Electronic Journal of Qualitative Theory of Differential Equations
2023, No. **37**, 1–19; https://doi.org/10.14232/ejqtde.2023.1.37 www.math.u-szeged.hu/ejqtde/

1/2-Laplacian problem with logarithmic and exponential nonlinearities

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Received 11 December 2022, appeared 16 August 2023
Communicated by Dimitri Mugnai

Abstract. In this paper, based on a suitable fractional Trudinger-–Moser inequality, we establish sufficient conditions for the existence result of least energy sign-changing solution for a class of one-dimensional nonlocal equations involving logarithmic and exponential nonlinearities. By using a main tool of constrained minimization in Nehari manifold and a quantitative deformation lemma, we consider both subcritical and critical exponential growths. This work can be regarded as the complement for some results of the literature.

Keywords: 1/2-Laplacian operator, logarithmic nonlinearity, exponential nonlinearity, sign-changing solutions.

2020 Mathematics Subject Classification: 35R11, 35J60, 35B33.

1 Introduction

In the present paper, we investigate the existence of least energy sign-changing solution for a Dirichlet problem driven by the 1/2-Laplacian operator of the following type:

$$\begin{cases} (-\Delta)^{1/2}u = |u|^{p-2}u\ln|u|^2 + \mu f(u) & \text{in } (0,1), \\ u = 0 & \text{in } \mathbb{R}\setminus(0,1), \end{cases} \tag{1.1}$$

where $2 < p < \infty$, μ is a positive parameter and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function with exponential subcritical or critical growth in the sense of the fractional Trudinger–Moser inequality. The nonlocal operator $(-\Delta)^{1/2}$ defined on smooth functions by

$$(-\Delta)^{1/2}u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^2}\,dy, \qquad \forall x \in \mathbb{R}. \tag{1.2}$$

Recently, a great attention has been focused on the study of nonlocal operators $(-\Delta)^s_p$, $p > 1$, $s \in (0,1)$. These arise in thin obstacle problems, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, water waves, etc. See for instance [8].

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It is natural to work on the *Sobolev–Slobodeckij* space

$$X := W_0^{1/2,2}(0,1) = \left\{ u \in H^{1/2}(\mathbb{R}) : u = 0 \text{ a.e. in } \mathbb{R}\backslash(0,1) \right\}$$

with respect to the Gagliardo semi-norm

$$\|u\| := [u]_{H^{1/2}(\mathbb{R})} = \left[\int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y \right]^{\frac{1}{2}}.$$

The problem of type (1.1) with exponential growth nonlinearity is motivated from the fractional Trudinger–Moser inequality, which specialized the results of Iannizzotto, Squassina [14, Corollary 2.4] to the space X: there exists $0 < \omega \leq \pi$ such that for all $0 < \alpha < 2\pi\omega$, we can find $K_\alpha > 0$ such that

$$\int_0^1 e^{\alpha u^2} \, \mathrm{d}x \leq K_\alpha, \qquad \text{for all } u \in X, \ \|u\| \leq 1. \tag{1.3}$$

For more information, we refer the readers to Ozawa [21, Theorem 1], and Kozono, Sato & Wadade [17, Theorem 1.1], and do Ó, Medeiros & Severo [11, Theorem 1.1]. Therefore, from this result we have naturally associated notions of subcriticality and criticality, namely: we say that a function $f : \mathbb{R} \to \mathbb{R}$ has *subcritical* growth if

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha|t|^2}} = 0, \qquad \forall \alpha > 0,$$

and f has *critical* growth if there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha|t|^2}} = 0, \qquad \forall \alpha > \alpha_0$$

and

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha|t|^2}} = +\infty, \qquad \forall \alpha < \alpha_0.$$

We assume the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is a function with exponential growth in the sense of Trudinger–Moser inequality. More precisely, the function f satisfies the following conditions:

(f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and there exists $\alpha_0 \geq 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha|t|^2}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0; \end{cases}$$

(f_2) $\lim_{t\to 0} \frac{|f(t)|}{|t|} = 0$;

(f_3) there exists $\theta > p$ such that

$$0 < \theta F(t) \leq t f(t) \quad \text{for } t \in \mathbb{R}\backslash\{0\},$$

where $F(t) = \int_0^t f(s) ds$;

(f_4) $t f'(t) \geq (p-1) f(t)$ for $t > 0$ and $t f'(t) \leq (p-1) f(t)$ for $t < 0$.

Similar conditions were also used in [28]. Here we'd like to highlight that the result in this work can be applied for the model nonlinearity $f(t) = |t|^{\theta-2} t e^{\alpha_0 t^2}$, $t \in \mathbb{R}$.

Remark 1.1. The condition (f_4) implies that $H(s) = sf(s) - pF(s)$ is a nonnegative function, increasing in $|s|$ with

$$sH'(s) = s^2 f'(s) - (p-1)f(s)s \geq 0, \quad \text{for any } |s| > 0.$$

The problem driven by the 1/2-Laplacian operator was earlier considered in [14] (see also [13]), where the authors studied the existence of mountain-pass weak solutions to the problem

$$-\frac{1}{2\pi}\int_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^2}\,\mathrm{d}y = f(u), \qquad u \in W_0^{1/2,2}(-1,1).$$

We also mention [10, 11] for other investigations in the one dimensional case on the whole space $\mathbb{R}$, facing the problem of the lack of compactness. In particular in [11], the existence of ground state solutions for the problem

$$-\frac{1}{2\pi}\int_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^2}\,\mathrm{d}y + u = f(u), \qquad u \in W_0^{1/2,2}(\mathbb{R})$$

was proved, where f is a Trudinger–Moser critical growth nonlinearity. In [7], Böer and Miyagaki investigated the existence and multiplicity of nontrivial solutions for the Choquard logarithmic equation

$$(-\Delta)^{1/2}u + u + \left(\ln|\cdot| * |u|^2\right)u = f(u), \quad \text{in } \mathbb{R},$$

for the nonlinearity f with exponential critical growth.

For local quasilinear problems of the following type

$$\begin{cases} -\Delta_N u = f(u), & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where the nonlinearity $f(u)$ behaves like $\exp\left(\alpha|u|^{N/N-1}\right)$, as $|u| \to \infty$, have been analyzed in literature, see [1,9,18,27] and the references therein.

On the other hand, the signed and sign-changing solutions for elliptic equations with logarithmic nonlinearities were investigated. There is an extensive bibliography on this subject. See, for instance, Ji, Szulkin [15], Alves, Ji [2–4], Tian [23], Wen, Tang & Chen [25], Truong [24], Liang, Rădulescu [19], and the references therein.

After a careful bibliography review, we have found only a paper is due to Zhang et al. [28], which is dealing with the existence of sign-changing solutions for the local quasilinear N-Laplacian problem with logarithmic and exponential critical nonlinearities

$$\begin{cases} -\Delta_N u = |u|^{p-2}u\ln|u|^2 + \mu f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In that interesting paper, the authors applied the constrained minimization in Nehari manifold and the quantitative deformation lemma, and obtained the existence of least energy sign-changing solution.

Motivated by above works, especially by [14,22,28], the main goal of this paper is to show the existence of least energy sign-changing solutions for problem (1.1). To the author's knowledge, in the framework of the Sobolev–Slobodeckij spaces $W_0^{1/2,2}(0,1)$, fractional counterparts of the local quasilinear N-Laplacian problem (1.4) were not previously tackled in the literature. This is precisely the goal of this manuscript.

