsilon, p, q) > 0$ satisfying

$$\|e^{t\Delta}\nabla\cdot\phi\|_{L^p(\Omega)} \le c_1\left(1+t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{p})}\right)\|\phi\|_{L^\mu(\Omega)}$$

for all $t > 0$ and each $\phi \in C^1(\bar{\Omega};\mathbb{R}^n)$ such that $\phi\cdot\nu = 0$ on $\partial\Omega$, which shows that for all $t \in (0, T_{\max})$,

$$\begin{aligned}
&\int_0^t \|e^{(t-s)(d_3\Delta-1)}\nabla\cdot(w(\cdot,s)\nabla u(\cdot,s))\|_{L^p(\Omega)}ds \\
&\qquad \le c_1\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{p})})e^{-(t-s)}\|w(\cdot,s)\nabla u(\cdot,s)\|_{L^\mu(\Omega)}ds. \qquad (2.23)
\end{aligned}$$

Hence due to the second relation in (2.20), we may employ the Hölder inequality shows that again writing $L(t) := 1+\sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^p(\Omega)}$ and $M(t) := \sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}$, $t \in (0, T_{\max})$, for any such t we have

$$\begin{aligned}
\|w(\cdot,s)\nabla u(\cdot,s)\|_{L^\mu(\Omega)} &\le \|w(\cdot,s)\|^{\alpha}_{L^p(\Omega)}\|w(\cdot,s)\|^{1-\alpha}_{L^1(\Omega)}\|\nabla u(\cdot,s)\|_{L^q(\Omega)} \\
&\le m_1^{1-\alpha}\|w(\cdot,s)\|^{\alpha}_{L^p(\Omega)}(c_6 + c_6\|\nabla v(\cdot,s)\|_{L^q(\Omega)}) \\
&\le c_6 m_1^{1-\alpha}L^{\alpha}(t) + c_6 m_1^{1-\alpha}L^{\alpha}(t)M(t), \qquad s \in (0,t)
\end{aligned}$$

with $\alpha = \alpha(\varepsilon, p, q) := \frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}} \in (0,1)$.

The relation (2.23) indicates that with some $c_7 = c_7(\varepsilon, p, q) > 0$,

$$\int_0^t \|e^{(t-s)(d_3\Delta-1)}\nabla\cdot(w(\cdot,s)\nabla u(\cdot,s))\|_{L^p(\Omega)}ds \le c_7 L^{\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}}}(t) + c_7 L^{\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}}}(t)M(t) \tag{2.24}$$

for all $t \in (0, T_{\max})$. In order to make appropriate use of this, we observe that from the third equation of (1.4),

$$w_t \le d_3\Delta w - w + \xi\cdot\nabla(w\nabla u) + (r+1)w \quad \text{in } \Omega\times(0,T_{\max}).$$

In view of the nonnegativity of w and an associated variation-of-constants formula, one can obtain that

$$\begin{aligned}
&\|w(\cdot,t)\|_{L^p(\Omega)}\\
&\le \left\| e^{t(d_3\Delta-1)}w_0 + \xi\int_0^t e^{(t-s)(d_3\Delta-1)}\nabla\cdot(w(\cdot,s)\nabla u(\cdot,s))ds + (r+1)\int_0^t e^{(t-s)(d_3\Delta-1)}w ds\right\|_{L^p(\Omega)}\\
&\le e^{-t}\|w_0\|_{L^p(\Omega)} + |\xi|\int_0^t \|e^{(t-s)(d_3\Delta-2)}\nabla\cdot(w(\cdot,s)\nabla u(\cdot,s))\|_{L^p(\Omega)}ds\\
&\quad + (r+1)c_8\int_0^t (1+t^{-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{p})})e^{-(t-s)}\|w(\cdot,s)\|_{L^\mu(\Omega)}ds, \qquad t\in(0,T_{\max}).
\end{aligned} \tag{2.25}$$

Using the Hölder inequality, we have

$$\|w(\cdot,s)\|_{L^\mu(\Omega)} \le \|w(\cdot,s)\|^{\beta}_{L^p(\Omega)}\|w(\cdot,s)\|^{1-\beta}_{L^1(\Omega)} \le m_1^{1-\beta}\|w(\cdot,s)\|^{\beta}_{L^p(\Omega)},$$

where $\beta = \frac{1-\frac{1}{\mu}}{1-\frac{1}{p}}$. Therefore, by the Young inequality, we obtain that

$$\begin{aligned}
&(r+1)c_8\int_0^t (1+t^{-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{p})})e^{-(t-s)}\|w(\cdot,s)\|_{L^\mu(\Omega)}ds\\
&\le (r+1)c_8 m_1^{1-\beta}\left\{\sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^p(\Omega)}\right\}^{\beta}\left(1+\Gamma\left(1-\frac{n}{2}\left(\frac{1}{\mu}-\frac{1}{p}\right)\right)\right)\\
&\le \frac{1}{2}\sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^p(\Omega)} + c_9
\end{aligned}$$

where $c_9 = \frac{1}{2}(r+1)c_8 m_1^{1-\beta}(1+\Gamma(1-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{p})))$ and $\Gamma(1-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{p}))$ is positive and real-valued due to $\frac{1}{\mu} < \frac{2}{n}+\frac{1}{p}$.

