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Existence of nontrivial weak solutions for nonuniformly elliptic equation with mixed boundary condition in a variable exponent Sobolev space

Junichi Aramaki ✉

Division of Science, Faculty of Science and Engineering, Tokyo Denki University, Hatoyama-machi, Saitama, 350-0394, Japan

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Abstract. In this paper, we consider a mixed boundary value problem for nonuniformly elliptic equation in a variable exponent Sobolev space containing $p(\cdot)$-Laplacian and mean curvature operator. More precisely, we are concerned with the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show the existence of a nontrivial weak solution and at least two nontrivial weak solutions according to some hypotheses on given functions.

Keywords: $p(\cdot)$-Laplacian type equation, mean curvature operator, mixed boundary value problem, Ekeland variational principle.

2020 Mathematics Subject Classification: 35H30, 35D05, 35J60, 35J70.

1 Introduction

In this paper, we consider the following problem

$$\begin{cases} -\mathrm{div}\,[\boldsymbol{a}(x, \nabla u(x))] = f(x, u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma_1, \\ \boldsymbol{n}(x) \cdot \boldsymbol{a}(x, \nabla u(x)) = g(x, u(x)) & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz-continuous ($C^{0,1}$ for short) boundary Γ satisfying that

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are disjoint open subsets of } \Gamma \text{ such that } \overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma \text{ and } \Gamma_1 \neq \emptyset, \tag{1.2}$$

and the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. The function $\boldsymbol{a}(x, \boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^d$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$. Then the operator $u \mapsto \mathrm{div}\,[\boldsymbol{a}(x, \nabla u(x))]$ is more general than the $p(\cdot)$-Laplacian $\Delta_{p(x)} u(x) = \mathrm{div}\,[|\nabla u(x)|^{p(x)-2} \nabla u(x)]$ and the mean curvature

✉ Email: aramaki@hctv.ne.jp

operator div $[(1+|\nabla u(x)|^2)^{(p(x)-2)/2}\nabla u(x)]$. These generalities bring about difficulties and requires some conditions.

We impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $g:\Gamma_2\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions satisfying some conditions.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [28]), in electrorheological fluids (Diening [7], Halsey [15], Mihăilescu and Rădulescu [18], Růžička [20]).

Over the last two decades, there are many articles on the existence of weak solutions for the Dirichlet boundary condition, that is, in the case $\Gamma_2=\emptyset$ in (1.1), (for example, see Mashiyev et al. [17], Duc and Vu [10], Wei and Chen [22], Yücedağ [25], Nápoli and Mariani [19]).

However, since we can only find a few of papers associate with the problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1). See Aramaki [1–3]. We are convinced of the reason for existence of this paper.

In particular, the authors in [10] considered the problem (1.1) when $p(x)=p=$ const. and $\Gamma_2=\emptyset$, and derived the existence of a nontrivial weak solution to (1.1). This paper is an extension of the article [10] to the case of variable exponent and mixed boundary value problem. In the paper [10], the authors derived the weakly continuous differentiability of the corresponding energy functional and then applied a version of the Mountain-pass lemma introduced in Duc [9]. However, in this paper we show that the corresponding energy functional is of class C^1, and so it suffices to apply the standard Mountain-pass lemma.

The paper is organized as follows. Section 2 consists of two subsections. In Subsection 2.1, we recall some results on variable exponent Lebesgue-Sobolev spaces. In Subsection 2.2, we give the assumptions to the main theorems. In Section 3, we state the main theorems (Theorem 3.3 and Theorem 3.5) on the existence of at least one and two nontrivial weak solutions. The proofs of the main theorems are given in Section 4.

2 Preliminaries and the main theorems

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d\geq 2)$ with a $C^{0,1}$-boundary Γ. Moreover, we assume that Γ satisfies (1.2).

Throughout this paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^d by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a}=(a_1,\dots,a_d)$ and $\boldsymbol{b}=(b_1,\dots,b_d)$ in $\mathbb{R}^d$ by $\boldsymbol{a}\cdot\boldsymbol{b}=\sum_{i=1}^d a_ib_i$ and $|\boldsymbol{a}|=(\boldsymbol{a}\cdot\boldsymbol{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle\cdot,\cdot\rangle_{B^*,B}$.

2.1 Variable exponent Lebesgue and Sobolev spaces

In this subsection, we recall some well-known results on variable exponent Lebesgue–Sobolev spaces. See Diening et al. [8], Fan and Zhang [12], Kováčik and Rákosník [16] and references therein for more detail. Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^d$ with a $C^{0,1}$-boundary Γ and Ω is locally on the same side of Γ. Define $\mathcal{P}(\Omega)=\{p:\Omega\to[1,\infty);p$ is a measurable function$\}$, and for any $p\in\mathcal{P}(\Omega)$, put

$$p^+=\operatorname*{ess\,sup}_{x\in\Omega}p(x)\quad\text{and}\quad p^-=\operatorname*{ess\,inf}_{x\in\Omega}p(x).$$

For any measurable function u on Ω, a modular $\rho_{p(\cdot)} = \rho_{p(\cdot),\Omega}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\}$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0; \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define, for any integer $m \geq 0$,

$$W^{m,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); \partial^\alpha u \in L^{p(\cdot)}(\Omega) \text{ for } |\alpha| \leq m\},$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index, $|\alpha| = \sum_{i=1}^d \alpha_i$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ and $\partial_i = \partial/\partial x_i$, endowed with the norm

$$\|u\|_{W^{m,p(\cdot)}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^{p(\cdot)}(\Omega)}.$$

Of course, $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$. Define

$$W_0^{m,p(\cdot)}(\Omega) = \text{ the closure of the set of } W^{m,p(\cdot)}(\Omega)\text{-functions with compact supports in } \Omega.$$

The following three propositions are well known (see Fan et al. [14,22], Fan and Zhao [13], Zhao et al. [27], and [25]).

Proposition 2.1. *Let $p \in \mathcal{P}(\Omega)$ and let $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, \dots)$ Then we have*

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1, > 1)$.

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Longleftrightarrow \rho_{p(\cdot)}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let $p \in \mathcal{P}_+(\Omega)$, where*

$$\mathcal{P}_+(\Omega) = \{p \in \mathcal{P}(\Omega); 1 < p^- \leq p^+ < \infty\}.$$

For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have

$$\int_\Omega |u(x)v(x)| dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \leq 2\|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Here and from now on, $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$, that is, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

For $p \in \mathcal{P}(\Omega)$, define

$$p^*(x) = \begin{cases} \frac{dp(x)}{d-p(x)} & \text{if } p(x) < d, \\ \infty & \text{if } p(x) \geq d. \end{cases}$$

Proposition 2.3. *Let Ω be a bounded domain with $C^{0,1}$-boundary and let $p \in \mathcal{P}_+(\Omega)$ and $m \geq 0$ be an integer. Then we have the following:*

(i) *The spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.*

(i) *If $q(\cdot) \in \mathcal{P}_+(\Omega)$ and satisfies $q(x) \leq p(x)$ for all $x \in \Omega$, then $W^{m,p(\cdot)}(\Omega) \hookrightarrow W^{m,q(\cdot)}(\Omega)$, where $\hookrightarrow$ means that the embedding is continuous.*

(i) *If $q(x) \in \mathcal{P}_+(\Omega)$ satisfies that $q(x) \leq p^*(x)$ for all $x \in \Omega$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous. Moreover, if $q(x) < p^*(x)$ for all $x \in \Omega$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.*

We say that $p \in \mathcal{P}(\Omega)$ belongs to $\mathcal{P}^{\log}(\Omega)$ if p has the log-Hölder continuity in Ω, that is, $p : \Omega \to \mathbb{R}$ satisfies that there exists a constant $C_{\log}(p) > 0$ such that

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1/|x-y|)} \quad \text{for all } x, y \in \Omega.$$

We also write $\mathcal{P}^{\log}_+(\Omega) = \{p \in \mathcal{P}^{\log}(\Omega); 1 < p^- \leq p^+ < \infty\}$.

Proposition 2.4. *If $p \in \mathcal{P}^{\log}_+(\Omega)$ and $m \geq 0$ is an integer, then $\mathcal{D}(\Omega) := C^\infty_0(\Omega)$ is dense in $W^{m,p(\cdot)}_0(\Omega)$.*

For the proof, see [8, Corollary 11.2.4].

Next we consider the notion of trace. Let Ω be a domain of $\mathbb{R}^d$ with a $C^{0,1}$-boundary Γ and $p \in \mathcal{P}_+(\overline{\Omega})$. Since $W^{1,p(\cdot)}(\Omega) \subset W^{1,1}_{\text{loc}}(\Omega)$, the trace $\gamma(u) = u\big|_\Gamma$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1_{\text{loc}}(\Gamma)$. We define

$$\text{Tr}(W^{1,p(\cdot)}(\Omega)) = (\text{Tr}\, W^{1,p(\cdot)})(\Gamma) = \{f; f \text{ is the trace to } \Gamma \text{ of a function } F \in W^{1,p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|f\|_{(\text{Tr}\, W^{1,p(\cdot)})(\Gamma)} = \inf\{\|F\|_{W^{1,p(\cdot)}(\Omega)}; F \in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F\big|_\Gamma = f\}$$

for $f \in (\text{Tr}\, W^{1,p(\cdot)})(\Gamma)$, where the infimum can be achieved. Then $(\text{Tr}\, W^{1,p(\cdot)})(\Gamma)$ is a Banach space. More precisely, see [8, Chapter 12]. In the later we also write $F\big|_\Gamma = f$ by $F = f$ on Γ. Moreover, we denote

$$(\text{Tr}\, W^{1,p(\cdot)})(\Gamma_i) = \{f\big|_{\Gamma_i}; f \in (\text{Tr}\, W^{1,p(\cdot)})(\Gamma)\} \quad \text{for } i = 1, 2$$

equipped with the norm

$$\|g\|_{(\text{Tr}\, W^{1,p(\cdot)})(\Gamma_i)} = \inf\{\|f\|_{(\text{Tr}\, W^{1,p(\cdot)})(\Gamma)}; f \in (\text{Tr}\, W^{1,p(\cdot)})(\Gamma) \text{ satisfying } f\big|_{\Gamma_i} = g\},$$

where the infimum can also be achieved, so for any $g \in (\text{Tr}\, W^{1,p(\cdot)})(\Gamma_i)$, there exists $F \in W^{1,p(\cdot)}(\Omega)$ such that $F\big|_{\Gamma_i} = g$ and $\|F\|_{W^{1,p(\cdot)}(\Omega)} = \|g\|_{(\text{Tr}\, W^{1,p(\cdot)})(\Gamma_i)}$.

