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A class of singularly perturbed Robin boundary value problems in critical case

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Abstract. This paper discusses a class of nonlinear singular perturbation problems with Robin boundary values in critical cases. By using the boundary layer function method and successive approximation theory, the corresponding asymptotic expansions of small parameters are constructed, and the existence of uniformly efficient smooth solutions is proved. Meanwhile, we give a concrete example to prove the validity of our results.

Keywords: critical case, singular perturbation, boundary function method, approximate solution, diagonalization technique.

2020 Mathematics Subject Classification: 34B15, 34E10, 34E15.

1 Introduction

Singular perturbation problems with small parameters have been used in many fields such as chemical kinetics [14], semiconductor simulation [16], and radio engineering [5, 18, 20]. The singular perturbed differential equation considered in this paper is obtained by transforming and dimensionless the differential equation controlling the enzyme kinetic reaction [4, 15, 18]. Using boundary layer function method [18], the first or higher order approximate solution of the problem can provide stronger theoretical support for obtaining the approximate value of enzyme concentration, substrate concentration and intermediate enzyme mixture concentration.

It is well known that the reduction equations for singularly perturbed systems usually have isolated roots. However, the degenerate equation of singularly perturbed problem considered in this paper has no isolated root. Instead, it has a series of solutions that depend on one or more parameters. This case will be called the critical case [17]. Compared with the singularly perturbed problem in the non-critical case, the singularly perturbed problem in the critical case is not only difficult to find, but also very complicated in the calculation process. Its complexity lies in the need to solve the following three difficulties in the calculation process: first, the zero-degree regular approximation solution is an unknown arbitrary function,

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which needs to be obtained through the following conditions; secondly, the solution process of the zero-order boundary layer is also very complicated. Finally, appropriate diagonalization should be found for the subsequent k-order boundary layer part to reduce the coupling degree of the equation.

The research methods of singular perturbation critical problems mainly include boundary layer function method [17, 18], that is, the solution is separated by fast scale and slow scale. The fast scale is the boundary layer part, and the slow scale is the regular part. The two parts are solved separately to construct the formal asymptotic solution. In addition, unlike the boundary layer function method, in the literature [9], the authors use orthogonal projectors on $\ker A(t)$ and $\ker A(t)'$(the prime denotes the transposition) for an asymptotic approximation, where $A(t)$ is a singular matrix in front of the unknown function at the right end of the singular perturbation equation. Through the theory of boundary layer function method, Vasil'eva and Butuzov [17] were the first to study initial value problems for singularly perturbed systems in the critical case. Subsequently, Vasil'eva and Adelaida [19], Dontchev and Veliov [3], Wang [21], Karandjulov [8], Kurina and Thi Hoai [10] generalized the results of singularly perturbed in the critical case. As far as we know, only Dirichlet or Neumann boundary value conditions are discussed in the above problems, and Robin boundary value conditions are not studied. In general, Robin boundary value conditions are a combination of Dirichlet and Neumann boundary value conditions. The singularly perturbed Robin boundary value problem in the critical case will result in that the initial value or boundary value corresponding to the differential equation of any order asymptotic term cannot be given directly, but must be obtained indirectly through certain techniques. In the past several decades, authors of [1, 2, 6, 7, 11, 12, 22, 23] discussed the singularly perturbed Robin boundary value problem for various noncritical cases.

However, until now, from what we understand, there is no literature talking about the singular perturbation problem in critical cases with Robin boundary value conditions seriously so far. Motivated by these issues, we fill in the gaps of this class of problems, giving corresponding asymptotic expansions and numerical examples in this paper.

The structure of the paper is as follows. In Section 2 we discuss singularly perturbed critical cases with Robin boundary value conditions. In the next section, we determine all terms of the asymptotic expansion of the system (2.1)–(2.2) using boundary layer function theory. Based on the successive approximation principle, section 4 proves the existence, uniqueness, and remainder estimation of the solutions to problems (2.1)–(2.2). Section 5 illustrates our results with an example. The last section gives concluding remarks.

2 Problem formulation

We consider a class of nonlinear singularly perturbed systems in critical case

$$\begin{cases} \mu \dfrac{dx}{dt} = A(y, \mu z, t)z + \mu B(y, z, t), \\ \mu \dfrac{dy}{dt} = C(x, y, t) + \mu D(y, z, t), \qquad a \le t \le b, \\ \mu \dfrac{dz}{dt} = F(\mu x, y, t)z + \mu H(y, z, t), \end{cases} \tag{2.1}$$

then, add the corresponding Robin initial and boundary value conditions as follows

$$\begin{gathered} y(a,\mu)-\mu y'(a,\mu)=y^0, \qquad x(b,\mu)+x'(b,\mu)=x^1, \\ z(a,\mu)-\mu z'(a,\mu)=z^0. \end{gathered} \tag{2.2}$$

Where x, y, and z are scalar functions, and $0 < \mu \ll 1$ is a small parameter. y^0, x^1, and z^0 are given known initial boundary values.

The following assumptions are theoretically some basic assumptions of questions (2.1)–(2.2).

$[H_1]$ Suppose that the functions A, B, C, D, F, and H are sufficiently smooth for $a \le t \le b, |x| \le l, |y| \le l$, and $|z| \le l$, where l is some real numbers.

$[H_2]$ Suppose that $C_x \neq 0, C_y < 0, F < 0$.

$[H_3]$ Suppose that the degradation equation of the system (2.1) is

$$\begin{cases} A(\bar{y},0,t)\bar{z}=0, \\ C(\bar{x},\bar{y},t)=0, \\ F(0,\bar{y},t)\bar{z}=0, \end{cases} \tag{2.3}$$

we suppose that the system (2.3) has a series of solutions

$$\bar{x}(t)=\omega(t), \qquad \bar{y}(t)=\beta(\omega(t),t), \qquad \bar{z}(t)=0, \tag{2.4}$$

where $\omega(t)$ is an arbitrary scalar function and $\beta(\omega(t),t)$ is a function with respect to $\omega(t)$.

Set

$$W=\begin{pmatrix} A(y,\mu z,t)z+\mu B(y,z,t) \\ C(x,y,t)+\mu D(y,z,t) \\ F(\mu x,y,t)z+\mu H(y,z,t) \end{pmatrix}, \qquad u=(x,y,z)^T;$$

Then the Jacobian matrix W_u at the equilibrium point $x = x^*, y = y^*, z = z^*$ has the following form

$$W_u=\begin{pmatrix} 0 & 0 & A \\ C_x & C_y & 0 \\ 0 & 0 & F \end{pmatrix}.$$

Therefore, it is easy to get the eigenvalue $\lambda \equiv 0$ of the matrix W_u and the other two eigenvalues $\lambda_1 = C_y < 0, \lambda_2 = F < 0$. According to the eigenvalues corresponding to the equilibrium point, $\lambda \equiv 0$ corresponds to the critical case, while the other two eigenvalues $\lambda_{1,2} < 0$ correspond to the stable equilibrium position. At the same time, the above case is called critically stable case in singular perturbation problems. Therefore, we can use the boundary layer function method [18] in the stable case, and because $\lambda_{1,2} < 0$, it is easy to find that the boundary function decays exponentially. Given the assumption that the condition $[H_2]$ and the initial value condition (2.2) hold, the solutions $x(t,\mu), y(t,\mu)$ and $z(t,\mu)$ generally only produce boundary layers near $t = a$.