In conjunction with (2.25) and (2.24), this infers the existence of $c_{10} = c_{10}(\varepsilon, p, q) > 0$ such that

$$L(t) \le c_{10} + c_{10}L^{\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}}}(t) + c_{10}L^{\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}}}(t)M(t), \qquad t\in(0,T_{\max}),$$

where the third inequality in (2.20) ensures that $\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}} < 1$ and Young inequality so as to provide

$$c_{10}L^{\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}}} \le \frac{1}{4}L(t) + c_{11},$$

and

$$c_{10}L^{\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}}}M(t) \le \frac{1}{4}L(t) + c_{12}M^{\frac{1-\frac{1}{p}}{\frac{1}{\mu}-\frac{1}{p}-\frac{1}{q}}}(t), \qquad t \in (0, T_{\max}).$$

In light of (2.22), this yields (2.17). □

3 Proof of Theorem 1.1

3.1 Boundedness when $n = 2$

Lemma 3.1. *Let $n = 2$. Then there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C, \qquad t \in (0, T_{\max}). \tag{3.1}$$

Proof. Without loss of generality assuming that $p < n$. Let

$$\theta(\zeta,\varepsilon) := \left\{\frac{1-\frac{1}{p}}{\frac{2}{n}-\zeta} + \varepsilon\right\}\left\{\frac{(n-1-n\zeta)(n+1-n\zeta)}{n(1-\frac{1}{p})(5-n-n\zeta)} + \varepsilon\right\},$$
$$\zeta \in J := \left(0, \frac{n-1}{n}\right], \qquad \varepsilon > 0,$$

noting that θ is well-defined because $\frac{n-1}{n} < \frac{5-n}{n}$. Since evidently $\theta(\frac{n-1}{n}, 0) = 0$, and since apart from that clearly $\frac{1}{p} - \frac{1}{n} < \frac{n-1}{n}$, by means of a continuity argument we can choose $\zeta \in J$ and $\varepsilon > 0$ such that $\zeta < \frac{n-1}{n}$ and

$$\zeta > \frac{1}{p} - \frac{1}{n} \tag{3.2}$$

and that

$$\theta(\zeta,\varepsilon) < 1, \tag{3.3}$$

and thus $\zeta < \frac{1}{n}$. Writing $q := \frac{1}{\zeta}$, therefore one can find that $q > \frac{n}{n-1}$ and $(n-p)q < np$ as well as $q > n$, where the latter relation together with the inequality $p > \frac{n}{n-1}$ enables us to invoke Lemma 2.5, thus inferring the existence of $c_1 > 0$ such that for $L(t) := 1 + \sup_{s\in(0,t)} \|w(\cdot,s)\|_{L^p(\Omega)}$ and $M(t) := \sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|_{L^q(\Omega)}$, $t \in (0, T_{\max})$, we have

$$L(t) \le c_1 + c_1 M^{\frac{1-\frac{1}{p}}{\frac{2}{n}-\frac{1}{q}}+\varepsilon}(t), \qquad t \in (0, T_{\max}). \tag{3.4}$$

On the other hand, using that $(n-p)q < np$ and $q > \frac{n}{n-1}$, and that thus also $q > \frac{n}{5-n}$, we may employ Lemma 2.4 to find $c_2 > 0$ such that

$$M(t) \le c_2 L^{\frac{(n+1-\frac{n}{q})(n-1-\frac{n}{q})}{n(1-\frac{1}{p})(5-n-\frac{n}{q})}+\varepsilon}(t), \qquad t \in (0, T_{\max}). \tag{3.5}$$

Combined with (3.4), this provides that

$$L(t) \le c_1 + c_1 c_2^{\frac{1-\frac{1}{p}}{\frac{2}{n}-\frac{1}{q}}} L^{\theta(\frac{1}{q},\varepsilon)}(t), \qquad t \in (0, T_{\max})$$

and thus shows that with some $c_3 > 0$ we have

$$L(t) \le c_3, \qquad t \in (0, T_{\max}),$$

because $\theta(\frac{1}{q}, \varepsilon) < 1$ by (3.3). Through (3.5), the latter entails boundedness of $(0, T_{\max}) \ni t \mapsto \|\nabla v(\cdot,t)\|_{L^q(\Omega)}$, so that Lemma 2.2 establishes the claim. □

Lemma 3.2. *Let $n = 2$. Then for all $q > n$ there exists $C(q) > 0$ fulfilling*

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le C(q), \qquad t \in (0, T_{\max}). \tag{3.6}$$

Proof. For each fixed $q > n$,

$$\frac{n-1-\frac{n}{q}}{n(\frac{2}{n}-\frac{1}{q})} = \frac{n-1-\frac{n}{q}}{2-\frac{n}{q}} < 1,$$

by a continuity argument we can pick $\varepsilon = \varepsilon(q) > 0$ appropriately small such that still

$$\theta := \left\{ \frac{1}{\frac{2}{n}-\frac{1}{q}} + \varepsilon \right\} \cdot \left\{ \frac{n-1-\frac{n}{q}}{n} + \varepsilon \right\} < 1.$$