Let $q \in \mathcal{P}_+(\Gamma) := \{q \in \mathcal{P}(\Gamma); q^- > 1\}$ and denote the surface measure on Γ induced from the Lebesgue measure dx on Ω by $d\sigma$. We define

$$L^{q(\cdot)}(\Gamma) = \Big\{ u; u : \Gamma \to \mathbb{R} \text{ is a measurable function with respect to } d\sigma \\ \text{satisfying } \int_\Gamma |u(x)|^{q(x)} d\sigma < \infty \Big\}$$

equipped with the norm

$$\|u\|_{L^{q(\cdot)}(\Gamma)} = \inf \left\{ \lambda > 0; \int_\Gamma \left| \frac{u(x)}{\lambda} \right|^{q(x)} d\sigma \le 1 \right\},$$

and we also define a modular on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u) = \int_\Gamma |u(x)|^{q(x)} d\sigma.$$

Proposition 2.5. *We have the following properties.*

(i) $\|u\|_{L^{q(\cdot)}(\Gamma)} \ge 1 \Longrightarrow \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}$.

(ii) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 \Longrightarrow \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}$.

Proposition 2.6. *Let Ω be a bounded domain with a $C^{0,1}$-boundary Γ and let $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$. If $f \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma)$ and there exists a constant $C > 0$ such that*

$$\|f\|_{L^{p(\cdot)}(\Gamma)} \le C\|f\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)}.$$

In particular, if $f \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma_i)$ and $\|f\|_{L^{p(\cdot)}(\Gamma_i)} \le C\|f\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)}$.

For $p \in \mathcal{P}_+(\overline{\Omega})$, define

$$p^\partial(x) = \begin{cases} \frac{(d-1)p(x)}{d-p(x)} & \text{if } p(x) < d, \\ \infty & \text{if } p(x) \ge d. \end{cases}$$

Proposition 2.7. *Let $p \in \mathcal{P}_+(\overline{\Omega})$. Then if $q(x) \in \mathcal{P}_+(\Gamma)$ satisfies $q(x) < p^\partial(x)$ for all $x \in \Gamma$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$ is well defined and compact. In particular, the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Gamma)$ is compact and there exists a constant $C > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Gamma)} \le C\|u\|_{W^{1,p(\cdot)}(\Omega)} \quad \text{for } u \in W^{1,p(\cdot)}(\Omega).$$

For the proof, see Yao [24, Proposition 2.6].

Define a space by

$$X = \{v \in W^{1,p(\cdot)}(\Omega); v = 0 \text{ on } \Gamma_1\}. \tag{2.1}$$

Then it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable Banach space. We show the following Poincaré type inequality (cf. Ciarlet and Dinca [6]).

Lemma 2.8. *Let $p \in \mathcal{P}_+^{\log}(\Omega)$. Then there exists a constant $C = C(\Omega, d, p) > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{\mathbf{L}^{p(\cdot)}(\Omega)} \quad \text{for all } u \in X,$$

where $\|\nabla u\|_{\mathbf{L}^{p(\cdot)}(\Omega)} := \||\nabla u|\|_{L^{p(\cdot)}(\Omega)}$.

In particular, the norm $\|\nabla u\|_{\mathbf{L}^{p(\cdot)}(\Omega)}$ is equivalent to $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ for $u \in X$.

For the direct proof, see Aramaki [4, Lemma 2.5].

Thus we can define the norm on the space X defined by (2.1) so that

$$\|v\|_X = \|\nabla v\|_{L^{p(\cdot)}(\Omega)} \quad \text{for } v \in X, \tag{2.2}$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$ from Lemma 2.8.

2.2 Assumptions to the main theorems

In this subsection, we state the assumptions to the main theorems. Let $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$ be fixed.

Let $A : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a function satisfying that for a.e. $x \in \Omega$, the function $A(x,\cdot) : \mathbb{R}^d \ni \boldsymbol{\xi} \mapsto A(x,\boldsymbol{\xi})$ is of C^1-class, and for all $\boldsymbol{\xi} \in \mathbb{R}^d$, the function $A(\cdot,\boldsymbol{\xi}) : \Omega \ni x \mapsto A(x,\boldsymbol{\xi})$ is measurable. Moreover, suppose that $A(x,\mathbf{0}) = 0$ and put $\boldsymbol{a}(x,\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} A(x,\boldsymbol{\xi})$. Then $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function. Assume that there exist constants $c_0, k_0, k_1 > 0$ and nonnegative functions $h_0 \in L^{p'(\cdot)}(\Omega)$ and $h_1 \in L^1_{\text{loc}}(\Omega)$ with $h_1(x) \geq 1$ a.e. $x \in \Omega$ such that the following conditions hold.

(A1) $|\boldsymbol{a}(x,\boldsymbol{\xi})| \leq c_0(h_0(x) + h_1(x)|\boldsymbol{\xi}|^{p(x)-1})$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $x \in \Omega$.

(A2) A is $p(\cdot)$-uniformly convex, that is,

$$A\left(x, \frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right) + k_1 h_1(x)|\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)} \leq \frac{1}{2}A(x,\boldsymbol{\xi}) + \frac{1}{2}A(x,\boldsymbol{\eta})$$
$$\text{for all } \boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^d \text{ and a.e. } x \in \Omega.$$

(A3) A is $p(\cdot)$-subhomogeneous, that is,

$$0 \leq \boldsymbol{a}(x,\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \leq p(x)A(x,\boldsymbol{\xi}) \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d \text{ and a.e. } x \in \Omega.$$

(A4) $A(x,\boldsymbol{\xi}) \geq k_0 h_1(x)|\boldsymbol{\xi}|^{p(x)}$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and a.e. $x \in \Omega$.

Example 2.9.

(i) $A(x,\boldsymbol{\xi}) = \frac{h(x)}{p(x)}|\boldsymbol{\xi}|^{p(x)}$ with $p^- \geq 2$, $h \in L^1_{\text{loc}}(\Omega)$ satisfying $h(x) \geq 1$.

(ii) $A(x,\boldsymbol{\xi}) = \frac{h(x)}{p(x)}((1+|\boldsymbol{\xi}|^2)^{p(x)/2} - 1)$ with $p^- \geq 2$, $h \in L^{p'(\cdot)}(\Omega)$ satisfying $h(x) \geq 1$ a.e. $x \in \Omega$.

Then $A(x,\boldsymbol{\xi})$ and $\boldsymbol{a}(x,\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} A(x,\boldsymbol{\xi})$ satisfy (A1)–(A4).

Remark 2.10. When $h(x) \equiv 1$, (i) corresponds to the $p(\cdot)$-Laplacian and (ii) corresponds to the prescribed mean curvature operator for nonparametric surface.

For the function $h_1 \in L^1_{\text{loc}}(\Omega)$ with $h_1(x) \geq 1$ a.e. $x \in \Omega$, we define a modular

$$\rho_{p(\cdot),h_1(\cdot)}(\nabla v) = \int_\Omega h_1(x)|\nabla v(x)|^{p(x)}dx \quad \text{for } v \in W^{1,p(\cdot)}(\Omega).$$

Define our basic space

$$Y = \{v \in X; \rho_{p(\cdot),h_1(\cdot)}(\nabla v) < \infty\} \tag{2.3}$$

equipped with the norm

$$\|v\|_Y = \inf\left\{\lambda > 0; \rho_{p(\cdot),h_1(\cdot)}\left(\frac{\nabla v}{\lambda}\right) \leq 1\right\},$$

then Y is a Banach space (see Lemma 2.12 below). We note that $C_0^\infty(\Omega) \subset Y$. Since

$$\rho_{p(\cdot),h_1(\cdot)}(\nabla v) = \rho_{p(\cdot)}(h_1^{1/p(\cdot)}\nabla v),$$

we have

$$\|v\|_Y = \|h_1^{1/p(\cdot)}\nabla v\|_{L^{p(\cdot)}(\Omega)}. \tag{2.4}$$

Then we have the following lemma.

Lemma 2.11.

