Divulgaciones Matemáticas Vol. 11 No. 1(2003), pp. 19–38

Exact Controllability for the Equation of the One Dimensional Plate in Domains with Moving Boundary

Controlabilidad Exacta para la Ecuación de la Placa Unidimensional en Dominios con Frontera Móvil

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Abstract

This work studies the problem of the exact controlability in the boundary of the equation $u_{tt} + u_{xxxx} = 0$ in a domain with moving boundary.
Key words and phrases: Exact control, moving boundary, HUM method.

Resumen

Este trabajo estudia el problema de la controlabilidad exacta en la frontera de la ecuación $u_{tt} + u_{xxxx} = 0$ en un dominio con frontera móvil.
Palabras y frases clave: Control exacto, frontera m'ovil, m'etodo HUM.

Recibido 2002/06/15. Aceptado 2003/06/09.
MSC (2000): 35B37, 49K20.

1 Introduction

Let $\alpha : [0, +\infty) \rightarrow \mathbf{R}$ and $\beta : [0, +\infty) \rightarrow \mathbf{R}$ be two functions of class C^3. Let us consider the noncylindrical domain $\widehat{Q}$, defined by:

$$\widehat{Q} = \left\{(x,t) \in \mathbf{R}^2;\ \alpha(t) < x < \beta(t),\ 0 < t < T\right\}.$$

The lateral boundary of $\widehat{Q}$ is given by

$$\bigcup_{0<t<T} [(\alpha(t) \times t) \cup (\beta(t) \times t)].$$

This work investigates the following problem of exact control: given $T > T_0$, for some fix $T_0 > 0$ and inicial data

$$\{u^o, u^1\} \in L^2(\alpha(0), \beta(0)) \times H^{-2}(\alpha(0), \beta(0)),$$

we find the controls $g_1 \in L^2(0,T)$, $\quad g_2 \in L^2(0,T)$, such that the solution u of the following system:

$$\text{(I)} \quad \left| \begin{array}{lll} u_{tt} + u_{xxxx} = 0 & \text{in} & \widehat{Q} \\ u(\alpha(t),t) = u(\beta(t),t) = 0 & \text{in} &]0,T[\\ u_x(\alpha(t),t) = g_1(t), \quad u_x(\beta(t),t) = g_2(t) & \text{in} &]0,T[\\ u(x,0) = u^0, \qquad u_t(x,0) = u^1 & \text{in} &]\alpha(0), \beta(0)[\end{array} \right.$$

verifies that $u(x,T) = 0$, $u_t(x,T) = 0$ in $]\alpha(T), \beta(T)[$.
Different authors have already studied the problem of the exact controllability of problem (I), in the case of bounded n-dimensional cylindrical domains. We can mention among them D. L. Russel [12], J. Lagnese and J. L. Lions [3], J. L. Lions [5], [6], [7]. This last author introduced a new method to solve this kind of problems, called *Hilbert Uniquenees Method* (HUM), which is a simple and direct way to deal with these problems. Other authors followed this way of thinking, for instance E. Zuazua [11], [14], [15], who studies problem (I) in unidimensional cylindrical domains. In [13], he studied the exact controllability of (I) in an arbitrarily small time. Our work is focused on studying problem (I) in non-cylindrical domains of a broad generality, which was not so much studied, with a restriction, for enough long time.

2 Notations and main results

Let us consider the real functions $\alpha(t)$ and $\beta(t)$ satisfying the following conditions:

(H1) $\alpha, \beta \in C^3([0, +\infty)\mathbf{R})$, with α', α'', β', $\beta'' \in L^1(0, \infty)$.

(H2) $\alpha(t) < \beta(t)$ for all $t \geq 0$ and $0 < \gamma_0 = \inf_{t\geq 0} \gamma(t)$, where $\gamma(t) = \beta(t) - \alpha(t)$.

Remark 2.1. It follows from $(H1)$ and $(H2)$ that α, α', β, and β' are bounded. In fact,

$$\begin{aligned} ||\alpha(t)| - |\alpha(0)|| &\leq |\alpha(t) - \alpha(0)| = \left| \int_0^t \alpha'(s)\, ds \right| \\ &\leq \int_0^t |\alpha'(s)|\, ds \leq \int_0^\infty |\alpha'(s)|\, ds. \end{aligned}$$

Thus

$$|\alpha(t)| \leq |\alpha(0)| + \int_0^\infty |\alpha'(s)|\, ds.$$

hence, α is bounded. The proof is similar for α', β, and β'.

In what follows we use the notations:

$$\begin{aligned} \gamma_1 &= \sup_{t\geq 0} \gamma(t), & s_0 &= \sup_{t\geq 0}\{|\alpha'(t)|, |\beta'(t)|\}, \\ l_1 &= \int_0^\infty |\gamma'(t)|\, dt < \infty, & l_2 &= \int_0^\infty |\gamma''(t)|\, dt < \infty, \\ l_3 &= \int_0^\infty |\alpha'(t)|\, dt < \infty, & l_4 &= \int_0^\infty |\alpha''(t)|\, dt < \infty. \end{aligned}$$

Notice that when (x, t) varies in $\widehat{Q}$ the point (y, t) with $y = (x - \alpha)/\gamma$ varies in the cylinder $Q =]0, 1[\times]0, T[$. The application $\tau : \widehat{Q} \to Q$ given by $\tau(x, t) = (y, t)$ is a diffeomorphism.
By the change of variables $u(x, t) = v(y, t)$ with $y = (x - \alpha)/\gamma$ we transform the operator

$$\widehat{L}\, u = u_{tt} + u_{xxxx} \quad \text{in } \widehat{Q}$$

into the operator

$$L\, v = v_{tt} + a(t)\, v_{yyyy} + b(y, t)\, v_{yy} + c(y, t)\, v_{yt} + d(y, t)\, v_y \quad \text{in } Q$$

where

$$(1)\quad \left|\begin{array}{ll} a(t) = \frac{1}{\gamma^4(t)}; & b(y,t) = (\frac{\alpha'+\gamma' y}{\gamma})^2; \\ & \\ c(y,t) = -2(\frac{\alpha'+\gamma' y}{\gamma}); & d(y,t) = \frac{-\gamma\alpha''-y\gamma\gamma''+2\gamma'\alpha'+2y(\gamma')^2}{\gamma^2}. \end{array}\right.$$