3 Construction of asymptotic solution

The asymptotic approximation of problems (2.1)–(2.2) constructed by the boundary layer function method [18] will have the following form

$$u(t,\mu)=\begin{cases} x=\sum_{k=0}^{\infty}\mu^k[\bar{x}_k(t)+L_kx(\tau_0)],\\ y=\sum_{k=0}^{\infty}\mu^k[\bar{y}_k(t)+L_ky(\tau_0)],\\ z=\sum_{k=0}^{\infty}\mu^k[\bar{z}_k(t)+L_kz(\tau_0)],\end{cases} \tag{3.1}$$

where $\tau_0=\frac{t-a}{\mu}$; $\bar{x}_k(t)$, $\bar{y}_k(t)$, and $\bar{z}_k(t)(a<t\le b)$ are coefficients of regular terms; $L_kx(\tau_0)$, $L_ky(\tau_0)$, and $L_kz(\tau_0)(\tau_0\ge 0)$ are coefficients of boundary layer terms at $t=a$; By the initial and boundary value conditions, we obtain

$$\begin{aligned} \bar{x}_0(b)+\bar{x}_0'(b)&=x^1, & \bar{x}_k(b)+\bar{x}_k'(b)&=0,\\ \bar{y}_0(a)+L_0y(0)-L_0y'(0)&=y^0, & \bar{y}_k(a)+L_ky(0)&=\bar{y}_{k-1}'(a)+L_ky'(0),\\ \bar{z}_0(a)+L_0z(0)-L_0z'(0)&=z^0, & \bar{z}_k(a)+L_kz(0)&=\bar{z}_{k-1}'(a)+L_kz'(0).\end{aligned}$$

Set

$$\begin{aligned} \bar{x}(t,\mu)&=\sum_{i=0}^{k}\mu^i\bar{x}_k(t), & Lx(\tau_0,\mu)&=\sum_{i=0}^{k}\mu^iL_kx(\tau_0);\\ \bar{y}(t,\mu)&=\sum_{i=0}^{k}\mu^i\bar{y}_k(t), & Ly(\tau_0,\mu)&=\sum_{i=0}^{k}\mu^iL_ky(\tau_0);\\ \bar{z}(t,\mu)&=\sum_{i=0}^{k}\mu^i\bar{z}_k(t), & Lz(\tau_0,\mu)&=\sum_{i=0}^{k}\mu^iL_kz(\tau_0).\end{aligned} \tag{3.2}$$

Substituting equations (3.2) into equations (2.1), the following form can be obtained according to scale separation:

$$\begin{aligned} \mu\frac{d\bar{x}}{dt}+\frac{dLx}{d\tau_0}&=\bar{A}+\mu\bar{B}+LA+\mu LB,\\ \mu\frac{d\bar{y}}{dt}+\frac{dLy}{d\tau_0}&=\bar{C}+\mu\bar{D}+LC+\mu LD,\\ \mu\frac{d\bar{z}}{dt}+\frac{dLz}{d\tau_0}&=\bar{F}+\mu\bar{H}+LF+\mu LH,\end{aligned}$$

with

$$\begin{aligned} \bar{A}+\mu\bar{B}&=A(\bar{y}(t,\mu),\mu\bar{z}(t,\mu),t)\bar{z}(t,\mu)+\mu B(\bar{y}(t,\mu),\bar{z}(t,\mu),t),\\ \bar{C}+\mu\bar{D}&=C(\bar{x}(t,\mu),\bar{y}(t,\mu),t)+\mu D(\bar{y}(t,\mu),\bar{z}(t,\mu),t),\\ \bar{F}+\mu\bar{H}&=F(\mu\bar{x}(t,\mu),\bar{y}(t,\mu),t)\bar{z}(t,\mu)+\mu H(\bar{y}(t,\mu),\bar{z}(t,\mu),t),\\ LA&=A(\bar{y}(\mu\tau_0+a,\mu)+Ly(\tau_0,\mu),\mu\bar{z}(\mu\tau_0+a,\mu)+\mu Lz(\tau_0,\mu),\mu\tau_0+a)(Lz(\tau_0,\mu)\\ &\quad+\bar{z}(\mu\tau_0+a,\mu))-A(\bar{y}(\mu\tau_0+a,\mu),\mu\bar{z}(\mu\tau_0+a,\mu),\mu\tau_0+a)\bar{z}(\mu\tau_0+a,\mu),\\ \mu LB&=\mu B(\bar{y}(\mu\tau_0+a,\mu)+Ly(\tau_0,\mu),\bar{z}(\mu\tau_0+a,\mu)+Lz(\tau_0,\mu),\mu\tau_0+a)\\ &\quad-\mu B(\bar{y}(\mu\tau_0+a,\mu),\bar{z}(\mu\tau_0+a,\mu),\mu\tau_0+a),\end{aligned}$$

$$\begin{aligned}
LC &= (\bar{x}(\mu\tau_0 + a, \mu) + Lx(\tau_0, \mu), \bar{y}(\mu\tau_0 + a, \mu) + Ly(\tau_0, \mu), \mu\tau_0 + a) \\
&\quad - C(\bar{x}(\mu\tau_0 + a, \mu), \bar{y}(\mu\tau_0 + a, \mu), \mu\tau_0 + a), \\
\mu LD &= \mu D(\bar{y}(\mu\tau_0 + a, \mu) + Ly(\tau_0, \mu), \bar{z}(\mu\tau_0 + a, \mu) + Lz(\tau_0, \mu), \mu\tau_0 + a) \\
&\quad - \mu D(\bar{y}(\mu\tau_0 + a, \mu), \bar{z}(\mu\tau_0 + a, \mu), \mu\tau_0 + a), \\
LF &= F(\mu\bar{x}(\mu\tau_0 + a, \mu) + \mu Lx(\tau_0, \mu), \bar{y}(\mu\tau_0 + a, \mu) + Ly(\tau_0, \mu), \mu\tau_0 + a)(Lz(\tau_0, \mu) \\
&\quad + \bar{z}(\mu\tau_0 + a, \mu)) - F(\mu\bar{x}(\mu\tau_0 + a, \mu), \bar{y}(\mu\tau_0 + a, \mu), \mu\tau_0 + a)\bar{z}(\mu\tau_0 + a, \mu), \\
\mu LH &= \mu H(\bar{y}(\mu\tau_0 + a, \mu) + Ly(\tau_0, \mu), \bar{z}(\mu\tau_0 + a, \mu) + Lz(\tau_0, \mu), \mu\tau_0 + a) \\
&\quad - \mu H(\bar{y}(\mu\tau_0 + a, \mu), \bar{z}(\mu\tau_0 + a, \mu), \mu\tau_0 + a).
\end{aligned}$$

Secondly, according to the two scales t and τ_0, the equations of the regular part and the boundary layer part are written respectively:

$$\begin{cases}
\mu \dfrac{d\bar{x}}{dt} = \bar{A} + \mu\bar{B}, \\
\mu \dfrac{d\bar{y}}{dt} = \bar{C} + \mu\bar{D}, \\
\mu \dfrac{d\bar{z}}{dt} = \bar{F} + \mu\bar{H},
\end{cases} \tag{3.3}$$

$$\begin{cases}
\dfrac{dLx}{d\tau_0} = LA + \mu LB, \\
\dfrac{dLy}{d\tau_0} = LC + \mu LD, \\
\dfrac{dLz}{d\tau_0} = LF + \mu LH.
\end{cases} \tag{3.4}$$

Finally, the right-hand sides of equations (3.3) and (3.4) are expanded into a power series of μ, and then, according to the same power of μ at both ends of equations (3.3) and (3.4), the equations for the regular terms $\bar{u}_k(t)$ $(k \geqslant 0)$ and the boundary layer terms $L_k u(t)$ $(k \geqslant 0)$ are written, respectively.

We consider the zero-order regular part of the asymptotic solution of the form of problems (2.1)–(2.2) and obtain the zero-order regular parts $\bar{u}_0(t)$ is the same as the degenerate problem (2.3)–(2.4)

$$\begin{aligned}
A(\bar{y}_0, 0, t)\bar{z}_0 &= 0, \\
C(\bar{x}_0, \bar{y}_0, t) &= 0, \\
F(0, \bar{y}_0, t)\bar{z}_0 &= 0,
\end{aligned} \tag{3.5}$$

the root of the system of degradation equations (3.5) is

$$\bar{x}_0(t) = \omega(t), \qquad \bar{y}_0(t) = \beta(\omega(t), t), \qquad \bar{z}_0(t) = 0. \tag{3.6}$$

The equations for $L_0 u(\tau_0)$ are:

$$\begin{aligned}
\frac{dL_0 x(\tau_0)}{d\tau_0} &= A(\beta(\omega(a), a) + L_0 y(\tau_0), 0, a) L_0 z(\tau_0), \\
\frac{dL_0 y(\tau_0)}{d\tau_0} &= C(\omega(a) + L_0 x(\tau_0), \beta(\omega(a), a) + L_0 y(\tau_0), a) - C(\omega(a), \beta(\omega(a), a), a), \\
\frac{dL_0 z(\tau_0)}{d\tau_0} &= F(0, \beta(\omega(a), a) + L_0 y(\tau_0), a) L_0 z(\tau_0),
\end{aligned} \tag{3.7}$$

with the initial and boundary conditions

$$\begin{aligned} L_0y(0) &= L_0y'(0) - \beta(\omega(a), a) + y^0, \qquad & \omega(b) + \omega'(b) = x^1, \\ L_0z(0) &= L_0z'(0) + z^0, & L_0u(+\infty) = 0, \end{aligned} \tag{3.8}$$

where $\omega(a)$ is unknown, and the initial value of $L_0u(0)$ is arbitrary. We will use this arbitrariness to ensure that $L_0u(\tau_0)$ decays exponentially and satisfies $L_0u(+\infty) = 0$.