We give our problem a variational formulation by setting for all $u \in X$

$$I_\mu(u) = \frac{1}{2}\|u\|^2 + \frac{2}{p^2}\int_0^1 |u|^p \,\mathrm{d}x - \frac{1}{p}\int_0^1 |u|^p \ln|u|^2 \,\mathrm{d}x - \mu\int_0^1 F(u)\,\mathrm{d}x.$$

Observe that

$$\lim_{|t|\to 0}\frac{|t|^{p-1}\ln|t|^2}{|t|} = 0,$$
$$\lim_{|t|\to \infty}\frac{|t|^{p-1}\ln|t|^2}{|t|^{r-1}} = 0, \qquad \text{for all } r \in (p,\infty),$$

since $p > 2$. Then for any $\epsilon > 0$, there exists a positive constant $C_1 = C_1(\epsilon)$ such that

$$|t|^{p-1}\ln|t|^2 \le \epsilon|t| + C_1|t|^{r-1}, \qquad \text{for all } t \in \mathbb{R}. \tag{1.5}$$

By (f_1), for all $\alpha \ge \alpha_0$ there exists $c_2 > 0$ such that

$$|f(t)| \le c_2 e^{\alpha t^2}, \qquad \text{for all } t \in \mathbb{R}. \tag{1.6}$$

For given $\epsilon > 0$, (f_2) implies that there exists $\delta > 0$ such that for all $|t| < \delta$ we have $F(t) \le \frac{\epsilon}{2}|t|^2$. Fix $q > 2, 0 < \alpha < 2\pi\omega$ and $r > 1$ such that $r\alpha < 2\pi\omega$ as well. By (1.6) there exists $C_\epsilon > 0$ such that for all $|t| \ge \delta$ we have $F(t) \le C_\epsilon|t|^q e^{\alpha t^2}$. Summarizing, we obtain

$$|F(t)| \le \frac{\epsilon}{2}|t|^2 + C_\epsilon|t|^q e^{\alpha t^2}, \qquad \forall t \in \mathbb{R}. \tag{1.7}$$

Using (1.5), (1.7), the Sobolev embedding theorem and the fractional Trudinger–Moser inequality (1.3), one can verify that I_μ is well defined, of class $C^1(X,\mathbb{R})$ and

$$\begin{aligned}\langle I'_\mu(u), v\rangle &= \langle u, v\rangle_X - \int_0^1 |u|^{p-2}uv\ln|u|^2\,\mathrm{d}x - \mu\int_0^1 f(u)v\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y - \int_0^1 |u|^{p-2}uv\ln|u|^2\,\mathrm{d}x - \mu\int_0^1 f(u)v\,\mathrm{d}x\end{aligned}$$

for all $u, v \in X$. From now on, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X' and X. Clearly, the critical points of I_μ are exactly the weak solutions of problem (1.1).

We call u a *least energy sign-changing solution* to problem (1.1) if $u^\pm \ne 0$ and

$$I_\mu(u) = \inf\left\{I_\mu(v) : v^\pm \ne 0, I'_\mu(v) = 0\right\},$$

where $v^+ = \max\{v(x), 0\}$ and $v^- = \min\{v(x), 0\}$. By a simple calculation, for any $u = u^+ + u^-$ with $u^\pm \ne 0$, we obtain

$$\begin{aligned}\|u\|^2 &= \|u^+\|^2 + \|u^-\|^2 + 2H(u),\\ I_\mu(u) &= I_\mu(u^+) + I_\mu(u^-) + H(u) > I_\mu(u^+) + I_\mu(u^-),\\ \langle I'_\mu(u), u^\pm\rangle &= \langle I'_\mu(u^\pm), u^\pm\rangle + H(u) > \langle I'_\mu(u^\pm), u^\pm\rangle,\end{aligned}$$

where

$$H(u) = -\int_0^1\int_0^1 \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y > 0.$$

Therefore, the methods used to seek sign-changing solutions of the local problems do not work to problem (1.1) due to the presence of the nonlocal operator $(-\Delta)^{1/2}$. And so, a careful analysis is necessary in a lot of estimates. Inspired by [6], our strategy consists in finding sign-changing solutions which minimize the corresponding energy functional I_μ among the set of all sign-changing solutions to problem (1.1). To this end, we define the sign-changing Nehari set as

$$\mathcal{M}_\mu := \left\{ u \in X : \langle I'_\mu(u), u^+ \rangle = \langle I'_\mu(u), u^- \rangle = 0, u^\pm \neq 0 \right\}.$$

Note that $u^\pm \in X$ and $u = u^+ + u^-$. Clearly, any sign-changing solution of problem (1.1) lies in the set $\mathcal{M}_\mu$.

Here are our main results.

Theorem 1.2. (Subcritical case). *Assume that conditions (f_2)–(f_4) and (f_1) with $\alpha_0 = 0$ hold. Then problem (1.1) admits a least energy sign-changing solution $u_\mu \in \mathcal{M}_\mu$ for $\mu > 0$ satisfying $I_\mu(u_\mu) = m_\mu$, where $m_\mu = \inf_{u\in\mathcal{M}_\mu} I_\mu(u)$.*

Theorem 1.3. (Critical case). *Assume that conditions (f_2)–(f_4) and (f_1) with $\alpha_0 > 0$ hold. Then there exists $\mu^* > 0$ such that problem (1.1) has a least energy sign-changing solution $u_\mu \in \mathcal{M}_\mu$ for $\mu \geq \mu^*$ satisfying $I_\mu(u_\mu) = m_\mu$.*

The inequality (3.4) or (4.3) plays a crucial role to show that the minimum m_μ of the associated energy functional I_μ is achieved. In the subcritical case, (3.4) holds due to the positive number α can take arbitrary small, thus we can conclude that Lemma 3.2 for all $\mu > 0$. However, in the critical case, we can't prove directly that (4.3) holds by the fractional Trudinger–Moser inequality (1.3) since $\alpha > \alpha_0$ for some positive number α_0. Based on this reason, we need to further analyze the asymptotic property of m_μ, by utilize Lemmas 2.3(ii) and 2.4, we can find a threshold $\mu^* > 0$ such that (4.3) holds for all $\mu \geq \mu^*$. Thus, we can conclude that Lemma 4.2 for all $\mu \geq \mu^*$. It is quite natural to ask whether in the critical case a least energy sign-changing solution exists even for $\mu \in (0, \mu^*)$. This is the issue we need to further consider in the future. Our initial idea is below: to do that, based on works such as [29], we insert an additional condition that makes possible to get a boundedness for the integral involving the exponential term. By utilize an argument similar to [29], we will try to pull the energy of sign-changing solutions down below some critical value to recover the compactness which urges us to prove that m_μ can be achieved by some $u_\mu \in \mathcal{M}_\mu$. Finally, followed the idea used in [30, Theorem 1.1], we shall prove that u_μ is indeed a least energy sign-changing solution of problem (1.1).

This paper is organized as follows. In Section 2, we show some technical lemmas and estimates in both subcritical and critical cases. Then we give the proofs of Theorem 1.2 and Theorem 1.3 in Section 3 and 4, respectively.

2 Technical lemmas

In this section, we present some extra framework information and provide very useful technical results.

We start remembering the operator $(-\Delta)^{1/2}$, of a smooth function $u : \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathcal{F}\left((-\Delta)^{1/2}u\right)(\xi) = |\xi|\mathcal{F}(u)(\xi),$$

where $\mathcal{F}$ denotes the Fourier transform, that is,

$$\mathcal{F}(\phi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi\cdot x} \phi(x) \, \mathrm{d}x$$

for functions ϕ in the Schwartz class. Also $(-\Delta)^{1/2}u$ can be equivalently represented as (1.2).

Now, we turn our attention to the Hilbert space

$$H^{1/2}(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) : \int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y < \infty \right\}$$

endowed with the norm

$$\|u\|_{H^{1/2}(\mathbb{R})} = \left(\|u\|^2_{L^2(\mathbb{R})} + [u]^2_{H^{1/2}(\mathbb{R})} \right)^{\frac{1}{2}},$$

where $\|\cdot\|_{L^s(\mathbb{R})}$ denotes the standard $L^s(\mathbb{R})$ norm for any $s \geq 1$. We know that $(H^{1/2}(\mathbb{R}), \|\cdot\|_{H^{1/2}(\mathbb{R})})$ is a Hilbert space. Also, in light of [8, Proposition 3.6], we have

$$\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} = (2\pi)^{-\frac{1}{2}} [u]_{H^{1/2}(\mathbb{R})}, \quad \text{for all } u \in H^{1/2}(\mathbb{R}),$$

and, sometimes, we identify these two quantities by omitting the normalization constant $1/2\pi$.