(i) *$Y \hookrightarrow X$ and $\|v\|_X \leq \|v\|_Y$ for all $v \in Y$.*

(ii) *Let $v \in Y$. Then $\|v\|_Y > 1(=1, <1) \Longleftrightarrow \rho_{p(\cdot),h_1(\cdot)}(\nabla v) > 1(=1, <1)$.*

(iii) *Let $v \in Y$. Then $\|v\|_Y > 1 \Longrightarrow \|v\|_Y^{p^-} \leq \rho_{p(\cdot),h_1(\cdot)}(\nabla v) \leq \|v\|_Y^{p^+}$.*

(iv) *Let $v \in Y$. Then $\|v\|_Y < 1 \Longrightarrow \|v\|_Y^{p^+} \leq \rho_{p(\cdot),h_1(\cdot)}(\nabla v) \leq \|v\|_Y^{p^-}$.*

(v) *Let $u_n, u \in Y$. Then $\lim_{n\to\infty} \|u_n - u\|_Y = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot),h_1(\cdot)}(\nabla u_n - \nabla u) = 0$.*

(vi) *Let $u_n \in Y$. Then $\|u_n\|_Y \to \infty$ as $n \to \infty \Longleftrightarrow \rho_{p(\cdot),h_1(\cdot)}(\nabla u_n) \to \infty$ as $n \to \infty$.*

When $q \in \mathcal{P}_+^{\log}(\overline{\Omega})$ satisfies $q(x) \leq p^*(x)$ for all $x \in \Omega$, define

$$\lambda_q = \inf\left\{\frac{\|v\|_Y}{\|v\|_{L^{q(\cdot)}(\Omega)}}; v \in Y \setminus \{0\}\right\}. \tag{2.5}$$

By Proposition 2.3 and Lemma 2.11, there exists a constant $c > 0$ such that $\|v\|_{L^{q(\cdot)}(\Omega)} \leq c\|v\|_X \leq c\|v\|_Y$ for all $v \in Y$, so we can see that $\lambda_q > 0$.

When $q \in \mathcal{P}_+^{\log}(\overline{\Omega})$ satisfies $q(x) \leq p^\partial(x)$ for all $x \in \Gamma_2$, define

$$\mu_q = \inf\left\{\frac{\|v\|_Y}{\|v\|_{L^{q(\cdot)}(\Gamma_2)}}; v \in Y \text{ with } v \neq 0 \text{ on } \Gamma_2\right\}. \tag{2.6}$$

By Proposition 2.7 and Lemma 2.11, there exists a constant $c > 0$ such that $\|v\|_{L^{q(\cdot)}(\Gamma_2)} \leq c\|v\|_X \leq c\|v\|_Y$ for all $v \in Y$, so we can see that $\mu_q > 0$.

Lemma 2.12. *The space $(Y, \|\cdot\|_Y)$ is a reflexive Banach space.*

Proof. Since $\|v\|_Y = \|h_1^{1/p(\cdot)}\nabla v\|_{L^{p(\cdot)}(\Omega)}$ for $v \in Y(\subset X)$, it is clear that Y is a normed linear space. Let $\{v_n\}$ be a Cauchy sequence in Y. Then $\{\|v_n\|_Y\}$ is bounded, so $\{\rho_{p(\cdot),h_1(\cdot)}(\nabla v_n)\}$ is bounded from Lemma 2.11 (vi) and we have

$$\lim_{n\to\infty} \liminf_{j\to\infty} \int_\Omega h_1(x)|\nabla u_j(x) - \nabla u_n(x)|^{p(x)} dx = 0.$$

Since $\|v\|_X \leq \|v\|_Y$ for all $v \in Y$, $\{v_n\}$ is also a Cauchy sequence in X. Hence there exists $v \in X$ such that $v_n \to v$ in X, that is, $\nabla v_n \to \nabla v$ in $\boldsymbol{L}^{p(\cdot)}(\Omega)$. So there exists a subsequence $\{v_{n'}\}$ of $\{v_n\}$ such that $\nabla v_{n'}(x) \to \nabla v(x)$ a.e. in Ω. By the Fatou lemma,

$$\int_\Omega h_1(x)|\nabla v(x)|^{p(x)}dx \leq \liminf_{n'\to\infty} \int_\Omega h_1(x)|\nabla v_{n'}(x)|^{p(x)}dx < \infty.$$

Thereby $v \in Y$. Applying again the Fatou lemma,

$$\lim_{n'\to\infty} \int_\Omega h_1(x)|\nabla v(x) - \nabla v_{n'}(x)|^{p(x)}dx \leq \lim_{n'\to\infty} \liminf_{j'\to\infty} \int_\Omega h_1(x)|\nabla v_{j'}(x) - \nabla v_{n'}(x)|^{p(x)}dx = 0.$$

This implies $v_{n'} \to v$ in Y. Since $\{v_n\}$ is a Cauchy sequence in Y, we see that $v_n \to v$ in Y, so $(Y, \|\cdot\|_Y)$ is a Banach space.

We claim that $(Y, \|\cdot\|_Y)$ is a uniformly convex Banach space. Since $\boldsymbol{L}^{p(\cdot)}(\Omega)$ is uniformly convex, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$ satisfy $\|\boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \leq 1$, $\|\boldsymbol{v}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \leq 1$ and $\|\boldsymbol{u} - \boldsymbol{v}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} > \varepsilon$, then $\|(\boldsymbol{u}+\boldsymbol{v})/2\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} < 1 - \delta$. Thus if $u, v \in Y$ satisfy $\|u\|_Y \leq 1$, $\|v\|_Y \leq 1$ and $\|u - v\|_Y > \varepsilon$, then $\|h_1^{1/p(\cdot)}\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \leq 1$, $\|h_1^{1/p(\cdot)}\nabla v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \leq 1$ and $\|h_1^{1/p(\cdot)}\nabla u - h_1^{1/p(\cdot)}\nabla v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} > \varepsilon$ from (2.4). Hence we have

$$\|(h_1^{1/p(\cdot)}\nabla u + h_1^{1/p(\cdot)}\nabla v)/2\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \leq 1 - \delta.$$

Therefore we get $\|(u+v)/2\|_Y \leq 1 - \delta$. This implies the uniform convexity of Y. So it follows from the Milman theorem (cf. Brezis [5, Theorem III.29]) that Y is reflexive. □

We continue to state the assumptions of f and g in (1.1).

Let f is a real Carathéodory function on $\Omega \times \mathbb{R}$ having the following properties.

(F1) $|f(x,t)| \leq c_1(1 + |t|^{q(x)-1})$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$, where c_1 is a positive constant and $q \in \mathcal{P}_+^{\log}(\overline{\Omega})$ such that $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $p^+ < q^-$.

(F2) There exist $\theta > p^+$ and $t_0 > 0$ such that

$$0 < \theta F(x,t) \leq f(x,t)t \quad \text{for all } t \in \mathbb{R} \setminus (-t_0, t_0) \text{ and a.e. } x \in \Omega,$$

where

$$F(x,t) = \int_0^t f(x,s)ds. \tag{2.7}$$

(F3) Let λ_{p^+} be defined by (2.5). There exist $\lambda \in (0, k_0 p^+(\lambda_{p^+})^{p^+}/4)$ and $0 < \delta < 1$ such that

$$\frac{f(x,t)}{|t|^{p^+-2}t} \leq \lambda \quad \text{for all } t \in (-\delta, \delta) \setminus \{0\} \text{ and a.e. } x \in \Omega.$$

Let g be a real Carathéodory function on $\Gamma_2 \times \mathbb{R}$ having the following properties.

(G1) $|g(x,t)| \leq c_2(1 + |t|^{r(x)-1})$ for all $t \in \mathbb{R}$ and a.e. $x \in \Gamma_2$, where c_2 is a positive constant and $r \in \mathcal{P}_+^{\log}(\overline{\Omega})$ such that $r(x) < p^\partial(x)$ for all $x \in \overline{\Gamma_2}$ and $p^+ < r^-$.

(G2) Let θ and t_0 be as in (F2). That is, there exist $\theta > p^+$ and $t_0 > 0$ such that

$$0 < \theta G(x,t) \leq g(x,t)t \quad \text{for all } t \in \mathbb{R} \setminus (-t_0, t_0) \text{ and a.e. } x \in \Gamma_2,$$

where

$$G(x,t) = \int_0^t g(x,s)ds. \tag{2.8}$$

(G3) Let μ_{p^+} be defined by (2.6). There exist $\mu \in (0, k_0 p^+ (\mu_{p^+})^{p^+}/4)$ and $0 < \delta < 1$ such that

$$\frac{g(x,t)}{|t|^{p^+-2}t} \leq \mu \quad \text{for all } t \in (-\delta, \delta) \setminus \{0\} \text{ and a.e. } x \in \Gamma_2.$$

3 Main theorems

In this section, we state the main theorems.

Definition 3.1. We say $u \in Y$ is a weak solution of (1.1) if u satisfies that

$$\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla v(x) dx = \int_\Omega f(x, u(x)) v(x) dx + \int_{\Gamma_2} g(x, u(x)) v(x) d\sigma \quad \text{for all } v \in Y. \tag{3.1}$$

Remark 3.2. Since $\{\varphi \in C^\infty(\overline{\Omega}); \varphi = 0 \text{ on } \Gamma_1\} \subset Y$, if $u \in Y$ satisfies (3.1), then the equation (1.1) holds in the distribution sense.

Then we obtain the following two theorems.

Theorem 3.3. *Let Ω be a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a $C^{0,1}$-boundary Γ satisfying* (1.2)*. Under the hypotheses (A1)-(A4), (F1)-(F3) and (G1)-(G3), the problem* (1.1) *has a nontrivial weak solution.*

Remark 3.4. This theorem extends the result of [10] in which the authors considered the case where $p(x) = p = \text{const.}$ and $\Gamma_2 = \emptyset$.

We impose one more assumption.

(F4) There exist a constant $c > 0$ and $0 < m < 1$ such that $f(x,t) \geq ct^{m-1}$ for $0 < t \leq \delta$ and a.e. $x \in \Omega$, where $\delta > 0$ is as in (F3).

