



Homoclinic solutions for a class of asymptotically autonomous Hamiltonian systems with indefinite sign nonlinearities

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Abstract. In this paper, we obtain the multiplicity of homoclinic solutions for a class of asymptotically autonomous Hamiltonian systems with indefinite sign potentials. The concentration-compactness principle is applied to show the compactness. As a byproduct, we obtain the uniqueness of the positive ground state solution for a class of autonomous Hamiltonian systems and the best constant for Sobolev inequality which are of independent interests.

Keywords: multiple homoclinic solutions, asymptotically autonomous Hamiltonian systems, indefinite sign nonlinearities, best constant for Sobolev inequality, the Concentration-Compactness Principle.

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1 Introduction and main results

In this paper, we consider the following second-order planar Hamiltonian systems

$$\ddot{u}(t) + \nabla V(t, u(t)) = 0, \quad (1.1)$$

where $V : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a C^1 -map. We say that a solution $u(t)$ of problem (1.1) is nontrivial homoclinic (to 0) if $u \not\equiv 0$, $u(t) \rightarrow 0$ and $\dot{u}(t) \rightarrow 0$ as $t \rightarrow \pm\infty$. Subsequently, $\nabla V(t, x)$ denotes the gradient with respect to the x variable, $(\cdot, \cdot) : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ denotes the standard inner product in \mathbb{R}^2 and $|\cdot|$ is the induced norm.

Hamiltonian system is a classical model in celestial mechanics, fluid mechanics and so on. Since its importance in physic fields, searching for the solutions of the Hamiltonian systems has attracted much attention of mathematicians since Poincaré. In a remarkable paper [31], the periodic solutions are firstly obtained for (1.1) with prescribed energy and prescribed period cases respectively via variational methods by Rabinowitz. However, to show homoclinic solutions via variational methods seems difficult since the lack of compactness for

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the Sobolev embedding. In order to regain the compactness, different strategies are adopted. In 1990, Rabinowitz [32] considered (1.1) with the following potentials

$$V(t, x) = -\frac{1}{2}(a(t)x, x) + W(t, x),$$

where $a(t)$ and $W(t, x)$ are T -periodic in t and homoclinic solution are obtained as the limit of a sequence of $2kT$ -periodic solutions. Without periodic hypothesis, Rabinowitz and Tanaka [23] assumed the least eigenvalues of $a(t)$ go to infinity as $t \rightarrow \infty$. Under this condition, Omana and Willem [28] obtained a compact embedding theorem and showed the multiplicity of homoclinic solutions for problem (1.1). Without periodic or coercive hypothesis, there are still some other conditions proposed to obtain the nontrivial homoclinic solutions. In 2007, Lv and Tang [24] assumed that $V(t, x)$ is even in t and obtained one homoclinic solution for (1.1) as the limit of the solutions of nil-boundary-value problems. In 2010, Wu, Wu and Tang [43] showed that (1.1) possesses at least one nontrivial homoclinic solution if there is a nontrivial perturbation. In detail, they considered the following systems

$$\ddot{u}(t) - L(t)u(t) - \nabla W(t, u(t)) = f(t). \quad (1.2)$$

When $f \not\equiv 0$, the authors showed the existence of nontrivial homoclinic solutions for (1.2) without periodic nor coercive conditions on L and W .

As we know, the growth of W is crucial in determining the geometric structure of the corresponding functional and the boundedness of the almost critical sequence. Three typical growth cases are superquadratic, subquadratic and asymptotically quadratic cases. The following Ambrosetti–Rabinowitz-type condition is a classical superquadratic condition.

(AR) there exists a constant $v > 2$ such that

$$0 < vW(t, x) \leq (\nabla W(t, x), x)$$

for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \setminus \{0\}$.

In 1991, Rabinowitz and Tanaka [33] also obtained the homoclinic solutions for (1.1) under the following non-quadratic condition

(RT) $s^{-1}(\nabla W(t, sx), x)$ is an increasing function of $s \in (0, 1]$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

As shown in [25], condition (RT) implies that

(MS) there exists $\theta \geq 1$ such that

$$\theta \tilde{W}(t, x) \geq \tilde{W}(t, sx)$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ and $s \in [0, 1]$, where $\tilde{W}(t, x) = (\nabla W(t, x), x) - 2W(t, x)$.

With (MS) Lv and Tang [25] obtained infinitely many homoclinic solutions for (1.1). Besides, many superquadratic conditions are introduced. In 2004, Ou and Tang [29] considered the following superquadratic condition

(OT) $W(t, x)/|x|^2 \rightarrow +\infty$ as $|x| \rightarrow \infty$ uniformly in $t \in \mathbb{R}$.

Based on above results, Ding and Lee [8] introduced the following superquadratic condition

(DL) $\tilde{W}(t, x) > 0$ if $x \neq 0$, and there exist $\epsilon \in (0, 1)$ and $c > 0$ such that

$$\tilde{W}(t, x) \geq c \frac{(\nabla W(t, x), x)}{|x|^{2-\epsilon}} \quad \text{for } |x| \text{ large enough.}$$

There are also some other superquadratic growth conditions introduced by many mathematicians. The readers are referred to [6, 15, 18, 22, 23, 29, 30, 39, 40, 43–48] for more details.

In this paper, we mainly consider the asymptotically autonomous potentials without periodic, coercive, even assumption or perturbations. In 1999, Carrião and Miyagaki [5] showed the existence of homoclinic for problem (1.1) by assuming that $V(t, x)$ converges to $V_\infty(x)$ as $|t| \rightarrow +\infty$ and $V_\infty(x)$ satisfies the (AR) condition. The asymptotically autonomous Hamiltonian systems has also been considered by Lv, Xue and Tang [26] with asymptotically quadratic potentials. They showed the existence of homoclinic solutions for systems (1.1) with $a(t) \equiv \text{const.}$ being small enough. In another paper, Lv, etc. [27] also obtained ground state homoclinic orbits for a class of asymptotically periodic second-order Hamiltonian systems. Their results generalized the conclusions in [1, 5] by replacing the (AR) condition with strict monotonic conditions on $W(t, x)$.

In this paper, we mainly consider the combined nonlinearities. In [6, 26, 36, 37, 44, 46], the authors also considered the following concave-convex potentials

$$V(t, x) = -\frac{1}{2}(a(t)x, x) + \lambda F(t, x) + G(t, x),$$

where $a(t)$ is coercive, i.e. $a(t) \rightarrow +\infty$ as $t \rightarrow \infty$, $F(t, x)$ is subquadratic and $G(t, x)$ is superquadratic in $x \in \mathbb{R}^N$. The coercivity of $a(t)$ is an important assumption which can guarantee the compactness of Sobolev embedding.

In [46], Yang, Chen and Sun assumed that $a(t)$ is coercive, $F(t, x) = m(t)|x|^\gamma$ and $G(t, x) = d|x|^p$ with $m \in L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R}^+)$ and $1 < \gamma < 2$, $d \geq 0$, $p > 2$. This result is generalized by Chen and He [6] with the following generalized superquadratic condition

(CH) There exist $\rho > 2$ and $1 < \delta < 2$ such that

$$\rho G(t, x) - (\nabla G(t, x), x) \leq h(t)|x|^\delta, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N$$

where $h : \mathbb{R} \rightarrow \mathbb{R}^+$ is a positive continuous function such that $h \in L^{\frac{2}{2-\delta}}(\mathbb{R}, \mathbb{R}^+)$.

Obviously, (CH) is weaker than (AR) since $h(t) > 0$ for all $t \in \mathbb{R}$. In [42], Wu, Tang and Wu generalized the above results by relaxing the conditions on G . However, $a(t)$ is also required to be coercive.

Without coercive assumption, there are also some other papers concerning on this case with the steep well potentials (see [36, 37]). In [36], the nonlinearities are the combination of subquadratic and asymptotic quadratic nonlinearities. While in [37], the nonlinearities are the combination of superquadratic and subquadratic nonlinearities. In [46], Ye and Tang obtained infinitely many homoclinic solutions for systems (1.1) with

$$V(t, x) = -\frac{1}{2}(a(t)x, x) + \frac{h(t)}{p}|x|^p + \frac{d(t)}{v}|x|^v, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $a(t) \geq 0$ and

$$\begin{cases} h \in L^{2/(2-p)}(\mathbb{R}, \mathbb{R}^+) \\ d \in L^\infty(\mathbb{R}, \mathbb{R}) \\ 1 < p < 2 < v. \end{cases}$$

By assuming $h(t) > 0$, the authors in [46] constructed a sequence of negative critical values. However, in [5, 26, 27], $W(t, x)$ is assumed to be non-negative in $\mathbb{R} \times \mathbb{R}^N$. A natural question is whether (1.1) possesses homoclinic solutions if $W(t, x)$ change signs without periodic or coercive assumptions. In this paper, we partially give some answers to this question. Precisely, we consider the sign-changing and asymptotically autonomous potentials, which have not been considered before as we know. Hence, we cannot obtain our results as the authors did in [6, 26, 36, 37, 44, 46]. Concentration-compactness principle (CCP) is adopted to show the compactness. The crucial step in using the (CCP) is to exclude the dichotomy case by estimating the critical values. This can be easily done if W satisfies the following monotonic condition

(MC) the mapping $\tau \rightarrow (\frac{\nabla W(t, \tau x)}{\tau}, x)$ is strictly increasing in $\tau \in (0, 1]$ for all $x \neq 0$ and $t \in \mathbb{R}$.

However, condition (MC) is not valid for our potentials. Hence we need more delicate estimates for the critical values to show the contradictions. The constant for the Sobolev inequality plays an important role in obtaining our results. In the next section, we show the best constant for the Sobolev inequality.

2 Best constant for the Sobolev inequality

Let's make it clear that $L^p(\mathbb{R}, \mathbb{R}^m)$ and $H^1(\mathbb{R}, \mathbb{R}^m)$ are the Banach spaces of functions on \mathbb{R} valued in \mathbb{R}^m under the norms

$$\|u\|_p := \left(\int_{\mathbb{R}} |u|^p dt \right)^{1/p}$$

and

$$\|u\| = \|u\|_{H^1} = \left(\int_{\mathbb{R}} (|\dot{u}|^2 + |u|^2) dt \right)^{1/2}.$$

Moreover, let $L^\infty(\mathbb{R}, \mathbb{R}^m)$ be the Banach space of essentially bounded measurable functions from \mathbb{R} into \mathbb{R}^m under the norm

$$\|u\|_\infty := \text{ess sup}\{|u(t)| : t \in \mathbb{R}\}.$$

As we know, for any $m > 1$, $H^1(\mathbb{R}, \mathbb{R}^m)$ can be embedded into $L^\nu(\mathbb{R}, \mathbb{R}^m)$ continuously for any $\nu \in [2, +\infty]$. Then we have the following Sobolev inequality

$$\|u\|_\nu \leq C_\nu \|u\| \quad \text{for all } u \in H^1(\mathbb{R}, \mathbb{R}^m), \quad (2.1)$$

where C_ν is the best constant which is defined in the following proof. This inequality is important in using variational methods to show the existence and multiplicity of differential equations. However, since the best constant for the Sobolev inequality seems not important in previous studies of Hamiltonian systems, as we know, there is no paper concerning on the best constant of Sobolev inequality for (2.1). In this section, we show the best constant for (2.1).