From (3.7)–(3.8), we have

$$\begin{aligned} L_0z(\tau_0) &= \frac{z^0 e^{\int_0^{\tau_0} F(0,\beta(\omega(a),a)+L_0y(s),a)ds}}{1 - F(0, \beta(\omega(a), a) + L_0y(0), a)}; \\ L_0x(\tau_0) &= \int_{+\infty}^{\tau_0} A(\beta(\omega(a), a) + L_0y(s), 0, a) \frac{z^0 e^{\int_0^{s} F(0,\beta(\omega(a),a)+L_0y(p),a)dp}}{1 - F(0, \beta(\omega(a), a) + L_0y(0), a)} ds \\ &= \varphi(L_0y(\tau_0), \omega(a), L_0y(0), a). \end{aligned} \tag{3.9}$$

At this time, substituting (3.9) into (3.7) the second differential equation and adding the initial boundary value condition (3.8), the following form can be obtained

$$\begin{cases} \dfrac{dL_0y(\tau_0)}{d\tau_0} = C(\omega(a) + \varphi, \beta(\omega(a), a) + L_0y(\tau_0), a) - C(\omega(a), \beta(\omega(a), a), a), \\ L_0y(0) = L_0y'(0) - \beta + y^0, \qquad L_0y(+\infty) = 0. \end{cases} \tag{3.10}$$

$[H_4]$ Suppose that equation (3.10) has a root $L_0y(\tau_0)$, which is denoted by

$$L_0y(\tau_0) = \Phi_0(\tau_0, \omega(a), L_0y(0), a). \tag{3.11}$$

However, neither $\omega(a)$ nor $L_0y(0)$ is known. Therefore, $L_0y(\tau_0)$ is not know. After $\omega(t)$ is found, $\omega(a)$ and $L_0y(0)$ can be determined and thus known. For the currently unknown functions $\omega(t)$ and $\beta(\omega, t)$, we need to determine them in the first approximation equation of the regular part $\bar{u}_1(t)$.

For $\bar{u}_1(t)$, we get

$$\begin{aligned} 0 &= F(0, \beta(\omega, t), t)\bar{z}_1 + H(\beta(\omega, t), 0, t), \\ \frac{d\omega(t)}{dt} &= A(\beta(\omega, t), 0, t)\bar{z}_1 + B(\beta(\omega, t), 0, t), \\ \frac{d\beta(\omega, t)}{dt} &= C_x(\omega, \beta(\omega, t), t)\bar{x}_1 + D(\beta(\omega, t), 0, t) + C_y(\omega, \beta(\omega, t), t)\bar{y}_1, \end{aligned} \tag{3.12}$$

the system of differential equations (3.12) can be rewritten in the following expression

$$\begin{aligned} \bar{z}_1(t) &= -\frac{\bar{H}}{\bar{F}} = \zeta(\omega, t), \qquad \frac{d\omega(t)}{dt} = \bar{A}\bar{z}_1 + \bar{B}, \\ \bar{x}_1(t) &= \frac{\beta' - C_y\bar{y}_1 - D}{C_x}, \end{aligned} \tag{3.13}$$

where $\bar{A}, \bar{B}$ and $\bar{H}$ are all taken value at the point $(\beta(\omega, t), 0, t)$, $\bar{F}$ is taken value at the point $(0, \beta(\omega, t), t)$. Using (3.13) and by taking (3.5) into account, we can determine $\omega(b) = \omega^1$. According to the existence of the solution of the boundary value problem, the second first-order differential equation in (3.13), and the known boundary value condition $\omega(b) = \omega^1$, there is a solution $\omega(t)$ for $a \le t \le b$. Hence, both $\omega(a)$ and $\bar{y}_0 = \beta(\omega, t)$ can be determined. At this time, $\bar{u}_0(t)$ can be completely determined.

Substituting (3.10) into (3.8) the first initial value condition, we can determine $L_0y(0)=\varphi^0$. Therefore, $L_0y(\tau_0)$ can be obtained by $[H_4]$. By (3.9) and $[H_4]$, we obtain $L_0z(\tau_0)$ and $L_0x(\tau_0)$. At this time, $L_0u(\tau_0)$ are completely determined. For the first approximation of the regular part of the system (2.1)–(2.2), equations (3.12) has only been determined $\bar{z}_1(t)$, while $\bar{x}_1(t)$ and $\bar{y}_1(t)$ need to be determined by the equation system of the regular parts $\bar{u}_2(t)$ and the corresponding boundary value conditions. Next, we need to first determine the first-order boundary layer terms $L_1u(\tau_0)$ of the asymptotic solution.

The equations for $L_1u(\tau_0)$ are:

$$\begin{aligned}\frac{dL_1x}{d\tau_0}&=\widehat{A}(\beta(\omega,a)+L_0y(\tau_0),0,a)L_1z(\tau_0)+\widehat{A}_yL_0z(\tau_0)L_1y(\tau_0)\\&\quad+\widehat{A}_yL_0z\bar{y}_1(a)+\phi_1,\\\frac{dL_1y}{d\tau_0}&=\tilde{C}_x(\omega(a)+L_0x(\tau_0),\beta(\omega,a)+L_0y(\tau_0),a)L_1x(\tau_0)\\&\quad+\tilde{C}_yL_1y(\tau_0)+(\tilde{C}_x-\bar{C}_x)\bar{x}_1(a)+(\tilde{C}_y-\bar{C}_y)\bar{y}_1(a)+\phi_2,\\\frac{dL_1z}{d\tau_0}&=\breve{F}(0,\beta(\omega,a)+L_0y(\tau_0),a)L_1z(\tau_0)+\breve{F}_yL_0z(\tau_0)L_1y(\tau_0)\\&\quad+\breve{F}_yL_0z\bar{y}_1(a)+\phi_3,\end{aligned}\tag{3.14}$$

where

$$\begin{aligned}\phi_1&=\left(\widehat{A}-\bar{A}\right)\bar{z}_1+\left[\widehat{A}_zL_0z+\widehat{A}_y\beta'\tau_0+\widehat{A}_t\tau_0\right]L_0z+\hat{B}-\bar{B},\\\phi_2&=\left[\left(\tilde{C}_x-\bar{\bar{C}}_x\right)\omega'(a)+\left(\tilde{C}_y-\bar{\bar{C}}_y\right)\beta'+\left(\tilde{C}_t-\bar{\bar{C}}_t\right)\right]\tau_0+\hat{D}-\bar{D},\\\phi_3&=\left(\breve{F}-\bar{\bar{\bar{F}}}\right)\bar{z}_1+\left[\breve{F}_z(\omega(a)+L_0x)+\left(\breve{F}_y\beta'+\breve{F}_t\right)\tau_0\right]L_0z+\hat{H}-\bar{H}.\end{aligned}$$

Here $\bar{B},\bar{A},\bar{D},\bar{H}$ take values at the point $(\beta(\omega,a),0,a)$, $\bar{\bar{C}}_x,\bar{\bar{C}}_y,\bar{\bar{C}}_t$ take values at the point $(\omega(a),\beta(\omega,a),a)$, $\bar{\bar{\bar{F}}}$ take values at the point $(0,\beta(\omega,a),a)$,$\breve{F},\breve{F}_y,\breve{F}_z,\breve{F}_t$ take values at the point $(0,\beta(\omega,a)+L_0y(\tau_0),a)$,$\hat{B},\hat{D},\hat{H}$ take values at the point $(\beta(\omega,a)+L_0y(\tau_0),L_0z(\tau_0),a)$, $\tilde{C}_x$, $\tilde{C}_y$, $\tilde{C}_t$ take values at the point $(\omega(a)+L_0x(\tau_0),\beta(\omega,a)+L_0y(\tau_0),a)$ and $\widehat{A},\widehat{A}_y,\widehat{A}_z,\widehat{A}_t$ take values at the point $(\beta(\omega,a)+L_0y(\tau_0),0,a)$.