Then from Lemma 3.1, we may employ Lemma 2.3 with $p := \infty$ to find $c_1 = c_1(q) > 0$ such that $L(t) := 1 + \sup_{s\in(0,t)} \|w(\cdot,s)\|_{L^p(\Omega)}$ and $M(t) := \sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|_{L^q(\Omega)}$, $t \in (0, T_{\max})$, satisfy

$$M(t) \le c_1 L^{\frac{n-1-\frac{n}{q}}{n}}(t), \qquad t \in (0, T_{\max}) \tag{3.7}$$

which we combine with the outcome of Lemma 2.5, applicable since the inequality $q > n$, which namely yields $c_2 = c_2(q) > 0$ fulfilling

$$L(t) \le c_2 + c_2 M^{\frac{1}{\frac{2}{n}-\frac{1}{q}}+\varepsilon}(t), \qquad t \in (0, T_{\max}).$$

Therefore

$$L(t) \le c_2 + c_1^{\frac{1}{\frac{2}{n}-\frac{1}{q}}+\varepsilon} c_2 L^\theta(t), \qquad t \in (0, T_{\max}),$$

so that the inequality $\theta < 1$ guarantees boundedness of L and thus, by (3.7), also derives boundedness of M. □

3.2 Boundedness in the one-dimensional case

Lemma 3.3. *Let $n = 1$. Then for all $q > 1$ there exists $C(q) > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot,t)\|_{L^q(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \le C(q), \qquad t \in (0, T_{\max}). \tag{3.8}$$

Proof. In view of the boundedness of $(0, T_{\max}) \ni t \mapsto \|v(\cdot,t)\|_{L^1(\Omega)}$ asserted by Lemma 2.1, straightforward application of L^1–L^∞ smoothing estimates for the Neumann heat semigroup in the present one-dimensional situation entails $c_1 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le c_1, \qquad t \in (0, T_{\max}), \tag{3.9}$$

which again thanks to Lemma 2.1 ensures boundedness of $(0, T_{\max}) \ni t \mapsto \|u(\cdot,t)w(\cdot,t)\|_{L^1(\Omega)}$ and $(0, T_{\max}) \ni t \mapsto \|u(\cdot,t)v(\cdot,t)\|_{L^1(\Omega)}$. Accordingly, standard L^∞–$W^{1,q}$ regularization properties of $(e^{t\Delta})_{t\ge0}$ guarantee the existence of $c_2 = c_2(q) > 0$ fulfilling

$$\|v_x(\cdot,t)\|_{L^q(\Omega)} \le c_2, \qquad t \in (0, T_{\max}), \tag{3.10}$$

therefore $\|u_x(\cdot,t)\|_{L^q(\Omega)} \le c_3$.

To establish $L^\infty(\Omega)$ bound for w, we can find some $\mu = \mu(q) \in (1, q)$ for any q, and again combine the maximum principle with a known smoothing feature of the heat semigroup to fix $c_4, c_5 > 0$ such that

$$\begin{aligned}\|w(\cdot,t)\|_{L^\infty(\Omega)} \le \|e^{t(d_3\Delta-1)}w_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(d_3\Delta-1)}\partial_x(w(\cdot,s)u_x(\cdot,s)\|_{L^\infty(\Omega)}ds \\ + (r+1)\int_0^t \|e^{(t-s)(d_3\Delta-1)}w(\cdot,s)\|_{L^\infty(\Omega)}ds \\ \le e^{-t}\|w_0\|_{L^\infty(\Omega)} + c_4\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{1}{2\mu}})e^{-(t-s)}\|w(\cdot,s)u_x(\cdot,s)\|_{L^\mu(\Omega)}ds \\ + c_5\int_0^t (1+(t-s)^{-\frac{n}{2\mu}})e^{-(t-s)}\|w(\cdot,s)\|_{L^\mu(\Omega)}ds, \qquad t\in(0,T_{\max}), \end{aligned} \tag{3.11}$$

where by the Hölder inequality, for all $s \in (0, T_{\max})$ one can estimate

$$\begin{aligned}\|w(\cdot,s)u_x(\cdot,s)\|_{L^\mu(\Omega)} \le \|w(\cdot,s)\|_{L^{\frac{\mu q}{q-\mu}}(\Omega)}\|u_x(\cdot,s)\|_{L^q(\Omega)} \\ \le \|w(\cdot,s)\|^{\gamma}_{L^\infty(\Omega)}\|w(\cdot,s)\|^{1-\gamma}_{L^1(\Omega)}\|u_x(\cdot,s)\|_{L^q(\Omega)}\end{aligned}$$