Theorem 3.5. *Addition to the hypotheses of Theorem 3.3, assume that (F4) also holds. Then the problem* (1.1) *has at least two nontrivial weak solutions.*

Remark 3.6. The authors in [17] considered the equation

$$-\text{div}\,[\boldsymbol{a}(x, \nabla u(x))] = m(x)|u(x)|^{r(x)-2}u(x) + n(x)|u(x)|^{s(x)-2}u(x)$$

and $\Gamma_2 = \emptyset$. The authors got the same result of Theorem 3.5 under stronger hypotheses than (A1) and (A4), that is, $h_1(x) \equiv 1$. However, they use an inequality $A(x, t\boldsymbol{\xi}) \leq t^{p(x)} A(x, \boldsymbol{\xi})$ for small $t > 0$ which does not hold for the function in Example 2.9 (ii). To overcome their mistake, we assume a stronger condition (F4).

4 Proofs of Theorem 3.3 and Theorem 3.5

In this section, we give proofs of Theorem 3.3 and Theorem 3.5. In order to do so, we use the variational method. Define a functional on Y

$$I(u) = E(u) - J(u) - K(u) \tag{4.1}$$

where

$$E(u) = \int_\Omega A(x, \nabla u(x))dx, \tag{4.2}$$

$$J(u) = \int_\Omega F(x, u(x))dx, \qquad F \text{ is defined by (2.7)}, \tag{4.3}$$

$$K(u) = \int_{\Gamma_2} G(x, u(x))d\sigma, \qquad G \text{ is defined by (2.8)}. \tag{4.4}$$

The proof of Theorem 3.3 consists of several lemmas and propositions.

Lemma 4.1.

(i) $|A(x,\xi)| \le c_0(h_0(x)|\xi| + h_1(x)|\xi|^{p(x)})$ *for all* $\xi \in \mathbb{R}^d$ *and a.e.* $x \in \Omega$.

(ii)
$$E\left(\frac{u+v}{2}\right) + k_1\rho_{p(\cdot),h_1(\cdot)}(\nabla u - \nabla v) \le \frac{1}{2}E(u) + \frac{1}{2}E(v) \quad \text{for all } u, v \in Y$$

and

$$E((1-\tau)u + \tau v) \le (1-\tau)E(u) + \tau E(v) \quad \text{for all } u, v \in Y \text{ and } \tau \in [0,1].$$

(iii) *There exists a constant* $c_3 > 0$ *such that* $|F(x,t)| \le c_3(1 + |t|^{q(x)})$ *for all* $t \in \mathbb{R}$ *and a.e.* $x \in \Omega$.

(iv) *There exists* $\gamma \in L^\infty(\Omega)$ *such that* $\gamma(x) > 0$ *a.e.* $x \in \Omega$ *and* $F(x,t) \ge \gamma(x)t^\theta$ *for all* $t \in [t_0, \infty)$ *and a.e.* $x \in \Omega$.

(v) *There exists a constant* $c_4 > 0$ *such that* $|G(x,t)| \le c_4(1 + |t|^{r(x)})$ *for all* $t \in \mathbb{R}$ *and a.e.* $x \in \Gamma_2$.

(vi) *There exists* $\delta \in L^\infty(\Gamma_2)$ *such that* $\delta(x) > 0$ *a.e.* $x \in \Gamma_2$ *and* $G(x,t) \ge \delta(x)t^\theta$ *for all* $t \in [t_0, \infty)$ *and a.e.* $x \in \Gamma_2$.

Proof. (i) Using (A1), we have

$$\begin{aligned}
|A(x,\xi)| &= |A(x,\xi) - A(x,\mathbf{0})| \\
&= \left| \int_0^1 \frac{d}{dt} A(x, t\xi)dt \right| \\
&= \left| \int_0^1 \boldsymbol{a}(x, t\xi) \cdot \xi dt \right| \\
&\le c_0 \int_0^1 (h_0(x) + h_1(x)t^{p(x)-1}|\xi|^{p(x)-1})|\xi|dt \\
&\le c_0(h_0(x)|\xi| + h_1(x)|\xi|^{p(x)}).
\end{aligned}$$

(ii) The first inequality easily follows from (A2). Since $A(x,\xi)$ is continuous with respect to ξ, it follows from (A2) that $A(x, (1-\tau)\xi + \tau\eta) \le (1-\tau)A(x,\xi) + \tau A(x,\eta)$ for all $\xi, \eta \in \mathbb{R}^d$ and $\tau \in [0,1]$, so the second inequality follows from this inequality.

(iii) From (F1),

$$|F(x,t)| = \left| \int_0^t f(x,\tau)d\tau \right| \le c_1 \left| \int_0^t (1 + |\tau|^{q(x)-1})d\tau \right| \le c_1 \left(|t| + \frac{1}{q(x)}|t|^{q(x)} \right).$$

Since $q(x) > 1$, we have $|t| \le 1 + |t|^{q(x)}$, so (iii) follows.

(iv) From (F2), for $t \ge t_0$,

$$0 < \theta F(x,t) \le f(x,t)t. \tag{4.5}$$

Put $\gamma(x) = F(x, t_0)t_0^{-\theta}$. Then $\gamma(x) > 0$ a.e. $x \in \Omega$ and it follows from (iii) that

$$\gamma(x) \le c_3(1 + t_0^{q(x)})t_0^{-\theta} \le c_3(1 + \max\{t_0^{q^+}, t_0^{q^-}\})t_0^{-\theta} < \infty.$$

So $\gamma \in L^\infty(\Omega)$. From (4.5),

$$\frac{\theta}{\tau} \le \frac{f(x,\tau)}{F(x,\tau)} = \frac{\frac{\partial F}{\partial \tau}(x,\tau)}{F(x,\tau)}.$$

Integrating this inequality over (t_0, t), we have

$$\theta \log \frac{t}{t_0} \le \log \frac{F(x,t)}{F(x,t_0)} \quad \text{for all } t \ge t_0.$$

This implies $F(x,t) \ge \gamma(x)t^\theta$ for all $t \ge t_0$.

(v) and (vi) follow from the same arguments as (iii) and (iv), respectively. □

Proposition 4.2. *The functionals $E, J, K \in C^1(Y, \mathbb{R})$ and the Fréchet derivatives E', J' and K' satisfy the following equalities.*

$$\langle E'(u), v\rangle_{Y^*,Y} = \int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla v(x) dx, \tag{4.6}$$

$$\langle J'(u), v\rangle_{Y^*,Y} = \int_\Omega f(x, u(x))v(x) dx, \tag{4.7}$$

$$\langle K'(u), v\rangle_{Y^*,Y} = \int_{\Gamma_2} g(x, u(x))v(x) d\sigma \tag{4.8}$$

for all $u, v \in Y$.

Proof. Step 1. We show that E is continuous on Y. Let $u_n \to u$ in Y as $n \to \infty$. Then from (2.4),

$$\|h_1^{1/p(\cdot)}\nabla u_n - h_1^{1/p(\cdot)}\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \to 0 \quad \text{as } n \to \infty. \tag{4.9}$$

From [2, Proposition A.1], there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $k \in L^{p(\cdot)}(\Omega)$ such that $h_1(x)^{1/p(x)}\nabla u_{n'}(x) \to h_1(x)^{1/p(x)}\nabla u(x)$ a.e. $x \in \Omega$, and since $h_1(x) \ge 1$ a.e. $x \in \Omega$,

$$|\nabla u_{n'}(x)| \le |h_1(x)^{1/p(x)}\nabla u_{n'}(x)| \le k(x) \quad \text{a.e. } x \in \Omega.$$

In particular, $\nabla u_{n'}(x) \to \nabla u(x)$ a.e. $x \in \Omega$. Since $A(x, \boldsymbol{\xi})$ is a Carathéodory function, $A(x, \nabla u_{n'}(x)) \to A(x, \nabla u(x))$ a.e. $x \in \Omega$ as $n' \to \infty$. By Lemma 4.1 (i),

$$|A(x, \nabla u_{n'}(x))| \le c_0(h_0(x)|\nabla u_{n'}(x)| + h_1(x)|\nabla u_{n'}(x)|^{p(x)}) \le c_0(h_0(x)k(x) + k(x)^{p(x)}).$$

Since $h_0 \in L^{p'(\cdot)}(\Omega)$ and $k \in L^{p(\cdot)}(\Omega)$, taking the Hölder inequality (Proposition 2.2) into consideration, we see that the last term is an integrable function independent of n'. By the Lebesgue dominated convergence theorem, we have

$$\lim_{n' \to \infty} \int_\Omega A(x, \nabla u_{n'}(x)) dx = \int_\Omega A(x, \nabla u(x)) dx.$$

By the convergent principle (cf. Zeidler [26, Proposition 10.13 (i)], for the full sequence $\{u_n\}$,

$$\lim_{n \to \infty} \int_\Omega A(x, \nabla u_n(x)) dx = \int_\Omega A(x, \nabla u(x)) dx.$$

This means that $E(u_n) \to E(u)$ as $n \to \infty$, so the functional E is continuous in Y.