So, the problem of the noncylindrical control (I) is transformed into the following problem of cilyndrical control:

Given $T > T_0$, $T_0 > 0$ fix, and the initial data $\upsilon^0 \in L^2(0,1)$, $\upsilon^1 \in H^{-2}(0,1)$, find the controls $f_1 \in L^2(0,T)$, $\quad f_2 \in L^2(0,T)$, such that the solution υ of the system below

$$(\mathrm{II})\quad \left|\begin{array}{lll} L\upsilon = 0 & \text{in} & Q \\ & & \\ \upsilon(0,t) = \upsilon(1,t) = 0 & \text{in} &]0,T[\\ & & \\ \upsilon_y(0,t) = f_1(t)\ ,\ \upsilon_y(1,t) = f_2(t) & \text{in} &]0,T[\\ & & \\ \upsilon(y,0) = \upsilon^0(y)\ ,\ \upsilon_t(y,0) = \upsilon^1(y) & \text{in} &]0,1[\end{array}\right.$$

verifies $\upsilon(y,T) = 0$, $\upsilon_t(y,T) = 0$ in]0,1[.

Our main results are:

Theorem 2.1*Let us assume that the hypotheses (H1) and (H2) are satisfied. Then there exists $T_0 > 0$, such that for $T > T_0$ and the initial data $\{u^0, u^1\} \in L^2(\alpha(0), \beta(0)) \times H^{-2}(\alpha(0), \beta(0))$, there exists a pair of controls $\{g_1, g_2\} \in L^2(0,T) \times L^2(0,T)$ such that the ultra weak solution of (I) satisfies $u(x,T) = u_t(x,T) = 0$ for all $\alpha(T) < x < \beta(T)$.*

Theorem 2.2*Let us assume that the hypotheses (H1) and (H2) are satisfied. Then there exists $T_0 > 0$, such that for $T > T_0$ and the inicial data $\{\upsilon^0, \upsilon^1\} \in L^2(0,1) \times H^{-2}(0,1)$. there exists a pair of controls $\{f_1, f_2\} \in [L^2(0,T)]^2$ such that the ultra weak solution of (II) satisfies $\upsilon(y,T) = \upsilon_t(y,T) = 0$ for all $0 < y < 1$.*

3 The problem of exact control in the cylinder

3.1 The Homogeneus Problem

Let us consider the operator R, given by:

$$\begin{aligned} Rw &= w_{tt} + a(t)w_{yyyy} + b(y,t)w_{yy} + c(y,t)w_{yt} \\ &\quad + e(y,t)w_y + g(y,t)w + h(y,t)w_t, \end{aligned}$$

where

$$a_1 \geq a(t) \geq a_0 > 0 \quad \text{for all } t \geq 0,$$

$$b, c, e, g, h \in W^{1,\infty}(0,T; L^\infty(0,1)), \qquad c_y \in L^\infty(Q).$$

Our objetive in this section is to find a solution for the following problem:

$$\text{(III)} \quad \left| \begin{array}{lll} Rw = G & \text{in} & Q \\ w(0,t) = 0,\ w(1,t) = 0 & \text{in} &]0,T[\\ w_y(0,t) = w_y(1,t) = 0 & \text{in} &]0,T[\\ w(0) = w^0\ ,\ w_t(0) = w^1 & \text{in} &]0,1[. \end{array} \right.$$

Theorem 3.1 *(Strong solution) Given* $w^0 \in H_0^2(0,1) \cap H^4(0,1)$, $w^1 \in H_0^2(0,1)$, $G, G_t \in L^1(0,T; L^2(0,1))$, *then there exists a unique solution* w *of (III) in the class* $w \in C([0,T]; H_0^2(0,1) \cap H^4(0,1)) \cap C^1([0,T]; H_0^2(0,1))$ *verifying* $Rw = G$ in $L^1(0,T; L^2(0,1))$.

Theorem 3.2 *(Weak solution) Given* $w^0 \in H_0^2(0,1)$, $w^1 \in L^2(0,1)$, $G \in L^1(0,T; L^2(0,1))$, *then there exists a unique function* $w : Q \to R$, *solution of (III) in the class* $w \in C([0,T]; H_0^2(0,1)) \cap C^1([0,T]; L^2(0,1))$ *satisfying*

(i) $$-(w_t, z_t)_{L^2(Q)} + (a(t)\, w_{yy}, z_{yy})_{L^2(Q)} + (b\, w_{yy}, z)_{L^2(Q)} - (c\, w_t, z_y)_{L^2(Q)}$$

$$+(e\, w_y, z)_{L^2(Q)} + (g\, w_t, z)_{L^2(Q)} + (h\, w, z)_{L^2(Q)} = (G, z)_{L^2(Q)},$$

for all $z \in L^2(0,T; H_0^2(0,1))$ *with* $z_t \in L^2(Q)$, $z(0) = z(T) = 0$ *and* $w(0) = w^0$, $w_t(0) = w^1$.

(ii) $Rw = G$ *in* $L^1(0,T; H^{-2}(0,1))$.

(iii) $$E(t) = E(0) + \frac{1}{2}\int_0^t a'(s)|w_{yy}(s)|^2 ds - \int_0^t (b\, w_{yy}(s), w_t(s))\, ds$$

$$-\int_0^t (e\,w_y(s), w_t(s))\,ds - \int_0^t (h\,w_t(s), w_t(s))\,ds - \int_0^t (g\,w(s), w_t(s))\,ds$$

$$+\frac{1}{2}\int_0^t (c_y\,w_t(s), w_t(s))\,ds + \int_0^t (G, w_t(s))\,ds$$

(iv) $\quad E(t) \leq c\Big[E(0) + \Big(\int_0^t |G(s)|\,ds\Big)^2\Big]\,e^{c_1 T}$, *being*

$E(t) = \frac{1}{2}|w_t(t)|^2 + \frac{1}{2}a(t)|w_{yy}(t)|^2$ *the energy of the system.*

It can be observed that the Theorems 3.1 and 3.2 are valid for the operators L and L^*(adjoint of L) given by

$$L^*w = w_{tt}+a(t)w_{yyyy}+b(y,t)w_{yy}+c(y,t)w_{yt}+e(y,t)w_y+g(y,t)w+h(y,t)w_t,$$

where a, b and c are given in (1) and $e(y,t) = 2b_y + c_t - d,\ g(y,t) = b_{yy} + c_{yt} - d_y,\quad h(y,t) = c_y$.