with $\gamma := \frac{\mu q - q + \mu}{\mu q} \in (0,1)$ since $q > \mu$. And

$$\|w(\cdot,s)\|_{L^\mu(\Omega)} \le \|w(\cdot,s)\|^{\delta}_{L^\infty(\Omega)}\|w(\cdot,s)\|^{1-\delta}_{L^1(\Omega)} \le c_6\|w(\cdot,s)\|_{L^\infty(\Omega)} + c_7$$

where $c_6 := \frac{1}{2c_5 m^{1-\delta}(1+\Gamma(1-\frac{n}{2\mu}))}$, $c_7 := \frac{1}{4c_6}$. In view of (3.10) and Lemma 2.1, from (3.11) we thus infer the existence of $c_8, c_9 > 0$ such that if now we let $L(t) := 1 + \sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^\infty(\Omega)}$, $t \in (0, T_{\max})$, then

$$L(t) \le c_8 + c_8 \cdot \left\{\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{1}{2\mu}})e^{-(t-s)}ds\right\} \cdot L^\gamma(t) + \frac{1}{2}L(t)$$

thus

$$L(t) \le 2c_8 + 2c_8c_9L^\delta(t), \qquad t \in (0, T_{\max}),$$

where $c_9 \le \int_0^\infty (1+\sigma^{-\frac{1}{2}-\frac{1}{2\mu}})e^{-\sigma}d\sigma = 1 + \Gamma(\frac{1}{2} - \frac{1}{2\mu})$ is finite since $\mu > 1$. As $\gamma < 1$, this indicates boundedness of w and hence completes the proof. □

3.3 Proof of Theorem 1.1

Proof of Theorem 1.1. Using (2.3)–(2.5) and Lemma 3.3 when $n = 1$; combining Lemma 3.1 and Lemma 3.2 when $n = 2$, the conclusion of Theorem 1.1 is obtained immediately. □

4 Dynamical behavior of prey-evasion system

In this section, we investigate the dynamic behavior of the system (1.4). We first consider the local stability of the constant equilibrium solutions by linearized stability analysis. According to the principle of linearized stability for quasi-linear parabolic problems (see [21] Th 8.6, [6] Th 5.2), we know that the constant equilibrium $(\tilde{u}, \tilde{v}, \tilde{w})$ is locally asymptotically stable with respect to (1.4) if and only if all the eigenvalues of the linearized elliptic problem of (1.4) at an equilibrium are of negative real parts. To this end, we introduce the asymptotic stability of $(\tilde{u}, \tilde{v}, \tilde{w})$ of kinetic system (1.2) in [3].

Proposition 4.1. *Suppose that $ar > c$. Let*

$$f(\tilde{w}) = a(a+1)\tilde{w}^3 + (a^2+3a+1)\tilde{w}^2 + (a+1-ac-ar)\tilde{w} - c. \tag{4.1}$$

Then there exists a unique γ^*, such that $\tilde{\mathbf{u}}$ is asymptotically stable if $\gamma > \gamma^*$ and is unstable if $0 < \gamma < \gamma^*$, where $\gamma^* = \frac{a\tilde{w}-c}{r-\tilde{w}}$, $f(\tilde{w}) = 0$.

Linearizing the system (1.4) at an equilibrium solution (u, v, w), we obtain that

$$\begin{pmatrix} \varphi_t \\ \phi_t \\ \psi_t \end{pmatrix} = \mathcal{L}(\xi) \begin{pmatrix} \varphi \\ \phi \\ \psi \end{pmatrix} = D \begin{pmatrix} \Delta\varphi \\ \Delta\phi \\ \Delta\psi \end{pmatrix} + J_{(u,v,w)} \begin{pmatrix} \varphi \\ \phi \\ \psi \end{pmatrix} \tag{4.2}$$

where

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ \xi w & 0 & d_3 \end{pmatrix}, \qquad J_{(u,v,w)} = \begin{pmatrix} -1 & 1 & 0 \\ aw - \gamma v & -\gamma u - cv & au \\ -w & 0 & r - 2w - u \end{pmatrix}. \tag{4.3}$$

The stability of $\tilde{\mathbf{u}}$ is determined by the following eigenvalue problem

$$\mathcal{L} \begin{pmatrix} \varphi \\ \phi \\ \psi \end{pmatrix} = \lambda \begin{pmatrix} \varphi \\ \phi \\ \psi \end{pmatrix},$$

that is

$$\begin{cases} d_1\Delta\varphi - \varphi + \phi = \lambda\varphi, & x \in \Omega, \\ d_2\Delta\phi + (aw - \gamma v)\varphi - (\gamma u + c)\phi + au\psi = \lambda\phi, & x \in \Omega, \\ \xi w\Delta\varphi + d_3\Delta\psi - w\varphi + (r - 2w - u)\psi = \lambda\psi, & x \in \Omega, \\ \dfrac{\partial\varphi}{\partial\nu} = \dfrac{\partial\phi}{\partial\nu} = \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.4}$$