There have been many papers concerning on the best constant for the Sobolev inequality in $H^1(\mathbb{R}, \mathbb{R})$ (see [2–4, 12]). In a remarkable paper, Talenti [38] obtained the best constant for

Sobolev inequality in $H^1(\mathbb{R}^N, \mathbb{R})$ with $N > 1$. In 1983, Weinstein obtained the best constant for the following Gagliardo-Nirenberg-Sobolev inequalities

$$\|u\|_v^v \leq C_* \|\nabla u\|_2^{\frac{N(v-2)}{2}} \|u\|_2^{v - \frac{N(v-2)}{2}} \quad \text{for } u \in H^1(\mathbb{R}^N, \mathbb{R}), \quad (2.2)$$

where $N \geq 2$, $2 < v < \frac{2N}{N-2}$, $C_* = \frac{v}{2\|\mathcal{G}\|_2^{v-2}}$ is the best constant for (2.2) and \mathcal{G} is the unique positive solution for the following scalar field equation

$$-\frac{N(v-2)}{4} \Delta u + \left(1 + \frac{v-2}{4}(2-N)\right) u = |u|^{v-2} u, \quad x \in \mathbb{R}^N.$$

In a recent paper, Dolbeault, etc. [10] considered the best constant for the one-dimensional Gagliardo-Nirenberg-Sobolev inequalities in $H^1(\mathbb{R}, \mathbb{R}^m)$ ($m = 1$) and obtained

$$\frac{1}{M_{GN}(v)} = \inf_{y \in H^1(\mathbb{R}, \mathbb{R}) \setminus \{0\}} \frac{\left(\int_{\mathbb{R}} |y'|^2 dt\right)^{\frac{v-2}{4v}} \left(\int_{\mathbb{R}} |y|^2 dt\right)^{\frac{v+2}{4v}}}{\left(\int_{\mathbb{R}} |y|^v dt\right)^{\frac{1}{v}}}, \quad (2.3)$$

where $M_{GN}(v)$ is defined as

$$M_{GN}(v) = 4^{-\frac{1}{v}} \left(\frac{(v+2)^{v+2}}{(v-2)^{v-2}}\right)^{\frac{1}{4v}} \left(\frac{2\sqrt{\pi}\Gamma\left(\frac{2}{v-2}\right)}{(v+2)\Gamma\left(\frac{2}{v-2} + \frac{1}{2}\right)}\right)^{\frac{v-2}{2v}}. \quad (2.4)$$

Moreover, $M_{GN}(v)$ is attained at v_* , which is the unique optimal function up to translations, multiplication by a constant and scalings, defined as

$$v_*(t) = \frac{1}{(\cosh t)^{\frac{2}{v-2}}}.$$

The following computation is made by the authors in [10]. For the reader's convenience, we write them here.

$$\int_{\mathbb{R}} |v_*|^2 dt = \frac{\sqrt{\pi}\Gamma\left(\frac{2}{v-2}\right)}{\Gamma\left(\frac{2}{v-2} + \frac{1}{2}\right)}, \quad \int_{\mathbb{R}} |v_*|^v dt = \frac{4}{v+2} \int_{\mathbb{R}} |v_*|^2 dt$$

and

$$\int_{\mathbb{R}} |v_*'|^2 dt = \frac{4}{(v-2)(v+2)} \int_{\mathbb{R}} |v_*|^2 dt.$$

Subsequently, we consider the case $m > 1$. For any $u(t) = (u_1(t), \dots, u_m(t)) \in H^1(\mathbb{R}, \mathbb{R}^m) \setminus \{0\}$, set

$$y(t) = |u(t)| = \sqrt{\sum_{i=1}^m u_i^2(t)} \in H^1(\mathbb{R}, \mathbb{R}) \setminus \{0\}. \quad (2.5)$$

Then we have

$$[y']^2 = \frac{(\sum_{i=1}^m u_i u_i')^2}{\sum_{i=1}^m u_i^2}. \quad (2.6)$$

For any $v > 2$, let

$$\mathfrak{R} = \inf_{u_1, \dots, u_m \in H^1(\mathbb{R}, \mathbb{R}) \setminus \{0\}} \frac{\left(\int_{\mathbb{R}} (\sum_{i=1}^m u_i'^2) dt\right)^{\frac{v-2}{4v}}}{\left(\int_{\mathbb{R}} \frac{(\sum_{i=1}^m u_i u_i')^2}{\sum_{i=1}^m u_i^2} dt\right)^{\frac{v-2}{4v}}}. \quad (2.7)$$

On one hand, if we choose $u_1 = \dots = u_m$, it is easy to see that $\mathfrak{R} \leq 1$. On the other hand, we can also deduce that $\mathfrak{R} \geq 1$ since

$$\left(\sum_{i=1}^m u_i u'_i \right)^2 \leq \left(\sum_{i=1}^m u_i^2 \right) \left(\sum_{i=1}^m u_i'^2 \right),$$

which implies $\mathfrak{R} = 1$. Therefore, by (2.3), (2.5)–(2.7), one has

$$\begin{aligned} & \inf_{u \in H^1(\mathbb{R}, \mathbb{R}^m) \setminus \{0\}} \frac{\left(\int_{\mathbb{R}} |\dot{u}|^2 dt \right)^{\frac{\nu-2}{4\nu}} \left(\int_{\mathbb{R}} |u|^2 dt \right)^{\frac{\nu+2}{4\nu}}}{\left(\int_{\mathbb{R}} |u|^\nu dt \right)^{\frac{1}{\nu}}} \\ &= \inf_{u_1, \dots, u_m \in H^1(\mathbb{R}, \mathbb{R}) \setminus \{0\}} \frac{\left(\int_{\mathbb{R}} \left(\sum_{i=1}^m u_i'^2 \right) dt \right)^{\frac{\nu-2}{4\nu}} \left(\int_{\mathbb{R}} \frac{\left(\sum_{i=1}^m u_i u'_i \right)^2}{\sum_{i=1}^m u_i'^2} dt \right)^{\frac{\nu-2}{4\nu}} \left(\int_{\mathbb{R}} \left(\sum_{i=1}^m u_i^2 \right) dt \right)^{\frac{\nu+2}{4\nu}}}{\left(\int_{\mathbb{R}} \frac{\left(\sum_{i=1}^m u_i u'_i \right)^2}{\sum_{i=1}^m u_i'^2} dt \right)^{\frac{\nu-2}{4\nu}} \left(\int_{\mathbb{R}} \left(\sum_{i=1}^m u_i^2 \right)^{\frac{\nu}{2}} dt \right)^{\frac{1}{\nu}}} \\ &\geq \inf_{u_1, \dots, u_m \in H^1(\mathbb{R}, \mathbb{R}) \setminus \{0\}} \frac{\left(\int_{\mathbb{R}} \left(\sum_{i=1}^m u_i'^2 \right) dt \right)^{\frac{\nu-2}{4\nu}}}{\left(\int_{\mathbb{R}} \frac{\left(\sum_{i=1}^m u_i u'_i \right)^2}{\sum_{i=1}^m u_i'^2} dt \right)^{\frac{\nu-2}{4\nu}}} \inf_{y \in H^1(\mathbb{R}, \mathbb{R}) \setminus \{0\}} \frac{\left(\int_{\mathbb{R}} |y'|^2 dt \right)^{\frac{\nu-2}{4\nu}} \left(\int_{\mathbb{R}} |y|^2 dt \right)^{\frac{\nu+2}{4\nu}}}{\left(\int_{\mathbb{R}} |y|^\nu dt \right)^{\frac{1}{\nu}}} \\ &= \frac{1}{M_{GN}(\nu)}. \end{aligned}$$

Hence, for any $\nu > 2$

$$\left(\int_{\mathbb{R}} |u|^\nu dt \right)^{\frac{1}{\nu}} \leq M_{GN}(\nu) \left(\int_{\mathbb{R}} |\dot{u}|^2 dt \right)^{\frac{\nu-2}{4\nu}} \left(\int_{\mathbb{R}} |u|^2 dt \right)^{\frac{\nu+2}{4\nu}} \quad \text{for all } u \in H^1(\mathbb{R}, \mathbb{R}^m),$$

where $M_{GN}(\nu)$ is the best constant defined in (2.4) and attained at

$$\mathcal{V} = (k_1, \dots, k_m) v_\star \tag{2.8}$$

with $k_i \geq 0$ and $k_1^2 + \dots + k_m^2 = 1$. Moreover, for any $\Delta \subset \mathbb{R}$, there holds

$$\left(\int_{\Delta} |u|^\nu dt \right)^{\frac{1}{\nu}} \leq M_{GN}(\nu) \left(\int_{\Delta} |\dot{u}|^2 dt \right)^{\frac{\nu-2}{4\nu}} \left(\int_{\Delta} |u|^2 dt \right)^{\frac{\nu+2}{4\nu}} \quad \text{for all } u \in H_0^1(\Delta, \mathbb{R}^m) \tag{2.9}$$

and $M_{GN}(\nu)$ is the best constant which can be attained if and only if $\Delta = \mathbb{R}$. For any $u \in H^1(\mathbb{R}, \mathbb{R}^m) \setminus \{0\}$ and $\tau > 0$, let $q_\tau(t) = u(\tau t)$ with

$$\mathcal{Q}_\tau(u) = \frac{\left(\int_{\mathbb{R}} (|\dot{q}_\tau|^2 + |q_\tau|^2) dt \right)^{\frac{1}{2}}}{\left(\int_{\mathbb{R}} |q_\tau|^\nu dt \right)^{\frac{1}{\nu}}}$$

and

$$\tau_u = \sqrt{\frac{(\nu-2) \int_{\mathbb{R}} |u|^2 dt}{(\nu+2) \int_{\mathbb{R}} |\dot{u}|^2 dt}}.$$

It is easy to see that

$$\inf_{\tau > 0} \mathcal{Q}_\tau(u) = \mathcal{Q}_{\tau_u}(u) \leq \mathcal{Q}_1(u) = \frac{\|u\|}{\|u\|_\nu} \tag{2.10}$$

and

$$\begin{aligned} \inf_{u \in H^1(\mathbb{R}, \mathbb{R}^m) \setminus \{0\}} \mathcal{Q}_{\tau_u}(u) &= \inf_{u \in H^1(\mathbb{R}, \mathbb{R}^m) \setminus \{0\}} \left(\frac{\nu+2}{\nu-2} \right)^{\frac{\nu-2}{4\nu}} \left(\frac{2\nu}{\nu+2} \right)^{\frac{1}{2}} \frac{(\int_{\mathbb{R}} |\dot{u}|^2 dt)^{\frac{\nu-2}{4\nu}} (\int_{\mathbb{R}} |u|^2 dt)^{\frac{\nu+2}{4\nu}}}{(\int_{\mathbb{R}} |u|^\nu dt)^{\frac{1}{\nu}}} \\ &= \frac{1}{M_{GN}(\nu)} \left(\frac{\nu+2}{\nu-2} \right)^{\frac{\nu-2}{4\nu}} \left(\frac{2\nu}{\nu+2} \right)^{\frac{1}{2}}. \end{aligned} \quad (2.11)$$