3.2 Direct inequality

Theorem 3.3 *Let be w a weak solution of the problem (III). Then there exists a positive constant c_0, such that*

(i) $\quad G = 0 \Rightarrow E(0)\,e^{-c_0} \leq E(t) \leq E(0)e^{c_0}$, for all $t \geq 0$.

(ii) $\quad G \neq 0 \Rightarrow E(t) \leq c\Big\{E(0)+\Big(\int_0^t |G(t)|dt\Big)^2\Big\}\,e^{c_0}$ for $t \in [0,T]$.

Proof. From $(\mathrm{III})_1$, we have :

$$\frac{d}{dt}E(t) = \frac{a'(t)}{2}|w_{yy}(t)|^2 - (bw_{yy}(t)w_t(t)) - (ew_y(t), w_t(t)) - (hw_t(t), w_t(t))$$
$$- (g\,w(t), w_t(t)) + \frac{1}{2}(c_y\,w_t(t), w_t(t)) + (G, w_t(t)).$$

Then we obtain

$$\left|\frac{d\,E(t)}{dt}\right| \leq \Big(\frac{|a'(t)|}{a(t)}+\frac{|b|}{\sqrt{a(t)}}+\frac{\lambda_0|e|}{\sqrt{a(t)}}+2|h|+\frac{\lambda_0^2|g|}{\sqrt{a(t)}}+|c_y|\Big)E(t)+|G(t)||w_t(t)|,$$

where λ_0 is Poincare's constant. Then, from (H1)-(H2), we get

$$\left|\frac{d}{dt}E(t)\right| \leq K(t)\,E(t)+|G(t)|\,|w_t(t)|, \tag{5}$$

where

$$K(t) = c_1(|\gamma'|+|\gamma''|+|\alpha''|+|\gamma'|^2+|\alpha'|^2), \text{ with}$$

$$c_1 = \frac{1}{\gamma_0^2}\max\{\frac{2}{\gamma_0^3}(2\gamma_1^4+3\gamma_0^2),\, 2\gamma_1^2(1+6\lambda_0+3\lambda_0^2),\, 3\lambda_0\gamma_1^3(1+\lambda_0)\}.$$

If $G=0$ then, from (5), we have

$$-K(t)E(t) \leq \frac{d}{dt}E(t) \leq K(t)E(t).$$

Then, from Remark 2.1, it can be concluded that (i) is verified with $c_0 = c_1[(1+2s_0)\,l_1+l_2+s_0l_3+l_4]$. If $G\neq 0$ then, from (5), we conclude that

$$E(t) \leq E(0) + \int_0^t \{K(s)E(s)^{1/2}+\sqrt{2}|G(s)|\}E(s)^{1/2}ds.$$

From the definition of $K(t)$ it can be concluded that (ii) is verified.

Next, we will obtain the Fundamental Identity which will allow us to have estimates for $w_{yy}(0,.)$ and $w_{yy}(1,.)$.

Lemma 3.4. *If $q\in C^3([0,1])$ and w is the strong solution of problem (III), then w satisfies the identity,*

$$\frac{1}{2}\int_0^T a(t)\,q\,|w_{yy}|^2\Big|_0^1\,dt = (w_t+\frac{c}{2}\,w_y, q\,w_y)\Big|_0^T \tag{6}$$

$$+\frac{1}{2}\int_0^T\int_0^1 q_y|w_t|^2dy\,dt + \frac{3}{2}\int_0^T\int_0^1 a(t)q_y|w_{yy}|^2dy\,dt$$

$$+\int_0^T\int_0^1 gqw_tw_y\,dy\,dt + \int_0^T\int_0^1 (\frac{b}{2}q_y+\frac{b_y}{2}q-a(t)q_{yyy}-\frac{1}{2}c_tq+eq)|w_y|^2dy\,dt$$

$$-\int_0^T\int_0^1 (h\,q_y+h_y\,q)|w|^2dydt - \int_0^T\int_0^1 q\,G\,w_y\,dy\,dt,$$

named Fundamental Identity.

Besides this, when $q(y)=y-\frac{1}{2}$, we have

$$\begin{aligned}(7)\qquad &\int_0^T a(t)|w_{yy}(1,t)|^2 dt + \int_0^T a(t)|w_{yy}(0,t)|^2 dt \\ &\leq C\left\{E(0) + \left(\int_0^T |G(t)| dt\right)^2\right\}\end{aligned}$$

Proof. Multiplying $(\mathrm{III})_1$ by qw_y and integrating in Q , we obtain :

$$\begin{aligned}(8)\qquad &\int_0^T\int_0^1 w_{tt} q w_y \, dy\, dt + \int_0^T\int_0^1 a(t) w_{yyyy} q w_y \, dy\, dt \\ &+ \int_0^T\int_0^1 b w_{yy} q w_y \, dy\, dt + \int_0^T\int_0^1 c\, w_{yt}\; q\, w_y \, dy\, dt + \int_0^T\int_0^1 e\, w_y^2\, q \, dy\, dt \\ &+ \int_0^T\int_0^1 g w_t q w_y \, dy\, dt + \int_0^T\int_0^1 h w q w_y \, dy\, dt = \int_0^T\int_0^1 G q w_y \, dy\, dt.\end{aligned}$$