It follows from Proposition 2.2 in [14] to that there exists $\lambda_1 > 0$ such that for all $u \in X$

$$\|u\|_{L^2(0,1)} \leq \lambda_1^{-\frac{1}{2}} \|u\|. \tag{2.1}$$

Moreover, equality holds for some $u \in X$ with $\|u\|_{L^2(0,1)} = 1$. Due to the inequality (2.1), we can prove further $(X, \|\cdot\|)$ is a Hilbert space, where $\|\cdot\|$ is induced by an inner product, defined for all $u, v \in X$ by

$$\langle u, v \rangle_X = \int_{\mathbb{R}^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y.$$

Hereafter, we assume throughout, unless otherwise mentioned, that the function f satisfies conditions (f_1) to (f_4). Now, fix $u \in X$ with $u^{\pm} \neq 0$, and we define the function $\Psi_u : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and mapping $T_u : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ as

$$\Psi_u(a,b) = I_\mu\left(au^+ + bu^-\right) \tag{2.2}$$

and

$$T_u(a,b) = \left(\langle I'_\mu\left(au^+ + bu^-\right), au^+ \rangle, \langle I'_\mu\left(au^+ + bu^-\right), bu^- \rangle \right) = (g_1(a,b), g_2(a,b)). \tag{2.3}$$

Lemma 2.1. *For each $u \in X$ with $u^{\pm} \neq 0$, there exists an unique pair $(a_u, b_u) \in (0,\infty) \times (0,\infty)$ such that*

$$a_u u^+ + b_u u^- \in \mathcal{M}_\mu.$$

In particular, the set $\mathcal{M}_\mu$ is nonempty. Moreover, for all $a, b \geq 0$ with $(a,b) \neq (a_u, b_u)$

$$I_\mu\left(au^+ + bu^-\right) < I_\mu\left(a_u u^+ + b_u u^-\right)$$

holds.

Proof. First we will work to obtain the existence result. From (f_1) and (f_2), given $\epsilon > 0$, there exists a positive constant $C_2 = C_2(\epsilon)$ such that

$$f(t)t \leq \epsilon|t|^2 + C_2|t|^q e^{\alpha t^2} \quad \text{for all } \alpha > \alpha_0, q > 2. \tag{2.4}$$

Now, given $u \in X$ with $u^\pm \neq 0$, it follows from (1.5), (2.4), the Sobolev embedding theorem, the Hölder inequality and the fractional Trudinger–Moser inequality (1.3) that when $s, s' > 1$ with $1/s + 1/s' = 1$ and small $a > 0$ with $\alpha s \|au^+\|^2 \leq 2\pi\omega$

$$\begin{aligned}
g_1(a,b) &= \langle I'_\mu\left(au^+ + bu^-\right), au^+\rangle \\
&= \|au^+\|^2 + abH(u) - \int_0^1 |au^+|^p \ln|au^+|^2 \, \mathrm{d}x - \mu\int_0^1 f\left(au^+\right)au^+ \, \mathrm{d}x \\
&\geq \|au^+\|^2 - \epsilon\int_0^1 |au^+|^2 \, \mathrm{d}x - C_1\int_0^1 |au^+|^r \, \mathrm{d}x \\
&\quad - \mu\epsilon\int_0^1 |au^+|^2 \, \mathrm{d}x - \mu C_2\int_0^1 |au^+|^q e^{\alpha|au^+|^2} \, \mathrm{d}x \\
&\geq \|au^+\|^2 - \epsilon C_3\|au^+\|^2 - C_1C_4\|au^+\|^r - \mu\epsilon C_3\|au^+\|^2 \\
&\quad - \mu C_2\left(\int_0^1 |au^+|^{qs'} \, \mathrm{d}x\right)^{\frac{1}{s'}}\left(\int_0^1 e^{\alpha s\|au^+\|^2(|au^+|/\|au^+\|)^2} \, \mathrm{d}x\right)^{\frac{1}{s}} \\
&\geq (1 - \epsilon C_3 - \mu\epsilon C_3)\|au^+\|^2 - C_1C_4\|au^+\|^r - \mu C_2K_{\alpha s\|au^+\|^2}C_5\|au^+\|^q
\end{aligned} \tag{2.5}$$

holds. Choose $\epsilon > 0$ sufficiently small such that $1 - \epsilon C_3 - \mu\epsilon C_3 > 0$ and then it is easy to see that $\langle I'_\mu\left(au^+ + bu^-\right), au^+\rangle > 0$ for small $a > 0$ and all $b > 0$ by $r, q > 2$. In turn, we can also obtain that $\langle I'_\mu\left(au^+ + bu^-\right), bu^-\rangle > 0$ for $b > 0$ small enough and all $a > 0$. Hence, it is evident that there exists $\delta_1 > 0$ such that

$$\langle I'_\mu\left(\delta_1u^+ + bu^-\right), \delta_1u^+\rangle > 0, \qquad \langle I'_\mu\left(au^+ + \delta_1u^-\right), \delta_1u^-\rangle > 0 \tag{2.6}$$

for all $a, b > 0$.

On the other hand, recall the elementary inequality

$$2t^p - pt^p\ln t^2 \leq 2 \tag{2.7}$$

for all $t \in (0,\infty)$. From (f_3), we can deduce that there exist $C_{\theta,1}, C_{\theta,2} > 0$ such that

$$F(t) \geq C_{\theta,1}|t|^\theta - C_{\theta,2}. \tag{2.8}$$

Now, choose $a = \delta_2^* > \delta_1$ with δ_2^* large enough and it follows from (2.7), (2.8) and $2 < p < \theta$ that

$$\begin{aligned}
g_1(\delta_2^*, b) &= \langle I'_\mu\left(\delta_2^*u^+ + bu^-\right), \delta_2^*u^+\rangle \\
&\leq \|\delta_2^*u^+\|^2 + \delta_2^*bH(u) + \int_0^1\left(\frac{2}{p} - \frac{2}{p}|\delta_2^*u^+|^p\right)\mathrm{d}x - \mu\theta\int_0^1 C_{\theta,1}|\delta_2^*u^+|^\theta \, \mathrm{d}x + \mu\theta C_{\theta,2} \\
&\leq 0
\end{aligned}$$

for $b \in [\delta_1, \delta_2^*]$. With the similar argument, we can choose sufficiently large $b = \delta_2^* > \delta_1$ such that $\langle I'_\mu\left(au^+ + \delta_2^*u^-\right), \delta_2^*u^-\rangle \leq 0$ holds for $a \in [\delta_1, \delta_2^*]$.

Hence, let $\delta_2 > \delta_2^*$ be large enough. Then we obtain that

$$\langle I'_\mu\left(\delta_2u^+ + bu^-\right), \delta_2u^+\rangle < 0, \qquad \langle I'_\mu\left(au^+ + \delta_2u^-\right), \delta_2u^-\rangle < 0 \tag{2.9}$$

for all $a, b \in [\delta_1, \delta_2]$. Combining (2.6) and (2.9) with Miranda's Theorem [5], there exists at least one point pair $(a_u, b_u) \in (0,\infty) \times (0,\infty)$ such that $T_u(a_u, b_u) = (0,0)$, that is, $a_u u^+ + b_u u^- \in \mathcal{M}_\mu$.

Next we will prove the uniqueness of the pair (a_u, b_u). In fact, it is sufficient to show that if $u \in \mathcal{M}_\mu$ and $a_0 u^+ + b_0 u^- \in \mathcal{M}_\mu$ with $a_0 > 0$ and $b_0 > 0$, then $(a_0, b_0) = (1,1)$. Assume that $u \in \mathcal{M}_\mu$ and $a_0 u^+ + b_0 u^- \in \mathcal{M}_\mu$. We thus obtain that $\langle I'_\mu(a_0 u^+ + b_0 u^-), a_0 u^+\rangle = 0$, $\langle I'_\mu(a_0 u^+ + b_0 u^-), b_0 u^-\rangle = 0$, and $\langle I'_\mu(u), u^\pm\rangle = 0$, namely

$$\left\|a_0 u^+\right\|^2 + a_0 b_0 H(u) = \int_0^1 \left|a_0 u^+\right|^p \ln\left|a_0 u^+\right|^2 \,\mathrm{d}x + \mu \int_0^1 f\left(a_0 u^+\right) a_0 u^+ \,\mathrm{d}x, \tag{2.10}$$

$$\left\|b_0 u^-\right\|^2 + b_0 a_0 H(u) = \int_0^1 \left|b_0 u^-\right|^p \ln\left|b_0 u^-\right|^2 \,\mathrm{d}x + \mu \int_0^1 f\left(b_0 u^-\right) b_0 u^- \,\mathrm{d}x, \tag{2.11}$$

$$\left\|u^+\right\|^2 + H(u) = \int_0^1 \left|u^+\right|^p \ln\left|u^+\right|^2 \,\mathrm{d}x + \mu \int_0^1 f\left(u^+\right) u^+ \,\mathrm{d}x, \tag{2.12}$$

$$\left\|u^-\right\|^2 + H(u) = \int_0^1 \left|u^-\right|^p \ln\left|u^-\right|^2 \,\mathrm{d}x + \mu \int_0^1 f\left(u^-\right) u^- \,\mathrm{d}x. \tag{2.13}$$