It follows from (2.10) and (2.11) that

$$\inf_{u \in H^1(\mathbb{R}, \mathbb{R}^m) \setminus \{0\}} \frac{\|u\|}{\|u\|_\nu} \geq \frac{1}{M_{GN}(\nu)} \left(\frac{\nu+2}{\nu-2} \right)^{\frac{\nu-2}{4\nu}} \left(\frac{2\nu}{\nu+2} \right)^{\frac{1}{2}} \quad (2.12)$$

and

$$\frac{\|\mathcal{V}_{\tau_\nu}\|}{\|\mathcal{V}_{\tau_\nu}\|_\nu} = \frac{1}{M_{GN}(\nu)} \left(\frac{\nu+2}{\nu-2} \right)^{\frac{\nu-2}{4\nu}} \left(\frac{2\nu}{\nu+2} \right)^{\frac{1}{2}}. \quad (2.13)$$

Then, we infer that (2.1) holds and

$$C_\nu = M_{GN}(\nu) \left(\frac{\nu-2}{\nu+2} \right)^{\frac{\nu-2}{4\nu}} \left(\frac{\nu+2}{2\nu} \right)^{\frac{1}{2}} \quad (2.14)$$

is the best constant. Moreover, we also need to consider the best constant when $\nu = +\infty$. It follows from (2.14) that $C_\nu \rightarrow \frac{1}{\sqrt{2}}$ as $\nu \rightarrow +\infty$. It has been shown by Janczewska in [16] that $C_\infty = \frac{1}{\sqrt{2}}$, which is the best constant for (2.1) when $\nu = \infty$.

3 Solutions for the limit systems

In this section, we consider the solutions for the limit systems of (1.1). In the rest of this paper, we only consider the systems in \mathbb{R}^2 . The potential V is defined as

$$V(t, x) = -\frac{1}{2}a(t)|x|^2 + \lambda F(t, x) + d(t)|x|^\nu,$$

where $a, d \in C(\mathbb{R}, \mathbb{R})$, $\lambda > 0$, $\nu > 2$ and the following conditions hold

- (V1) there exists $a_0 > 0$ such that $a(t) \geq a_0$ for all $t \in \mathbb{R}$;
- (V2) there exist $a_\infty, d_\infty > 0$ such that $a(t) \rightarrow a_\infty$ and $d(t) \rightarrow d_\infty$ as $|t| \rightarrow +\infty$;
- (V3) $\|a\|_\infty \geq 1$, $d(0) = \|d\|_\infty$;
- (V4) $F(t, 0) = 0$ and $F(t, x) \in C^1(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$;
- (V5) for any $(t, x) \in \mathbb{R} \times \mathbb{R}^2$, there exist $1 < r_1 \leq r_2 < 2$ such that

$$|\nabla F(t, x)| \leq b_1(t)|x|^{r_1-1} + b_2(t)|x|^{r_2-1},$$

where $b_1(t) \in L^{\beta_1}(\mathbb{R}, \mathbb{R}^+)$ and $b_2(t) \in L^{\beta_2}(\mathbb{R}, \mathbb{R}^+)$ for some $\beta_1 \in (1, \frac{2}{2-r_1})$ and $\beta_2 \in (1, \frac{2}{2-r_2})$;

- (V6) there exist $\bar{t} \in \mathbb{R}$, $\bar{r} \in (1, 2)$ and $b_0 > 0$ such that $F(\bar{t}, x) > b_0|x|^{\bar{r}}$ for all $x \in \mathbb{R}^2$.

Here $Q_\nu : \mathbb{R} \rightarrow \mathbb{R}^+$ is the **unique positive ground state** solution (up to translations) for the following equation

$$\ddot{u}(t) - u(t) + u^{\nu-1}(t) = 0 \quad \text{for } t \in \mathbb{R}. \quad (3.1)$$

Let us consider the following systems

$$\begin{cases} \Delta u_j - a_\infty u_j + \nu d_\infty \left(\sum_{i=1}^m u_i^2 \right)^{\frac{\nu-2}{2}} u_j = 0 \text{ in } \mathbb{R}^n, \\ \dot{u}_j(0) = 0, \quad j = 1, \dots, m, \\ u_j(y) \rightarrow 0 \text{ as } |y| \rightarrow +\infty. \end{cases} \quad (3.2)$$

The existence of solutions for systems (3.2) has been considered by many mathematicians via the variational methods. A solution (u_1, \dots, u_m) for (3.2) is said to be **positive** if $u_1, \dots, u_m > 0$. When $m = 1$, (3.2) reduces to a differential equation and the uniqueness of positive ground state solution for (3.2) has been shown by M. K. Kwong [19] with $\nu > 2$. The readers are also referred to [17, 34] for more general cases.

When $m > 1$, (3.2) is related to the coupled nonlinear Schrödinger equations. In last decades, there have been many mathematicians devoting themselves to the uniqueness of positive solutions for the coupled nonlinear Schrödinger equations and obtained many significant results (see [9, 19, 26, 41]). In a recent paper [41], Wei and Yao considered the following systems

$$\begin{cases} \ddot{u}(r) + \frac{n-1}{r} \dot{u}(r) - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0, & \text{in } [0, \infty) \\ \ddot{v}(r) + \frac{n-1}{r} \dot{v}(r) - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0, & \text{in } [0, \infty) \\ u(r), v(r) > 0 & \text{in } [0, \infty) \\ \dot{u}(0) = \dot{v}(0) = 0, \text{ and } u(r), v(r) \rightarrow 0 \text{ as } r \rightarrow \infty. \end{cases} \quad (3.3)$$

When $\lambda_1 = \lambda_2 = \lambda$ with $0 \leq \beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$, they showed the uniqueness of positive solutions for system (3.3), defined as

$$(u_0, v_0) = \left(\sqrt{\frac{\lambda(\beta - \mu_2)}{\beta^2 - \mu_1 \mu_2}} w_0(\sqrt{\lambda}x), \sqrt{\frac{\lambda(\beta - \mu_1)}{\beta^2 - \mu_1 \mu_2}} w_0(\sqrt{\lambda}x) \right)$$

where w_0 is the unique positive solution of

$$\Delta w - w + w^3 = 0 \quad \text{in } \mathbb{R}, \quad w(0) = \max_{x \in \mathbb{R}^N} w(x), \quad w(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

q When $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \beta$, it has also been shown in [41] that all the positive solutions of system (3.3) have the following form

$$(u(x), v(x)) = \left(\sqrt{\frac{\lambda}{\beta}} w(\sqrt{\lambda}x) \cos \theta, \sqrt{\frac{\lambda}{\beta}} w(\sqrt{\lambda}x) \sin \theta \right), \quad \theta \in (0, \pi/2).$$

For the high dimension cases, i.e. $n = 2, 3$ and $m = 2$, the readers are referred to another paper by Dai, Tian and Zhang [11]. However, the case $n = 1$ is not considered. We can see that (3.2) reduces to (3.3) if $\nu = 4$. Motivated by above papers, we obtain the uniqueness of solutions for (3.2) when $m = 2$, $n = 1$ and $\nu > 2$. More precisely, we obtain the following lemma.

Lemma 3.1. *Suppose $m = 2$, $n = 1$, $a_\infty, d_\infty > 0$ and $v > 2$. Then system (3.2) possesses at least one positive solution. Let $\mathcal{U}_v : \mathbb{R} \rightarrow \mathbb{R}^+ \times \mathbb{R}^+$ be a positive solution for systems (3.2), then there exists $\omega \in (0, \pi/2)$ such that*

$$\mathcal{U}_v = \left(\frac{a_\infty}{vd_\infty} \right)^{\frac{1}{v-2}} Q_v(\sqrt{a_\infty}t) (\cos \omega, \sin \omega) \quad (3.4)$$

and \mathcal{U}_v is the ground state solution for (3.2).

Proof. Since $n = 1$, the critical exponent equals to $+\infty$. The existence of positive solutions for the subcritical problems have been considered in [7, 13, 35, 41]. Subsequently, we only show (3.4) holds and \mathcal{U}_v is the ground state solution for (3.2). Let

$$\mathcal{M}_v(t) = \left(\frac{a_\infty}{vd_\infty} \right)^{-\frac{1}{v-2}} \mathcal{U}_v \left(\frac{t}{\sqrt{a_\infty}} \right).$$

Then $\mathcal{M}_v = (\mathcal{M}_1(t), \mathcal{M}_2(t))$ is the positive solution for the following system

$$\begin{cases} \ddot{u}_j(t) - u_j(t) + (u_1^2(t) + u_2^2(t))^{\frac{v-2}{2}} u_j(t) = 0, & j = 1, 2, \text{ for } t \in \mathbb{R}, \\ \dot{u}_1(0) = \dot{u}_2(0) = 0, \\ u_1(t), u_2(t) \rightarrow 0 & \text{as } |t| \rightarrow \infty, \end{cases}$$

which implies

$$\ddot{\mathcal{M}}_1 - \mathcal{M}_1 + (\mathcal{M}_1^2 + \mathcal{M}_2^2)^{\frac{v-2}{2}} \mathcal{M}_1 = 0, \quad (3.5)$$

$$\ddot{\mathcal{M}}_2 - \mathcal{M}_2 + (\mathcal{M}_1^2 + \mathcal{M}_2^2)^{\frac{v-2}{2}} \mathcal{M}_2 = 0. \quad (3.6)$$

Subtracting (3.5) by (3.6), one infers that

$$\frac{d}{dt} (\dot{\mathcal{M}}_1 \mathcal{M}_2 - \mathcal{M}_1 \dot{\mathcal{M}}_2) = 0,$$

which implies

$$\dot{\mathcal{M}}_1 \mathcal{M}_2 - \mathcal{M}_1 \dot{\mathcal{M}}_2 = C \quad \text{for some } C \in \mathbb{R}.$$

Since $\dot{\mathcal{M}}_1(0) = \dot{\mathcal{M}}_2(0) = 0$, we obtain

$$\dot{\mathcal{M}}_1 \mathcal{M}_2 - \mathcal{M}_1 \dot{\mathcal{M}}_2 = 0 \quad \text{for all } t \in \mathbb{R}.$$

By the ordinary differential equation theory, one can deduce

$$\mathcal{M}_1 = K \mathcal{M}_2 \quad \text{for some } K > 0. \quad (3.7)$$

Combining (3.6) and (3.7), we obtain

$$\ddot{\mathcal{M}}_2 - \mathcal{M}_2 + (K^2 + 1)^{\frac{v-2}{2}} \mathcal{M}_2^{v-1} = 0.$$