Observe that

$$(9)\qquad \int_0^T\int_0^1 w_{tt}\, q\, w_y\, dy\, dt = (w_t, q\, w_y)|_0^T + \frac{1}{2}\int_0^T\int_0^1 q_y |w_t(y,t)|^2 dy\, dt$$

$$\begin{aligned}(10)\qquad &\int_0^T\int_0^1 a(t)\, w_{yyyy}\, q\, w_y\, dy\, dt = -\frac{1}{2}\int_0^T a(t)\, q |w_{yy}|^2|_0^1 dt \\ &+\frac{3}{2}\int_0^T\int_0^1 a(t)\, q_y |w_{yy}|^2 dy\, dt - \int_0^T\int_0^1 a(t)\, q_{yyy} |w_y|^2 dy\, dt\end{aligned}$$

$$(11)\qquad \int_0^T\int_0^1 b q w_y w_{yy}\, dy\, dt = -\frac{1}{2}\int_0^T\int_0^1 (b q_y + b_y q)|w_y|^2 dy\, dt$$

$$(12)\qquad \int_0^T\int_0^1 c w_{yt} q w_y dy dt = \frac{1}{2}(c q w_y, w_y)|_0^T - \frac{1}{2}\int_0^T\int_0^1 c_t q |w_y|^2 dy\, dt$$

$$(13) \qquad \int_0^T \int_0^1 hqww_y dydt = -\frac{1}{2}\int_0^T \int_0^1 (hq_y + h_y q)|w|^2 dy\, dt.$$

From (8)-(13), we deduced (6).

As a consequence of the Lemma 3.4, we obtain the following result:

Theorem 3.5 If $q \in C^3([0,1])$ and $\{w^0, w^1, G\} \in H_0^2(0,1) \times L^2(0,1) \times L^1(0,T; L^2(0,1))$ then a weak solution w of problem (III) verifies the inequality

$$(14) \qquad \int_0^T a(t)|w_{yy}(1,t)|^2 dt + \int_0^T a(t)|w_{yy}(0,t)|^2 dt$$
$$\leq C\left\{E(0) + (\int_0^T |G(t)|dt)^2\right\}.$$

Proof. Consider F as being the vector space consisting of weak solutions of (III), with initial data $\{w^0, w^1\} \in H_0^2(0,1)\times L^2(0,1)\}$ and $G \in L^1(0,T; L^2(0,1))$. We define the application

$$\xi : H_0^2(0,1)\times L^2(0,1)\times L^1(0,T; L^2(0,1)) \to F \quad \text{by} \quad \xi(\{w^0, w^1, G\}) = w.$$

Observe that ξ is bijective. We define also the aplication $||\ .\ ||_\xi$: $F \to \mathbf{R}$, given by

$$\|w\|_\xi = |w^0_{yy}| + |w^1| + |G|_{L^1(0,T;L^2(0,1))}, \text{that is a norm in } F.$$

Observe also that $(F, \|.\|_\xi)$ is a Banach space. Consider now H as being the vector space consisting of strong solutions of (III), with initial data $w^0 \in H_0^2(0,1) \cap H^4(0,1), w^1 \in H_0^2(0,1)$, and $G \in W^{1,1}(0,T; L^2(0,1))$. Note that H is dense in F. On the other side using Lemma 3.4, it results that,

$$|w_{yy}(1,.)|_{L^2(0,T)} \leq c\{|w^0_{yy}| + |w^1| + |G|_{L^1(0,T;L^2(0,1))}\}, \quad \text{for all}\, w \in H$$

or

$$(15) \qquad |w_{yy}(1,.)|_{L^2(0,T)} \leq c\|w\|_\xi, \quad \text{for all}\, w \in H.$$

Analogously, we obtain

$$(16) \qquad |w_{yy}(0,.)|_{L^2(0,T)} \leq c||w||_\xi, \quad \text{for all}\, w \in H.$$

Motivated by (15) and (16), we define the continuous operators

$$\phi_0 : H \to L^2(0,T), \qquad \phi_1 : H \to L^2(0,T)$$

by

$$\phi_0(w) = w_{yy}(0,.) \quad \text{and} \quad \phi_1(w) = w_{yy}(1,.),$$

respectively. Since H is dense in F, there exist continuous operators

$$\tilde{\phi}_0 : F \to L^2(0,T), \quad \tilde{\phi}_1 : F \to L^2(0,T),$$

such that $\tilde{\phi}_0|_H = \phi_0$, $\tilde{\phi}_1|_H = \phi_1$. Therefore, if $w \in F$ then there exists a sequence $(w_m) \in H$, such that $w_m \to w$ in F. Then,

$$\lim \phi_0(w_m) = \lim \tilde{\phi}_0(w_m) = \tilde{\phi}_0(w)$$

or

$$w_{myy}(0,.) = \tilde{\phi}_0(w_m) \to \tilde{\phi}_0(w) \text{in } L^2(0,T).$$

Analogously, we have

$$w_{myy}(1,.) = \tilde{\phi}_1(w_m) \to \tilde{\phi}_1(w) \text{ in } L^2(0,T).$$

On the other hand, there exist $w_m^0 \in H_0^2(0,1) \cap H^4(0,1)$, $w_m^1 \in H_0^2(0,1)$, $G_m \in W^{1,1}(0,T;L^2(0,1))$, in the way that $\xi(\{w_m^0, w_m^1, G_m\}) = w_m$ and

(17) $\quad w_m^0 \to w^0 \quad$ in $\quad H_0^2(0,1)$

(18) $\quad w_m^1 \to w^1 \quad$ in $\quad L^2(0,1)$

(19) $\quad G_m \to G \quad$ in $\quad L^1(0,T;L^2(0,1))$

Besides this,

(20) $\quad w_m \to w \quad$ in $\quad C([0,T];H_0^2(0,1))$

(21) $\quad w_{mt} \to w_t \quad$ in $\quad C([0,T];L^2(0,1))$.

Then, $w_{myy}(0,.) \to w_{yy}(0,.)$ in $D'(0,T)$.

As $L^2(0,T) \hookrightarrow D'(0,T)$ then $\tilde{\phi}_0(w) = w_{yy}(0,.)$. Therefore

(22) $\quad w_{myy}(0,.) \to w_{yy}(0,.) \quad$ in $\quad L^2(0,1)$.