Without loss of generality, we may assume that $0 < a_0 \le b_0$. Thus, form (2.11), we get

$$\left\|b_0 u^-\right\|^2 + b_0^2 H(u) \ge \int_0^1 \left|b_0 u^-\right|^p \ln\left|b_0 u^-\right|^2 \,\mathrm{d}x + \mu \int_0^1 f\left(b_0 u^-\right) b_0 u^- \,\mathrm{d}x. \tag{2.14}$$

Combining (2.14) and (2.13), we deduce that

$$\int_0^1 \left|u^-\right|^p \ln\left|u^-\right|^2 \,\mathrm{d}x - \int_0^1 \frac{\left|b_0 u^-\right|^p \ln\left|b_0 u^-\right|^2}{b_0^2} \mathrm{d}x \ge \mu \int_0^1 \frac{f\left(b_0 u^-\right) b_0 u^-}{b_0^2} \,\mathrm{d}x - \mu \int_0^1 f\left(u^-\right) u^- \mathrm{d}x,$$

that is,

$$\int_0^1 \left(\left|u^-\right|^{p-2} \ln\left|u^-\right|^2 - \left|b_0 u^-\right|^{p-2} \ln\left|b_0 u^-\right|^2\right) \left|u^-\right|^2 \,\mathrm{d}x \ge \mu \int_0^1 \left(\frac{f\left(b_0 u^-\right)}{b_0 u^-} - \frac{f\left(u^-\right)}{u^-}\right) \left(u^-\right)^2 \mathrm{d}x.$$

It follows from (f_4) and $p > 2$ that $t \mapsto \frac{f(t)}{t}$ and $t \mapsto t^{p-2} \ln t^2$ are increasing for $t > 0$. If $b_0 > 1$, the left hand side of the above inequality is negative, which is absurd due to the right hand side is positive. Therefore, we obtain $a_0 \le b_0 \le 1$. Similarly, from (2.10), (2.12) and $0 < a_0 \le b_0$, one has

$$\int_0^1 \left(\left|u^+\right|^{p-2} \ln\left|u^+\right|^2 - \left|a_0 u^+\right|^{p-2} \ln\left|a_0 u^+\right|^2\right) \left|u^+\right|^2 \,\mathrm{d}x \le \mu \int_0^1 \left(\frac{f\left(a_0 u^+\right)}{a_0 u^+} - \frac{f\left(u^+\right)}{u^+}\right) \left(u^+\right)^2 \,\mathrm{d}x.$$

Thus, we can deduce that $a_0 \ge 1$. So $a_0 = b_0 = 1$.

To complete the proof of this lemma, it remains to show that (a_u, b_u) is the unique maximum point of Ψ_μ in $[0,\infty) \times [0,\infty)$. It follows from (2.7), (2.8), the Hölder inequality, the elementary inequality and $\theta > p > 2$ that

$$\begin{aligned}
\Psi_u(a,b) &= I_\mu\left(au^+ + bu^-\right) \\
&= \frac{1}{2}\left\|au^+ + bu^-\right\|^2 + \frac{2}{p^2} \int_0^1 \left|au^+ + bu^-\right|^p \,\mathrm{d}x \\
&\quad - \frac{1}{p} \int_0^1 \left|au^+ + bu^-\right|^p \ln\left|au^+ + bu^-\right|^2 \,\mathrm{d}x - \mu \int_0^1 F\left(au^+ + bu^-\right) \mathrm{d}x \\
&\le a^2 \left\|u^+\right\|^2 + b^2 \left\|u^-\right\|^2 + \frac{2}{p^2} - \mu C_{\theta,1} a^\theta \int_0^1 \left|u^+\right|^\theta \,\mathrm{d}x - \mu C_{\theta,1} b^\theta \int_0^1 \left|u^-\right|^\theta \,\mathrm{d}x + 2\mu C_{\theta,2},
\end{aligned}$$

which implies that $\lim_{|(a,b)|\to\infty}\Psi_u(a,b)=-\infty$. Therefore, it suffices to show that the maximum point of Ψ_u cannot be achieved on the boundary of $[0,\infty)\times[0,\infty)$. Suppose, by contradiction, that $(0,b)$ with $b\geq 0$ is a maximum point of Ψ_u. Then from (2.5), we have

$$a\frac{\mathrm{d}}{\mathrm{d}a}\left[I_\mu\left(au^+ + bu^-\right)\right]=\langle I'_\mu\left(au^+ + bu^-\right),au^+\rangle>0$$

for small $a>0$, which means that Ψ_u is increasing with respect to a if $a>0$ is small enough. This yields a contradiction. Similarly, we can deduce that Ψ_u cannot achieve its global maximum on $(a,0)$ with $a\geq 0$. $\square$

Lemma 2.2. *For any $u\in X$ with $u^\pm\neq 0$ such that $\langle I'_\mu(u),u^\pm\rangle\leq 0$, the unique maximum point (a_u,b_u) of Ψ_u on $[0,\infty)\times[0,\infty)$ satisfies $0<a_u,b_u\leq 1$.*

Proof. Here we will only prove $0<a_u\leq 1$. The proof of $0<b_u\leq 1$ is the same. For $u\in X$ with $u^\pm\neq 0$, by Lemma 2.1, there exist unique a_u and b_u such that $a_uu^+ + b_uu^-\in\mathcal{M}_\mu$. Without loss of generality, we may assume that $a_u\geq b_u>0$. Since $a_uu^+ + b_uu^-\in\mathcal{M}_\mu$. Then, we have that

$$\left\|a_uu^+\right\|^2+a_u^2H(u)\geq\int_0^1\left|a_uu^+\right|^p\ln\left|a_uu^+\right|^2\,\mathrm{d}x+\mu\int_0^1 f\left(a_uu^+\right)a_uu^+\,\mathrm{d}x. \tag{2.15}$$

Moreover, by $\langle I'_\mu(u),u^\pm\rangle\leq 0$, we have that

$$\left\|u^+\right\|^2+H(u)\leq\int_0^1\left|u^+\right|^p\ln\left|u^+\right|^2\,\mathrm{d}x+\mu\int_0^1 f\left(u^+\right)u^+\,\mathrm{d}x. \tag{2.16}$$

Therefore, from (2.15) and (2.16), it follows that

$$\int_0^1\frac{\left|a_uu^+\right|^p\ln\left|a_uu^+\right|^2}{a_u^2}\,\mathrm{d}x-\int_0^1\left|u^+\right|^p\ln\left|u^+\right|^2\mathrm{d}x\leq\mu\int_0^1 f\left(u^+\right)u^+\,\mathrm{d}x-\mu\int_0^1\frac{f\left(a_uu^+\right)a_uu^+}{a_u^2}\,\mathrm{d}x,$$

that is,

$$\int_0^1\left(\left|a_uu^+\right|^{p-2}\ln\left|a_uu^+\right|^2-\left|u^+\right|^{p-2}\ln\left|u^+\right|^2\right)\left|u^+\right|^2\mathrm{d}x\leq\mu\int_0^1\left(\frac{f\left(u^+\right)}{u^+}-\frac{f\left(a_uu^+\right)}{a_uu^+}\right)\left(u^+\right)^2\mathrm{d}x.$$

Now, we suppose, by contradiction, that $a_u>1$. Since (f_4) and $p>2$, then $t\mapsto\frac{f(t)}{t}$ and $t\mapsto t^{p-2}\ln t^2$ are increasing for $t>0$, which implies that the last inequality is impossible. Thus, we conclude $0<a_u\leq 1$. $\square$

Lemma 2.3. *For all $u\in\mathcal{M}_\mu$, there exists a positive number ρ independent of u such that*

(i) $\|u^\pm\|\geq\rho$;

(ii) $I_\mu(u)\geq\left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2$.