The initial and boundary value conditions corresponding to $L_1u(\tau_0)$ are

$$\begin{aligned}&\bar{y}_1(a)+L_1y(0)=\bar{y}_0'(a)+L_1y'(0),\qquad &&\bar{z}_1(a)+L_1z(0)=L_1z'(0),\\&L_1u(+\infty)=0,&&\bar{x}_1(b)+\bar{x}_1'(b)=0.\end{aligned}\tag{3.15}$$

Introduce a diagonal transformation

$$L_1x(\tau_0)=\delta_1+\frac{L_0x}{L_0z}\delta_3,\qquad L_1y(\tau_0)=\delta_2,\qquad L_1z(\tau_0)=\delta_3,\tag{3.16}$$

we can get

$$\begin{aligned}\frac{d\delta_1}{d\tau_0}&=(\widehat{A}_yL_0z-\breve{F}_yL_0x)\delta_2+\phi_4(\bar{y}_1(a),\tau_0),\\\frac{d\delta_2}{d\tau_0}&=\tilde{C}_x\left(\delta_1+\frac{L_0x}{L_0z}\delta_3\right)+\tilde{C}_y\delta_2+\left(\tilde{C}_x-\bar{\bar{C}}_x\right)\bar{x}_1(a)\\&\quad+\left(\tilde{C}_y-\bar{\bar{C}}_y\right)\bar{y}_1(a)+\phi_2,\\\frac{d\delta_3}{d\tau_0}&=\breve{F}\delta_3+\breve{F}_yL_0z\delta_2+\breve{F}_yL_0z\bar{y}_1(a)+\phi_3,\end{aligned}\tag{3.17}$$

where $\phi_4(\bar{y}_1(a),\tau_0) = (\overset{\frown}{A}_y L_0 z - \breve{F}_y L_0 x)\bar{y}_1(a) + \phi_1 - \frac{L_0 x}{L_0 z}\phi_3$. The initial and boundary conditions of δ_1, δ_2 and δ_3 are

$$\begin{aligned} \bar{y}_1(a) + \delta_2(0) = \bar{y}_0'(a) + \delta_2{}'(0), \qquad \bar{z}_1(a) + \delta_3(0) = \delta_3{}'(0), \\ \delta_i(+\infty) = 0 (i = 1,2,3), \qquad \bar{x}_1(b) + \bar{x}_1'(b) = 0. \end{aligned} \tag{3.18}$$

We introduce a new transformation $\delta_1 = \delta_4 + \frac{\overset{\frown}{A}_y L_0 z - \breve{F}_y L_0 x}{\breve{F}_y L_0 z}\delta_3$ so that the equation (3.17) changes to the following form

$$\begin{aligned} \frac{d\delta_2}{d\tau_0} &= \tilde{C}_y\delta_2 + \frac{\overset{\frown}{A}_y}{\breve{F}_y}\delta_3 + \tilde{C}_x\delta_4 + \phi_6, \\ \frac{d\delta_3}{d\tau_0} &= \breve{F}\delta_3 + \breve{F}_y L_0 z\delta_2 + \breve{F}_y L_0 z\bar{y}_1(a) + \phi_3, \\ \frac{d\delta_4}{d\tau_0} &= \left[\left(\frac{\overset{\frown}{A}_y L_0 z - \breve{F}_y L_0 x}{\breve{F}_y L_0 z}\right)' + \frac{(\overset{\frown}{A}_y L_0 z - \breve{F}_y L_0 x)\breve{F}}{\breve{F}_y L_0 z}\right]\delta_3 + \phi_5, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} \phi_5 &= \phi_4 - (\overset{\frown}{A}_y L_0 z - \breve{F}_y L_0 x)\bar{y}_1(a) + \frac{\overset{\frown}{A}_y L_0 z - \breve{F}_y L_0 x}{\breve{F}_y L_0 z}\phi_3, \\ \phi_6 &= (\tilde{C}_x - \bar{\bar{C}}_x)\bar{x}_1(a) + (\tilde{C}_y - \bar{\bar{C}}_y)\bar{y}_1(a) + \phi_2. \end{aligned}$$

Let us introduce the transformation $\delta_2 = \delta_5 - \frac{\breve{F}}{\breve{F}_y L_0 z}\delta_3$ again, the system of differential equations (3.19) changed to the equations (3.20)

$$\begin{aligned} \frac{d\delta_3}{d\tau_0} &= \breve{F}_y L_0 z\delta_5 + \phi_8, \\ \frac{d\delta_4}{d\tau_0} &= \left[\left(\frac{\overset{\frown}{A}_y L_0 z - \breve{F}_y L_0 x}{\breve{F}_y L_0 z}\right)' + \frac{(\overset{\frown}{A}_y L_0 z - \breve{F}_y L_0 x)\breve{F}}{\breve{F}_y L_0 z}\right]\delta_3 + \phi_5, \\ \frac{d\delta_5}{d\tau_0} &= \left[\frac{\overset{\frown}{A}_y\tilde{C}_x L_0 z - L_0 z\breve{F}}{\breve{F}_y L_0 z} + \left(\frac{\breve{F}}{\breve{F}_y L_0 z}\right)'\right]\delta_3 + \tilde{C}_x\delta_4 + (\tilde{C}_y + \breve{F})\delta_5 + \phi_7, \end{aligned} \tag{3.20}$$

where $\phi_7 = \phi_6 + \breve{F}\bar{y}_1(a) + \frac{\breve{F}}{\breve{F}_y L_0 z}\phi_3, \phi_8 = \breve{F}_y L_0 z\bar{y}_1(a) + \phi_3$. From the above two transformations, we find that the initial value condition (3.20) has the following form

$$\begin{aligned} &\bar{y}_1(a) + \delta_5(0) = \bar{y}_0'(a) + \delta_5{}'(0) - \left(\frac{\breve{F}}{\breve{F}_y L_0 z}\right)'\delta_3(0) - \frac{\breve{F}}{\breve{F}_y L_0 z}\bar{z}_1(a), \\ &\bar{z}_1(a) + \delta_3(0) = \delta_3{}'(0), \delta_i(+\infty) = 0 \quad (i = 3,4,5), \qquad \bar{x}_1(b) + \bar{x}_1'(b) = 0. \end{aligned} \tag{3.21}$$

We find that the right end of the first two equations of equation system (3.20) contains only one unknown function, which greatly reduces the coupling of the right end of the original equation system (3.14). Thus, by the system of equations (3.20) and the initial value condition (3.21), $\delta_3(\tau_0)$ and $\delta_4(\tau_0)$ can be written as

$$\begin{aligned}\delta_3(\tau_0) &= \int_{+\infty}^{\tau_0} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds,\\ \delta_4(\tau_0) &= \int_{+\infty}^{\tau_0} \Bigg[\left(\frac{\widehat{A}_y(p)L_0z(p) - \breve{F}_y(p)L_0x(p)}{\breve{F}_y(p)L_0z(p)} \right)' \int_{+\infty}^{p} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \\ &\quad + \frac{(\widehat{A}_y(p)L_0z(p) - \breve{F}_y(p)L_0x(p))\breve{F}(p)}{\breve{F}_y(p)L_0z(p)} \int_{+\infty}^{p} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \\ &\quad + \phi_5(\bar{y}_1(a), p) \Bigg] dp.\end{aligned} \tag{3.22}$$

At the same time, we substitute the expression (3.22) into the last equation of the system of equations (3.20), and the right end contains only the unknown function $\delta_5(\tau_0)$. At this point, the entire equation is a first-order integral differential equation for $\delta_5(\tau_0)$. First, we get the integral equation for the initial value condition $\delta_5(0)$

$$\begin{aligned}\delta_5(0) &= \frac{(\widehat{A}_y(0)L_0z(0) - \breve{F}_y(0)L_0x(0))\breve{F}(0)}{\breve{F}_y(0)L_0z(0)(1 - \tilde{C}_y(0) - \breve{F}(0))} \int_{+\infty}^{0} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \\ &\quad + \int_{+\infty}^{0} \Bigg[\left(\frac{\widehat{A}_y(p)L_0z(p) - \breve{F}_y(p)L_0x(p)}{\breve{F}_y(p)L_0z(p)} \right)' \int_{+\infty}^{p} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \\ &\quad + \frac{(\widehat{A}_y(p)L_0z(p) - \breve{F}_y(p)L_0x(p))\breve{F}(p)}{\breve{F}_y(p)L_0z(p)} \int_{+\infty}^{p} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \\ &\quad + \phi_5(\bar{y}_1(a), p) \Bigg] dp \frac{\tilde{C}_x(0)}{(1 - \tilde{C}_y(0) - \breve{F}(0))} + \frac{\bar{y}_0'(a) - \bar{y}_1(a) + \phi_7(0, \bar{y}_1(a))}{(1 - \tilde{C}_y(0) - \breve{F}(0))}.\end{aligned} \tag{3.23}$$

$[H_5]$ Suppose that the integral equation (3.23) can be converted to $\delta_5(0) = \varsigma(a, \bar{y}_1(a))$.