Analogously, we have

(23) $\quad w_{myy}(1,.) \to w_{yy}(1,.) \quad$ in $\quad L^2(0,1)$.

Taking account of assertions (17)-(23), we can show the result.

3.3 Inverse Inequality

Theorem 3.6 *There exists $T_0 > 0$ such that for $T > T_0$ the weak solution of problem (III) with $G = 0$ verifies*

$$\int_0^T a(t)|w_{yy}(1,t)|^2 dt + \int_0^T a(t)|w_{yy}(0,t)|^2 dt \geq 4e^{-c_0}(T - T_0)E(0). \tag{24}$$

Proof. Taking $q(y) = y - \frac{1}{2}$ in the identity (6) of the Lemma 3.4. we obtain the following result

$$\frac{1}{2}\int_0^T a(t)(y-\frac{1}{2})|w_{yy}|^2|_0^1\, dt \;=\; \underbrace{\int_0^T\int_0^1 [\frac{1}{2}|w(t)|^2 + \frac{3}{2}a(t)|w_{yy}|^2]\, dy\, dt}_{I_1}$$

$$+\underbrace{(w_t + \frac{c}{2}w_y, (y-\frac{1}{2})w_y)|_0^T}_{I_2} + \underbrace{\int_0^T\int_0^1 g(y-\frac{1}{2})w_t w_y dy\, dt}_{I_3}$$

$$+\underbrace{\int_0^T\int_0^1 [\frac{b}{2} + \frac{b_y}{2}(y-\frac{1}{2}) - \frac{1}{2}c_t(y-\frac{1}{2}) + e(y-\frac{1}{2})]|w_{yy}|^2\, dy\, dt}_{I_4}$$

$$+\underbrace{-\int_0^T\int_0^1 [(h+h_y q)|w|^2 dy\, dt}_{I_5}.$$

Then we have the following estimates:

- $|I_1| \geq T\, e^{-c_0}\, E(0)$,
- $|I_2| \leq (\lambda_0\gamma_1^2 + \frac{6s_0\lambda_0^2\gamma_1^4}{\gamma_0})\, e^{c_0}\, E(0)$,
- $|I_3| \leq \frac{\lambda_0\gamma_1^2}{2\gamma_0^2}(6l_1 + 3\gamma_1 l_2)\, e^{c_0}\, E(0)$,
- $|I_4| \leq \frac{\lambda_0^2\gamma_1^4}{\gamma_0^2}(17l_1^2 + 7l_3^2 + 4\gamma_1 l_2 + 4\gamma_1 l_4)\, e^{c_0}\, E(0)$,
- $|I_5| \leq \frac{4\lambda_0^4\gamma_1^4 l_1}{\gamma_0}\, e^{c_0}\, E(0)$.

Combining the above estimates, the proof of the Theorem 3.6 is completed where

$$T_0 = e^{2c_0}\Big[\lambda_0\gamma_1^2 + \frac{6s_0\lambda_0^2\gamma_1^4}{\gamma_0} + \frac{\lambda_0\gamma_1^2}{2\gamma_0^2}(6l_1 + 3\gamma_1 l_2) + \frac{\lambda_0^2\gamma_1^4}{\gamma_0^2}(17l_1^2 + 7l_3^2 + 4\gamma_1 l_2 + 4\gamma_1 l_4) + \frac{4\lambda_0^4\gamma_1^4 l_1}{\gamma_0}\Big].$$

For the study of problem (II), we introduce the concept of ultra weak solution.

Definition 3.7 Let be $v^0 \in L^2(0,1)$, $v^1 \in H^{-2}(0,1)$, $f = \{f_1, f_2\} \in [L^2(0,1)]^2$. Let us say that $v \in L^\infty(0,T;L^2(0,1))$ is an ultra weak solution of (II) if it satisfies

$$\int_0^T (v,G)\,dt = \langle v^1, w(0)\rangle_{H^{-2}\times H_0^2} - (v^0, w_t(0)) + \Big(\frac{2\gamma'(0)}{\gamma(0)}v^0, w(0)\Big) + 2\Big(\frac{\alpha'(0)+\gamma'(0)\,y}{\gamma(0)}v^0, w_y(0)\Big) + \int_0^T a(t)\,f_1(t)\,w_{yy}(0)dt - \int_0^T a(t)\,f_2(t)\,w_{yy}(1)\,dt \tag{25}$$

for all $G \in L^1(0,T;L^2(0,1))$ *and* w *is the solution of (III) associated to* G.

Theorem 3.8 *Let* $\{v^0, v^1\} \in L^2(0,1)\times H^{-2}(0,1)$, and $\{f_1,f_2\} \in [L^2(0,T)]^2$. *Then there exists a unique ultra weak solution of problem (II).*

Proof. Define the application $S : L^1(0,T;L^2(0,1)) \to \mathbf{R}$ by

$$\langle S,G\rangle = \langle v^1, w(0)\rangle_{H^{-2}\times H_0^2} - (v^0, w_t(0)) + \Big(\frac{2\gamma'(0)}{\gamma(0)}v^0, w(0)\Big) + 2\Big(\frac{\alpha'(0)+\gamma'(0)y}{\gamma(0)}v^0, w_y(0)\Big) + \int_0^T a(t)\,f_1(t)\,w_{yy}(0,t)\,dt - \int_0^T a(t)f_2(t)w_{yy}(1,t)\,dt,$$

$G \in L^1(0,T;L^2(0,1))$, where w is a weak solution of (III).
From Theorem 3.2 we have that $w \in C([0,T];H_0^2(0,1)) \cap C^1([0,T];L^2(0,1))$.
From Theorem 3.5, it follows that

$$|w_t(0)|^2 + |w_{yy}(0)|^2 \le C\Big(\int_0^T |G|dt\Big)^2, \tag{26}$$

(27) $$\int_0^T |w_{yy}(0,t)|^2 dt \le C(\int_0^T |G| dt)^2,$$

(28) $$\int_0^T |w_{yy}(1,t)|^2 dt \le C(\int_0^T |G| dt)^2.$$

From (26)-(28), we obtain

(29) $$\frac{|\langle S, G\rangle|}{|G|_{L^1(0,T;L^2(0,1))}} \le C\{|v^0| + |v^1|_{H^{-2}(0,1)} + |f_1|_{L^2(0,T)} + |f_2|_{L^2(0,T)}\}$$

or

(30) $$|S|_{(L^1(0,T;L^2(0,1)))'} \le C\{|v^0| + |v^1|_{H^{-2}(0,1)} + |f_1|_{L^2(0,T)} + |f_2|_{L^2(0,T)}\}.$$