Proof. (i) We only prove that there exists a positive constant ρ independent of u such that $\|u^+\|\geq\rho$ for all $u\in\mathcal{M}_\mu$ and the result for $\|u^-\|$ is similar. By contradiction, for arbitrary small $\varepsilon>0$, there exists $\{u_\varepsilon\}\subset\mathcal{M}_\mu$ such that $\|u_\varepsilon^+\|<\varepsilon$. Letting $\varepsilon=1/n$ for large enough $n\in\mathbb{N}$, thus, we can suppose that there exists a sequence $\{u_n\}\subset\mathcal{M}_\mu$ such that $u_n^+\to 0$ in X. Since $\langle I'_\mu\left(u_n\right),u_n^+\rangle=0$ holds. Then it follows from (1.5) and (2.4) that

$$\begin{aligned}\left\|u_n^+\right\|^2&\leq\left\|u_n^+\right\|^2+H(u_n)=\int_0^1\left|u_n^+\right|^p\ln\left|u_n^+\right|^2\,\mathrm{d}x+\mu\int_0^1 f\left(u_n^+\right)u_n^+\,\mathrm{d}x\\&\leq\epsilon\int_0^1\left|u_n^+\right|^2\,\mathrm{d}x+C_1\int_0^1\left|u_n^+\right|^r\,\mathrm{d}x+\mu\epsilon\int_0^1\left|u_n^+\right|^2\,\mathrm{d}x+\mu C_2\int_0^1\left|u_n^+\right|^q e^{\alpha\left|u_n^+\right|^2}\,\mathrm{d}x.\end{aligned} \tag{2.17}$$

Let $s > 1$ with $1/s + 1/s' = 1$. Since $u_n^+ \to 0$ in X, then there exists $n_0 \in \mathbb{N}$ such that $\alpha s \|u_n^+\|^2 \le 2\pi\omega$ for all $n \ge n_0$. From Hölder's inequality and the fractional Trudinger–Moser inequality (1.3), we have

$$\begin{aligned}\int_0^1 |u_n^+|^q \exp\left(\alpha |u_n^+|^2\right) \,\mathrm{d}x &\le \left(\int_0^1 |u_n^+|^{qs'} \,\mathrm{d}x\right)^{\frac{1}{s'}} \left(\int_0^1 e^{\alpha s \|u_n^+\|^2 (|u_n^+|/\|u_n^+\|)^2} \,\mathrm{d}x\right)^{\frac{1}{s}} \\ &\le K_{\alpha s\|u_n^+\|^2} \left(\int_0^1 |u_n^+|^{qs'} \,\mathrm{d}x\right)^{\frac{1}{s'}}.\end{aligned}$$

Combining (2.17) with the last inequality, we can deduce from the Sobolev embedding theorem that when $n \ge n_0$

$$\|u_n^+\|^2 \le (\epsilon + \mu\epsilon) C_6 \|u_n^+\|^2 + C_1 C_7 \|u_n^+\|^r + \mu C_2 K_{\alpha s \|u_n^+\|^2} C_8 \|u_n^+\|^q. \tag{2.18}$$

Choose appropriate $\epsilon > 0$ such that $1 - \mu\epsilon C_6 - \epsilon C_6 > 0$. Noticing that $2 < p < r$ and $2 < q$, we can deduce that (2.18) contradicts $u_n^+ \to 0$ in X.

(ii) Given $u \in \mathcal{M}_\mu$, by the definition of $\mathcal{M}_\mu$ and (f_3) we obtain

$$\begin{aligned} I_\mu(u) &= I_\mu(u) - \frac{1}{p}\langle I_\mu'(u), u\rangle \\ &= \frac{1}{2}\|u\|^2 + \frac{2}{p^2}\int_0^1 |u|^p \,\mathrm{d}x - \mu\int_0^1 F(u)\,\mathrm{d}x - \frac{1}{p}\|u\|^2 + \mu\frac{1}{p}\int_0^1 f(u)u\,\mathrm{d}x \\ &\ge \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2. \end{aligned}$$

Thus, we finish the proof. □

Lemma 2.3 tells that $I_\mu(u) > 0$ for all $u \in \mathcal{M}_\mu$. Therefore, I_μ is bounded below in $\mathcal{M}_\mu$, which means that $m_\mu = \inf_{u \in \mathcal{M}_\mu} I_\mu(u)$ is well-defined. The following lemma is about the asymptotic property of m_μ.

Lemma 2.4. *Let $m_\mu = \inf_{u\in\mathcal{M}_\mu} I_\mu(u)$, then $\lim_{\mu\to\infty} m_\mu = 0$.*

Proof. Fix $u \in X$ with $u^\pm \ne 0$. Then, by Lemma 2.1, for each $\mu > 0$ there exists a point pair (a_μ, b_μ) such that $a_\mu u^+ + b_\mu u^- \in \mathcal{M}_\mu$. Let

$$\mathcal{T}_u := \left\{(a_\mu, b_\mu) \in [0,\infty) \times [0,\infty) : T_u(a_\mu, b_\mu) = (0,0), \mu > 0\right\},$$

where T_u is defined in (2.3).

Since $a_\mu u^+ + b_\mu u^- \in \mathcal{M}_\mu$, by assumption (f_3), (2.7) and (2.8), we have

$$\begin{aligned} \|a_\mu u^+\|^2 + \|b_\mu u^-\|^2 + 2a_\mu b_\mu H(u) &= \int_0^1 |a_\mu u^+ + b_\mu u^-|^p \ln |a_\mu u^+ + b_\mu u^-|^2 \,\mathrm{d}x \\ &\quad + \mu\int_0^1 f\left(a_\mu u^+ + b_\mu u^-\right)\left(a_\mu u^+ + b_\mu u^-\right)\,\mathrm{d}x \\ &\ge \frac{2a_\mu^p}{p}\int_0^1 |u^+|^p \,\mathrm{d}x + \frac{2b_\mu^p}{p}\int_0^1 |u^-|^p \,\mathrm{d}x - \frac{2}{p} - \mu\theta C_{\theta,2} \\ &\quad + \mu\theta C_{\theta,1} a_\mu^\theta \int_0^1 |u^+|^\theta \,\mathrm{d}x + \mu\theta C_{\theta,1} b_\mu^\theta \int_0^1 |u^-|^\theta \,\mathrm{d}x. \end{aligned}$$

From $\theta > p > 2$, it follows that the set $\mathcal{T}_\mu$ is bounded. Hence, if $\{\mu_n\} \subset (0, \infty)$ satisfies $\mu_n \to \infty$ as $n \to \infty$, then, up to a subsequence, there exist $\bar{a}, \bar{b} \geq 0$ such that $a_{\mu_n} \to \bar{a}$ and $b_{\mu_n} \to \bar{b}$.

Claim that $\bar{a} = \bar{b} = 0$. Suppose, by contradiction, that $\bar{a} > 0$ or $\bar{b} > 0$. For each $n \in \mathbb{N}$, $a_{\mu_n}u^+ + b_{\mu_n}u^- \in M_{\mu_n}$, we have $\langle I'_{\mu_n}\left(a_{\mu_n}u^+ + b_{\mu_n}u^-\right), a_{\mu_n}u^+ + b_{\mu_n}u^-\rangle = 0$, namely

$$\begin{aligned}\left\|a_{\mu_n}u^+ + b_{\mu_n}u^-\right\|^2 = &\int_0^1 \left|a_{\mu_n}u^+ + b_{\mu_n}u^-\right|^p \ln\left|a_{\mu_n}u^+ + b_{\mu_n}u^-\right|^2 \mathrm{d}x \\ &+ \mu_n \int_0^1 f\left(a_{\mu_n}u^+ + b_{\mu_n}u^-\right)\left(a_{\mu_n}u^+ + b_{\mu_n}u^-\right) \mathrm{d}x.\end{aligned} \tag{2.19}$$

Note that $a_{\mu_n}u^+ \to \bar{a}u^+$ and $b_{\mu_n}u^- \to \bar{b}u^-$ in X, by (1.5) and the Lebesgue dominated convergence theorem, we have that

$$\int_0^1 \left|a_{\mu_n}u^+ + b_{\mu_n}u^-\right|^p \ln\left|a_{\mu_n}u^+ + b_{\mu_n}u^-\right|^2 \mathrm{d}x \to \int_0^1 \left|\bar{a}u^+ + \bar{b}u^-\right|^p \ln\left|\bar{a}u^+ + \bar{b}u^-\right|^2 \mathrm{d}x. \tag{2.20}$$

Once $\mu_n \to \infty$ as $n \to \infty$ and $\left\{a_{\mu_n}u^+ + b_{\mu_n}u^-\right\}$ is bounded in X, from (2.19), (2.20) and (f_3), it follows that

$$\begin{aligned}\left\|\bar{a}u^+ + \bar{b}u^-\right\|^2 = &\int_0^1 \left|\bar{a}u^+ + \bar{b}u^-\right|^p \ln\left|\bar{a}u^+ + \bar{b}u^-\right|^2 \mathrm{d}x \\ &+ \left(\lim_{n\to\infty} \mu_n\right) \lim_{n\to\infty} \int_0^1 f\left(a_{\mu_n}u^+ + b_{\mu_n}u^-\right)\left(a_{\mu_n}u^+ + b_{\mu_n}u^-\right) \mathrm{d}x,\end{aligned}$$

which is impossible. Thus, $\bar{a} = \bar{b} = 0$, i.e., $a_{\mu_n} \to 0$ and $b_{\mu_n} \to 0$ as $n \to \infty$. Finally, by (f_3) and (2.19), we have $0 \leq m_\mu = \inf_{\mathcal{M}_\mu} I_\mu(u) \leq I_{\mu_n}\left(a_{\mu_n}u^+ + b_{\mu_n}u^-\right) \to 0$, from which we conclude the fact that $m_\mu \to 0$ as $\mu \to \infty$. □

Subsequently, we will prove that if the minimum of I_μ on $\mathcal{M}_\mu$ is achieved in some $u_0 \in \mathcal{M}_\mu$, then u_0 is a critical point of I_μ. The proof of this lemma follows from some arguments used in [12, 19], including the quantitative deformation lemma and Brouwer degree in $\mathbb{R}$.