Let $-\Delta$ have eigenvalues $0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots$ and $\lim_{i\to\infty} = \infty$ under the Neumann boundary condition, and let $y_i(x)$ be the normalized eigenfunction corresponding to μ_i. Suppose that λ is an eigenvalue of (4.4) with corresponding eigenfunction (φ, ϕ, ψ), therefore according to the Fourier expansion, there exists $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ such that

$$\varphi(x) = \sum_{i=0}^{\infty} a_i\varphi_i(x), \qquad \phi(x) = \sum_{i=0}^{\infty} b_i\phi_i(x), \qquad \psi(x) = \sum_{i=0}^{\infty} c_i\psi_i(x).$$

By a straightforward computation, we have

$$\mathcal{L}_i(\xi) \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \lambda \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}, \qquad i = 0, 1, 2, \ldots$$

with

$$\mathcal{L}_i(\xi) = \begin{pmatrix} -d_1\mu_i - 1 & 1 & 0 \\ aw - \gamma v & -d_2\mu_i - \gamma u - c & au \\ -\xi w\mu_i - w & 0 & -d_3\mu_i + r - 2w - u \end{pmatrix}. \tag{4.5}$$

Therefore, the local stability of positive constant steady states of the system (1.4) is given by the following lemma.

Lemma 4.2. *Assume that* $ar > c, \gamma > \gamma^*, d_i > 0\ (i = 1, 2, 3),\ \xi > 0$. *Then for system* (1.4), $(\tilde{u}, \tilde{v}, \tilde{w})$ *is locally asymptotically stable if* $0 < \xi < \xi_0$ *and is unstable if* $\xi > \xi_0$, *where*

$$\xi_0 = \frac{1}{a\tilde{u}\tilde{w}\mu_i}(\beta_1\mu_i^3 + \beta_2\mu_i^2 + \beta_3\mu_i + \beta_4) > 0,$$

$\beta_i\ (i = 1, 2, 3, 4)$ *will be given in the following proof.*

Proof. If constant equilibrium solution $(u, v, w) = (\tilde{u}, \tilde{v}, \tilde{w})$, then

$$\mathcal{L}_i(\xi) = \begin{pmatrix} -d_1\mu_i - 1 & 1 & 0 \\ c & -d_2\mu_i - a\tilde{w} & a\tilde{u} \\ -\xi\tilde{w}\mu_i - \tilde{w} & 0 & -d_3\mu_i - \tilde{w} \end{pmatrix}, \tag{4.6}$$

and the characteristic equation of $\mathcal{L}_i$ is

$$\Phi(\lambda) = |\lambda I - \mathcal{L}_i| = \lambda^3 + \alpha_1(\xi)\lambda^2 + \alpha_2(\xi)\lambda + \alpha_3(\xi) = 0 \tag{4.7}$$

with

$$\begin{aligned} \alpha_1 &= (d_1 + d_2 + d_3)\mu_i + a\tilde{w} + \tilde{w} + 1, \\ \alpha_2 &= (d_1d_2 + d_1d_3 + d_2d_3)\mu_i^2 + ((d_1 + d_3)a\tilde{w} + (d_1 + d_2)\tilde{w} + d_2 + d_3)\mu_i + a\tilde{w}^2 + \gamma\tilde{u} + \tilde{w}, \quad (4.8) \\ \alpha_3 &= d_1d_2d_3\mu_i^3 + (d_1d_3a\tilde{w} + d_1d_2\tilde{w} + d_2d_3)\mu_i^2 + (d_1a\tilde{w}^2 + a\tilde{u}\tilde{w}\xi + d_3\gamma\tilde{u} + d_2\tilde{w})\mu_i + (ar - c)\tilde{w}. \end{aligned}$$

Obviously, $\alpha_j > 0\ (j = 1, 2, 3)$ for all $i = 0, 1, 2, \ldots,$ and

$$B(\xi) := \alpha_1\alpha_2 - \alpha_3 = \beta_1\mu_i^3 + \beta_2\mu_i^2 + (\beta_3 - a\tilde{u}\tilde{w}\xi)\mu_i + \beta_4,$$

where

$$\begin{aligned} \beta_1 &= (d_1 + d_3)(d_1 + d_2)(d_2 + d_3), \\ \beta_2 &= (d_1 + d_3)(d_1 + 2d_2 + d_3)a\tilde{w} + (d_1 + d_2)(d_1 + d_2 + 2d_3)\tilde{w} + (d_2 + d_3)(2d_1 + d_2 + d_3), \\ \beta_3 &= (d_1 + d_3)a^2\tilde{w}^2 + 2(d_1 + d_2 + d_3)a\tilde{w}^2 + (d_1 + d_2 + 2d_3)a\tilde{w} + (d_1 + d_2)(\gamma\tilde{u} + \tilde{w}^2) \\ &\quad + 2(d_1 + d_2 + d_3)\tilde{w} + d_2 + d_3, \\ \beta_4 &= a(a + 1)\tilde{w}^3 + (a^2 + 3a + 1)\tilde{w}^2 + (a + 1 - ac - ar)\tilde{w} - c. \end{aligned}$$