From (30), we conclude that $S \in (L^1(0,T;L^2(0,1)))'$. Therefore, by the Riez Theorem there exists a unique $v \in L^\infty(0,T;L^2(0,1))$ such that

$$\langle S, G\rangle = \int_0^T (v, G)\, dt \quad \text{for all } G \in L^1(0,T;L^2(0,1))$$

and also, $|v|_{L^\infty(0,T;L^2(0,1))} = |S|_{(L^1(0,T;L^2(0,1)))'}$.
Then, v is the ultra weak solution of problem (II) and it satisfies :

(31) $$|v|_{L^\infty(0,T;L^2(0,1))} \le C\{|v^0|+|v^1|_{H^{-2}(0,1)}+|f_1|_{L^2(0,T)}+|f_2|_{L^2(0,T)}\}.$$

This completes the proof of the theorem.

Our objetive now is to know the regularity of the ultra weak solution of problem (II).

Lemma 3.9 *Let* $v^0 \in H_0^2(0,1)$, $v^1 \in L^2(0,1)$, and $\{f_1, f_2\} \in [H_0^2(0,1)]^2$. *Then problem (II) admits a unique solution*

$$v \in C([0,T]; H_0^2(0,1)) \cap C^1([0,T]; L^2(0,1)).$$

Proof. Let $\overline{v}(y,t) = y^2(y-1)f_2(t) + y(y-1)^2 f_1(t)$. Since $f_1, f_2 \in H_0^2(0,1)$ then $\overline{v} \in H_0^2(0,T; H_0^1(0,1) \cap H^4(0,1))$. Besides, we have

$$\overline{v}(0,t) = \overline{v}(1,t) = 0,\ \overline{v}_y(0,t) = f_1,\ \overline{v}_y(1,t) = f_2.$$

The solution of (II) will be $v = \widehat{v} + \overline{v}$ where

$$\widehat{v} \in C([0,T]; H_0^2(0,1)) \cap C^1([0,T]; L^2(0,1))$$

is the solution of

$$\left| \begin{array}{lll} L\widehat{v} = -L\overline{v} & \text{in} & Q \\ \widehat{v}(0,t) = \widehat{v}(1,t) = 0 & \text{in} &]0,T[\\ \widehat{v}_y(0,t) = \widehat{v}_y(1,t) = 0 & \text{in} &]0,T[\\ \widehat{v}(y,0) = v^0, \quad \widehat{v}_t(y,0) = v^1, & \text{in} &]0,1[. \end{array} \right.$$

This complete the proof of this lemma.

Theorem 3.10 *If* $v^0 \in L^2(0,1)$, $v^1 \in H^{-2}(0,1)$,$\{f_1, f_2\} \in [L^2(0,1)]^2$. *Then the ultra weak solution* v *of (II) is such that* $v \in C([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-2}(0,1))$.

Proof. Let be the sequences $(v_n^0) \in H_0^2(0,1)$, $(v_n^1) \in L^2(0,1)$, $\{f_{1n}, f_{2n}\} \in [H_0^2(0,1)]^2$, such that

$$v_n^0 \to v^0 \text{ in } L^2(0,1)$$

$$v_n^1 \to v^1 \text{ in } H^{-2}(0,1)$$

$$(f_{1n}, f_{2n}) \to (f_1, f_2) \text{ in } [L^2(0,T)]^2.$$

Consider the problem

$$\left| \begin{array}{l} Lv_n = 0 \text{ in } Q \\ v_n(0,t) = v_n(1,t) = 0 \text{ in }]0,T[\\ v_{ny}(0,t) = f_{1n}(t) \ , \ v_{ny}(1,t) = f_{2n}(t) \text{ in }]0,T[\\ v_n(0) = v_n^0 \ , \ v_{nt}(0) = v_n^1 \text{ in }]0,1[. \end{array} \right.$$

From Lemma 3.9, we deduce that $v_n \in C([0,T]; H^2(0,1)) \cap C^1([0,T]; L^2(0,1))$. By the linearity of the system above we have

$$\left| \begin{array}{l} Lv_n - Lv_m = 0 \\ (v_n - v_m)(0,t) = (v_n - v_m)(1,t) = 0 \\ (v_n - v_m)_y(0,t) = f_{1n}(t) - f_{1m}(t) \ , \ (v_n - v_m)_y(1,t) = f_{2n}(t) - f_{2m}(t) \\ (v_n - v_m)(0) = v_n^0 - v_m^0 \ , \ (v_{nt} - v_{mt})(0) = v_n^1 - v_m^1 \ . \end{array} \right.$$

From (31), it follows that

$$(32) \qquad |v_n - v_m|_{L^\infty(0,T;L^2(0,1))} \le C\Big(|v_n^0 - v_m^0| + |v_n^1 - v_m^1|_{H^{-2}(0,1)} + |f_{1n} - f_{1m}|_{L^2(0,T)} + |f_{2n} - f_{2m}|_{L^2(0,T)}\Big).$$

Since (v_n^0) , (v_n^1) , (f_{1n}^0) , (f_{2n}^1) are Cauchy sequences, it result from (32) that (v_n) is a Cauchy sequence in $C([0,T];L^2(0,1))$. Then, there exists $\widetilde{v} \in C([0,T];L^2(0,1))$, such that $v_n \to \widetilde{v}$ in $C([0,T];L^2(0,1))$. It is proved that $\widetilde{v}$ is the ultra weak solution of (II). By uniqueness, we conclude that $\widetilde{v} = v$ and so $v \in C^1([0,T];L^2(0,1))$. In analogous form it is verified that $v \in C^1([0,T];H^{-2}(0,1))$.