Letting $T(t) = (K^2 + 1)^{\frac{1}{2}} \mathcal{M}_2(t)$, we see $T(t) > 0$ satisfies (3.1). By the uniqueness, one has $T = Q_v$, which implies $\mathcal{M}_2 = (K^2 + 1)^{-\frac{1}{2}} Q_v$. Then it follows that

$$\mathcal{M}_v(t) = Q_v(t) (K^2 + 1)^{-\frac{1}{2}} (K, 1),$$

which implies (3.4). We also show that \mathcal{U}_v is a ground state solution for systems (3.2). Actually, the corresponding functional of (3.2) is defined as

$$I_\infty(u) = \frac{1}{2} \int_{\mathbb{R}} (|\dot{u}|^2 + a_\infty |u|^2) dt - d_\infty \int_{\mathbb{R}} |u|^\nu dt.$$

Set $\mathcal{N} = \{u \in H^1(\mathbb{R}, \mathbb{R}^2) \setminus \{0\} : \langle I'_\infty(u), u \rangle = 0\}$ and $c_\infty = \inf_{u \in \mathcal{N}} I_\infty(u)$. Moreover, the corresponding functional of (3.1) is defined as

$$J_\infty(q) = \frac{1}{2} \int_{\mathbb{R}} (|\dot{q}|^2 + |q|^2) dt - \frac{1}{\nu} \int_{\mathbb{R}} |q|^\nu dt.$$

Let $\mathfrak{N} = \{q \in H^1(\mathbb{R}, \mathbb{R}) \setminus \{0\} : \langle J'_\infty(q), q \rangle = 0\}$ and $\mathfrak{C}_\infty = \inf_{q \in \mathfrak{N}} J_\infty(q)$. By the definition of Q_ν , we deduce that

$$J_\infty(Q_\nu) = \mathfrak{C}_\infty.$$

Obviously, for any $q(t) \in \mathfrak{N}$ and $e \in \mathbb{R}^2$ with $|e| = 1$, we have that $(\frac{a_\infty}{vd_\infty})^{\frac{1}{\nu-2}} q(\sqrt{a_\infty}t)e \in \mathcal{N}$. In turn, for any $u(t) \in \mathcal{N}$ we have $(\frac{vd_\infty}{a_\infty})^{\frac{1}{\nu-2}} |u(\frac{t}{\sqrt{a_\infty}})| \in \mathfrak{N}$. Therefore, we infer that $c_\infty = a^{\frac{\nu+2}{2(\nu-2)}} (\frac{1}{vd_\infty})^{\frac{2}{\nu-2}} \mathfrak{C}_\infty$. Moreover, it follows from (3.4) and the definition of Q_ν that

$$\int_{\mathbb{R}} (|\dot{Q}_\nu|^2 + |Q_\nu|^2) dt = \int_{\mathbb{R}} |Q_\nu|^\nu dt \quad (3.8)$$

and

$$c_\infty = I_\infty(\mathcal{U}_v) = \frac{\nu-2}{2\nu} \left(\frac{1}{vd_\infty} \right)^{\frac{2}{\nu-2}} a_\infty^{\frac{\nu+2}{2(\nu-2)}} \int_{\mathbb{R}} |Q_\nu|^\nu dt. \quad (3.9)$$

□

Remark 3.2. When $\nu = 4$, Theorem 3.1 reduces to the results in [41].

4 Main results

In this section, we prove our main result.

Theorem 4.1. *Suppose that $\nu > 2$, (V1)–(V5) hold. Then there exist $\lambda_0, d_0 > 0$ such that problem (1.1) possesses at least one homoclinic solution for all $\lambda \in (0, \lambda_0)$ and $d_\infty \in (0, d_0)$. Moreover, (1.1) possesses another homoclinic solution if (V6) holds.*

Remark 4.2. In [36, 37], Sun and Wu also considered (1.1) with mixed nonlinearities. In both papers, the infimum of $a(t)$ cannot be attained at infinity, which is different from our result.

Remark 4.3. In Theorem 4.1, there are no periodic, coercive or symmetric assumptions on $a(t)$, which is different from the results in [6, 14, 32, 39, 44]. According to our conditions, both of the superquadratic and subquadratic parts of V can change signs, then we can not obtain the compactness as the authors did in [46].

Remark 4.4. In [26, 27], $W(t, x)$ is required to satisfy

$$(\nabla W(t, x), x) \geq (\nabla W^\infty(x), x) \geq 0 \quad \text{for all } t \in \mathbb{R}, x \in \mathbb{R}^N, \quad (4.1)$$

and

$$(\nabla W(t, x), x) \geq 2W(x) \quad \text{for all } t \in \mathbb{R}, x \in \mathbb{R}^N, \quad (4.2)$$

where W^∞ is the limit function of W as $t \rightarrow \infty$. In our theorem, we have

$$W(t, x) = \lambda F(t, x) + d(t)|x|^\nu. \quad (4.3)$$

Since $F(t, x)$ and $d(t)$ can change signs, we infer that (4.1) and (4.2) are not valid for (4.3). Moreover, since (4.1), (4.2) and (MC) hold in [27], the authors can show that for any $u \in H^1(\mathbb{R}, \mathbb{R}^2)$, there exists unique $s_u > 0$ such that $s_u u \in \mathcal{L}$ and $\sup_{s \geq 0} I(su) = I(s_u u)$, where $\mathcal{L} = \{u \in H^1(\mathbb{R}, \mathbb{R}^2) \setminus \{0\} : \langle I'(u), u \rangle = 0\}$. This conclusion is crucial in using the (CCP) to show the contradictions. However, we can not obtain this conclusion by our conditions. Therefore, the Nehari-manifold method is not applicable for our theorem.

4.1 Preliminaries

The corresponding functional of (1.1) is defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}} (|\dot{u}|^2 + a(t)|u|^2) dt - \lambda \int_{\mathbb{R}} F(t, u) dt - \int_{\mathbb{R}} d(t)|u|^\nu dt. \quad (4.4)$$

Lemma 4.5. *Under (V1)–(V5), I is of C^1 class and weakly lower semi-continuous. Moreover, we have*

$$\langle I'(u), v \rangle = \int_{\mathbb{R}} ((\dot{u}, \dot{v}) + a(t)(u, v)) dt - \lambda \int_{\mathbb{R}} (\nabla F(t, u), v) dt - \nu \int_{\mathbb{R}} d(t)|u|^{\nu-2}(u, v) dt,$$

which implies that

$$\langle I'(u), u \rangle = \int_{\mathbb{R}} (|\dot{u}|^2 + a(t)|u|^2) dt - \lambda \int_{\mathbb{R}} (\nabla F(t, u), u) dt - \nu \int_{\mathbb{R}} d(t)|u|^\nu dt.$$

Proof. The proof is similar to Lemma 2.3 in [6]. □

Lemma 4.6. *The critical points of I are homoclinic solutions for problem (1.1).*

Proof. Since $\|a\|_\infty \geq a(t) > a_0 > 0$, the proof is similar to Lemma 3.1 in [49]. □

We will show the existence of two critical points of I by the Mountain Pass Theorem and the following critical point lemma respectively.

Lemma 4.7 (Lu [22]). *Let X be a real reflexive Banach space and $\Omega \subset X$ be a closed bounded convex subset of X . Suppose that $\varphi : X \rightarrow \mathbb{R}$ is a weakly lower semi-continuous (w.l.s.c. for short) functional. If there exists a point $x_0 \in \Omega \setminus \partial\Omega$ such that*

$$\varphi(x) > \varphi(x_0), \quad \forall x \in \partial\Omega,$$

then there must be an $x^* \in \Omega \setminus \partial\Omega$ such that

$$\varphi(x^*) = \inf_{x \in \Omega} \varphi(x).$$

4.2 The Mountain Pass Structure

In this section, we mainly show the Mountain Pass structure of I and obtain some crucial estimates.

Lemma 4.8. *Suppose the conditions of Theorem 4.1 hold, then there exist $\varrho_0, \bar{\alpha} > 0$ such that $I|_{\partial\mathcal{S}_{\varrho_0}} \geq \bar{\alpha}$, where $\mathcal{S}_{\varrho_0} = \{u \in H^1 : \|u\| \leq \varrho_0\}$.*

Proof. By (V4) and (V5), we can deduce that

$$|(\nabla F(t, x), x)| \leq b_1(t)|x|^{r_1} + b_2(t)|x|^{r_2} \quad (4.5)$$

and

$$|F(t, x)| \leq \frac{1}{r_1}b_1(t)|x|^{r_1} + \frac{1}{r_2}b_2(t)|x|^{r_2} \quad (4.6)$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^2$. By (2.1), (4.4), (4.6) and (V1), for all $u \in \partial\mathcal{S}_{\varrho}$, we have

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\mathbb{R}} (|\dot{u}|^2 + a(t)|u|^2) dt - \lambda \int_{\mathbb{R}} F(t, u) dt - \int_{\mathbb{R}} d(t)|u|^\nu dt \\ &\geq \frac{\min\{1, a_0\}}{2} \|u\|^2 - \lambda \left(\frac{1}{r_1} \int_{\mathbb{R}} b_1(t)|u|^{r_1} dt + \frac{1}{r_2} \int_{\mathbb{R}} b_2(t)|u|^{r_2} dt \right) - C_\nu^v \|d\|_\infty \|u\|^\nu \\ &\geq \frac{\min\{1, a_0\}}{2} \|u\|^2 - \lambda \left(\frac{1}{r_1} C_{r_1\beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|u\|^{r_1} + \frac{1}{r_2} C_{r_2\beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|u\|^{r_2} \right) - C_\nu^v \|d\|_\infty \|u\|^\nu. \end{aligned}$$

For any $\varrho > 0$, set

$$h(\varrho) = \frac{\min\{1, a_0\}}{2} \varrho^2 - \|d\|_\infty C_\nu^v \varrho^\nu.$$

It is easy to see that $h'(\varrho_0) = 0$ and ϱ_0 is the unique critical point of h defined as

$$\varrho_0 = \left(\frac{\min\{1, a_0\}}{\nu C_\nu^v \|d\|_\infty} \right)^{\frac{1}{\nu-2}}.$$

Then there exists $\bar{\lambda}_1 > 0$ such that for any $\lambda \in (0, \bar{\lambda}_1)$ with $\|u\| = \varrho_0$, we have

$$I(u) \geq \frac{1}{2} h(\varrho_0) \doteq \bar{\alpha}.$$

We obtain our conclusion. □

Lemma 4.9. *Suppose the conditions of Theorem 4.1 hold, then for λ small enough, there exists $e_0 \in H^1$ such that $\|e_0\| > \varrho_0$ and $I(e_0) \leq \bar{\alpha}$, where $\varrho_0, \bar{\alpha}$ are defined in Lemma 4.8.*