It is easy to see that $B(\xi)$ is monotonically decreasing with respect to ξ, that is $B(\xi) > 0$ if $\xi < \xi_0$, on the contrary $B(\xi) < 0$ if $\xi > \xi_0$, where $B(\xi_0) = 0$ with

$$\xi_0 = \frac{1}{a\tilde{u}\tilde{w}\mu_i}(\beta_1\mu_i^3 + \beta_2\mu_i^2 + \beta_3\mu_i + \beta_4) > 0 \tag{4.9}$$

thanks to $\beta_4 = f(\tilde{w}) > 0$ when $\gamma > \gamma^*$. By the Routh–Hurwitz criterion or Corollary 2.2 in [16], the proof is completed, that is $(\tilde{u}, \tilde{v}, \tilde{w})$ is locally asymptotically stable if $0 < \xi < \xi_0$ and is unstable if $\xi > \xi_0$. □

To illustrate our analysis of Lemma 4.2, we present the following numerical example.

Example 4.3. For (1.4), let $n = 1, \Omega = (0, 7)$ and set

$$a = 2, \quad c = 1, \quad r = 2, \gamma = 0.5, \quad d_1 = 0.3, \quad d_2 = 0.2, \quad d_3 = 0.3.$$

Then the equilibrium point $(\tilde{u}, \tilde{v}, \tilde{w}) = (1.2, 1.2, 0.8)$. According to the Lemma 4.2, $(\tilde{u}, \tilde{v}, \tilde{w})$ is asymptotically stable if $\xi < \xi_0 = 8.06\ (k = 3)$, see Figure 4.1, and $(\tilde{u}, \tilde{v}, \tilde{w})$ is unstable if $\xi > \xi_0 = 8.06\ (k = 3)$, see Figure 4.2.

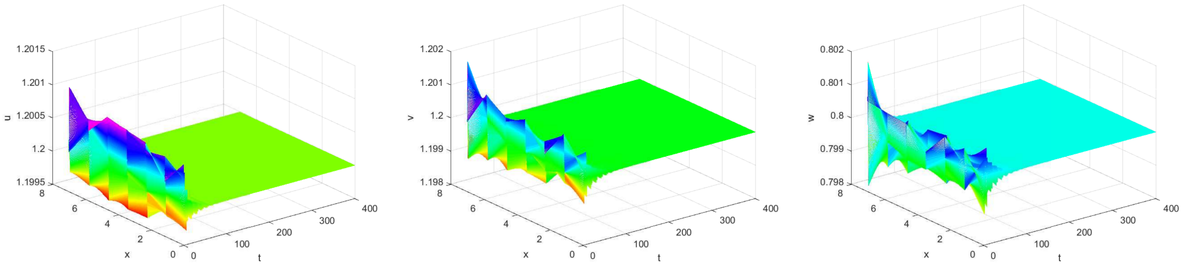

Figure 4.1: Stable behavior with $\chi = 7 < \chi_0 = 8.06$ for the model (1.4).

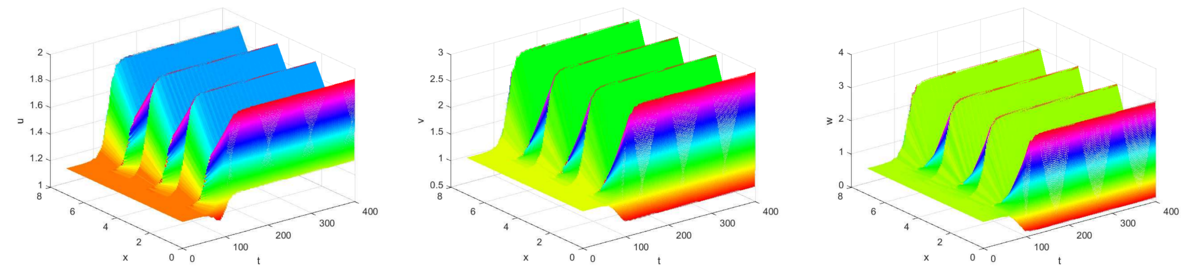

Figure 4.2: Unstable behavior with $\chi = 9 > \chi_0 = 8.06$ for the model (1.4).

Remark 4.4. Lemma 4.2 illustrates that prey-evasion has a destabilizing effect.

Remark 4.5. Lemma 4.2 implies that there is no steady state bifurcation curve near $(\tilde{u}, \tilde{v}, \tilde{w})$ since $\alpha_3 > 0$.

According to the proof of Lemma 4.2, we know that the linearized equation (4.4) has a pair of purely imaginary eigenvalues at $\xi = \xi_0$, then a Hopf bifurcation generating a family of periodic orbits of (1.4) occurs if some transversality conditions are met. We next show that the existence of periodic orbits of (1.4) for a certain parameter range.