Lemma 2.5. *If $u_0 \in \mathcal{M}_\mu$ satisfies $I_\mu(u_0) = m_\mu$, then $I'_\mu(u_0) = 0$.*

Proof. Since $u_0 \in \mathcal{M}_\mu$, we have $\langle I'_\mu(u_0), u_0^+\rangle = \langle I'_\mu(u_0), u_0^-\rangle = 0$. By Lemma 2.1, for $(\alpha, \beta) \in (\mathbb{R}_+ \times \mathbb{R}_+) \setminus (1, 1)$, we have

$$I_\mu\left(\alpha u_0^+ + \beta u_0^-\right) < I_\mu\left(u_0^+ + u_0^-\right) = m_\mu. \tag{2.21}$$

Arguing by contradiction. We assume that $I'_\mu(u_0) \neq 0$. For the continuity of I'_μ, there exists $\iota, \delta > 0$ such that

$$\left\|I'_\mu(v)\right\| \geq \iota, \quad \text{for all } \|v - u_0\| \leq 3\delta. \tag{2.22}$$

Choose $\tau \in (0, \min\{1/2, \delta/(\sqrt{2}\|u_0\|)\})$. Let $D = (1-\tau, 1+\tau) \times (1-\tau, 1+\tau)$ and $g(\alpha, \beta) = \alpha u_0^+ + \beta u_0^-$ for all $(\alpha, \beta) \in D$. By virtue of (2.21), it is easy to see that

$$\overline{m}_\mu := \max_{\partial D} I_\mu \circ g < m_\mu. \tag{2.23}$$

Indeed, let $\epsilon := \min\{(m_\mu - \overline{m}_\mu)/3, \iota\delta/8\}$, $S_\delta := B(u_0, \delta)$ and $I_\mu^c := I_\mu^{-1}((-\infty, c])$. And according to the quantitative deformation lemma [26, Lemma 2.3], there exists a deformation $\eta \in C([0,1] \times X, X)$ such that:

(i) $\eta(1,v) = v$, if $v \notin I_\mu^{-1}\left([m_\mu - 2\epsilon, m_\mu + 2\epsilon]\right) \cap S_{2\delta}$,

(ii) $\eta\left(1, I_\mu^{m_\mu+\epsilon} \cap S_\delta\right) \subset I_\mu^{m_\mu-\epsilon}$,

(iii) $I_\mu(\eta(1,v)) \le I_\mu(v)$, for all $v \in X$.

Since $I_\mu(g(\alpha,\beta)) \le m_\mu$ and $g(\alpha,\beta) \in S_\delta$ for $(\alpha,\beta) \in \overline{D}$, then it follows from (ii) that

$$\max_{(\alpha,\beta)\in\overline{D}} I_\mu(\eta(1,g(\alpha,\beta))) \le m_\mu - \epsilon. \tag{2.24}$$

In this way, we obtain a contradiction to (2.24) from the definition of m_μ if we could prove that $\eta(1,g(D)) \cap \mathcal{M}_\mu$ is nonempty. Thus we complete the proof of this lemma. To do this, we first define

$$\begin{aligned} \bar{g}(\alpha,\beta) &:= \eta(1,g(\alpha,\beta)), \\ \Psi_0(\alpha,\beta) &= \left(\langle I'_\mu(g(\alpha,\beta)), u_0^+\rangle, \langle I'_\mu(g(\alpha,\beta)), u_0^-\rangle\right) \\ &= \left(\langle I'_\mu\left(\alpha u_0^+ + \beta u_0^-\right), u_0^+\rangle, \langle I'_\mu\left(\alpha u_0^+ + \beta u_0^-\right), u_0^-\rangle\right) \\ &:= \left(\varphi_u^1(\alpha,\beta), \varphi_u^2(\alpha,\beta)\right), \end{aligned}$$

and

$$\Psi_1(\alpha,\beta) := \left(\frac{1}{\alpha}\langle I'_\mu(\bar{g}(\alpha,\beta)), (\bar{g}(\alpha,\beta))^+\rangle, \frac{1}{\beta}\langle I'_\mu(\bar{g}(\alpha,\beta)), (\bar{g}(\alpha,\beta))^-\rangle\right).$$

Moreover, a straightforward calculation, based on $u_0 \in \mathcal{M}_\mu$, shows that

$$\begin{aligned} \left.\frac{\partial \varphi_u^1(\alpha,\beta)}{\partial \alpha}\right|_{(1,1)} &= \|u_0^+\|^2 - (p-1)\int |u_0^+|^p \ln|u_0^+|^2 \, \mathrm{d}x - 2\int |u_0^+|^p \, \mathrm{d}x \\ &\quad - \mu \int_0^1 f'\left(u_0^+\right)|u_0^+|^2 \, \mathrm{d}x \\ &= (2-p)\int_0^1 |u_0^+|^p \ln|u_0^+|^2 \, \mathrm{d}x + \mu\int_0^1 f\left(u_0^+\right)u_0^+ \, \mathrm{d}x \\ &\quad - 2\int_0^1 |u_0^+|^p \, \mathrm{d}x - \mu\int_0^1 f'\left(u_0^+\right)\left(u_0^+\right)^2 \, \mathrm{d}x - H(u) \end{aligned}$$

and

$$\left.\frac{\partial \varphi_u^1(\alpha,\beta)}{\partial \beta}\right|_{(1,1)} = H(u).$$

Similarly,

$$\left.\frac{\partial \varphi_u^2(\alpha,\beta)}{\partial \alpha}\right|_{(1,1)} = H(u)$$

and

$$\begin{aligned} \left.\frac{\partial \varphi_u^2(\alpha,\beta)}{\partial \beta}\right|_{(1,1)} &= \|u_0^-\|^2 - (p-1)\int |u_0^-|^p \ln|u_0^-|^2 \, \mathrm{d}x - 2\int |u_0^-|^p \, \mathrm{d}x \\ &\quad - \mu \int_0^1 f'\left(u_0^-\right)|u_0^-|^2 \, \mathrm{d}x \\ &= (2-p)\int_0^1 |u_0^-|^p \ln|u_0^-|^2 \, \mathrm{d}x + \mu\int_0^1 f\left(u_0^-\right)u_0^- \, \mathrm{d}x \\ &\quad - 2\int_0^1 |u_0^-|^p \, \mathrm{d}x - \mu\int_0^1 f'\left(u_0^-\right)\left(u_0^-\right)^2 \, \mathrm{d}x - H(u). \end{aligned}$$

Let

$$H=\begin{bmatrix} \left.\frac{\partial \varphi_u^1(\alpha,\beta)}{\partial \alpha}\right|_{(1,1)}, & \left.\frac{\partial \varphi_u^2(\alpha,\beta)}{\partial \alpha}\right|_{(1,1)} \\ \left.\frac{\partial \varphi_u^1(\alpha,\beta)}{\partial \beta}\right|_{(1,1)}, & \left.\frac{\partial \varphi_u^2(\alpha,\beta)}{\partial \beta}\right|_{(1,1)} \end{bmatrix}.$$

Then we deduce that $\det H \neq 0$. Therefore, Ψ_0 is a C^1 function with the point pair $(1,1)$ being the unique isolated zero point in D. By using the Brouwer's degree in $\mathbb{R}$, we deduce that $\deg(\Psi_0, D, 0)=1$.