Proof. it follows from the definition of Q_ν , (3.8) and (2.14) that

$$\begin{aligned} J_\infty(Q_\nu) = \mathcal{E}(\nu) &= \inf_{u \in H^1(\mathbb{R}, \mathbb{R}) \setminus \{0\}} \frac{(\nu-2) (\|\dot{u}\|_2^2 + \|u\|_2^2)^{\frac{\nu}{\nu-2}}}{2\nu \|u\|_\nu^{\frac{2\nu}{\nu-2}}} \\ &= \frac{(\nu-2) (\|\dot{Q}_\nu\|_2^2 + \|Q_\nu\|_2^2)^{\frac{\nu}{\nu-2}}}{2\nu \|Q_\nu\|_\nu^{\frac{2\nu}{\nu-2}}} \\ &= \frac{\nu-2}{2\nu} \int_{\mathbb{R}} |Q_\nu|^\nu dt \\ &\geq \frac{\nu-2}{2\nu} C_\nu^{-\frac{2\nu}{\nu-2}}, \end{aligned}$$

which implies

$$\int_{\mathbb{R}} |Q_v|^v dt \geq C_v^{-\frac{2v}{v-2}}.$$

It follows from (V3) that, there exist $T > 0$ such that $|d(t) - \|d\|_{\infty}| \leq \varepsilon_0$ for all $t \in (-T, T)$. For any $u \in H_0^1((-T, T), \mathbb{R}^2)$, let

$$L(u) = \frac{\left(\int_{-T}^T (|\dot{u}|^2 + |u|^2) dt \right)^{\frac{v}{v-2}}}{\left(\int_{-T}^T |u|^v dt \right)^{\frac{2}{v-2}}}.$$

Let $\chi \in H_0^1((-T, T), \mathbb{R}^2)$ and

$$\bar{u}(t) = (\chi(\sqrt{\|a\|_{\infty} + \varepsilon_0 t}), 0),$$

which implies $\bar{u} \in H^1(\mathbb{R}, \mathbb{R}^2)$. For any $t \in (-T, T)$, it follows from $\|a\|_{\infty} \geq 1$ that $\frac{t}{\sqrt{\|a\|_{\infty} + \varepsilon_0}} \in (-T, T)$. Then

$$\begin{aligned} I(\theta\bar{u}) &\leq \frac{\theta^2 \sqrt{\|a\|_{\infty} + \varepsilon_0}}{2} \int_{-T}^T (|\dot{\chi}|^2 + |\chi|^2) dt - \frac{\theta^v}{\sqrt{\|a\|_{\infty} + \varepsilon_0}} \int_{-T}^T d \left(\frac{t}{\sqrt{\|a\|_{\infty} + \varepsilon_0}} \right) |\chi|^v dt \\ &\quad - \lambda \int_{-T}^T F(t, \theta\bar{u}) dt \\ &\leq \frac{\theta^2 \sqrt{\|a\|_{\infty} + \varepsilon_0}}{2} \int_{-T}^T (|\dot{\chi}|^2 + |\chi|^2) dt - \frac{(\|d\|_{\infty} - \varepsilon_0)\theta^v}{\sqrt{\|a\|_{\infty} + \varepsilon_0}} \int_{-T}^T |\chi|^v dt \\ &\quad + \lambda \left(\frac{\theta^{r_1}}{r_1} \|a\|_{\infty}^{-\frac{1}{2\beta_1^*}} C_{r_1\beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|\chi\|^{r_1} + \frac{\theta^{r_2}}{r_2} \|a\|_{\infty}^{-\frac{1}{2\beta_2^*}} C_{r_2\beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|\chi\|^{r_2} \right). \end{aligned}$$

Choose $\theta_0 > 0$ large enough such that $I(\theta_0\bar{u}) < 0$ and $\theta_0\|\bar{u}\| > \varrho_0$. Letting $e_0 = \theta_0\bar{u}$, we see that there exists $\bar{\lambda}_2 \in (0, \bar{\lambda}_1)$ such that for any $\lambda \in (0, \bar{\lambda}_2)$, $I(e_0) < 0$ and $\|e_0\| > \varrho_0$. We obtain the conclusion of this lemma. \square

By the Mountain Pass theorem, there exists a sequence $\{u_n\}$ and $c \geq \bar{a}$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) \mid g(0) = O, \quad g(1) = e_0\}.$$

such that

$$I(u_n) \rightarrow c \tag{4.7}$$

and for any $v \in H^1(\mathbb{R}, \mathbb{R}^2)$

$$\begin{aligned} o(1)\|v\| &= \langle I'(u_n), v \rangle = \int_{\mathbb{R}} ((\dot{u}_n, \dot{v}) + a(t)(u_n, v)) dt - \lambda \int_{\mathbb{R}} (\nabla F(t, u_n), v) dt \\ &\quad - v \int_{\mathbb{R}} d(t)|u_n|^{v-2}(u_n, v) dt. \end{aligned} \tag{4.8}$$

Next, we show an important relation between c and c_{∞} , which is crucial in the following concentration compactness study.

Lemma 4.10. *Suppose λ and d_∞ are small enough, then*

$$c_\infty - c \geq 2\lambda \left(\frac{r_1 + \nu}{\nu r_1} C_{r_1 \beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|v_n\|^{r_1} + \frac{r_1 + \nu}{\nu r_2} C_{r_2 \beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|v_n\|^{r_2} \right). \quad (4.9)$$

Proof. First, we estimate the critical value of I along the sequence $\{u_n\}$. For $s \in [0, 1]$, set

$$g_0(s) = s\varepsilon_0 = s\theta_0 \bar{u},$$

which implies $g_0(s) \in \Gamma$. It follows from the definition of c that

$$\begin{aligned} c &= \inf_{g \in \Gamma} \max_{s \in [0, 1]} I(g(s)) \\ &\leq \max_{s \in [0, 1]} I(g_0(s)) \\ &\leq \max_{s \in [0, 1]} \left[\frac{(s\theta_0)^2 \sqrt{\|a\|_\infty + \varepsilon_0}}{2} \int_{-T}^T (|\dot{\chi}|^2 + |\chi|^2) dt - \frac{(\|d\|_\infty - \varepsilon_0)(s\theta_0)^\nu}{\sqrt{\|a\|_\infty + \varepsilon_0}} \int_{-T}^T |\chi|^\nu dt \right] \\ &\quad + \lambda \left(\frac{\theta_0^{r_1}}{r_1} \|a\|_\infty^{-\frac{1}{2\beta_1^*}} C_{r_1 \beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|\chi\|^{r_1} + \frac{\theta_0^{r_2}}{r_2} \|a\|_\infty^{-\frac{1}{2\beta_2^*}} C_{r_2 \beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|\chi\|^{r_2} \right) \\ &\leq \frac{\nu - 2}{2\nu} \left(\frac{1}{\nu(\|d\|_\infty - \varepsilon_0)} \right)^{\frac{2}{\nu-2}} (\|a\|_\infty + \varepsilon_0)^{\frac{\nu+2}{2(\nu-2)}} L(\chi) \\ &\quad + \lambda \left(\frac{\theta_0^{r_1}}{r_1} \|a\|_\infty^{-\frac{1}{2\beta_1^*}} C_{r_1 \beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|\chi\|^{r_1} + \frac{\theta_0^{r_2}}{r_2} \|a\|_\infty^{-\frac{1}{2\beta_2^*}} C_{r_2 \beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|\chi\|^{r_2} \right). \end{aligned} \quad (4.10)$$

Moreover, there exists $d_0 > 0$ small enough such that for any $d_\infty \in (0, d_0)$, one has

$$\begin{aligned} a_\infty^{\frac{\nu+2}{2(\nu-2)}} \left(\frac{1}{d_\infty} \right)^{\frac{2}{\nu-2}} \int_{\mathbb{R}} |Q_\nu|^\nu dt &\geq a_\infty^{\frac{\nu+2}{2(\nu-2)}} \left(\frac{1}{d_\infty} \right)^{\frac{2}{\nu-2}} C_\nu^{-\frac{2\nu}{\nu-2}} \\ &= a_\infty^{\frac{\nu+2}{2(\nu-2)}} \left(\frac{1}{d_\infty} \right)^{\frac{2}{\nu-2}} \left(M_{GN}(\nu) \left(\frac{\nu-2}{\nu+2} \right)^{\frac{\nu-2}{4\nu}} \left(\frac{\nu+2}{2\nu} \right)^{\frac{1}{2}} \right)^{-\frac{2\nu}{\nu-2}} \\ &> \left(\frac{1}{\|d\|_\infty - \varepsilon_0} \right)^{\frac{2}{\nu-2}} (\|a\|_\infty + \varepsilon_0)^{\frac{\nu+2}{2(\nu-2)}} L(\chi). \end{aligned}$$

By (3.9) and (4.10), there exists $\bar{\lambda}_3 \in (0, \bar{\lambda}_2)$ such that for any $\lambda \in (0, \bar{\lambda}_3)$ and $\varepsilon_0 > \text{small enough}$

$$\begin{aligned} c_\infty - c &\geq \frac{\nu - 2}{2\nu} \left(\frac{1}{\nu} \right)^{\frac{2}{\nu-2}} \left(a_\infty^{\frac{\nu+2}{2(\nu-2)}} \left(\frac{1}{d_\infty} \right)^{\frac{2}{\nu-2}} \int_{\mathbb{R}} |Q_\nu|^\nu dt \right. \\ &\quad \left. - \left(\frac{1}{\|d\|_\infty - \varepsilon_0} \right)^{\frac{2}{\nu-2}} (\|a\|_\infty + \varepsilon_0)^{\frac{\nu+2}{2(\nu-2)}} L(\chi) \right) \\ &\quad - \lambda \left(\frac{\theta_0^{r_1}}{r_1} a_\infty^{-\frac{1}{2\beta_1^*}} C_{r_1 \beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|\chi\|^{r_1} + \frac{\theta_0^{r_2}}{r_2} a_\infty^{-\frac{1}{2\beta_2^*}} C_{r_2 \beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|\chi\|^{r_2} \right) \\ &> 2\lambda \left(\frac{r_1 + \nu}{\nu r_1} C_{r_1 \beta_1^*}^{r_1} \|b_1\|_{\beta_1} (4\mathfrak{D})^{r_1} + \frac{r_1 + \nu}{\nu r_2} C_{r_2 \beta_2^*}^{r_2} \|b_2\|_{\beta_2} (4\mathfrak{D})^{r_2} \right) \\ &\geq 2\lambda \left(\frac{r_1 + \nu}{\nu r_1} C_{r_1 \beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|v_n\|^{r_1} + \frac{r_1 + \nu}{\nu r_2} C_{r_2 \beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|v_n\|^{r_2} \right). \end{aligned}$$

We obtain our conclusion. \square

4.3 The compactness property

In this section, we show that $\{u_n\}$ converges to a nontrivial solution for problem (1.1). We will utilize the concentration-compactness principle by P. L. Lions [20] to obtain the compactness.