To apply the Hopf bifurcation theorem (Theorem 6.1 of [16]), we first let the three roots of (4.6) be $\theta_{1,2} = \sigma(\xi) \pm i\delta(\xi)$ and θ_3 satisfying $\sigma(\xi_0) = 0$, $\delta(\xi_0) > 0$ when $\xi \in (\xi_0 - \varepsilon, \xi_0 + \varepsilon)$. From (4.7), we have

$$\begin{cases} -\alpha_1(\xi) = 2\sigma(\xi) + \theta_3(\xi), \\ \alpha_2(\xi) = \sigma^2(\xi) + \delta^2(\xi) + 2\sigma(\xi)\theta_3(\xi), \\ -\alpha_3(\xi) = (\sigma^2(\xi) + \delta^2(\xi))\theta_3(\xi). \end{cases} \tag{4.10}$$

Differentiating (4.10) with respect to ξ and using (4.8), we obtain

$$\begin{aligned} 2\sigma'(\xi) + \theta_3'(\xi) &= 0, \\ 2\sigma(\xi)\sigma'(\xi) + 2\delta(\xi)\delta'(\xi) + 2\sigma'(\xi)\theta_3(\xi) + 2\sigma(\xi)\theta_3'(\xi) &= 0, \\ (2\sigma(\xi)\sigma'(\xi) + 2\delta(\xi)\delta'(\xi))\theta_3(\xi) + (\sigma^2(\xi) + \delta^2(\xi))\theta_3'(\xi) &= -a\tilde{u}\tilde{w}\mu_i. \end{aligned} \tag{4.11}$$

Solving (4.11) with $\xi = \xi_0$ by Cramer's rule, we derive that

$$\theta_3'(\xi_0) = -\frac{a\tilde{u}\tilde{w}\mu_i}{\delta^2 + \theta_3^2} < 0,$$

and

$$\sigma'(\xi_0) = -\frac{1}{2}\theta_3'(\xi_0) > 0. \tag{4.12}$$

Moreover, it is easy to see that $\alpha_3 > 0$ for all $i \in \mathbb{N}$ if $\xi > 0$, then 0 cannot be an eigenvalue for (4.4) when $\xi = \xi_0$. Besides, in order to illustrate that $\theta = \pm i\delta(\xi_0)$ are a pair of simple eigenvalues of (4.4) for $\delta(\xi_0) > 0$, we need to assume that $\xi_{0k} \neq \xi_{0j}$, $j \neq k$. Then this shows that (4.4) has no eigenvalues of the form $k\delta(\xi_0)i$ for $k \in \mathbb{Z} \setminus \{\pm 1\}$.

Therefore the existence of nontrivial periodic orbits of (1.4) would be stated in the following theorem.

Theorem 4.6. *Let $ar > c$, $\gamma > \gamma^*$ and $\xi_{0k} \neq \xi_{0j}, j \neq k$. For some $i \in \mathbb{N}$, assume that μ_i is a simple eigenvalue of $-\Delta$ in Ω with Neumann boundary condition, and the corresponding eigenfunction is $y_i(x)$. Then*

i) (1.4) *has a unique one-parameter family $\{p(\tau) : 0 < \tau < \varepsilon\}$ of nontrivial periodic orbits near $(\xi, u, v, w) = (\xi_0, \tilde{u}, \tilde{v}, \tilde{w})$. More precisely, there exist $\varepsilon > 0$ and C^∞ function $\tau \mapsto (\mathbf{u}_i(\tau), T_i(\tau), \xi_i(\tau))$ from $\tau \in (-\varepsilon, \varepsilon)$ to $C^1(\mathbb{R}, X^3) \times (0, \infty, \mathbb{R})$ satisfying*

$$(\mathbf{u}_i(0), T_i(0), \xi_i(0)) = ((\tilde{u}, \tilde{v}, \tilde{w}), 2\pi/\delta_0, \xi_0),$$

and

$$\mathbf{u}_i(\tau, x, t) = (\tilde{u}, \tilde{v}, \tilde{w}) + \tau y_i(x) \left(V_i^+ e^{i\delta_0 t} + V_i^- e^{-i\delta_0 t} \right) + o(\tau), \tag{4.13}$$

where

$$\delta_0 = \sqrt{(d_1d_2 + d_1d_3 + d_2d_3)\mu_i^2 + ((d_1 + d_3)a\tilde{w} + (d_1 + d_2)\tilde{w} + d_2 + d_3)\mu_i + a\tilde{w}^2 + \gamma\tilde{u} + \tilde{w}},$$

and $V_i^\pm$ is an eigenvector satisfying $\mathcal{L}_i(\xi)V_i^\pm = i\delta_0 V_i^\pm$;

ii) for $0 < |\tau| < \varepsilon$, $p(\tau) = p(\mathbf{u}_i(\tau)) = \{\mathbf{u}_i(\tau, \cdot, t) : t \in \mathbb{R}\}$ is a nontrivial periodic orbit of (1.4) *of period $T_i(\tau)$;*

iii) if $0 < \tau_1 < \tau_2 < \varepsilon$, then $p(\tau_1) \neq p(\tau_2)$;

iv) there exists $\iota > 0$ such that if (1.4) *has a nontrivial periodic solution $\bar{\mathbf{u}}(x, t)$ of period T for some $\xi \in \mathbb{R}$ with*

$$|\xi - \xi_{0i}| < \iota, \qquad |T - 2\pi/\delta_0| < \iota, \qquad \max_{t\in\mathbb{R}, x\in\bar{\Omega}} |\bar{\mathbf{u}}(x, t) - (\tilde{u}, \tilde{v}, \tilde{w})| < \iota,$$

then $\xi = \xi_0(\tau)$ and $\bar{\mathbf{u}}(x, t) = \mathbf{u}_i(\tau, x, t + \omega)$ for some $\tau \in (0, \varepsilon)$ and some $\omega \in \mathbb{R}$.