Step 2. We derive that E is Gateaux differentiable in Y. Let $u, v \in Y$ and $0 < |t| \le 1$. By the mean value theorem,

$$\begin{aligned}\frac{E(u+tv)-E(u)}{t} &= \int_\Omega \frac{A(x,\nabla u(x)+t\nabla v(x))-A(x,\nabla u(x)}{t}dx\\ &= \int_\Omega\int_0^1 \boldsymbol{a}(x,\nabla u(x)+\tau t\nabla v(x))\cdot\nabla v(x)d\tau dx.\end{aligned}$$

From (A1), we have

$$\begin{aligned}&|\boldsymbol{a}(x,\nabla u(x)+\tau t\nabla v(x))\cdot\nabla v(x)|\\ &\quad= c_0(h_0(x)+h_1(x)|\nabla u(x)+\tau t\nabla v(x)|^{p(x)-1})|\nabla v(x)|\\ &\quad\le c_0(h_0(x)|\nabla v(x)|+h_1(x)^{1/p(x)}|\nabla v(x)|h_1(x)^{(p(x)-1)/p(x)}(|\nabla u(x)|+|\nabla v(x)|)^{p(x)-1})\\ &\quad\le c_0(h_0(x)|\nabla v(x)|+h_1(x)^{1/p(x)}|\nabla v(x)|\big((h_1(x)^{1/p(x)}(|\nabla u(x)|+|\nabla v(x)|)\big)^{p(x)-1}.\end{aligned}$$

Here since $u, v \in Y, h_0 \in L^{p'(\cdot)}(\Omega), h_1^{1/p(\cdot)}|\nabla v| \in L^{p(\cdot)}(\Omega)$ and

$$\big((h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)|+h_1(\cdot)^{1/p(\cdot)}|\nabla v(x)|)\big)^{p(\cdot)-1} \in L^{p'(\cdot)}(\Omega),$$

it follows from the Hölder inequality (Proposition 2.2), the last term of the above inequality is an integrable function independent of t. On the other hand, $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function, we have

$$\boldsymbol{a}(x,\nabla u(x)+\tau t\nabla v(x))\cdot\nabla v(x) \to \boldsymbol{a}(x,\nabla u(x))\cdot\nabla v(x)$$

as $t \to 0$. Using again the Lebesgue dominated convergence theorem, we have

$$\frac{E(u+tv)-E(u)}{t} \to \int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla v(x)dx \quad \text{as } t\to 0.$$

Thus E is Gateaux differentiable at u and the Gateaux derivative DE satisfies

$$DE(u)(v) = \int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla v(x)dx.$$

Clearly $DE(u)$ is linear in Y.

Step 3. We show that for every $u \in Y$, we have $DE(u) \in Y^*$. For any $v \in Y$,

$$\begin{aligned}DE(u)(v) &= \int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla v(x)dx\\ &= \int_\Omega h_1(x)^{-1/p(x)}\boldsymbol{a}(x,\nabla u(x))\cdot h_1(x)^{1/p(x)}\nabla v(x)dx.\end{aligned}$$

We note that $\|v\|_Y = \|h_1^{1/p(\cdot)}\nabla v\|_{L^{p(\cdot)}(\Omega)}$ from (2.4). On the other hand, from (A1),

$$\begin{aligned}&\rho_{p'(\cdot)}(h_1^{-1/p(\cdot)}\boldsymbol{a}(\cdot,\boldsymbol{u}(\cdot)))\\ &\quad= \int_\Omega h_1(x)^{-p'(x)/p(x)}|\boldsymbol{a}(x,\nabla u(x))|^{p'(x)}dx\\ &\quad\le \int_\Omega h_1(x)^{-p'(x)/p(x)}\big(c_0(h_0(x)+h_1(x)|\nabla u(x)|^{p(x)-1}\big)^{p'(x)}dx\\ &\quad\le \max\left\{c_0^{(p')^+}, c_0^{(p')^-}\right\}2^{(p')^+-1}\int_\Omega(h_0(x)^{p'(x)}+h_1(x)|\nabla u(x)|^{p(x)})dx < \infty.\end{aligned}$$

Hence $h_1^{-1/p(\cdot)}\boldsymbol{a}(\cdot,\nabla u)\in \boldsymbol{L}^{p'(\cdot)}(\Omega)$. By the Hölder inequality (Proposition 2.2), we have

$$|DE(u)(v)|\le 2\|h_1^{-1/p(\cdot)}\boldsymbol{a}(\cdot,\nabla u(\cdot))\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}\|v\|_Y \quad \text{for all } v\in Y.$$

Hence we see that $DE(u)\in Y^*$ and

$$\|DE(u)\|_{Y^*}\le 2\|h_1^{-1/p(\cdot)}\boldsymbol{a}(\cdot,\nabla u(\cdot))\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}. \tag{4.10}$$

Step 4. We derive that the map $Y\ni u\mapsto DE(u)\in Y^*$ is continuous. Let $u_n\to u$ in Y as $n\to\infty$. Then (4.9) holds. So there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $\widetilde{k}\in L^{p(\cdot)}(\Omega)$ such that $\nabla u_{n'}(x)\to\nabla u(x)$ a.e. $x\in\Omega$ and $h_1(x)^{1/p(x)}|\nabla u_{n'}(x)|\le\widetilde{k}(x)$ a.e. $x\in\Omega$ and all n'. By (4.10),

$$\|DE(u_{n'})-DE(u)\|_{Y^*}\le 2\|h_1(\cdot)^{-1/p(\cdot)}\big(\boldsymbol{a}(\cdot,\nabla u_{n'}(\cdot))-\boldsymbol{a}(\cdot,\nabla u(\cdot))\big)\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}.$$

In order to show that the right-hand side converges to zero, taking Proposition 2.1 into consideration, it suffices to derive that

$$\rho_{p'(\cdot)}\big(h_1(\cdot)^{-1/p(\cdot)}(\boldsymbol{a}(\cdot,\nabla u_{n'}(\cdot))-\boldsymbol{a}(\cdot,\nabla u(\cdot)))\big)\to 0 \quad \text{as } n'\to\infty,$$

that is,

$$\int_\Omega h_1(x)^{-p'(x)/p(x)}|\boldsymbol{a}(x,\nabla u_{n'}(x))-\boldsymbol{a}(x,\nabla u(x))|^{p'(x)}dx\to 0 \quad \text{as } n'\to\infty. \tag{4.11}$$

Since $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function, and $\nabla u_{n'}(x)\to\nabla u(x)$ a.e. $x\in\Omega$, we have

$$h_1(x)^{-p'(x)/p(x)}|\boldsymbol{a}(x,\nabla u_{n'}(x))-\boldsymbol{a}(x,\nabla u(x))|^{p'(x)}\to 0 \quad \text{a.e. } x\in\Omega.$$

As in the argument in Step 3, we have

$$h_1(x)^{-p'(x)/p(x)}|\boldsymbol{a}(x,\nabla u_{n'}(x))|^{p'(x)}\le\max\left\{c_0^{(p')^+},c_0^{(p')^-}\right\}2^{(p')^+-1}(h_0(x)^{p'(x)}+\widetilde{k}(x)^{p(x)}).$$

The right-hand side is an integrable function in Ω independent of n'. By the Lebesgue dominated convergence theorem, (4.11) holds. Thus $\|DE(u_{n'})-DE(u)\|_{Y^*}\to 0$ as $n'\to\infty$. By the convergent principle (cf. [26, Proposition 10.13 (i)], for full sequence $\{u_n\}$ we have $\|DE(u_n)-DE(u)\|_{Y^*}\to 0$ as $n\to\infty$. Therefore, since the Gateaux differential DE is continuous in Y, we see that E is Fréchet differentiable and the Fréchet derivative E' is equal to the Gateaux derivative DE. Hence $E\in C^1(Y,\mathbb{R})$ and (4.6) holds.

Step 5. We show that J and K belong to $C^1(Y,\mathbb{R})$ and (4.7) and (4.8) hold. By Lemma 4.1 (iii) and [2, Proposition 2.12], the Nemytskii operator $N_F: L^{q(\cdot)}(\Omega)\ni u\mapsto F(\cdot,u(\cdot))\in L^1(\Omega)$ is continuous. From (F1), we have $Y\hookrightarrow X\hookrightarrow L^{q(\cdot)}(\Omega)$, so N_F is continuous in Y, so we see that J is continuous in Y. Since $F(x,t)$ is a C^1-function with respect to t, clearly J is Gateaux differentiable in Y and

$$DJ(u)(v)=\int_\Omega f(x,u(x))v(x)dx \quad \text{for all } u,v\in Y.$$

By the Hölder inequality (Proposition 2.2),

$$|DJ(u)(v)|\le 2\|f(\cdot,u(\cdot))\|_{L^{q'(\cdot)}(\Omega)}\|v\|_{L^{q(\cdot)}(\Omega)}\le C\|f(\cdot,u(\cdot))\|_{L^{q'(\cdot)}(\Omega)}\|v\|_Y \quad \text{for all } v\in Y.$$

Hence $DJ(u) \in Y^*$ and $\|DJ(u)\|_{Y^*} \leq C\|f(\cdot,u(\cdot))\|_{L^{q'(\cdot)}(\Omega)}$. Since $|f(x,t)| \leq c_1(1+|t|^{q(x)-1}) = c_1(1+|t|^{q(x)/q'(x)})$ from (F1), Nemytskii operator $N_f : u \mapsto f(\cdot,u(\cdot))$ is continuous from $L^{q(\cdot)}(\Omega)$ to $L^{q'(\cdot)}(\Omega)$ (cf. [1, Proposition 2.9]). Thus if $u_n \to u$ in $L^{q(\cdot)}(\Omega)$, then

$$\|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\|_{L^{q'(\cdot)}(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Since $Y \hookrightarrow X \hookrightarrow L^{q(\cdot)}(\Omega)$, we can see that $J \in C^1(Y,\mathbb{R})$ and (4.7) holds. Similarly, we can prove that $K \in C^1(Y,\mathbb{R})$ and (4.8) holds. □

Remark 4.3. When $p(\cdot) = p =$ const. and $\Gamma_2 = \emptyset$, the authors of [10] only prove the weakly continuously differentiable on Y, and so they must use a version of the Mountain-pass lemma introduced in [9]. However, since we derived that E belongs to $C^1(Y,\mathbb{R})$, it suffices to use the standard Mountain-pass lemma later.

Proposition 4.4.