Lemma 4.11 (See [20, Lemma1.1]). *Let $\{\rho_n\}$ be a sequence of nonnegative L^1 functions on \mathbb{R} satisfying $\int_{\mathbb{R}} \rho_n(t) dt = \kappa$, where κ is a fixed constant. Then there exists a subsequence which we still denote by $\{\rho_n\}$, satisfying one of the three following possibilities:*

(i) (Vanishing): for all $R > 0$, it follows

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{B_R(y)} \rho_n dt = 0;$$

(ii) (Compactness): there exists $\{y_n\} \subset \mathbb{R}$ such that, for any $\varepsilon > 0$, there exists $R > 0$ satisfying

$$\int_{B_R(y_n)} \rho_n dt \geq \kappa - \varepsilon;$$

(iii) (Dichotomy): there exist $\alpha \in (0, \kappa)$, $\rho_n^1 \geq 0$, $\rho_n^2 \geq 0$, and $\rho_n^1, \rho_n^2 \in L^1(\mathbb{R})$ such that

- (a) $\|\rho_n - (\rho_n^1 + \rho_n^2)\|_{L^1} \rightarrow 0$ as $n \rightarrow \infty$;
- (b) $\int_{\mathbb{R}} \rho_n^1 dt \rightarrow \alpha$ as $n \rightarrow \infty$;
- (c) $\int_{\mathbb{R}} \rho_n^2 dt \rightarrow \kappa - \alpha$ as $n \rightarrow \infty$;
- (d) $\text{dist}(\text{supp } \rho_n^1, \text{supp } \rho_n^2) \rightarrow \infty$ as $n \rightarrow \infty$.

Lemma 4.12 (See [21]). *Let $\{u_n\}$ be bounded sequence in $L^q(\mathbb{R})$ for $1 \leq q < +\infty$ such that $\{\dot{u}_n\}$ is bounded in $L^p(\mathbb{R})$ for $1 < p \leq +\infty$. If there exists $R > 0$ such that*

$$\sup_{y \in \mathbb{R}} \int_{B_R(y)} |u_n|^q dt \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

then $u_n \rightarrow 0$ in $L^r(\mathbb{R})$ for all $r \in (q, +\infty)$.

First, we show the boundedness of $\|u_n\|$. It follows from (4.4), (4.5), (4.7), (4.8) and (4.10) that

$$\begin{aligned} & \nu c + o(1) \\ & \geq \nu I(u_n) - \langle I'(u_n), u_n \rangle \\ & = \left(\frac{\nu}{2} - 1\right) \int_{\mathbb{R}} (|\dot{u}_n|^2 + a(t)|u_n|^2) dt - \lambda \int_{\mathbb{R}} ((\nabla F(t, u_n), u_n) - \nu F(t, u_n)) dt \\ & \geq \min\{1, a_0\} \left(\frac{\nu}{2} - 1\right) \|u_n\|^2 - \lambda \left(\frac{r_1 + \nu}{r_1} C_{r_1 \beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|u_n\|^{r_1} + \frac{r_1 + \nu}{r_2} C_{r_2 \beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|u_n\|^{r_2} \right). \end{aligned}$$

Hence there exists $\mathfrak{D} > 0$ such that

$$\|u_n\| \leq \mathfrak{D} \quad \text{for all } n \in \mathbb{N}. \quad (4.11)$$

Without loss of generality, we assume that

$$\lim_{n \rightarrow \infty} \|u_n\| = \sqrt{\kappa}. \quad (4.12)$$

We have that $\kappa > 0$. If not, assuming by contradiction that $\|u_n\| \rightarrow 0$, there will be a contradiction. It follows from $\|u_n\| \rightarrow 0$ that $\|u_n\|_\infty \rightarrow 0$. It is easy to see that

$$\left| \lambda \int_{\mathbb{R}} F(t, u_n) dt + \int_{\mathbb{R}} d(t) |u_n|^v dt \right| dt \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which contradicts to (4.7). Then (4.12) holds.

Lemma 4.13. *The sequence $\{u_n\}$ converges to a nontrivial function u_0 in $H^1(\mathbb{R}, \mathbb{R}^2)$, which is the homoclinic solution for systems (1.1).*

Proof. In order to prove this lemma, we consider three cases of behavior for $\{u_n\}$, which are classified in Lemma 4.11. Set $\rho_n(t) = |\dot{u}_n(t)|^2 + |u_n(t)|^2$. The proof is divided into three steps.

Step 1: Vanishing does not occur.

Suppose by contradiction, for all $R > 0$,

$$\limsup_{n \rightarrow \infty} \int_{B_R(y)} \rho_n dt = 0.$$

We deduce from Lemma 4.12 that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |u_n|^v dt = 0. \quad (4.13)$$

By (4.8), for n large enough, we can conclude that

$$\int_{\mathbb{R}} (|\dot{u}_n|^2 + a(t)|u_n|^2) dt \leq \lambda \int_{\mathbb{R}} (\nabla F(t, u_n), u_n) dt + v \int_{\mathbb{R}} d(t) |u_n|^v dt + \frac{1}{2}c. \quad (4.14)$$

It follows from (4.4), (4.5), (4.6), (4.7), (4.8), (4.13) and (4.14) that there exists $\bar{\lambda}_4 \in (0, \bar{\lambda}_3)$ such that for any $\lambda \in (0, \bar{\lambda}_4)$

$$\begin{aligned} & 0 < \frac{1}{2}c \\ & \leq I(u_n) \\ & = \frac{1}{2} \int_{\mathbb{R}} (|\dot{u}_n|^2 + a(t)|u_n|^2) dt - \lambda \int_{\mathbb{R}} F(t, u_n) dt - \int_{\mathbb{R}} d(t) |u_n|^v dt \\ & \leq \frac{1}{2} \left(\lambda \int_{\mathbb{R}} (\nabla F(t, u_n), u_n) dt + v \int_{\mathbb{R}} d(t) |u_n|^v dt + \frac{1}{2}c \right) - \lambda \int_{\mathbb{R}} F(t, u_n) dt - \int_{\mathbb{R}} d(t) |u_n|^v dt \\ & \leq \lambda \left(\frac{r_1 + 2}{2r_1} \|b_1\|_{\beta_1} \|u_n\|_{r_1 \beta_1^*}^{r_1} + \frac{r_1 + 2}{2r_2} \|b_2\|_{\beta_2} \|u_n\|_{r_2 \beta_2^*}^{r_2} \right) + \|d\|_\infty \left(\frac{v}{2} - 1 \right) \int_{\mathbb{R}} |u_n|^v dt + \frac{1}{4}c \\ & \rightarrow \frac{1}{4}c \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which is a contradiction. Then we see that vanishing case does not occur.

Step 2: Dichotomy does not occur.

There exist $R_0 > 0$ and sequences $\{y_n\} \subset \mathbb{R}, \{R_n\} \subset \mathbb{R}^+$, with $R_0 < R_1 < \dots < R_n < R_{n+1} \rightarrow \infty$, $\Omega_n = B_{R_n}(y_n) \setminus B_{R_0}(y_n)$ such that

$$\int_{\Omega_n} \rho_n dt \rightarrow 0, \quad \int_{B_{R_0}(y_n)} \rho_n dt \rightarrow \alpha \quad \text{and} \quad \int_{\mathbb{R} \setminus B_{2R_n}(y_n)} \rho_n dt \rightarrow \kappa - \alpha \quad (4.15)$$

as $n \rightarrow \infty$. Set $\zeta \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ with $0 \leq \zeta \leq 1$, $\zeta(s) \equiv 1$ for $s \leq 1$; $\zeta(s) \equiv 0$ for $s \geq 2$ and $|\dot{\zeta}(s)| \leq 2$. Let

$$v_n(t) = \zeta \left(\frac{|t - y_n|}{R_0} \right) u_n(t) \quad \text{and} \quad w_n(t) = \left(1 - \zeta \left(\frac{|t - y_n|}{R_n} \right) \right) u_n(t).$$

On one hand, we can easily deduce that

$$\begin{aligned} \|w_n\|^2 &= \int_{\mathbb{R}} |\dot{w}_n|^2 dt + \int_{\mathbb{R}} |w_n|^2 dt \\ &= \int_{\mathbb{R}} \left(\frac{1}{R_n^2} \left| \dot{\zeta} \left(\frac{|t - y_n|}{R_n} \right) u_n \right|^2 + \left| \left(1 - \zeta \left(\frac{|t - y_n|}{R_n} \right) \right) \dot{u}_n \right|^2 \right) dt \\ &\quad - \frac{2}{R_n} \int_{\mathbb{R}} \dot{\zeta} \left(\frac{|t - y_n|}{R_n} \right) \left(1 - \zeta \left(\frac{|t - y_n|}{R_n} \right) \right) (u_n, \dot{u}_n) dt + \int_{\mathbb{R}} \left| \left(1 - \zeta \left(\frac{|t - y_n|}{R_n} \right) \right) u_n \right|^2 dt \\ &\geq \int_{\mathbb{R} \setminus B_{2R_n}(y_n)} \rho_n dt - \frac{2}{R_n} \|u_n\|^2, \end{aligned}$$

which implies that

$$\liminf_{n \rightarrow \infty} \|w_n\|^2 \geq \kappa - \alpha.$$

On the other hand, it can be easily deduce from (4.15) that

$$\int_{\Omega_n} d(t) |u_n|^v dt \rightarrow 0, \quad \int_{\mathbb{R}} [(\dot{u}_n, \dot{w}_n) - |\dot{w}_n|^2] dt \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (4.16)$$

and

$$\int_{B_{R_n}(y_n)} (\nabla F(t, u_n), w_n) dt \rightarrow 0, \quad \int_{B_{R_n}(y_n)} (\nabla F(t, w_n), w_n) dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.17)$$

Then one has

$$\begin{aligned} \|w_n\|^2 &= \int_{\mathbb{R}} \left(\frac{1}{R_n^2} \left| \dot{\zeta} \left(\frac{|t - y_n|}{R_n} \right) u_n \right|^2 + \left| \left(1 - \zeta \left(\frac{|t - y_n|}{R_n} \right) \right) \dot{u}_n \right|^2 \right) dt \\ &\quad - \frac{2}{R_n} \int_{\mathbb{R}} \dot{\zeta} \left(\frac{|t - y_n|}{R_n} \right) \left(1 - \zeta \left(\frac{|t - y_n|}{R_n} \right) \right) (u_n, \dot{u}_n) dt + \int_{\mathbb{R}} \left| \left(1 - \zeta \left(\frac{|t - y_n|}{R_n} \right) \right) u_n \right|^2 dt \\ &\leq \frac{4}{R_n^2} \int_{\Omega_n} |u_n|^2 dt + \frac{2}{R_n} \|u_n\|^2 + \int_{\Omega_n} \rho_n dt + \int_{\mathbb{R} \setminus B_{2R_n}(y_n)} \rho_n dt, \end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} \|w_n\|^2 = \kappa - \alpha. \quad (4.18)$$

Subsequently, for any $u \in H^1(\mathbb{R}, \mathbb{R}^2)$ and $t \in \mathbb{R}$, set

$$G(t, u) = |\dot{u}|^2 + a(t)|u|^2 - \lambda(\nabla F(t, u), u) - v d(t)|u|^v.$$