Next, we write the integral differential equation for $\delta_5(\tau_0)$

$$\begin{aligned}\delta_5(\tau_0) &= \int_{0}^{\tau_0} \Bigg\{ \left[\frac{\widehat{A}_y\tilde{C}_xL_0z - L_0z\breve{F}}{\breve{F}_yL_0z} + \left(\frac{\breve{F}}{\breve{F}_yL_0z} \right)' \right] \int_{+\infty}^{q} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \\ &\quad + \tilde{C}_x \int_{+\infty}^{q} \Bigg[\left(\frac{\widehat{A}_y(p)L_0z(p) - \breve{F}_y(p)L_0x(p)}{\breve{F}_y(p)L_0z(p)} \right)' \int_{+\infty}^{p} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \\ &\quad + \frac{(\widehat{A}_y(p)L_0z(p) - \breve{F}_y(p)L_0x(p))\breve{F}(p)}{\breve{F}_y(p)L_0z(p)} \int_{+\infty}^{p} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \\ &\quad + \phi_5(\bar{y}_1(a), p) \Bigg] dp + (\tilde{C}_y(q) + \breve{F}(q))\delta_5(q) + \phi_7(q) \Bigg\} dq + \varsigma(a, \bar{y}_1(a)).\end{aligned} \tag{3.24}$$

$[H_6]$ Suppose that the integral equation (3.24) has a unique solution and can be expressed as

$$\delta_5(\tau_0) = \Phi_1(\tau_0, \bar{y}_1(a), a), \tag{3.25}$$

where $\bar{y}_1(a)$ is unknown. At this time, we can determine $\bar{y}_1(t)$ from the equation of the regular parts $\bar{u}_2(t)$.

For $\bar{u}_2(t)$, we get

$$\begin{aligned} \frac{d\bar{x}_1}{dt} &= \bar{A}\bar{z}_2 + \bar{A}_y\bar{y}_1\bar{z}_1 + \bar{B}_y\bar{y}_1 + \bar{B}_z\bar{z}_1, \\ \frac{d\bar{y}_1}{dt} &= \bar{\bar{\bar{C}}}_x\bar{x}_2 + \bar{\bar{\bar{C}}}_y\bar{y}_2 + g_2, \\ \frac{d\bar{z}_1}{dt} &= \bar{\bar{\bar{\bar{F}}}}\bar{z}_2 + \bar{\bar{\bar{\bar{F}}}}_y\bar{y}_1\bar{z}_1 + \bar{\bar{\bar{\bar{F}}}}_x\bar{x}_0\bar{z}_1 + \bar{H}_y\bar{y}_1 + \bar{H}_z\bar{z}_1, \end{aligned} \tag{3.26}$$

where g_2 is known to be a composite function. As for the unknown function $\bar{x}_1(t)$, the determination of $\bar{x}_1(t)$ is entirely similar to the of $\bar{x}_0(t) = \omega(t)$. Therefore, utilizing (3.26) and by taking (3.21) into account, we can determine $\bar{x}_1(b) = \alpha^1$. According to the existence of the solution of the boundary value problem, the first differential equation in (3.26), and the known boundary value condition $\bar{x}_1(b) = \alpha^1$, there is a solution $\bar{x}_1(t)$ for $a \leq t \leq b$, thence, both $\bar{x}_1(t)$ and $\bar{y}_1(t)$ can be determined. So far, $\bar{u}_1(t)$ can be completely determined.

Therefore, $\bar{y}_1(a)$ is known, $\delta_5(\tau_0)$ can be obtained by $[H_6]$. At the same time, $\delta_3(\tau_0)$ and $\delta_4(\tau_0)$ are determined. At this point, we go backwards along the diagonalization transformation to determine $L_1x(\tau_0)$, $L_1y(\tau_0)$, and $L_1z(\tau_0)$.

Next, the coefficients $\bar{u}_k(t)$ and $L_ku(\tau_0)(k \geq 2)$ of the higher-order asymptotic solutions are similar to the first-order asymptotic solutions $\bar{u}_1(t)$ and $L_1u(\tau_0)$. At this point, we need to write the equations and initial-boundary value conditions to determine $\bar{u}_{k+1}(t)$ and $L_ku(\tau_0)$.

The equations for $L_ku(\tau_0)$ are:

$$\begin{aligned} \frac{dL_kx}{d\tau_0} &= \widehat{A}L_kz(\tau_0) + \widehat{A}_yL_0z(\tau_0)L_ky(\tau_0) + \widehat{A}_yL_0z\bar{y}_k(a) + Q_k, \\ \frac{dL_ky}{d\tau_0} &= \tilde{C}_xL_kx(\tau_0) + \tilde{C}_yL_ky(\tau_0) + (\tilde{C}_x - \bar{C}_x)\bar{x}_k(a) + (\tilde{C}_y - \bar{C}_y)\bar{y}_k(a) + I_k, \\ \frac{dL_kz}{d\tau_0} &= \breve{F}L_kz(\tau_0) + \breve{F}_yL_0z(\tau_0)L_ky(\tau_0) + \breve{F}_yL_0z\bar{y}_k(a) + S_k, \end{aligned} \tag{3.27}$$

where Q_k, I_k, and S_k are all known composite functions. The initial and boundary conditions of $L_ku(\tau_0)$ are:

$$\begin{aligned} \bar{x}_k(b) + \bar{x}_k'(b) = 0, \qquad L_ku(+\infty) = 0,& \\ L_ky(0) = \bar{y}_{k-1}(a) + L_ky'(0) - \bar{y}_k(a),& \\ L_kz(0) = \bar{z}_{k-1}(a) + L_kz'(0) - \bar{z}_k(a).& \end{aligned} \tag{3.28}$$

The equations for $\bar{u}_{k+1}(t)$ are:

$$\begin{aligned} \frac{d\bar{x}_k}{dt} &= \bar{A}\bar{z}_{k+1} + \bar{A}_y\bar{y}_k\bar{z}_1 + \bar{B}_y\bar{y}_k + m_{k+1}, \\ \frac{d\bar{y}_k}{dt} &= \bar{\bar{\bar{C}}}_x\bar{x}_{k+1} + \bar{\bar{\bar{C}}}_y\bar{y}_{k+1} + g_{k+1}, \\ \frac{d\bar{z}_k}{dt} &= \bar{\bar{\bar{\bar{F}}}}\bar{z}_{k+1} + \bar{\bar{\bar{\bar{F}}}}_y\bar{y}_k\bar{z}_1 + \bar{H}_y\bar{y}_k + f_{k+1}, \end{aligned} \tag{3.29}$$

where m_{k+1}, g_{k+1}, and f_{k+1} are all known composite functions. The process of solving these problems is almost identical to the case of $k = 1$, so we will not repeat it here. In this way, the asymptotic expansion (3.1) can be completely determined.

4 The existence of the solution and the remainder estimate

We first introduce a curve L_0 in the space of the variables (u, t). The curve L_0 is composed of the following

$$L_{01} = \{(u,t) : \bar{u}_0(a) + L_0 u(\tau_0), \tau_0 \geq 0, t = a\}, \qquad L_{02} = \{(u,t) : \bar{u}_0(t), a \leq t \leq b\},$$

we denote the projection of L_0 onto the space of the variables (u, t) by $\tilde{L}_0$.