(i) *The functionals J and K are weakly continuous in Y, that is, if $u_n \to u$ weakly in Y as $n \to \infty$, then $J(u_n) \to J(u)$ and $K(u_n) \to K(u)$ as $n \to \infty$.*

(ii) *The functional E is weakly lower semi-continuous in Y, that is, if $u_n \to u$ weakly in Y as $n \to \infty$, then $E(u) \leq \liminf_{n\to\infty} E(u_n)$.*

(iii) *$E(u) - E(v) \geq \langle E'(v), u-v\rangle_{Y^*,Y}$ for all $u,v \in Y$.*

Proof. (i) Let $u_n \to u$ weakly in Y as $n \to \infty$. Since the embedding $Y \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact, we see that $u_n \to u$ strongly in $L^{q(\cdot)}(\Omega)$. Since J and K are continuous on $L^{q(\cdot)}(\Omega)$, we see that $J(u_n) \to J(u)$ and $K(u_n) \to K(u)$ as $n \to \infty$.

(ii) $A(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^d$ and $A(x,\boldsymbol{\xi}) \geq 0$ by (A4). Moreover, from (A2), $A(x,\boldsymbol{\xi})$ is convex with respect to $\boldsymbol{\xi}$ for a.e. $x \in \Omega$. If $u_n \to u$ weakly in Y, then $u_n, u \in W^{1,1}(\Omega)$ and $u_n \to u$ strongly in $L^1(\Omega)$ and $\nabla u_n \to \nabla u$ weakly in $\boldsymbol{L}^1(\Omega)$. Hence it follows from Struwe [21, Theorem 1.6, p. 9] that $E(u) \leq \liminf_{n\to\infty} E(u_n)$.

(iii) Since E is convex function in Y, for $u,v \in Y$ and $0 < \tau < 1$,

$$\begin{aligned}\frac{E(v+\tau(u-v)) - E(v)}{\tau} &= \frac{E((1-\tau)v+\tau u) - E(v)}{\tau} \\ &\leq \frac{(1-\tau)E(v)+\tau E(u) - E(v)}{\tau} \\ &= E(u) - E(v).\end{aligned}$$

Letting $\tau \to +0$, we get $\langle E'(v), u-v\rangle_{Y^*,Y} \leq E(u) - E(v)$, so (iii) holds. □

Lemma 4.5.

(i) *There exist constants $k_3 > 0$ and $c_3 > 0$ such that*

$$I(u) \geq \|u\|_Y^{p^+}\left(k_3 - c_3\left(\|u\|_Y^{q^- - p^+} + \|u\|_Y^{r^- - p^+}\right)\right) \text{ for all } u \in Y \text{ with } \|u\|_Y < 1.$$

(ii) *There exist constants $c_3 > 0$ and $k_4 \in \mathbb{R}$ such that*

$$I(u) \geq \|u\|_Y\left(c_4 \min\left\{\|u\|_Y^{p^+-1}, \|u\|_Y^{p^--1}\right\} - \frac{1}{\theta}\|I'(u)\|_{Y^*}\right) + k_4 \quad \text{for all } u \in Y.$$

Proof. (i) From (F3), for a.e. $x \in \Omega$,

$$F(x,t) = \int_0^t f(x,s)dx \le \frac{\lambda}{p^+}|t|^{p^+} \quad \text{for all } t \in (-\delta,\delta).$$

On the other hand, by Lemma 4.1 (iii), there exists $c_3' > 0$ such that $|F(x,t)| \le c_3'|t|^{q(x)}$ for all $t \in \mathbb{R} \setminus (-\delta,\delta)$. Hence

$$F(x,t) \le \frac{\lambda}{p^+}|t|^{p^+} + c_3'|t|^{q(x)} \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

Therefore, we have

$$\begin{aligned} J(u) = \int_\Omega F(x,u(x))dx &\le \frac{\lambda}{p^+}\int_\Omega |u(x)|^{p^+}dx + c_3'\int_\Omega |u(x)|^{q(x)}dx \\ &\le \frac{\lambda}{p^+}\|u\|^{p^+}_{L^{p^+}(\Omega)} + c_3' \max\left\{\|u\|^{q^+}_{L^{q(\cdot)}(\Omega)}, \|u\|^{q^-}_{L^{q(\cdot)}(\Omega)}\right\}. \end{aligned}$$

Similarly, there exists $c_4' > 0$ such that

$$K(u) \le \frac{\mu}{p^+}\|u\|^{p^+}_{L^{p^+}(\Gamma_2)} + c_4' \max\left\{\|u\|^{p^+}_{L^{r(\cdot)}(\Gamma_2)}, \|u\|^{r^-}_{L^{r(\cdot)}(\Gamma_2)}\right\}.$$

Since $p^+ < q^- \le q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ from (F1), we have $Y \hookrightarrow X \hookrightarrow L^{p^+}(\Omega)$, $L^{q(\cdot)}(\Omega)$. By (2.5), $\|u\|_{L^{p^+}(\Omega)} \le \frac{1}{\lambda_{p^+}}\|u\|_Y$ and $\|u\|_{L^{q(\cdot)}(\Omega)} \le \frac{1}{\lambda_q}\|u\|_Y$ for all $u \in Y$. Since we have $p^+ < r^- \le r(x) < p^\partial(x)$ for all $x \in \overline{\Gamma_2}$ from (G2), it follows from (2.6) that we can see that $Y \hookrightarrow X \hookrightarrow L^{p^+}(\Gamma_2), L^{r(\cdot)}(\Gamma_2)$. Thus we have $\|u\|_{L^{p^+}(\Gamma_2)} \le \frac{1}{\mu_{p^+}}\|u\|_Y$ and $\|u\|_{L^{r(\cdot)}(\Gamma_2)} \le \frac{1}{\mu_r}\|u\|_Y$ for all $u \in Y$. When $\|u\|_Y < 1$, there exist positive constants c_5 and c_6 such that

$$\begin{aligned} J(u) &\le \frac{\lambda}{p^+}\frac{1}{(\lambda_{p^+})^{p^+}}\|u\|_Y^{p^+} + c_5\|u\|_Y^{q^-}, \\ K(u) &\le \frac{\mu}{p^+}\frac{1}{(\mu_{p^+})^{p^+}}\|u\|_Y^{p^+} + c_6\|u\|_Y^{r^-}. \end{aligned}$$

On the other hand, from (A4),

$$E(u) = \int_\Omega A(x,\nabla u(x))dx \ge k_0\int_\Omega h_1(x)|\nabla u(x)|^{p(x)}dx \ge k_0\|u\|_Y^{p^+}.$$

Thus we have

$$\begin{aligned} I(u) = E(u) - J(u) - K(u) &\ge \frac{k_0}{2}\|u\|_Y^{p^+} - c_5\|u\|_Y^{q^-} - c_6\|u\|_Y^{r^-} \\ &= \|u\|_Y^{p^+}\left(k_3 - c_3(\|u\|_Y^{q^- - p^+} + \|u\|_Y^{r^- - p^+})\right), \end{aligned}$$

where $k_3 = k_0/2$ and $c_3 = \max\{c_5, c_6\}$ for all $u \in Y$ with $\|u\|_Y < 1$.

(ii) From (A3) and (A4), for any $u \in Y$,

$$\begin{aligned} E(u) - \frac{1}{\theta}\langle E'(u),u\rangle_{Y^*,Y} &= \int_\Omega A(x,\nabla u(x))dx - \frac{1}{\theta}\int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla u(x)dx \\ &\ge \int_\Omega A(x,\nabla u(x))dx - \frac{1}{\theta}\int_\Omega p(x)A(x,\nabla u(x))dx \\ &\ge \left(1 - \frac{p^+}{\theta}\right)\int_\Omega A(x,\nabla u(x))dx \\ &\ge \left(1 - \frac{p^+}{\theta}\right)k_0\int_\Omega h_1(x)|\nabla u(x)|^{p(x)}dx \\ &\ge c_4\min\left\{\|u\|_Y^{p^+}, \|u\|_Y^{p^-}\right\}, \end{aligned}$$

where $c_4 = k_0(1 - p^+/\theta) > 0$. Put $\Omega_u = \{x \in \Omega; |u(x)| > t_0\}$ and $\Gamma_u = \{x \in \Gamma_2; |u(x)| > t_0\}$. From (F2) and (G2),

$$\begin{aligned} &\frac{1}{\theta} f(x,u(x))u(x) - F(x,u(x)) \geq 0 \quad \text{for a.e. } x \in \Omega_u, \\ &\frac{1}{\theta} g(x,u(x))u(x) - G(x,u(x)) \geq 0 \quad \text{for a.e. } x \in \Gamma_u, \end{aligned}$$

and there exists a constant $M > 0$ such that

$$\begin{aligned} &\left| \frac{1}{\theta} f(x,u(x))u(x) - F(x,u(x)) \right| \leq M \quad \text{for a.e. } x \in \Omega \setminus \Omega_u, \\ &\left| \frac{1}{\theta} g(x,u(x))u(x) - G(x,u(x)) \right| \leq M \quad \text{for a.e. } x \in \Gamma_2 \setminus \Gamma_u. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\frac{1}{\theta}\langle J'(u), u\rangle_{Y^*,Y} - J(u) \\ &\quad = \int_{\Omega_u} \left(\frac{1}{\theta} f(x,u(x))u(x) - F(x,u(x)) \right) dx + \int_{\Omega\setminus\Omega_u} \left(\frac{1}{\theta} f(x,u(x))u(x) - F(x,u(x)) \right) dx \\ &\quad \geq -M|\Omega \setminus \Omega_u| \geq -M|\Omega| \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{\theta}\langle K'(u), u\rangle_{Y^*,Y} - K(u) \\ &\quad = \int_{\Gamma_u} \left(\frac{1}{\theta} g(x,u(x))u(x) - G(x,u(x)) \right) d\sigma + \int_{\Gamma_2\setminus\Gamma_u} \left(\frac{1}{\theta} g(x,u(x))u(x) - G(x,u(x)) \right) d\sigma \\ &\quad \geq -M|\Gamma_2 \setminus \Gamma_u| \geq -M|\Gamma_2|. \end{aligned}$$