Hence, it follows from the definition of v , w and (4.16) that

$$\begin{aligned}
& \int_{\mathbb{R}} |F(t, u_n) - F(t, v_n) - F(t, w_n)| dt \\
&= \int_{\Omega_n} |F(t, u_n) - F(t, v_n) - F(t, w_n)| dt \\
&\leq \int_{\Omega_n} |F(t, u_n)| dt + \int_{\Omega_n} |F(t, v_n)| dt + \int_{\Omega_n} |F(t, w_n)| dt \\
&\leq \frac{1}{r_1} \int_{\Omega_n} b_1(t) |u_n|^{r_1} dt + \frac{1}{r_2} \int_{\Omega_n} b_2(t) |u_n|^{r_2} dt \\
&\quad + \frac{1}{r_1} \int_{\Omega_n} b_1(t) |v_n|^{r_1} dt + \frac{1}{r_2} \int_{\Omega_n} b_2(t) |v_n|^{r_2} dt \\
&\quad + \frac{1}{r_1} \int_{\Omega_n} b_1(t) |w_n|^{r_1} dt + \frac{1}{r_2} \int_{\Omega_n} b_2(t) |w_n|^{r_2} dt \\
&\leq \frac{3}{r_1} \int_{\Omega_n} b_1(t) |u_n|^{r_1} dt + \frac{3}{r_2} \int_{\Omega_n} b_2(t) |u_n|^{r_2} dt \\
&\leq 3 \left(\frac{\|b_1\|_{\beta_1}}{r_1} + \frac{\|b_2\|_{\beta_2}}{r_2} \right) \left(\left(\int_{\Omega_n} |u_n|^{r_1 \beta_1^*} dt \right)^{\frac{1}{\beta_1^*}} + \left(\int_{\Omega_n} |u_n|^{r_2 \beta_2^*} dt \right)^{\frac{1}{\beta_2^*}} \right) \\
&\rightarrow 0 \quad \text{as } n \rightarrow \infty
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
\left| \int_{\mathbb{R}} d(t) (|u_n|^v - |v_n|^v - |w_n|^v) dt \right| &= \int_{\Omega_n} d(t) (|u_n|^v - |v_n|^v - |w_n|^v) dt \\
&\leq 3 \|d\|_{\infty} \|u_n\|_{\infty}^{v-2} \int_{\Omega_n} |u_n|^2 dt \\
&\leq 3 \times 2^{-\frac{v-2}{2}} \mathfrak{D}^{v-2} \|d\|_{\infty} \int_{\Omega_n} |u_n|^2 dt \\
&\rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{4.20}$$

Furthermore, we can deduce that

$$\begin{aligned}
\| |u_n|^2 - |v_n|^2 - |w_n|^2 \| &\leq \int_{\Omega_n} (|\dot{u}_n|^2 - |\dot{v}_n|^2 - |\dot{w}_n|^2) dt + \int_{\Omega_n} (|u_n|^2 - |v_n|^2 - |w_n|^2) dt \\
&\rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{4.21}$$

Together with (4.19), (4.20) and (4.21), we have

$$I(u_n) \geq I(v_n) + I(w_n) - o(1). \tag{4.22}$$

The discussion for this step is divided into two cases.

Case 1. $\{y_n\} \subset \mathbb{R}$ is bounded.

First, we show the following claim.

Claim 1: $I(w_n) \geq c_{\infty} - o(1)$.

By (V2), for any $\varepsilon > 0$, there exists $r_{\infty} > 0$ such that

$$|a(t) - a_{\infty}| \leq \varepsilon$$

for all $|t| \geq r_\infty$. Since $\{y_n\}$ is bounded, then there exists $\bar{y} > \underline{y} > 0$ such that $\{y_n\} \subset [\underline{y}, \bar{y}]$ for all $n \in \mathbb{N}$ and $\min\{R_n - \bar{y}, R_n + \underline{y}\} \rightarrow +\infty$ as $n \rightarrow \infty$. By the definition of w_n , for n large enough, we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}} (a_\infty - a(t)) |w_n|^2 dt \right| &\leq \int_{\mathbb{R} \setminus B_{R_n}(y_n)} |a_\infty - a(t)| \left(1 - \zeta \left(\frac{|t - y_n|}{R_n} \right) \right)^2 |u_n|^2 dt \\ &\leq \left(\int_{-\infty}^{\bar{y} - R_n} + \int_{\underline{y} + R_n}^{+\infty} \right) |a_\infty - a(t)| |u_n|^2 dt \\ &\leq 2\varepsilon \int_{\mathbb{R}} |u_n|^2 dt \\ &\leq 2\varepsilon \mathfrak{D}^2. \end{aligned}$$

By the arbitrariness of ε , we can see that

$$\left| \int_{\mathbb{R}} (a_\infty - a(t)) |w_n|^2 dt \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.23)$$

Similarly, we have

$$\left| \int_{\mathbb{R}} (d_\infty - d(t)) |w_n|^v dt \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.24)$$

Moreover, we have

$$\begin{aligned} \left| \int_{\mathbb{R}} F(t, w_n) dt \right| &\leq \frac{1}{r_1} \int_{\mathbb{R}} b_1(t) |w_n|^{r_1} dt + \frac{1}{r_2} \int_{\mathbb{R}} b_2(t) |w_n|^{r_2} dt \\ &\leq \frac{1}{r_1} \left(\int_{\mathbb{R} \setminus B_{R_n}(y_n)} |b_1|^{\beta_1} dt \right)^{\frac{1}{\beta_1}} \left(\int_{\mathbb{R} \setminus B_{R_n}(y_n)} |w_n|^{r_1 \beta_1^*} dt \right)^{\frac{1}{\beta_1^*}} \\ &\quad + \frac{1}{r_2} \left(\int_{\mathbb{R} \setminus B_{R_n}(y_n)} |b_2|^{\beta_2} dt \right)^{\frac{1}{\beta_2}} \left(\int_{\mathbb{R} \setminus B_{R_n}(y_n)} |w_n|^{r_2 \beta_2^*} dt \right)^{\frac{1}{\beta_2^*}} \\ &\leq \frac{1}{r_1} \left(\left(\int_{-\infty}^{\bar{y} - R_n} + \int_{\underline{y} + R_n}^{+\infty} \right) |b_1|^{\beta_1} dt \right)^{\frac{1}{\beta_1}} \left(\int_{\mathbb{R} \setminus B_{R_n}(y_n)} |u_n|^{r_1 \beta_1^*} dt \right)^{\frac{1}{\beta_1^*}} \\ &\quad + \frac{1}{r_2} \left(\left(\int_{-\infty}^{\bar{y} - R_n} + \int_{\underline{y} + R_n}^{+\infty} \right) |b_2|^{\beta_2} dt \right)^{\frac{1}{\beta_2}} \left(\int_{\mathbb{R} \setminus B_{R_n}(y_n)} |u_n|^{r_2 \beta_2^*} dt \right)^{\frac{1}{\beta_2^*}} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (4.25)$$

Similarly,

$$\left| \int_{\mathbb{R}} (\nabla F(t, w_n), w_n) dt \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.26)$$

Combining (4.23) and (4.25), we can obtain

$$I(w_n) \geq I_\infty(w_n) - o(1). \quad (4.27)$$

It follows from (4.23), (4.24), (4.26) that

$$\begin{aligned} &|\langle I'(w_n), w_n \rangle - \langle I'_\infty(w_n), w_n \rangle| \\ &\leq \int_{\mathbb{R}} |a_\infty - a(t)| |w_n|^2 dt + \lambda \int_{\mathbb{R}} |(\nabla F(t, w_n), w_n)| dt + \nu \int_{\mathbb{R}} |d_\infty - d(t)| |w_n|^v dt \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (4.28)$$

We can also infer from (4.16), (4.17) that

$$\begin{aligned}
& |\langle I'(u_n), w_n \rangle - \langle I'(w_n), w_n \rangle| \\
& \leq \int_{\mathbb{R}} [(\dot{u}_n, \dot{w}_n) - |\dot{w}_n|^2] dt + \lambda \left(\left| \int_{\Omega_n} (\nabla F(t, w_n), w_n) dt \right| + \left| \int_{\Omega_n} (\nabla F(t, u_n), w_n) dt \right| \right) \\
& \quad + \nu \|d\|_{\infty} \left(\left| \int_{\Omega_n} (1 - \xi)^{\nu} |w_n|^{\nu} dt \right| + \left| \int_{\Omega_n} (1 - \xi) |u_n|^{\nu} dt \right| \right) \\
& \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{4.29}$$

Together with (4.8), (4.18), (4.28) and (4.29), one has

$$\langle I'_{\infty}(w_n), w_n \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{4.30}$$

It follows from (4.18) that

$$\int_{\mathbb{R}} (|\dot{w}_n|^2 + a_{\infty} |w_n|^2) dt \geq \frac{\min\{1, a_{\infty}\}}{2} (\kappa - \alpha) > 0$$

for n large enough. Letting

$$A_n = \frac{\langle I'_{\infty}(w_n), w_n \rangle}{\int_{\mathbb{R}} (|\dot{w}_n|^2 + a_{\infty} |w_n|^2) dt}$$

and

$$\sigma_n = \left(\frac{1}{1 - A_n} \right)^{\frac{1}{\nu-2}},$$

we deduce that $A_n \rightarrow 0$ and $\sigma_n \rightarrow 1$ as $n \rightarrow \infty$. Setting $z_n = \sigma_n w_n(t)$, we have

$$\begin{aligned}
\langle I'_{\infty}(z_n), z_n \rangle &= \sigma_n^2 \left(\int_{\mathbb{R}} |w_n|^2 dt + \int_{\mathbb{R}} a_{\infty} |w_n|^2 dt - \sigma_n^{\nu-2} \nu d_{\infty} \int_{\mathbb{R}} |w_n|^{\nu} dt \right) \\
&= \sigma_n^2 (1 - \sigma_n^{\nu-2} (1 - A_n)) \left(\int_{\mathbb{R}} |w_n|^2 dt + \int_{\mathbb{R}} a_{\infty} |w_n|^2 dt \right) \\
&= 0,
\end{aligned}$$

which implies $z_n \in \mathcal{N}$. Furthermore, we have

$$\begin{aligned}
I_{\infty}(z_n) &= \frac{\sigma_n^2}{2} \left(\int_{\mathbb{R}} |w_n|^2 dt + \int_{\mathbb{R}} a_{\infty} |w_n|^2 dt \right) - \sigma_n^{\nu} d_{\infty} \int_{\mathbb{R}} |w_n|^{\nu} dt \\
&= \frac{\sigma_n^2 - \sigma_n^{\nu}}{2} \left(\int_{\mathbb{R}} |w_n|^2 dt + \int_{\mathbb{R}} a_{\infty} |w_n|^2 dt \right) + \sigma_n^{\nu} I_{\infty}(w_n) \\
&\geq c_{\infty},
\end{aligned}$$

which implies

$$\begin{aligned}
I_{\infty}(w_n) &\geq \frac{\sigma_n^{\nu} - \sigma_n^2}{2\sigma_n^{\nu}} \left(\int_{\mathbb{R}} |w_n|^2 dt + \int_{\mathbb{R}} a_{\infty} |w_n|^2 dt \right) + \frac{1}{\sigma_n^{\nu}} c_{\infty} \\
&\geq c_{\infty} - o(1).
\end{aligned}$$

By (4.27), we can finish the proof of **Claim 1**.