$[H_7]$ Suppose that the functions A, B, C, D, F, and H have continuous partial derivatives concerning each argument up to order $(n + 2)$ inclusive in some δ-tube of $\tilde{L}_0$.

We use $U_n(t, \mu)$ to denote the first $n + 1$ terms of the series (3.1) and

$$U_n = \sum_{k=0}^{n} \mu^k [\bar{u}_k(t) + L_k u(\tau_0)]. \tag{4.1}$$

Theorem 4.1 ([18, 21]). *When conditions $[H_1] \sim [H_7]$ are met, there must be constants $\mu_0 > 0$ and $c > 0$, so that when $\mu \in (0, \mu_0]$, the solutions $x(t, \mu), y(t, \mu)$ and $z(t, \mu)$ of the problems* (2.1) *and* (2.2) *are lying in a $c\delta$-tube of L_0, is unique and satisfies the inequality*

$$|u(t, \mu) - U_n(t, \mu)| \leq c\mu^{n+1}, \qquad a \leq t \leq b. \tag{4.2}$$

Proof. Let $\xi = x - X_{n+1}, \eta = y - Y_{n+1}, \rho = z - Z_{n+1}$, where (x, y, z) is an exact solution of the problem (2.1), (2.2), and $(X_{n+1}, Y_{n+1}, Z_{n+1})$ is the partial sum of (4.1). Substituting $x = \xi + X_{n+1}, y = \eta + Y_{n+1}, z = \rho + Z_{n+1}$ into (2.1), (2.2), the equations of the remainder (ξ, η, ρ) is obtained

$$\begin{cases} \mu \dfrac{d\xi}{dt} = A(\eta + Y_{n+1}, \mu(\rho + Z_{n+1}), t)(\rho + Z_{n+1}) - \mu \dfrac{dX_{n+1}}{dt} \mu B(\eta + Y_{n+1}, \rho + Z_{n+1}, t), \\ \mu \dfrac{d\eta}{dt} = C(\xi + X_{n+1}, \eta + Y_{n+1}, t) - \mu \dfrac{dY_{n+1}}{dt} + \mu D(\eta + Y_{n+1}, \rho + Z_{n+1}, t), \\ \mu \dfrac{d\rho}{dt} = F(\mu(\xi + X_{n+1}), \eta + Y_{n+1}, t)(\rho + Z_{n+1}) - \mu \dfrac{dZ_{n+1}}{dt} + \mu H(\eta + Y_{n+1}, \rho + Z_{n+1}, t), \end{cases} \tag{4.3}$$

separating the linear part of the zeroth approximation, we obtain for (ξ, η, ρ) the boundary value problem on the intervals $[a, b]$, respectively, namely,

$$\begin{cases} \mu \dfrac{d\xi}{dt} = A(\bar{y}_0 + L_0 y, 0, t)\rho + A_y(\bar{y}_0 + L_0 y, 0, t)\eta + G_1(\eta, \rho, t, \mu), \\ \mu \dfrac{d\eta}{dt} = C_x(\bar{x}_0 + L_0 x, \bar{y}_0 + L_0 y, t)\xi + C_y(\bar{x}_0 + L_0 x, \bar{y}_0 + L_0 y, t)\eta + G_2(\xi, \eta, t, \mu), \\ \mu \dfrac{d\rho}{dt} = F(0, \bar{y}_0 + L_0 y, t)\rho + F_y(0, \bar{y}_0 + L_0 y, t)\eta + G_3(\xi, \eta, \rho, t, \mu). \end{cases} \tag{4.4}$$

The functions G_1, G_2 and G_3 are

$$\begin{aligned} G_1(\eta,\rho,t,\mu) &= A(\eta+Y_{n+1},\mu(\rho+Z_{n+1}),t)(\rho+Z_{n+1})-A(\bar{y}_0+L_0y,0,t)\rho \\ &\quad - A_y(\bar{y}_0+L_0y,0,t)\eta-\mu\frac{dX_{n+1}}{dt}+\mu B(\eta+Y_{n+1},\rho+Z_{n+1},t), \\ G_2(\xi,\eta,t,\mu) &= C(\xi+X_{n+1},\eta+Y_{n+1},t)+\mu D(\eta+Y_{n+1},\rho+Z_{n+1},t)-\mu\frac{dY_{n+1}}{dt} \\ &\quad - C_x(\bar{x}_0+L_0x,\bar{y}_0+L_0y,t)\xi-C_y(\bar{x}_0+L_0x,\bar{y}_0+L_0y,t)\eta, \\ G_3(\xi,\eta,\rho,t,\mu) &= F(\mu(\xi+X_{n+1}),\eta+Y_{n+1},t)(\rho+Z_{n+1})-F(0,\bar{y}_0+L_0y,t)\rho \\ &\quad +\mu H(\eta+Y_{n+1},\rho+Z_{n+1},t)-\mu\frac{dZ_{n+1}}{dt}-F_y(0,\bar{y}_0+L_0y,t)\eta. \end{aligned}$$

G_1, G_2 and G_3, which we define having the following two important properties:

I. $|G_{1,2}(0,0,t,\mu)| \le c\mu^{n+2}, |G_3(0,0,0,t,\mu)| \le c\mu^{n+2}$, where $a \le t \le b, 0<\mu\le\mu_0$;

II. For all $\varepsilon= O(\mu) > 0$, there are constants $\delta=\delta(\varepsilon)$ and $\mu_0=\mu_0(\varepsilon)$ so that as long as $|\xi_1|\le\delta, |\xi_2|\le\delta, |\eta_1|\le\delta, |\eta_2|\le\delta, |\rho_1|\le\delta, |\rho_2|\le\delta, 0<\mu\le\mu_0$, then for

$$\begin{aligned} |G_1(\eta_1,\rho_1,t,\mu)-G_1(\eta_2,\rho_2,t,\mu)| &\le \varepsilon(|\eta_1-\eta_2|+|\rho_1-\rho_2|), \\ |G_2(\xi_1,\eta_1,t,\mu)-G_2(\xi_2,\eta_2,t,\mu)| &\le \varepsilon(|\xi_1-\xi_2|+|\eta_1-\eta_2|), \\ |G_3(\xi_1,\eta_1,\rho_1,t,\mu)-G_3(\xi_2,\eta_2,\rho_2,t,\mu)| &\le \varepsilon(|\xi_1-\xi_2|+|\eta_1-\eta_2|+|\rho_1-\rho_2|). \end{aligned}$$

Let $\xi = v+\frac{L_0x}{L_0z}\rho$, convert ξ to v and substitute it in (4.4), we have

$$\begin{cases} \mu\dfrac{dv}{dt} = \Big(A_y(\bar{y}_0+L_0y,0,t)-\dfrac{L_0x}{L_0z}F_y(0,\bar{y}_0+L_0y,t)\Big)\eta \\ \qquad + G_1(\eta,\rho,t,\mu)-\dfrac{L_0x}{L_0z}G_3\Big(v+\dfrac{L_0x}{L_0z}\rho,\eta,\rho,t,\mu\Big), \\ \mu\dfrac{d\eta}{dt} = C_x(\bar{x}_0+L_0x,\bar{y}_0+L_0y,t)\dfrac{L_0x}{L_0z}\rho+C_x(\bar{x}_0+L_0x,\bar{y}_0+L_0y,t)v \\ \qquad + C_y(\bar{x}_0+L_0x,\bar{y}_0+L_0y,t)\eta+G_2\Big(v+\dfrac{L_0x}{L_0z}\rho,\eta,t,\mu\Big), \\ \mu\dfrac{d\rho}{dt} = F(0,\bar{y}_0+L_0y,t)\rho+F_y(0,\bar{y}_0+L_0y,t)\eta+G_3\Big(v+\dfrac{L_0x}{L_0z}\rho,\eta,\rho,t,\mu\Big). \end{cases} \tag{4.5}$$

The initial value condition for $v(t,\mu)$ is of the same type as that for $\xi(t,\mu)$. That is, $v(a,\mu)=O(\mu^{n+2})$. Then, the first differential equation in (4.5) can be rewritten as an integral equation

$$\begin{aligned} v(t,\mu) &= O(\mu^{n+2})+\int_0^t \mu^{-1}\Big(G_1(\eta,\rho,s,\mu)-\frac{L_0x}{L_0z}G_3(\eta,\rho,v,s,\mu)\Big)ds \\ &\quad +\int_0^t \mu^{-1}\Big(A_y(\bar{y}_0+L_0y,0,s)-\frac{L_0x}{L_0z}F_y(0,\bar{y}_0+L_0y,s)\Big)\eta(s)ds \\ &= H_1(\rho,\eta,v,t,\mu). \end{aligned} \tag{4.6}$$

It is not difficult to prove that the integral operator $H_1(\rho,\eta,v,t,\mu)$ has a compression coefficient of $O(u)$ for ρ,η and v, and satisfies $H_1(0,0,0,t,\mu)=O(\mu^{n+1})$.