Now, it follows from (2.24) and (i) that $g(\alpha,\beta)=\bar{g}(\alpha,\beta)$ on ∂D. For the boundary dependence of Brouwer's degree (see [20, Theorem 4.5]), there holds $\deg(\Psi_1, D, 0)=\deg(\Psi_0, D, 0)=1$. Therefore, there exists some $(\bar{\alpha},\bar{\beta})\in D$ such that

$$\eta(1, g(\bar{\alpha},\bar{\beta}))\in \mathcal{M}_\mu.$$

So we obtain a contradiction to (2.24). □

3 Subcritical case

Lemma 3.1 (Subcritical case)**.** *If $\{u_n\}\subset\mathcal{M}_\mu$ is a minimizing sequence for m_μ, then there exists some $u\in X$ such that*

$$\int_0^1 f\left(u_n^\pm\right)u_n^\pm\,\mathrm{d}x\to\int_0^1 f\left(u^\pm\right)u^\pm\,\mathrm{d}x \ \ \textit{and}\ \ \int_0^1 F\left(u_n^\pm\right)\,\mathrm{d}x\to\int_0^1 F\left(u^\pm\right)\,\mathrm{d}x.$$

Proof. We will only prove the first result. Since the second limit is a direct consequence of the first one, we omit it here.

Let sequence $\{u_n\}\subset\mathcal{M}_\mu$ be a minimizing sequence such that $\lim_{n\to\infty} I_\mu(u_n)=m_\mu$. Thus, $\{u_n\}$ is bounded in X by Lemma 2.3. It follows from Proposition 2.2 in [14] to that $\{u_n\}$ is bounded in $H^{1/2}(\mathbb{R})$ as well. By [8, Theorem 7.1 and Theorem 6.10], passing to a subsequence we may assume that $u_n\rightharpoonup u$ weakly in both X and $H^{1/2}(\mathbb{R})$, and that $u_n\to u$ in $L^q(0,1)$ for all $q\geq 1$ and $u_n(x)\to u(x)$ a.e. in $(0,1)$. Thus,

$$\begin{aligned} u_n^\pm &\rightharpoonup u^\pm \quad \text{weakly in } X,\\ u_n^\pm &\to u^\pm \quad \text{in } L^q(0,1) \text{ for } q\in[1,\infty),\\ u_n^\pm &\to u^\pm \quad \text{a.e. in } (0,1). \end{aligned} \tag{3.1}$$

Note that by (2.4), we have

$$f\left(u_n^\pm(x)\right)u_n^\pm(x)\leq \epsilon\left|u_n^\pm(x)\right|^2+C_2\left|u_n^\pm(x)\right|^q e^{\alpha|u_n^\pm(x)|^2}=:h\left(u_n^\pm(x)\right), \tag{3.2}$$

for all $\alpha>\alpha_0=0$ and $q>2$. It is sufficient to prove that sequence $\{h(u_n^\pm)\}$ is convergent in $L^1(0,1)$.

Choosing $s,s'>1$ with $1/s+1/s'=1$, by (3.1), we get that

$$|u_n^\pm|^q\to|u^\pm|^q \quad \text{in } L^{s'}(0,1). \tag{3.3}$$

In particular, there exists $c_5>0$ such that $\|u_n^\pm\|^2\leq c_5$ for all $n\in\mathbb{N}$. Choosing $0<\alpha<2\pi\omega/sc_5$, by the fractional Trudinger–Moser inequality (1.3), we get

$$\int_0^1 e^{\alpha s|u_n^\pm|^2}\,\mathrm{d}x\leq\int_0^1 e^{\alpha s c_5(u_n^\pm/\|u_n^\pm\|)^2}\,\mathrm{d}x\leq K_{\alpha s c_5}. \tag{3.4}$$

By reflexivity of $L^s(0,1)$, passing to a subsequence, we have

$$e^{\alpha|u_n^\pm|^2} \rightharpoonup e^{\alpha|u^\pm|^2} \quad \text{weakly in } L^s(0,1). \tag{3.5}$$

Hence, by (3.3), (3.5) and [16, Lemma 4.8, Chapter 1], we conclude that

$$\int_0^1 f\left(u_n^\pm\right) u_n^\pm \, \mathrm{d}x \to \int_0^1 f\left(u^\pm\right) u^\pm \, \mathrm{d}x.$$

This completes the proof. □

Lemma 3.2 (Subcritical case)**.** *There exists some $u_\mu \in \mathcal{M}_\mu$ such that $I_\mu(u_\mu) = m_\mu$.*

Proof. As indicated earlier that $m_\mu > 0$. In what follows, we only need to show that m_μ is achieved. By the definition of $m_\mu = \inf_{u\in\mathcal{M}_\mu} I_\mu(u)$, there exists a sequence $\{u_n\} \subset \mathcal{M}_\mu$ such that

$$\lim_{n\to\infty} I_\mu\left(u_n\right) = m_\mu.$$

On the one hand, (3.1) and the Vitali convergence theorem yield that

$$\lim_{n\to\infty} \int_0^1 |u_n|^p \ln|u_n|^2 \, \mathrm{d}x \to \int_0^1 |u|^p \ln|u|^2 \, \mathrm{d}x. \tag{3.6}$$

On the other hand, it follows from (3.1) that $u_n \to u$ in $L^p(0,1)$, we have

$$\lim_{n\to\infty} \int_0^1 |u_n|^p \, \mathrm{d}x \to \int_0^1 |u|^p \, \mathrm{d}x. \tag{3.7}$$

Lemma 2.1 implies $I_\mu\left(\alpha u_n^+ + \beta u_n^-\right) \le I_\mu\left(u_n\right)$ for all $\alpha, \beta \ge 0$. So, by using the Brezis–Lieb Lemma, Fatou's Lemma, (3.6), (3.7) and Lemma 3.1, we get

$$\begin{aligned}
\liminf_{n\to\infty} I_\mu\left(\alpha u_n^+ + \beta u_n^-\right) \ge{}& \frac{\alpha^2}{2} \lim_{n\to\infty}\left(\left\|u_n^+ - u^+\right\|^2 + \left\|u^+\right\|^2\right) + \frac{\beta^2}{2} \lim_{n\to\infty}\left(\left\|u_n^- - u^-\right\|^2 + \left\|u^-\right\|^2\right) \\
&+ \alpha\beta \liminf_{n\to\infty} H\left(u_n\right) - \mu \int_0^1 F\left(\alpha u^+\right) \mathrm{d}x - \mu \int_0^1 F\left(\beta u^-\right) \mathrm{d}x \\
&+ \frac{2}{p^2} \int_0^1 \left|\alpha u^+ + \beta u^-\right|^p \, \mathrm{d}x - \frac{1}{p} \int_0^1 \left|\alpha u^+ + \beta u^-\right|^p \ln\left|\alpha u^+ + \beta u^-\right|^2 \, \mathrm{d}x \\
\ge{}& I_\mu\left(\alpha u^+ + \beta u^-\right) + \frac{\alpha^2}{2} A_1 + \frac{\beta^2}{2} A_2,
\end{aligned}$$

where $A_1 = \lim_{n\to\infty} \left\|u_n^+ - u^+\right\|^2$, $A_2 = \lim_{n\to\infty} \left\|u_n^- - u^-\right\|^2$. So, for all $\alpha \ge 0$ and all $\beta \ge 0$, one has that

$$m_\mu \ge I_\mu\left(\alpha u^+ + \beta u^-\right) + \frac{\alpha^2}{2} A_1 + \frac{\beta^2}{2} A_2. \tag{3.8}$$

Firstly, we prove that $u^\pm \ne 0$. Here we only prove $u^+ \ne 0$ since $u^- \ne 0$ is analogous, by contradiction, we assume $u^+ = 0$. Hence, let $\beta = 0$ in (3.8) and we have that

$$m_\mu \ge \frac{\alpha^2}{2} A_1 \quad \text{for all } \alpha \ge 0. \tag{3.9}$$

If $A_1 = 0$, that is, $u_n^+ \to u^+$ in X. Lemma 2.3(i) implies $\|u^+\| > 0$, which contradicts supposition. If $A_1 > 0$, by (3.9) we get $m_\mu \ge \frac{\alpha^2}{2} A_1$ for all $\alpha \ge 0$, which is a contradiction by Lemma 2.4. That is, we deduce that $u^+ \ne 0$.