Now, the objetive is to prove the main result of exact control in the cylinder.

Proof of Theorem 2.2.

By the Theorems 3.2, 3.5 and 3.6, it is proved that for each pair $\{\varphi^0, \varphi^1\} \in H_0^2(0,1) \times L^2(0,1)$, there exists a unique solution φ of the mixed problem

$$(\mathrm{IV}) \quad \left| \begin{array}{lll} L^*\varphi = 0 & \text{in} & Q \\ \varphi(0,t) = \varphi(1,t) = 0 \ , \ \varphi_y(0,t) = \varphi_y(1,t) = 0 \ , & \text{in} &]0,T[\\ \varphi(y,0) = \varphi^0 \ , \ \varphi_t(y,0) = \varphi^1 & \text{in} &]0,1[\end{array} \right.$$

such that $\varphi \in C([0,T];H_0^2(0,1)) \cap C^1([0,T];L^2(0,1))$, $\varphi_{yy}(0,.), \varphi_{yy}(1,.) \in L^2(0,T)$.

On the other hand we have

$$(33) \qquad (T - T_0)C_0E(0) \le \int_0^T a(t)|\varphi_{yy}(1,t)|^2 dt + \int_0^T a(t)|\varphi_{yy}(0,t)|^2 dt \le C_1E(0).$$

With the function φ defined above we consider the following system :

$$
(\mathrm{V}) \quad \left| \begin{array}{ll}
L\psi = 0 & \text{in} \quad Q \\
\\
\psi(0,t) = \psi(1,t) = 0, \;\; \psi_y(0,t) = -\varphi_{yy}(0,t), & \\
\psi_y(1,t) = \varphi_{yy}(1,t) & \text{in} \quad]0,T[\\
\\
\psi(y,T) = 0, \;\; \psi_t(y,T) = 0 & \text{in} \quad]0,1[.
\end{array} \right.
$$

Considering $\xi = T - t$ e $\psi(y,t) = \widehat{\psi}(y,\xi)$ and using the Theorem 3.10, it is shown that there exists an ultra weak solution ψ of (V), such that $\psi \in C([0,T]; L^2(0,1) \cap C^1([0,T]; H^{-2}(0,1))$.
From the definition of the ultra weak solution we have the following identity :

$$
(34) \qquad \int_0^T a(t)|\varphi_{yy}(0,t)|^2 dt + \int_0^T a(t)|\varphi_{yy}(1,t)|^2 dt = -\langle \psi(0), \varphi^1 \rangle
$$

$$
-\langle \psi_t(0), \varphi^0 \rangle \;\; - \;\; 2\langle \frac{\alpha'(0) + \gamma'(0)y}{\gamma(0)} \psi_y(0), \varphi^0 \rangle.
$$

The identity (34) suggests the following operator :

$$
\Lambda : H_0^2(0,1) \times L^2(0,1) \to H^{-2}(0,1) \times L^2(0,1), \text{such that}
$$

$$
\Lambda\{\varphi^0, \varphi^1\} = \{\psi_t(0) - 2\frac{\alpha'(0) + \gamma'(0)y}{\gamma(0)} \psi_y(0), \psi(0)\}.
$$

Then

$$
(35) \quad \langle \Lambda\{\varphi^0, \varphi^1\}, \{\varphi^0, \varphi^1\} \rangle = \int_0^T a(t)|\varphi_{yy}(1,t)|^2 dt + \int_0^T a(t)|\varphi_{yy}(0,t)|^2 dt.
$$

From (H2), Remark 2.1 and remembering that $a(t) = \frac{1}{\gamma^4(t)}$, we have $(\gamma_1)^{-4} \leq a(t) \leq (\gamma_0)^{-4}$. From (31), we obtain

$$
(36) \quad (\gamma_0)^{-4}\{\int_0^T |\varphi_{yy}(1,t)|^2 dt + \int_0^T |\varphi_{yy}(0,t)|^2 dt\} \leq \langle \Lambda\{\varphi^0, \varphi^1\}, \{\varphi^0, \varphi^1\} \rangle
$$

$$
\leq (\gamma_1)^{-4}\{\int_0^T |\varphi_{yy}(1,t)|^2 dt + \int_0^T |\varphi_{yy}(0,t)|^2 dt\}.
$$

From (35) and (36) we conclude that Λ is an injective selfadjoint operator . Therefore, Λ is an isomorphism of $H_0^2(0,1)\times L^2(0,1)$ over $H^{-2}(0,1)\times L^2(0,1)$. Finally, considering $f_1(t) = -\varphi_{yy}(0,t)$ and $f_2(t) = \varphi_{yy}(1,t)$, $0 < t < T$, and initial data $\{v^0, v^1\} \in L^2(0,1) \times H^{-2}(0,1)$, problem (II) has an ultra weak solution v.
On the other side, considering problem (II) with $v^0 = \psi(0)$, $v^1 = \psi_t(0)$ where ψ is ultra weak solution of (V), it results that ψ is also ultra weak solution of (II). By uniqueness, it can be concluded that $v = \psi$. So from the system (V), we obtain $v(y,T) = 0$, $v_t(y,T) = 0$, which proves the Theorem 2.2.