Put $k_4 = -M|\Omega| - M|\Gamma_2|$. Summing up, we have

$$\begin{aligned} &I(u) - \frac{1}{\theta}\langle I'(u), u\rangle_{Y^*,Y} \\ &\quad = E(u) - \frac{1}{\theta}\langle E'(u), u\rangle_{Y^*,Y} - J(u) + \frac{1}{\theta}\langle J'(u), u\rangle_{Y^*,Y} - K(u) + \frac{1}{\theta}\langle K'(u), u\rangle_{Y^*,Y} \\ &\quad \geq c_4 \min\left\{ \|u\|_Y^{p^+}, \|u\|_Y^{p^-} \right\} + k_4. \end{aligned}$$

Hence

$$\begin{aligned} I(u) &\geq c_4 \min\left\{ \|u\|_Y^{p^+}, \|u\|_Y^{p^-} \right\} + \frac{1}{\theta}\langle I'(u), u\rangle_{Y^*,Y} + k_4 \\ &\geq c_4 \min\left\{ \|u\|_Y^{p^+}, \|u\|_Y^{p^-} \right\} - \frac{1}{\theta}\|I'(u)\|_{Y^*}\|u\|_Y + k_4. \end{aligned}$$ □

For a proof of Theorem 3.3, we apply the following standard Mountain-pass lemma (cf. Willem [23]).

Proposition 4.6. *Let $(V, \|\cdot\|_V)$ be a Banach space and $I \in C^1(V, \mathbb{R})$ be a functional satisfying the Palais–Smale condition, that is, if a sequence $\{u_n\} \subset V$ satisfies that $\lim_{n\to\infty} I(u_n) = \gamma$ exists and $\lim_{n\to\infty} \|I'(u_n)\|_{V^*} = 0$, then $\{u_n\}$ has a convergent subsequence. Assume that $I(0) = 0$, and there exist $\rho > 0$ and $z_0 \in V$ such that $\|z_0\|_V > \rho$, $I(z_0) \leq I(0) = 0$ and*

$$\alpha := \inf\{I(u); u \in V \text{ with } \|u\|_V = \rho\} > 0.$$

Put $G = \{\varphi \in C([0,1], V); \varphi(0) = 0, \varphi(1) = z_0\}$ and $\beta = \inf\{\max I(\varphi([0,1])); \varphi \in G\}$. Then $\beta \geq \alpha$ and β is a critical value of I.

We apply Proposition 4.6 with $(V, \|\cdot\|_V) = (Y, \|\cdot\|_Y)$. In order to do so, we must show the following proposition.

Proposition 4.7.

(i) *The functional I satisfies the Palais–Smale condition.*

(ii) $I(0) = 0$.

(iii) *There exists $\rho > 0$ such that* $\inf\{I(u); u \in Y \text{ with } \|u\|_Y = \rho\} > 0$.

(iv) *There exists $z_0 \in Y$ such that $\|z_0\|_Y > \rho$ and $I(z_0) \le 0$.*

(v) $G \neq \emptyset$.

Proof. (i) Assume that a sequence $\{u_n\} \subset Y$ satisfies that $\lim_{n\to\infty} I(u_n) = \gamma$ exists and $\lim_{n\to\infty} \|I'(u_n)\|_{Y^*} = 0$.

Step 1. The sequence $\{u_n\}$ is bounded in Y. Indeed, if $\{u_n\}$ is unbounded, there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\|u_{n'}\| \ge n'$ for any $n' \in \mathbb{N}$. By Lemma 4.5 (ii),

$$I(u_{n'}) \ge \|u_{n'}\|_Y \left(c_4 \|u_{n'}\|_Y^{p^- -1} - \frac{1}{\theta} \|I'(u_{n'})\|_{Y^*} \right) + k_4 \to \infty \quad \text{as } n' \to \infty.$$

This contradicts $\lim_{n'\to\infty} I(u_{n'}) = \gamma$.

Step 2. Since $\{u_n\}$ is bounded in Y and Y is a reflexive Banach space, passing to a subsequence, we may assume that $u_n \to u$ weakly in Y. By Proposition 4.4 (ii) and (iii),

$$E(u) \le \liminf_{n\to\infty} E(u_n) = \lim_{n\to\infty} (J(u_n) + K(u_n) + I(u_n)) = J(u) + K(u) + \gamma.$$

Since $\{\|u_n - u\|_Y\}$ is a bounded sequence and $\lim_{n\to\infty} \|I'(u_n)\|_{Y^*} = 0$, we see that $\langle I'(u_n), u - u_n\rangle_{Y^*,Y} \to 0$ as $n \to \infty$. By the Rellich–Kondrachov theorem, $u_n \to u$ strongly in $L^{q(\cdot)}(\Omega)$ and $u_n \to u$ strongly in $L^{r(\cdot)}(\Gamma_2)$. By (F1) and (G1), $|f(\cdot, u_n(\cdot))|$ is bounded in $L^{q'(\cdot)}(\Omega)$ and $|g(\cdot, u_n(\cdot))|$ is also bounded in $L^{r'(\cdot)}(\Gamma_2)$. Hence

$$\lim_{n\to\infty} \langle J'(u_n), u - u_n\rangle_{Y^*,Y} = \lim_{n\to\infty} \int_\Omega f(x, u_n(x))(u(x) - u_n(x))dx = 0$$

and

$$\lim_{n\to\infty} \langle K'(u_n), u - u_n\rangle_{Y^*,Y} = \lim_{n\to\infty} \int_{\Gamma_2} g(x, u_n(x))(u(x) - u_n(x))d\sigma = 0.$$

Therefore.

$$\begin{aligned} &\lim_{n\to\infty} \langle E'(u_n), u - u_n\rangle_{Y^*,Y} \\ &\qquad = \lim_{n\to\infty} \big(\langle I'(u_n), u - u_n\rangle_{Y^*,Y} + \langle J'(u_n), u - u_n\rangle_{Y^*,Y} + \langle K'(u_n), u - u_n\rangle_{Y^*,Y} \big) = 0. \end{aligned}$$

On the other hand, by Proposition 4.4 (iii) and the above equality,

$$E(u) - \limsup_{n\to\infty} E(u_n) = \liminf_{n\to\infty} (E(u) - E(u_n)) \ge \lim_{n\to\infty} \langle E'(u_n), u - u_n\rangle_{Y^*,Y} = 0.$$

Thus by Proposition 4.4 (ii), we have $\lim_{n\to\infty} E(u_n) = E(u)$.

Step 3. We show that $u_n \to u$ strongly in Y, that is, $\rho_{p(\cdot),h_1(\cdot)}(\nabla u_n - \nabla u) \to 0$ as $n \to \infty$. If this is not satisfied, there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $\varepsilon_0 > 0$ such that

$$\rho_{p(\cdot),h_1(\cdot)}(\nabla u_{n'} - \nabla u) \geq \varepsilon_0 \quad \text{for all } n' \in \mathbb{N}.$$

By Lemma 4.1,

$$\frac{1}{2}E(u_{n'}) + \frac{1}{2}E(u) - E\left(\frac{u_{n'}+u}{2}\right) \geq k_1 \rho_{p(\cdot),h_1(\cdot)}(\nabla u_{n'} - \nabla u) \geq k_1 \varepsilon_0.$$

Letting $n' \to \infty$ and using Step 2, we have

$$E(u) - \liminf_{n' \to \infty} E\left(\frac{u_{n'}+u}{2}\right) \geq k_1 \varepsilon_0. \tag{4.12}$$

On the other hand, since $\frac{u_{n'}+u}{2} \to u$ weakly in Y, it follows from Proposition 4.4 (ii) that

$$E(u) \leq \liminf_{n' \to \infty} E\left(\frac{u_{n'}+u}{2}\right).$$

This contradicts (4.12).

(ii) Since $E(0) = J(0) = K(0) = 0$, we have $I(0) = 0$.

(iii) Since $k_3 > 0$, $q^- > p^+$ and $r^- > p^+$, there exists $0 < \rho < 1$ such that

$$\rho^{p^+}(k_3 - c_3 \rho^{q^- - p^+} - c_3 \rho^{r^- - p^+}) > 0.$$

By Lemma 4.5 (i),

$$\begin{aligned} I(u) &\geq \|u\|_Y^{p^+}(k_3 - c_3\|u\|_Y^{q^- - p^+} - c_3\|u\|_Y^{r^- - p^+}) \\ &= \rho^{p^+}(k_3 - c_3 \rho^{q^- - p^+} - c_3 \rho^{r^- - p^+}) > 0 \end{aligned}$$

for all $u \in Y$ with $\|u\|_Y = \rho$.