Lastly, we prove that m_μ is achieved. By Lemma 2.1, there exists $(s_u, t_u) \in (0,\infty) \times (0,\infty)$ such that $u_\mu := s_u u^+ + t_u u^- \in \mathcal{M}_\mu$, that is,

$$\langle I'_\mu\left(s_u u^+ + t_u u^-\right), s_u u^+\rangle = 0 = \langle I'_\mu\left(s_u u^+ + t_u u^-\right), t_u u^-\rangle.$$

We now claim that $0 < s_u, t_u \leq 1$. Indeed, by $\{u_n\} \subset \mathcal{M}_\mu$, we have $\langle I'_\mu(u_n), u_n^\pm\rangle = 0$, that is,

$$\|u_n^\pm\|^2 + H(u_n) = \int_0^1 |u_n^\pm|^p \ln |u_n^\pm|^2 \, \mathrm{d}x + \mu \int_0^1 f\left(u_n^\pm\right) u_n^\pm \, \mathrm{d}x.$$

Therefore, by the weak lower semicontinuity of norm, Fatou's lemma, (3.6), and Lemma 3.1 we have

$$\|u^\pm\|^2 + H(u) \leq \int_0^1 |u^\pm|^p \ln |u^\pm|^2 \, \mathrm{d}x + \mu \int_{\mathbb{R}^3} f\left(u^\pm\right) u^\pm \, \mathrm{d}x.$$

That is,

$$\langle I'_\mu\left(u\right), u^\pm\rangle \leq \liminf_{n\to\infty} \langle I'_\mu\left(u_n\right), u_n^\pm\rangle = 0. \tag{3.10}$$

By (3.10) and similar to the proof in Lemma 2.2, we have $s_u, t_u \leq 1$.

Our next step is show that $I_\mu(u_\mu) = m_\mu$. Remark 1.1 shows that $H(s) := sf(s) - pF(s)$ is a nonnegative function, increasing in $|s|$. Hence, by the weaker lower semicontinuity of norm, (3.7), Remark 1.1, $\mu > 0$ and Lemma 3.1, we get

$$\begin{aligned}
m_\mu &\leq I_\mu\left(u_\mu\right) = I_\mu\left(u_\mu\right) - \frac{1}{p}\langle I'_\mu\left(u_\mu\right), u_\mu\rangle \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_\mu\|^2 + \frac{2}{p^2}\int_0^1 |u_\mu|^p \, \mathrm{d}x + \frac{\mu}{p}\int_0^1 \left[f(u_\mu)u_\mu - pF(u_\mu)\right] \, \mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\|s_u u^+\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\|t_u u^-\|^2 + 2\left(\frac{1}{2} - \frac{1}{p}\right) s_u t_u H(u) \\
&\quad + \frac{2}{p^2} s_u^p \int_0^1 |u^+|^p \, \mathrm{d}x + \frac{2}{p^2} t_u^p \int_0^1 |u^-|^p \, \mathrm{d}x \\
&\quad + \frac{\mu}{p}\int_0^1 \left(f\left(s_u u^+\right) s_u u^+ - pF\left(s_u u^+\right)\right) \, \mathrm{d}x \\
&\quad + \frac{\mu}{p}\int_0^1 \left(f\left(t_u u^-\right) t_u u^- - pF\left(t_u u^-\right)\right) \, \mathrm{d}x \\
&\leq \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 + \frac{2}{p^2}\int_0^1 |u|^p \, \mathrm{d}x + \frac{\mu}{p}\int_0^1 \left(f\left(u\right)u - pF\left(u\right)\right) \, \mathrm{d}x \\
&\leq \liminf_{n\to\infty}\left[\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + \frac{2}{p^2}\int_0^1 |u_n|^p \, \mathrm{d}x + \frac{\mu}{p}\int_0^1 \left(f\left(u_n\right)u_n - pF\left(u_n\right)\right) \, \mathrm{d}x\right] \\
&= \liminf_{n\to\infty}\left(I_\mu\left(u_n\right) - \frac{1}{p}\langle I'_\mu\left(u_n\right), u_n\rangle\right) \\
&= \liminf_{n\to\infty} I_\mu\left(u_n\right) = m_\mu,
\end{aligned}$$

and if $s_u < 1$ or $t_u < 1$, then the above inequality is strict. Hence, it follows that $s_u = t_u = 1$. Thus, $u_\mu \in \mathcal{M}_\mu$ and $I_\mu\left(u_\mu\right) = m_\mu$. This completes the proof. □

Proof of Theorem 1.2. From Lemma 2.5 and Lemma 3.2, we deduce that u_μ is a least energy sign-changing solution for problem (1.1). □

4 Critical case

Lemma 4.1 (Critical case). *There exists $\mu^* > 0$ such that if $\mu \geq \mu^*$ and $\{u_n\} \subset \mathcal{M}_\mu$ is a minimizing sequence for m_μ, then*

$$\int_0^1 f\left(u_n^{\pm}\right) u_n^{\pm}\, \mathrm{d}x \to \int_0^1 f\left(u^{\pm}\right) u^{\pm}\, \mathrm{d}x \quad \textit{and} \quad \int_0^1 F\left(u_n^{\pm}\right)\, \mathrm{d}x \to \int_0^1 F\left(u^{\pm}\right)\, \mathrm{d}x$$

hold for some $u \in X$.

Proof. Arguing as in Lemma 3.1, it is sufficient to prove that $\{h\left(u_n^{\pm}\right)\}$ is convergent in $L^1(0,1)$ for appropriate $\mu > 0$, where $\{h\left(u_n^{\pm}(x)\right)\}$ is defined in (3.2).

Let sequence $\{u_n\} \subset \mathcal{M}_\mu$ satisfy $\lim_{n\to\infty} I_\mu\left(u_n\right) = m_\mu$ and $\nu > 0$. Since Lemma 2.3(ii) and Lemma 2.4, there exists $\mu^* > 0$ such that when $\mu \geq \mu^*$, there holds

$$\limsup_{n\to\infty} \|u_n^{\pm}\|^2 < \frac{\pi\omega}{\alpha_0 + \nu}. \tag{4.1}$$

Now, considering $s, s' > 1$ with $1/s + 1/s' = 1$ and s close to 1 , we get that

$$|u_n^{\pm}|^q \to |u^{\pm}|^q \quad \text{in } L^{s'}(0,1). \tag{4.2}$$

Moreover, choosing $\alpha = \alpha_0 + \nu$, from (4.1), we get that

$$\int_0^1 e^{\alpha s|u_n^{\pm}(x)|^2}\, \mathrm{d}x = \int_0^1 e^{(\alpha_0+\nu)s|u_n^{\pm}(x)|^2}\, \mathrm{d}x \leq \int_0^1 e^{\pi\omega s(|u_n^{\pm}|/\|u_n^{\pm}\|)^2}\, \mathrm{d}x.$$

It follows from $s > 1$ close to 1 and the fractional Trudinger–Moser inequality (1.3) that there exists $K_{\pi\omega s} > 0$ such that

$$\int_0^1 e^{\alpha s|u_n^{\pm}(x)|^2}\, \mathrm{d}x \leq K_{\pi\omega s}. \tag{4.3}$$

Since $e^{\alpha|u_n^{\pm}(x)|^2} \to e^{\alpha|u^{\pm}(x)|^2}$ a.e. in $(0,1)$. From (4.3) and [16, Lemma 4.8, Chapter 1], we obtain that

$$e^{\alpha|u_n^{\pm}|^2} \rightharpoonup e^{\alpha|u^{\pm}|^2} \quad \text{weakly in } L^s(0,1). \tag{4.4}$$

Hence, by (4.2), (4.4) and [16, Lemma 4.8, Chapter 1] again, we conclude that

$$\int_0^1 f\left(u_n^{\pm}\right) u_n^{\pm}\, \mathrm{d}x \to \int_0^1 f\left(u^{\pm}\right) u^{\pm}\, \mathrm{d}x.$$

Hence, we complete the proof. □

Lemma 4.2 (Critical case). *If $\mu \geq \mu^*$, then there exists some $u_\mu \in \mathcal{M}_\mu$ such that $I_\mu(u_\mu) = m_\mu$.*

Proof. By an argument similar to Lemma 3.2, replacing Lemma 3.1 by Lemma 4.1, we can obtain the same conclusion. □

Proof of Theorem 1.3. From Lemma 2.5 and Lemma 4.2, we deduce that u_μ is a least energy sign-changing solution for problem (1.1). □

Acknowledgements

The author expresses his gratitude to the anonymous referees for useful comments and remarks.

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