At this time, we consider the right ends C_xv+G_2 and G_3 of the last two equations of equation (4.5) as non-homogeneous terms, and write them into the equivalent integral equations. We write the Green's function of the first two equations in (4.5) as $\gamma(t,s,\mu)$. Under the boundary condition $\eta(a,\mu) = \eta(b,\mu) = 0$, γ is satisfied with the estimate $\gamma(t,s,\mu) = O\big(\exp\big(\frac{\kappa|s-t|}{\mu}\big)\big), a \le t \le b$. By the conditions $\eta(a,\mu) = O(\mu^{n+2})$ and $\eta(b,\mu) = O(\mu^{n+2})$, thus, the last two equations of equation (4.5) can be replaced with the following integral equations

$$\begin{pmatrix} \eta(t,\mu) \\ \rho(t,\mu) \end{pmatrix} = O(\mu^{n+2}) + \frac{1}{\mu}\int_0^t \gamma(t,s,\mu)\begin{pmatrix} C_x v + G_2 \\ G_3 \end{pmatrix} ds = \begin{pmatrix} \lambda_1(\rho,\eta,v,t,\mu) \\ \lambda_2(\rho,\eta,v,t,\mu) \end{pmatrix}, \tag{4.7}$$

under (4.6) in (4.7), we get

$$\begin{aligned} \eta(t,\mu) &= \lambda_1(\rho,\eta,v,t,\mu) \equiv H_2(\rho,\eta,v,t,\mu), \\ \rho(t,\mu) &= \lambda_2(\rho,\eta,v,t,\mu) \equiv H_3(\rho,\eta,v,t,\mu). \end{aligned} \tag{4.8}$$

Among them, the integral operator H_2, H_3 is similar to H_1.

By using the successive approximation method for the system (4.6), (4.8), we can prove that when parameter μ_0 is sufficient small, there is a unique solution $\xi = \eta = \rho = 0$ in the δ-tube, and meet the estimation of the solution $\xi = O(\mu^{n+1}), \eta = O(\mu^{n+1})$, and $\rho = O(\mu^{n+1})$. Therefore, $\xi = x - X_{n+1}, \eta = y - Y_{n+1}$ and $\rho = z - Z_{n+1}$ are all of the order $O(\mu^{n+1})$.

As a result of $u(t,\mu) - U_n(t,\mu) = O(\mu^{n+1})$, we obtain the inequality (4.2). This completes the proof. □

5 Illustrative example

Example 5.1. Consider the following system:

$$\begin{aligned} &\mu\frac{dx}{dt} = (\mu z - 1)z + \mu y, \qquad \mu\frac{dy}{dt} = x - y + \mu y, \\ &\mu\frac{dz}{dt} = (\mu x - 2)z + \mu y, \qquad 0 \le t \le 1, \end{aligned} \tag{5.1}$$

with the initial and boundary conditions

$$\begin{aligned} &y(0,\mu) - \mu y'(0,\mu) = \frac{1}{\sqrt{e}} - \frac{3}{2}, \qquad x(1,\mu) + x'(1,\mu) = \frac{3}{2}, \\ &z(0,\mu) - \mu z'(0,\mu) = 3. \end{aligned} \tag{5.2}$$

According to (2.1)–(2.2), we have $A = \mu z - 1, C = x - y, F = \mu x - 2, B = D = H = y, y^0 = \frac{1}{\sqrt{e}} - \frac{3}{2}, x^1 = \frac{3}{2}$ and $z^0 = 3$. It is easy to see that $C_x \neq 0, C_y < 0, F < 0$. Then the condition $[H_2]$ is satisfied.

By calculation, we can get

$$W_u = \begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

therefore, it is easy to get the eigenvalue $\lambda \equiv 0$ of the matrix W_u and the other two eigenvalues $\lambda_1 = -1 < 0, \lambda_2 = -2 < 0$. Thus, we get the case where the critical condition is stable.

By taking $\mu = 0$, we obtain the solution of the degenerated problem on the interval $[0,1]$, that is,

$$\bar{x}_0(t) = \omega(t), \qquad \bar{y}_0(t) = \omega(t), \qquad \bar{z}_0(t) = 0.$$

For $L_0u(\tau_0)$ and $\bar{u}_1(t)$, we have

$$\frac{dL_0x}{d\tau_0} = -L_0z, \qquad \frac{dL_0y}{d\tau_0} = L_0x - L_0y, \qquad \frac{dL_0z}{d\tau_0} = -2L_0z, \tag{5.3}$$

$$\frac{d\omega}{dt} = \bar{y}_0 - \bar{z}_1, \qquad \frac{d\bar{y}_0}{dt} = \bar{x}_1 - \bar{y}_1 + \bar{y}_0, \qquad \bar{y}_0 = 2\bar{z}_1, \tag{5.4}$$

and zero-order initial boundary value conditions

$$\begin{aligned} \omega(1) + \omega'(1) = \frac{3}{2}, \qquad L_0z(0) = L_0z'(0) + 3, \\ L_0y(0) = L_0y'(0) - \bar{y}_0(0) + \frac{1}{\sqrt{e}} - \frac{3}{2}. \end{aligned} \tag{5.5}$$

After calculation, we can get

$$\begin{aligned} L_0x(\tau_0) = \frac{1}{2}L_0z(\tau_0) = \frac{1}{2}e^{-2\tau_0}, \qquad L_0y(\tau_0) = -\frac{1}{2}e^{-2\tau_0}, \\ \bar{x}_0(t) = \bar{y}_0(t) = e^{\frac{1}{2}(t-1)}, \qquad \bar{z}_1(t) = \frac{1}{2}e^{\frac{1}{2}(t-1)}. \end{aligned} \tag{5.6}$$

Therefore, $L_0u(\tau_0)$ satisfies an exponential decay estimation $|L_0u(\tau_0)| \le C_0e^{-\kappa\tau_0}$, for C_0 and κ are positive constants.

The system for $L_1u(\tau_0)$ and $\bar{u}_2(t)$, we have

$$\begin{aligned} &\frac{dL_1x}{d\tau_0} = L_0^2z - L_1z + L_0y, \qquad \frac{dL_1y}{d\tau_0} = L_1z - L_1y + L_0y, \\ &\frac{dL_1z}{d\tau_0} = L_0xL_0z - 2L_1z + L_0y. \end{aligned} \tag{5.7}$$

$$\begin{aligned} &\frac{d\bar{x}_1}{dt} = -\bar{z}_2 + \bar{y}_1, \qquad \frac{d\bar{y}_1}{dt} = \bar{x}_2 - \bar{y}_2 + \bar{y}_1, \\ &\frac{d\bar{z}_1}{dt} = \bar{x}_0\bar{z}_1 - 2\bar{z}_2 + \bar{y}_1, \end{aligned} \tag{5.8}$$

and first-order initial boundary value conditions

$$\begin{aligned} &\bar{y}_1(0) + L_1y(0) = \bar{y}_0'(0) + {L_1}'y(0), \qquad \bar{x}_1(1) + \bar{x}_1'(1) = 0, \\ &\bar{z}_1(0) + L_1z(0) = {L_1}'z(0). \end{aligned} \tag{5.9}$$