(iv) Let $t > 1$ and choose $v_0 \in C_0^\infty(\Omega)(\subset Y)$ such that $v_0(x) \geq 0$ and $W = \{x \in \Omega; v_0(x) \geq t_0\}$ has a positive measure. By (F2), $F(x, v_0(x)) > 0$ for a.e. $x \in \Omega$. If we put $W_t = \{x \in \Omega; tv_0(x) \geq t_0\}$, then $W \subset W_t$. By Lemma 4.1 (iv),

$$\int_{W_t} F(x, tv_0(x))dx \geq \int_{W_t} \gamma(x) t^\theta v_0(x)^\theta dx \geq t^\theta L(v_0),$$

where $L(v_0) = \int_W \gamma(x) v_0(x)^\theta dx > 0$. By Lemma 4.1 (iii), there exists a constant $M > 0$ such that $|F(x,t)| \leq M$ for $t \in [0, t_0]$ and a.e. $x \in \Omega$. We note that (F2) implies that

$$F(x, st) \geq F(x,t)s^\theta \quad \text{for all } t \in \mathbb{R} \setminus (-t_0, t_0), s > 1 \text{ and a.e. } x \in \Omega. \tag{4.13}$$

Indeed, if we define $g(s) = F(x, st)$, then

$$g'(s) = f(x, st)t = \frac{1}{s} f(x, st)st \geq \frac{\theta}{s} F(x, st) = \frac{\theta}{s} g(s).$$

Thus $g'(s)/g(s) \geq \theta/s$, so $\log g(s)/g(1) \geq \theta \log s$. This implies $g(s) \geq g(1)s^\theta$.

On the other hand, (A3) implies that

$$A(x, s\xi) \leq A(x, \xi)s^{p(x)} \quad \text{for all } \xi \in \mathbb{R}^d, \text{ a.e. } x \in \Omega \text{ and } s > 1. \tag{4.14}$$

In fact, if we define $g(s) = A(x, s\boldsymbol{\xi})$, then

$$g'(s) = \boldsymbol{a}(x, s\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \leq \frac{p(x)}{s} A(x, s\boldsymbol{\xi}) = \frac{p(x)}{s} g(s).$$

Hence $g'(s)/g(s) \leq p(x)/s$. We also get $g(s) \geq g(1)s^{p(x)}$. From (4.14), we have

$$\begin{aligned} E(tv_0) &= \int_\Omega A(x, t\nabla v_0(x))dx \leq \int_\Omega A(x, \nabla v_0(x))t^{p(x)}dx \\ &\leq t^{p^+} \int_\Omega A(x, \nabla v_0(x))dx = t^{p^+} E(v_0). \end{aligned}$$

Since $v_0 \in C_0^\infty(\Omega)$, we have $K(tv_0) = 0$. Therefore,

$$\begin{aligned} I(tv_0) &= E(tv_0) - J(tv_0) \\ &= E(tv_0) - \int_{W_t} F(x, tv_0(x))dx - \int_{\Omega \setminus W_t} F(x, tv_0(x))dx \\ &\leq t^{p^+} E(v_0) - t^\theta L(v_0) + M|\Omega|. \end{aligned}$$

Since $\theta > p^+$ and $L(v_0) > 0$, $I(tv_0) \to -\infty$ as $t \to \infty$. Hence there exists $t_1 > 1$ such that $\|t_1 v_0\|_Y > \rho$ and $I(t_1 v_0) \leq 0$. If we put $z_0 = t_1 v_0$, then the conclusion of (iv) holds.

(v) If we define $\varphi(t) = tz_0$, then $\varphi \in G$, so $G \neq \emptyset$. $\square$

Proof of Theorem 3.3. By Propositions 4.2 and 4.7, we see that all the hypotheses in Proposition 4.6 hold. Hence there exists $u_0 \in Y$ such that $0 < \alpha \leq I(u_0) = \beta$ and $I'(u_0) = 0$, so

$$\begin{aligned} \langle I'(u_0), v \rangle_{Y^*, Y} &= \int_\Omega \boldsymbol{a}(x, \nabla u_0(x)) \cdot \nabla v(x) dx \\ &= \int_\Omega f(x, u_0(x)) v(x) dx + \int_{\Gamma_2} g(x, u_0(x)) v(x) d\sigma \text{ for all } v \in Y. \end{aligned}$$

Thus u_0 is a weak solution of (1.1). Since $I(u_0) = \beta > I(0) = 0$, u_0 is a nontrivial weak solution of (1.1). $\square$

Proof of Theorem 3.5. By (F4),

$$F(x, t) = \int_0^t f(x, \tau) d\tau \geq \frac{c}{m} t^m \quad \text{for } 0 \leq t \leq \delta \text{ and a.e. } x \in \Omega.$$

Choose $\varphi \in C_0^\infty(\Omega)$ so that $0 \leq \varphi \leq 1$ and $\varphi \not\equiv 0$. Let $0 < t < \delta (< 1)$. Since $A(x, \boldsymbol{\xi})$ is convex with respect to $\boldsymbol{\xi}$ and $A(x, \mathbf{0}) = 0$, we have $A(x, t\boldsymbol{\xi}) = A(x, t\boldsymbol{\xi} + (1-t)\mathbf{0}) \leq tA(x, \boldsymbol{\xi})$. Thus

$$\begin{aligned} I(t\varphi) &= E(t\varphi) - J(t\varphi) \\ &= \int_\Omega A(x, t\nabla \varphi(x))dx - \int_\Omega F(x, t\varphi(x))dx \\ &\leq t \int_\Omega A(x, \nabla \varphi(x))dx - \frac{c}{m} t^m \int_\Omega \varphi(x)^m dx. \end{aligned}$$

Since $m < 1$ and $\frac{c}{m} \int_\Omega \varphi(x)^m dx > 0$, we see that $I(t\varphi) < 0$ for small $t > 0$. By Lemma 4.5 (i), I is bounded from below on $\overline{B_\rho(0)}$, where $B_\rho(0) = \{v \in Y; \|v\|_Y < \rho\}$. Hence

$$-\infty < \underline{c} := \inf_{v \in \overline{B_\rho(0)}} I(v) < 0.$$

Let $0 < \varepsilon < \inf_{v\in\partial B_\rho(0)} I(v) - \inf_{v\in\overline{B_\rho(0)}} I(v)$. Then there exists $u \in \overline{B_\rho(0)}$ such that

$$\inf_{v\in\overline{B_\rho(0)}} I(v) \le I(u) \le \inf_{v\in\overline{B_\rho(0)}} I(v) + \varepsilon^2.$$

Since $\inf_{v\in\overline{B_\rho(0)}} I(v) < 0$, we can choose $u \in \overline{B_\rho(0)}$ so that $I(u) < 0$. Applying the Ekeland variational principle [11, Theorem 1.1] to the complete metric space $\overline{B_\rho(0)}$, there exists $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$I(u_\varepsilon) \le I(u), \tag{4.15}$$

$$I(u_\varepsilon) \le I(v) + \varepsilon\|v - u_\varepsilon\|_Y \text{ for all } v \in \overline{B_\rho(0)}, \tag{4.16}$$

$$\|u - u_\varepsilon\|_Y \le \varepsilon. \tag{4.17}$$

Define $\Phi : \overline{B_\rho(0)} \to \mathbb{R}$ by $\Phi(v) = I(v) + \varepsilon\|v - u_\varepsilon\|_Y$ for $v \in \overline{B_\rho(0)}$. Since $I(u_\varepsilon) \le I(u) < 0$ and $I(v) > 0$ for all $v \in \partial B_\rho(0)$, we have $u_\varepsilon \in B_\rho(0)$. Choose $\rho' > 0$ small enough, so that if $w \in \overline{B_{\rho'}(0)}$, then $u_\varepsilon + w \in \overline{B_\rho(0)}$. From (4.16), since $\Phi(u_\varepsilon) \le \Phi(u_\varepsilon + w)$ for all $w \in \overline{B_{\rho'}(0)}$. We have

$$\begin{aligned}
&\frac{\langle I'(u_\varepsilon), w\rangle_{Y^*,Y} + \varepsilon\|w\|_Y}{\|w\|_Y} \\
&= \frac{\langle I'(u_\varepsilon), tw\rangle_{Y^*,Y} + \varepsilon t\|w\|_Y - (\Phi(u_\varepsilon + tw) - \Phi(u_\varepsilon))}{t\|w\|_Y} + \frac{\Phi(u_\varepsilon + tw) - \Phi(u_\varepsilon)}{t\|w\|_Y} \\
&\ge \frac{\langle I'(u_\varepsilon), tw\rangle_{Y^*,Y} - (I(u_\varepsilon + tw) - I(u_\varepsilon))}{t\|w\|_Y} \to 0 \quad \text{as } t \to +0.
\end{aligned}$$

Hence $\langle I'(u_\varepsilon), w\rangle_{Y^*,Y} + \varepsilon\|w\|_Y \ge 0$ for all $w \in \overline{B_\rho(0)}$, so $\langle I'(u_\varepsilon), w\rangle_{Y^*,Y} \ge -\varepsilon\|w\|_Y$. Replacing w with $-w$, we have $|\langle I'(u_\varepsilon), w\rangle_{Y^*,Y}| \le \varepsilon\|w\|_Y$ for all $w \in \overline{B_\rho(0)}$. Thus $\|I'(u_\varepsilon)\|_{Y^*} \le \varepsilon$. Letting $\varepsilon \to 0$, we see that $I(u_\varepsilon) \to \underline{c}$ and $I'(u_\varepsilon) \to 0$ in Y^*. Since I satisfies the Palais–Smale condition in Y and $I \in C^1(Y, \mathbb{R})$, there exist a subsequence $\{u_n\}$ of $\{u_\varepsilon\}$ and $u_2 \in \overline{B_\rho(0)}$ such that $u_n \to u_2$ in Y and $I'(u_2) = 0$. Therefore, u_2 is a weak solution of (1.1). Since $I(u_2) = \underline{c} < 0 = I(0)$, u_2 is a nontrivial weak solution of (1.1). Since $I(u_2) = \underline{c} < 0 < I(u_1)$, we have $u_1 \ne u_2$. This completes the proof of Theorem 3.5. □

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