We carry out numerical simulation in one-dimension to demonstrate the analytical results of Theorem 4.6.

Example 4.7. For (1.4), let $n = 1$, $\Omega = (0, 8)$, and choose $a = 2$, $r = 2$, $c = 0.1$, $\gamma = 0.5$, $d_1 = 0.3$, $d_2 = 0.2$, $d_3 = 0.3$. Then the equilibrium point $(\tilde{u}, \tilde{v}, \tilde{w}) = (1.56, 1.56, 0.44)$. It can be calculated that Hopf bifurcation value $\xi = 5.33 (k = 3)$. This parameter set shows that the occurrence of a Hopf bifurcation at $(\tilde{u}, \tilde{v}, \tilde{w}, \xi)$, and the expression (4.13) gives the oscillation frequency, the eigenfunction $y_i(x) = \cos\frac{\pi j x}{l}$ gives the spatial profile of the oscillation, see Figure 4.3.

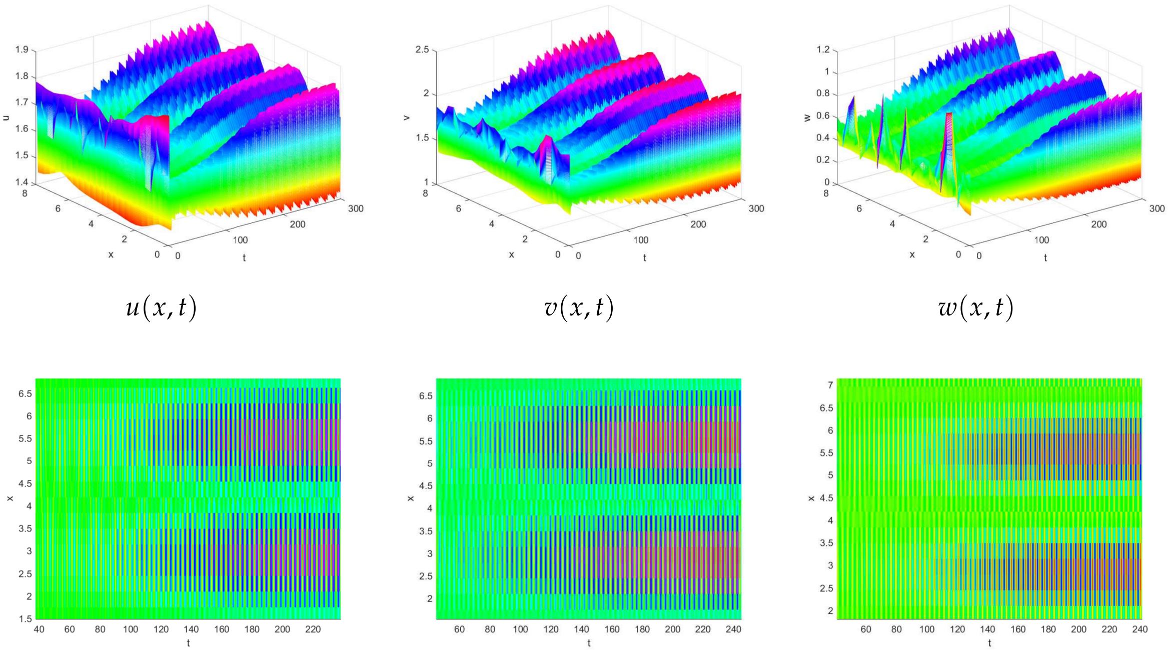

Figure 4.3: Spatiotemporal patterns of (1.4).

5 Conclusions

In this paper, a predator-prey system with both cannibalism and prey-evasion is considered. We first investigate the global existence and boundedness of the unique classical solution in 1D and 2D. The core steps are to establish some inequalities relating certain powers of the quantities

$$\sup_{s\in(0,t)} \|u(\cdot,s)\|_{L^\infty}, \qquad \sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|_{L^q}, \qquad \sup_{s\in(0,t)} \|w(\cdot,s)\|_{L^p}, \qquad t \in (0, T_{\max}),$$

for suitably wide ranges of the free parameters $p \in (1,\infty]$ and $q \in (1,\infty)$ when $n \geq 2$.

Then we obtain the result that Turing instability occurs when prey-evasion sensitive coefficient ξ surpasses the threshold value ξ_0. We also show the existence of periodic orbits of (1.4) by treating prey-evasion ξ as a bifurcation parameter, which gives spatiotemporal patterns. This means that prey-evasion is the decisive factor in destabilizing positive steady state and cannibalism is no longer a stabilizing effect.

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