Through calculation, we can get

$$\begin{aligned} \bar{x}_1(t) &= -\frac{1}{2}e^{(t-1)} + \frac{3}{8}te^{\frac{1}{2}(t-1)} - \frac{1}{24}e^{\frac{1}{2}(t-1)}, \\ \bar{y}_1(t) &= -\frac{1}{2}e^{(t-1)} + \frac{3}{8}te^{\frac{1}{2}(t-1)} + \frac{11}{24}e^{\frac{1}{2}(t-1)}, \qquad \bar{z}_1(t) = \frac{1}{2}e^{\frac{1}{2}(t-1)}, \\ L_1x(\tau_0) &= \left(\frac{1}{8} - \frac{1}{4\sqrt{e}}\right)e^{-2\tau_0} - \frac{1}{4}\tau_0e^{-2\tau_0} - \frac{5}{16}e^{-4\tau_0}, \\ L_1z(\tau_0) &= -\frac{1}{2\sqrt{e}}e^{-2\tau_0} - \frac{1}{2}\tau_0e^{-2\tau_0} - \frac{1}{4}e^{-4\tau_0}, \\ L_1y(\tau_0) &= \left(\frac{1}{4e} - \frac{5}{48\sqrt{e}} - \frac{103}{96}\right)e^{-\tau_0} + \left(\frac{5}{8} + \frac{1}{4\sqrt{e}} + \frac{1}{4}\tau_0\right)e^{-2\tau_0} + \frac{5}{48}e^{-4\tau_0}. \end{aligned} \tag{5.10}$$

It is easy to get that $L_1 u(+\infty) = 0$. Thus, $L_1 u(\tau_0)$ satisfies an exponential decay estimation. Therefore, we construct the first-order approximate solution $u = (x\ y\ z)^T$ of the system (5.1)–(5.2). Namely, we have

$$u(t,\mu) = \begin{cases} x(t,\mu) = \bar{x}_0(t) + L_0 x(\tau_0) + \mu(\bar{x}_1(t) + L_1 x(\tau_0)) + O(\mu^2), \\ y(t,\mu) = \bar{y}_0(t) + L_0 y(\tau_0) + \mu(\bar{y}_1(t) + L_1 y(\tau_0)) + O(\mu^2), \\ z(t,\mu) = \bar{z}_0(t) + L_0 z(\tau_0) + \mu(\bar{z}_1(t) + L_1 z(\tau_0)) + O(\mu^2), \end{cases} \tag{5.11}$$

where $\tau_0 = \dfrac{t-0}{\mu}, 0 \leq t \leq 1$.

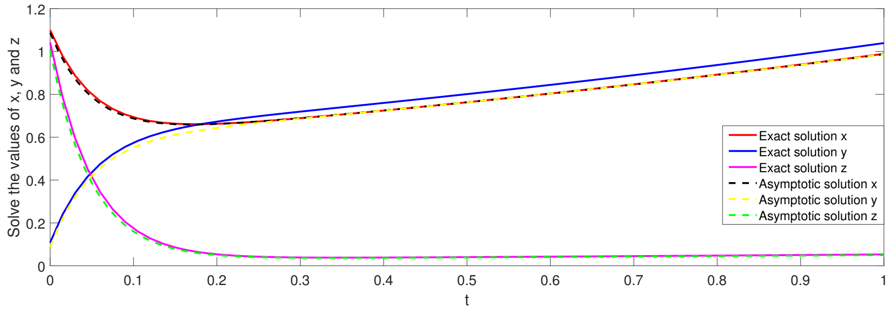

Figure 5.1: Contrast of the exact with the approximate solutions to (5.1) with boundary condition (5.2) for values of the perturbing parameter $\mu = 0.1$.

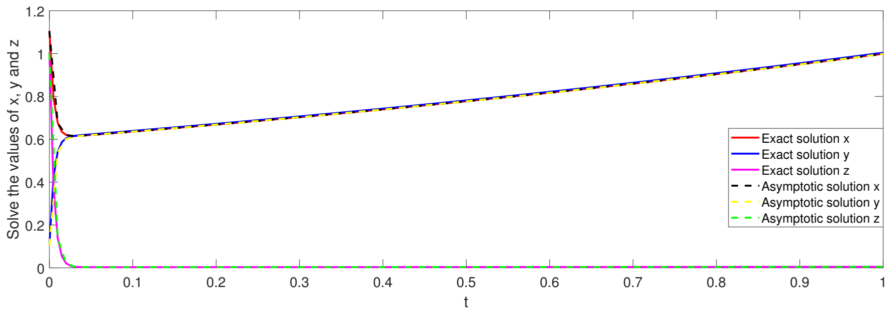

Figure 5.2: Contrast of the exact with the approximate solutions to (5.1) with boundary condition (5.2) for values of the perturbing parameter $\mu = 0.01$.

Therefore, these results were obtained using the boundary layer function method in [18]. The results obtained by Matlab are given in Figures 5.1 and 5.2. Different line types corresponding to exact and approximate solutions have been marked in the figure. These graphs show that an asymptotic solution is closer to the exact one if we use higher-order asymptotics. If we use the small parameter μ to be smaller, the formal asymptotic solution is more approximate to the exact solution. The image of the solution also better illustrates the nature of the exponential decay of the boundary function.

6 Conclusive remarks

This paper studies a class of nonlinear critical singular perturbation problems with Robin initial boundary value, the results show that how to obtain the zero-order approximate solution and simplify the first-order boundary layer term equation is the key to obtaining the approximate solution of the system. In this paper, the successive approximation method is used to prove the remainder estimation and the existence and uniqueness of the solution.

Finally, in the process of researching the system (2.1)–(2.2), the following situations were discovered.

Remark 6.1. When $F > 0$ and $F \neq 1$ in the interval $[a, b]$ of t, the system (2.1) only produces the right layer, but the solution method is similar to the left boundary layer. And when the F has a zero point in the interval $[a, b]$, the stability will be in the interval $[a, b]$ has changed, thus forming a very complicated nonlinear turning point problem. Since the linear turning point problem is already very complicated, as the nonlinear strength increases, the nonlinear turning point problem will become more complicated. So this type of problem will be very challenging. At this time, the boundary layer function method will no longer be applicable, and it is necessary to find a suitable method to solve it.

Remark 6.2. When the coefficients in front of $y'(t, \mu)$ and $z'(t, \mu)$ in Robin's initial boundary value condition (2.2) do not contain the small parameter μ when the derivative of t is calculated for the boundary layer function, the μ^{-1} term will appear. So the right end of the corresponding boundary condition should have the μ^{-1} term to perform matching, and in the process of using the boundary layer function method to solve the problem, it should be set to the following form

$$\begin{cases} y = \sum_{k=-1}^{\infty} \left[\mu^{k+1} \bar{y}_{k+1}(t, \mu) + \mu^k L_k y(\tau_0, \mu) \right], \\ z = \sum_{k=-1}^{\infty} \left[\mu^{k+1} \bar{z}_{k+1}(t, \mu) + \mu^k L_k z(\tau_0, \mu) \right]. \end{cases} \tag{6.1}$$

But when y and z are substituted into the system (2.1) at this time, A, B, C, D, H, and F can't perform Taylor expansion when $\mu = 0$, which contradicts the boundary layer function method. Therefore, the weak nonlinear problem generally does not produce an infinite initial boundary value problem.

Remark 6.3. When the initial boundary values (2.2) are all Robin initial conditions, the form is as follows

$$\begin{aligned} &y(a, \mu) - \mu y'(a, \mu) = y^0, \qquad x(a, \mu) + \mu x'(a, \mu) = x^0, \\ &z(a, \mu) - \mu z'(a, \mu) = z^0, \end{aligned} \tag{6.2}$$

the process of solving the n-order equations of problem (2.1)–(2.2) does not change, but the corresponding initial conditions and boundary value conditions change. Through calculations, we found that the basic process of the solution does not change significantly, but from first finding the value of $\bar{x}(t)$ at $t = b$, it becomes possible to find the value of $\bar{y}(t)$ at $t = a$, and $L_0 y(\tau_0)$ at the value of $\tau_0 = 0$.

Remark 6.4. For system 2.1, because the functions B, D and H at the right end are preceded by a small parameter μ. Therefore, no matter the form is $(z, y, t), (x, y, t)$ or (x, z, t), system 2.1 is still a critically stable situation, and the corresponding asymptotic solution form and solution method will not be changed.

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