4 The problem of exact controllability in noncylindrical domains

4.1 Weak solutions.

Let us consider the following problem :

$$
\text{(VI)}\quad \left|\begin{array}{lll}
\theta_{tt} + \theta_{xxxx} = \widehat{h} & \text{in} & \widehat{Q} \\[2mm]
\theta(\alpha(t),t) = \theta(\beta(t),t) = 0 & \text{in} &]0,T[\\[2mm]
\theta_x(\alpha(t),t) = \theta_x(\beta(t),t) = 0 & \text{in} &]0,T[\\[2mm]
\theta(x,0) = \theta^0, \quad \theta_t(x,0) = \theta^1 & \text{in} &]\alpha(0),\beta(0)[.
\end{array}\right.
$$

Definition 4.2 Let $\theta^0 \in H_0^2(\alpha(0),\beta(0))$, $\theta^1 \in L^2(\alpha(0),\beta(0))$, and $\widehat{h} \in L^1(0,T;L^2(\alpha(t),\beta(t)))$. We say that θ is a weak solution of problem (VI) if $\theta \in C([0,T];H_0^2(\alpha(t),\beta(t)))$, $\theta_t \in C([0,T];L^2(\alpha(t),\beta(t)))$ and satisfies

$$
\begin{aligned}
&-\int_0^T (\theta_t,\psi_t)_{L^2(\alpha(t),\beta(t))}dt + \int_0^T (\theta_{xx},\psi_{xx})_{L^2(\alpha(t),\beta(t))}dt \\
&= \int_0^T (\widehat{h},\psi)_{L^2(\alpha(t),\beta(t))}dt.
\end{aligned}
$$

for all $\psi \in L^2(0,T;H_0^2(\alpha(t),\beta(t)))$ and $\psi_t \in L^2(\widehat{Q})$ with $\psi(0) = \psi(T) = 0$ and initial data $\theta(0) = \theta^0$, $\theta_t(0) = \theta^1$.

Let us consider the following relations

(37) $u(x,t) = v(\frac{x-\alpha}{\gamma}, t)$

(38) $\theta(x,t) = \frac{1}{\gamma} w(\frac{x-\alpha}{\gamma}, t)$

(39) $g_1(t) = \frac{1}{\gamma} f_1(t)$

(40) $g_2(t) = \frac{1}{\gamma} f_2(t)$.

From (38) we conclude that problem (VI) is equivalent to the problem (III). Therefore by Theorem 3.2 we guarantee the existence of weak solutions of the problem (VI).
It can be noted that the previus results are also valid when we replace the initial conditions of problem (VI) by $\theta(x,T) = \theta^0$, $\theta_t(x,T) = \theta^1$.

4.2 Ultra weak solutions

Definition 4.3 Given $(u^0, u^1) \in L^2(\alpha(0), \beta(0)) \times H^{-2}(\alpha(0), \beta(0))$, $(g_1, g_2) \in [L^2(0,T)]^2$,we say that $u \in L^\infty(0,T; L^2(\alpha(t), \beta(t)))$ is an ultra weak solution of problem (I) if

$$\int_0^T (u, \widehat{h})_{L^2(\alpha(t),\beta(t))}\, dt = \langle u^1, \theta(0) \rangle_{H^{-2} \times H_0^2} - (u^0, \theta_t(0))_{L^2(\alpha(0),\beta(0))}$$
$$+ \int_0^T g_1(t) \theta_{xx}(\alpha(t), t)\, dt - \int_0^T g_2(t) \theta_{xx}(\beta(t), t)\, dt.$$

for all $\widehat{h} \in L^1(0,T; L^2(\alpha(t), \beta(t))$ where θ is a weak solution of problem (IV) with $\theta(x,T) = 0$, $\theta_t(x,T) = 0$ in the place of $(IV)_4$.
Using (37) we deduce that the ultra weak solution of problem (I) is equivalent to that of problem (II).

4.3 Proof of Theorem 2.1

As problems (I) e (II) are equivalent, like (III) and (IV), we can consider the following isomorphisms:
$G_1 : L^2(0,1) \times H^{-2}(0,1) \to L^2(\alpha(0), \beta(0)) \times H^{-2}(\alpha(0), \beta(0))$, such that $G_1\{v^0, v^1\} = \{u^0, u^1\}$, and
$G_2 : H_0^2(0,1) \times L^2(0,1) \to H_0^2(\alpha(0), \beta(0)) \times L^2(\alpha(0), \beta(0))$, such that $G_2\{w^0, w^1\} = \{\theta^0, \theta^1\}$.
Let us consider also the following isomorphisms :
$\sigma : L^2(0,1) \times H^{-2}(0,1) \to H^{-2}(0,1)) \times L^2(0,1)$, such that $\sigma\{v^0, v^1\} = \{v^1 - 2\frac{\alpha'(0)+\gamma'(0)y}{\gamma(0)} v_y^0 \ , \ -v^0\}$

$\Lambda: \quad H_0^2(0,1)\times L^2(0,1) \quad \to H^{-2}(0,1)\times L^2(0,1), \qquad$ such that
$\Lambda\{w^0, w^1\} = \{v^1 - \frac{2\alpha'(0)+\gamma'(0)y}{\gamma(0)} v_y^0,\ -v^0\}$.
Therefore, the composed function $\Lambda_1 = G_2 \circ \Lambda^{-1} \circ \sigma \circ G_1^{-1}$ is an isomorphism of $L^2(\alpha(0), \beta(0)) \times H^{-2}(\alpha(0), \beta(0))$ in $H_0^2(\alpha(0), \beta(0)) \times L^2(\alpha(0), \beta(0))$.
By that, for the initial data $\{u^0, u^1\} \in L^2(\alpha(0), \beta(0)) \times H^{-2}(\alpha(0), \beta(0))$, we determine $\{\theta^0, \theta^1\}$. Then with these data we find a weak solution to problem (IV). In a similar way with $\{w^0, w^1\} = G_2^{-1}\{\theta^0, \theta^1\}$, we find the weak solution w to problem (III).
Therefore we determine the ultra weak solution of the problem below:

$$
\text{(VII)} \quad \left| \begin{array}{lll}
Lv = 0 & \text{in} & Q \\
v(0,t) = v(1,t) = 0 & \text{in} &]0,T[\\
v_y(0,t) = -w_{yy}(0,t)\ ,\ v_y(1,t) = w_{yy}(1,t) & \text{in} &]0,T[\\
v(0) = v^0\ ,\ v_t(0) = v^1 & \text{in} &]0,1[
\end{array} \right.
$$

where $\{v^0, v^1\}$ and $\{w^0, w^1\}$ are related by the aplication Λ. Then by Theorem 2.2, $v(T) = v_t(T) = 0$.
Finally, from (37) we obtain that u satisfies the condition $u(T) = u_t(T) = 0$.

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