Similar to (4.28), (4.29) and (4.30), we get $\langle I'(v_n), v_n \rangle \rightarrow 0$ as $n \rightarrow \infty$. By the definition of v_n and (4.11), we have

$$\|v_n\| \leq 4\|u_n\| \leq 4\mathcal{Q}.$$

Therefore,

$$\begin{aligned}
I(v_n) &= \frac{1}{2} \int_{\mathbb{R}} |\dot{v}_n|^2 dt + \frac{1}{2} \int_{\mathbb{R}} a(t) |v_n|^2 dt - \lambda \int_{\mathbb{R}} F(t, v_n) dt - \int_{\mathbb{R}} d(t) |v_n|^v dt \\
&= \frac{1}{2} \left(\int_{\mathbb{R}} |\dot{v}_n|^2 dt + \int_{\mathbb{R}} a(t) |v_n|^2 dt \right) - \lambda \int_{\mathbb{R}} F(t, v_n) dt \\
&\quad - \frac{1}{v} \left(\left(\int_{\mathbb{R}} |\dot{v}_n|^2 dt + \int_{\mathbb{R}} a(t) |v_n|^2 dt \right) - \lambda \int_{\mathbb{R}} (\nabla F(t, v_n), v_n) dt - \langle I'(v_n), v_n \rangle \right) \\
&\geq \left(\frac{1}{2} - \frac{1}{v} \right) \left(\int_{\mathbb{R}} |\dot{v}_n|^2 dt + \int_{\mathbb{R}} a(t) |v_n|^2 dt \right) \\
&\quad + \lambda \left(\frac{1}{v} \int_{\mathbb{R}} (\nabla F(t, v_n), v_n) dt - \int_{\mathbb{R}} F(t, v_n) dt \right) + o(1) \\
&\geq -\lambda \left(\frac{r_1 + v}{vr_1} C_{r_1 \beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|v_n\|^{r_1} + \frac{r_1 + v}{vr_2} C_{r_2 \beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|v_n\|^{r_2} \right) + o(1). \tag{4.31}
\end{aligned}$$

It follows from (4.7), (4.22), (4.31) and **Claim 1** that

$$\lambda \left(\frac{r_1 + v}{vr_1} C_{r_1 \beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|v_n\|^{r_1} + \frac{r_1 + v}{vr_2} C_{r_2 \beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|v_n\|^{r_2} \right) \geq c_\infty - c - o(1). \tag{4.32}$$

This is an obvious contradiction to Lemma 4.10 when $\lambda > 0$ and $d_\infty > 0$ are small enough. Then the dichotomy does not occur when $\{y_n\}$ is bounded.

Case 2: $\{y_n\} \subset \mathbb{R}$ is unbounded. Then, passing to a subsequence if necessary, we can assume that $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$. In this case, we can choose a suitable sequence $\{R_n\} \subset \mathbb{R}$ such that $R_n \pm y_n \rightarrow +\infty$ as $n \rightarrow \infty$ and arguing similarly as above. Then we conclude that dichotomy does not occur when $\{y_n\}$ is unbounded.

Step 3: Compactness.

It can be seen from Theorem 4.1 that there exists $\{y_n\} \subset \mathbb{R}$ such that, for any $\varepsilon > 0$, there exists $R_1 > 0$ satisfying

$$\int_{B_{R_1}(y_n)} \rho_n dt \geq \kappa - \varepsilon. \tag{4.33}$$

Since $\int_{\mathbb{R}} \rho_n dt = \kappa$, then we have

$$\int_{\mathbb{R} \setminus B_{R_1}(y_n)} \rho_n dt \leq \varepsilon$$

for all $n \in N$. If $\{y_n\}$ is unbounded, similar to the arguments in Step 2, we can obtain a contradiction. Then we conclude that $\{y_n\}$ is bounded. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}, \mathbb{R}^2)$, there exists u_0 in H^1 such that $u_n \rightharpoonup u_0$. It follows from the continuity of the embedding $H^1(\mathbb{R}, \mathbb{R}^2) \hookrightarrow L^v(\mathbb{R}, \mathbb{R}^2)$ for any $v \in [2, +\infty]$ that there exists $R_2 > 0$ such that

$$\int_{\mathbb{R} \setminus B_{R_2}(0)} |u_n|^v dt \leq \varepsilon \quad \text{and} \quad \int_{\mathbb{R} \setminus B_{R_2}(0)} |u_0|^v dt \leq \varepsilon. \tag{4.34}$$

It is clear that $u_n \rightarrow u_0$ in $L^v(B_{R_2}(0), \mathbb{R}^2)$ and it follows from (4.33) and (4.34) that

$$\begin{aligned}
\int_{\mathbb{R}} |u_n - u_0|^v dt &= \int_{B_{R_2}(0)} |u_n - u_0|^v dt + \int_{\mathbb{R} \setminus B_{R_2}(0)} |u_n - u_0|^v dt \\
&\leq \int_{B_{R_2}(0)} |u_n - u_0|^v dt + 2^{v-1} \left(\int_{\mathbb{R} \setminus B_{R_2}(0)} |u_0|^v dt + \int_{\mathbb{R} \setminus B_{R_2}(0)} |u_n|^v dt \right) \\
&\leq \int_{B_{R_2}(0)} |u_n - u_0|^v dt + 2^v \varepsilon,
\end{aligned}$$

which implies that

$$u_n \rightarrow u_0 \quad \text{as } n \rightarrow \infty \quad \text{in } L^v(\mathbb{R}) \quad \text{for any } v \in [2, +\infty).$$

On one hand, by Lebesgue Dominated Convergence Theorem, we can deduce that

$$\int_{\mathbb{R}} |u_n|^{v-2}(u_n, u_0) dt \rightarrow \int_{\mathbb{R}} |u_0|^v dt \quad \text{as } n \rightarrow \infty.$$

By $\langle I'(u_n), u_0 \rangle \rightarrow 0$ as $n \rightarrow \infty$, we obtain

$$\begin{aligned} o(1) &= \langle I'(u_n), u_n - u_0 \rangle \\ &= \|u_n - u_0\|^2 + \int_{\mathbb{R}} (\dot{u}_0, \dot{u}_n - \dot{u}_0) dt + \int_{\mathbb{R}} a(t)(u_0, u_n - u_0) dt - \int_{\mathbb{R}} |u_n - u_0|^2 dt \\ &\quad - \lambda \int_{\mathbb{R}} (\nabla F(t, u_n), u_n - u_0) dt - \nu \int_{\mathbb{R}} d(t)|u_n|^{v-2}(u_n, u_n - u_0) dt. \end{aligned} \quad (4.35)$$

On one hand, for $i = 1, 2$, set

$$\Delta_{i,1} = (1, 2], \quad \Delta_{i,2} = \left(\frac{2}{3-r_i}, \frac{2}{2-r_i} \right).$$

It is easy to see that $(1, \frac{2}{2-r_i}) = \Delta_{i,1} \cup \Delta_{i,2}$ and $\Delta_{i,1} \cap \Delta_{i,2} \neq \emptyset$. Hence, we deduce that there exists $\eta_i \in [2, +\infty)$ such that $\frac{1}{\beta_i} + \frac{r_i-1}{\xi_i} + \frac{1}{\eta_i} = 1$. Moreover, let

$$\xi_i = \begin{cases} +\infty & \text{if } \beta_i \in \Delta_{i,1}, \\ 2 & \text{if } \beta_i \in \Delta_{i,2} \setminus \Delta_{i,1}. \end{cases}$$

By (V5), we show

$$\begin{aligned} \int_{\mathbb{R}} (\nabla F(t, u_n), u_n - u_0) dt &\leq \int_{\mathbb{R}} \sum_{i=1,2} b_i(t) (|u_n|^{r_i-1} + |u_0|^{r_i-1}) dt \\ &\leq \sum_{i=1,2} \|b_i\|_{\beta_i} (\|u_n\|_{\xi_i}^{r_i-1} + \|u_0\|_{\xi_i}^{r_i-1}) \|u_n - u_0\|_{\eta_i} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

On the other hand, it is easy to see

$$\begin{aligned} \left| \int_{\mathbb{R}} d(t)|u_n|^{v-2}(u_n, u_n - u_0) dt \right| &\leq \|d\|_{\infty} \int_{\mathbb{R}} |u_n|^{v-1} |u_n - u_0| dt \\ &\leq \|d\|_{\infty} \|u_n\|_{2(v-1)}^{v-1} \|u_n - u_0\|_2^2 \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

We conclude from (4.35) that $\|u_n - u_0\| \rightarrow 0$ as $n \rightarrow \infty$, which implies that u_0 is a homoclinic solution for problem (1.1). \square

4.4 Proof of Theorem 4.1

In this section, we look for the second homoclinic solution corresponding to negative critical value with the following lemma.

Lemma 4.14 (See [22]). *Let X be a real reflexive Banach space and $\Omega \subset X$ be a closed bounded convex subset of X . Suppose that $\varphi : X \rightarrow \mathbb{R}$ is a weakly lower semi-continuous (w.l.s.c. for short) functional. If there exists a point $x_0 \in \Omega \setminus \partial\Omega$ such that*

$$\varphi(x) > \varphi(x_0), \quad \forall x \in \partial\Omega$$

then there must be a $x^ \in \Omega \setminus \partial\Omega$ such that*

$$\varphi(x^*) = \inf_{x \in \Omega} \varphi(x).$$

It follows from (V4) and (V6) that there exists $\delta > 0$ such that

$$F(t, x) > \frac{1}{2}b_0|x|^{r_0} \tag{4.36}$$

for all $t \in (\bar{t} - \delta, \bar{t} + \delta)$ and $x \in \mathbb{R}^2$. Choose $\psi \in C_0^\infty((t_0 - \delta, t_0 + \delta), \mathbb{R}^2) \setminus \{0\}$. It follows from (4.36) and $r_0 \in (0, 2)$ that

$$\begin{aligned} I(\vartheta\psi) &= \frac{\vartheta^2}{2}\|\psi\|^2 - \lambda \int_{\mathbb{R}} F(t, \psi) dt - \vartheta^\nu \int_{\mathbb{R}} d(t)|\psi|^\nu dt \\ &\leq \frac{\vartheta^2}{2}\|\psi\|^2 - \lambda b_0 \vartheta^{r_0} \int_{t_0 - \delta}^{t_0 + \delta} |\psi|^{r_0} dt - \vartheta^\nu d_\infty \int_{\mathbb{R}} |\psi|^\nu dt \\ &< 0 \end{aligned}$$

for $\vartheta > 0$ small enough. By Lemma 4.14, we can see there exists a critical point of I corresponding to negative critical value. \square

Data availability statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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